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DISCRETE COMPACTNESS FOR P-VERSION OF TETRAHEDRAL EDGE ELEMENTS

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Abstract. We consider the first family of $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming Nedéléc finite elements on tetrahedral meshes. Spectral approximation (p-version) is achieved by keeping the mesh fixed and raising the polynomial degree p uniformly in all mesh cells. We prove that the associated subspaces of discretely weakly divergence free piecewise polynomial vector fields enjoy a long conjectured discrete compactness property as $p \to \infty$. This permits us to conclude asymptotic spectral correctness of spectral Galerkin finite element approximations of Maxwell eigenvalue problems.

Key words. Edge elements, Maxwell eigenvalue problem, discrete compactness, Poincaré lifting, projection based interpolation

AMS subject classifications. 65N30, 65N25, 78M10

1. Introduction. Identifying spectrally correct conforming Galerkin approximations of the Maxwell eigenvalue problem [15]: seek[1] $\mathbf{u} \in \boldsymbol{H}(\mathbf{curl},\Omega)$ and $\omega > 0$, such that

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}\right)_{L^2(\Omega)} = \omega^2\left(\boldsymbol{\epsilon}\mathbf{u},\mathbf{v}\right)_{L^2(\Omega)} \quad \forall \mathbf{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\;, \tag{1.1}$$

($\boldsymbol{\epsilon},\boldsymbol{\mu} \in (L^\infty(\Omega))^{3,3}$ uniformly positive definite material tensors) has turned out to be a highly inspiring challenge in numerical analysis. Obviously, eigenfunctions of (1.1) belong to $\boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}_0(\mathrm{div}_{\boldsymbol{\epsilon}}\,0,\Omega)$ and the compact embedding $\boldsymbol{L}^2(\Omega) \hookrightarrow \boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}_{\boldsymbol{\epsilon}}\,0,\Omega)$ [30] relates (1.1) to an eigenvalue problem for a compact selfadjoint operator. However, asymptotically dense families of finite elements in $\boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}_{\boldsymbol{\epsilon}}\,0,\Omega)$ are not known in general.

Let us assume that a merely $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming family $\left(\mathcal{W}^1_p(\mathcal{M})\right)_{p\in\mathbb{N}}$ of finite dimensional trial and test spaces $\mathcal{W}^1_p(\mathcal{M}) \subset \boldsymbol{H}(\mathbf{curl},\Omega)$ for (1.1) is employed for the Galerkin discretization of (1.1). The corresponding discrete eigenfunctions $\mathbf{u}_p \in \mathcal{W}^1_p(\mathcal{M})$, if they exist, will satisfy

$$\mathbf{u}_p \in \mathcal{X}^1_p(\mathcal{M}) := \{\mathbf{w}_p \in \mathcal{W}^1_p(\mathcal{M}) : (\boldsymbol{\epsilon}\mathbf{w}_p,\mathbf{v}_p)_{L^2(\Omega)} = 0\; \forall \mathbf{v}_p \in \mathrm{Ker}(\mathbf{curl}) \cap \mathcal{W}^1_p(\mathcal{M})\}\;. \tag{1.2}$$

We cannot expect $\mathcal{X}^1_p(\mathcal{M}) \subset \boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}_0(\mathrm{div}_{\boldsymbol{\epsilon}}\,0,\Omega)$ and, thus, a standard Galerkin approximation of (1.1) boils down to an outer approximation of the eigenvalue problem. Good approximation properties of the finite element space no longer automatically translate into convergence of eigenvalues and eigenfunctions. As investigated Caorsi, Fernandes and Rafetto in [12], an array of other requirements has to be met by the finite element spaces, the most prominent of which is the *discrete compactness property* [3].

DEFINITION 1.1. *The* discrete compactness property *holds for an asymptotically dense family* $\left(\mathcal{W}^1_p(\mathcal{M})\right)_{p\in\mathbb{N}}$ *of finite dimensional subspaces of* $\boldsymbol{H}(\mathbf{curl},\Omega)$*, if any*

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[1]We use the customary notations for Sobolev spaces like $H^s(\Omega)$, $\boldsymbol{H}(\mathbf{curl},\Omega)$, $\boldsymbol{H}(\mathrm{div},\Omega)$, etc., and write $\boldsymbol{H}(\mathbf{curl}\,0,\Omega)$, $\boldsymbol{H}(\mathbf{curl}\,0,\Omega)$, etc., for the kernels of differential operators. The reader is referred to [24, § I.2] and [29, Sect. 2.4] for more information.

bounded *sequence in* $\mathcal{X}_p^1(\mathcal{M}) \subset \boldsymbol{H}(\mathbf{curl}, \Omega)$ *contains a subsequence that* converges in $\boldsymbol{L}^2(\Omega)$.

The same notion applies in the case of homogeneous Dirichlet boundary conditions, when (1.1) is considered in $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$. In this case the eigenfunctions will belong to $\boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}_{\boldsymbol{\epsilon}}\, 0, \Omega)$ and zero tangential trace on $\partial\Omega$ has to be imposed on trial and test functions.

The discrete compactness property of $\mathcal{W}_p^1(\mathcal{M})$ is key to establishing spectral correctness and asymptotic optimality of Galerkin approximations of (1.1), see [11,12,15] for details. Small wonder, that substantial effort has been spent on proving this property for various asymptotically dense families of $\boldsymbol{H}(\mathbf{curl}, \Omega)/\boldsymbol{H}_0(\mathbf{curl}, \Omega)$-conforming finite elements. For the h-version of Nedéléc's edge elements Kikuchi [31–33] accomplished the first proof, which was later generalized in [7,23,36], see [35, Sect. 7.3.2], [29, Sect. 4], and [15] for a survey. Conversely, spectral edge element schemes in 3D have long defied all attempts to prove their discrete compactness property, though they perform well for Maxwell eigenvalue problems [15, 17, 39]. Partial success was reported for edge elements in 2D: In [9] the analysis of the discrete compactness property for triangular hp finite elements has been tackled, but the proof of the main result relied on a conjectured L^2 estimate, which had only been demonstrated numerically. The first fully rigorous analysis of 2D hp edge elements on rectangles was devised in [8].

In [29, Remark 15] an interpolation estimate was identified as crucial missing step in the analysis. Since then, two major advances have paved the way for closing the gap:

1. In [16] M. Costabel and M. McIntosh discovered a construction of $\boldsymbol{H}^1(\Omega)$-stable vector potentials by means of a smoothed Poincaré mapping. This will be reviewed in Sect. 2 of the present paper.
2. In the breakthrough paper [21] L. Demkowicz and A. Buffa achieved a comprehensive analysis of commuting projection based interpolation operators. To maintain the article self-contained, their approach will be explained in Sect. 4 and their interpolation error estimates will be presented in Sect. 5.

In addition, we exploit the possibility to construct high order versions of Nedéléc's first family of edge elements [38] by using Cartan's Poincaré map [4,27–29], see Sect. 3 for details. Another important tool are stable polynomial preserving extension operators developed, for example, in [1, 5, 22, 37]. In addition, we heavily rely on spectral polynomial approximation estimates, see [6, 37, 40].

Thus, standing on the shoulders of giants and combining all these profound theories of numerical analysis, this article manages to give the first proof for the discrete compactness property of the p-version for the first family of Nedéléc's edge elements on tetrahedral meshes of Lipschitz polyhedra Ω, consult Sect. 6 for the proof.

Theorem 1.2. *The sequence* $\left(\mathcal{W}_p^1(\mathcal{M})\right)_{p\in\mathbb{N}}$ *of trial spaces generated by the p-version of the first family of Nedéléc's* $\boldsymbol{H}(\mathbf{curl}, \Omega)$*- or* $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$*-conforming finite elements on a fixed tetrahedral mesh* $\mathcal{M}$ *of a bounded Lipschitz polyhedron* $\Omega \subset \mathbb{R}^3$ *satisfies the discrete compactness property.*

The idea of the proof is to inspect the $L^2(\Omega)$-orthogonal Helmholtz decomposition [24, § I.3]

$$\mathbf{w}_p = \widetilde{\mathbf{w}}_p \oplus_{L^2} \mathbf{w}_p^0 \,, \quad \mathbf{w}_p^0 \in \mathrm{Ker}(\mathbf{curl}) \,, \tag{1.3}$$

whose so-called solenoidal components $\widetilde{\mathbf{w}}_p$ belong to $\boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}\, 0, \Omega)$. The above mentioned compact embedding guarantees the existence of a subsequence of $(\widetilde{\mathbf{w}}_p)_{p\in\mathbb{N}}$ that converges in $\boldsymbol{L}^2(\Omega)$. Hence, it "merely" takes to show $\|\widetilde{\mathbf{w}}_p - \mathbf{w}_p\|_{L^2(\Omega)} \rightarrow$

0 for $p \to \infty$ in order to establish discrete compactness. Clever use of projection operators that enjoy a commuting diagram property, converts this task to a uniform interpolation estimate. The core of this paper is devoted to this seemingly humble program.

REMARK 1.1. *Generalizations of Thm. 1.2 to other families of tetrahedral edge elements, and corresponding hp-finite element schemes are straightforward [8]. For the sake of readability, these extensions will not be pursued in the present paper.*

Since the Poincaré map does not fit a tensor product structure, extending the results of this paper to 3D hexahedral edge elements will take some new ideas.

2. Poincaré lifting. Let $D \subset \mathbb{R}^3$ stand for a bounded domain that is star-shaped with respect to a subdomain $B \subset D$, that is,

$$\forall \boldsymbol{a} \in B,\, \boldsymbol{x} \in D: \quad \{t\boldsymbol{a} + (1-t)\boldsymbol{x},\ 0 < t < 1\} \subset D\,. \tag{2.1}$$

DEFINITION 2.1. *The* Poincaré lifting[2] $\mathsf{R}_{\boldsymbol{a}} : \boldsymbol{C}^0(\overline{\Omega}) \mapsto \boldsymbol{C}^0(\overline{\Omega})$, $\boldsymbol{a} \in B$, *is defined as*

$$\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})(\boldsymbol{x}) := \int_0^1 t\mathbf{u}(\boldsymbol{x} + t(\boldsymbol{x} - \boldsymbol{a}))\,\mathrm{d}t \times (\boldsymbol{x} - \boldsymbol{a})\,, \quad \boldsymbol{x} \in D\,, \tag{2.2}$$

where $\times$ designates the cross product of two vectors in $\mathbb{R}^3$.

This is a special case of the generalized path integral formula for differential forms, which is instrumental in proving the exactness of closed forms on star-shaped domains, the so-called "Poincaré lemma", see [13, Sect. 2.13].

The linear mapping $\mathsf{R}_{\boldsymbol{a}}$ provides a right inverse of the **curl**-operator on divergence-free vectorfields, see [25, Prop. 2.1] for the simple proof, and [13, Sect. 2.13] for a general proof based on differential forms.

LEMMA 2.2. *If* $\operatorname{div} \mathbf{u} = 0$, *then, for any* $\boldsymbol{a} \in B$, $\mathbf{curl}\,\mathsf{R}_{\boldsymbol{a}}\mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in \boldsymbol{C}^1(\overline{\Omega})$.

Unfortunately, the mapping $\mathsf{R}_{\boldsymbol{a}}$ cannot be extended to a continuous mapping $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$, *cf.* [25, Thm. 2.1]. As discovered in the breakthrough paper [16] based on earlier work of Bogovskiĭ [10], it takes a smoothed version to accomplish this: we introduce the *smoothed Poincaré lifting* [3]

$$\mathsf{R}(\mathbf{u}) := \int_B \Phi(\boldsymbol{a})\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})\,\mathrm{d}\boldsymbol{a}\,, \tag{2.3}$$

where

$$\Phi \in C^\infty(\mathbb{R}^3)\,, \quad \operatorname{supp}\Phi \subset B\,, \quad \int_B \Phi(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a} = 1\,. \tag{2.4}$$

The substitution

$$\boldsymbol{y} := \boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}) \quad , \quad \tau := \frac{1}{1-t}\,, \tag{2.5}$$

transforms the integral (2.4) into

$$\begin{aligned}\mathsf{R}(\mathbf{u})(\boldsymbol{x}) &= \int\limits_{\mathbb{R}^3}\int\limits_1^\infty \tau(1-\tau)\mathbf{u}(\boldsymbol{y}) \times (\boldsymbol{x} - \boldsymbol{y})\Phi(\boldsymbol{y} + \tau(\boldsymbol{y} - \boldsymbol{x}))\,\mathrm{d}\tau\mathrm{d}\boldsymbol{y}\\ &= \int\limits_{\mathbb{R}^3} \mathbf{k}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}) \times \mathbf{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,,\end{aligned} \tag{2.6}$$

[2] Bold symbols will generally be used to tag vector valued functions and spaces of such.
[3] The dependence of R on Φ is dropped from the notation.

that is, R is a convolution-type integral operator with kernel

$$\begin{aligned}\mathbf{k}(\boldsymbol{x},\boldsymbol{z}) &= \int_1^\infty \tau(1+\tau)\Phi(\boldsymbol{x}+\tau\boldsymbol{z})\boldsymbol{z}\,\mathrm{d}\tau \\ &= \frac{\boldsymbol{z}}{|\boldsymbol{z}|^2}\int_1^\infty \zeta\Phi(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|})\,\mathrm{d}\zeta + \frac{\boldsymbol{z}}{|\boldsymbol{z}|^3}\int_1^\infty \zeta^2\Phi(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|})\,\mathrm{d}\zeta\ .\end{aligned} \tag{2.7}$$

The kernel can be bounded by $|\mathbf{k}(\boldsymbol{x},\boldsymbol{z})| \leq K(\boldsymbol{x})|\boldsymbol{z}|^{-2}$, where $K \in C^\infty(\mathbb{R}^3)$ depends only on Φ and is locally uniformly bounded. As a consequence, (2.6) exists as an improper integral.

The intricate but elementary analysis of [16, Sect. 3.3] further shows, that $\mathbf{k}$ belongs to the Hörmander symbol class $S^{-1}_{1,0}(\mathbb{R}^3)$, see [41, Ch. 7]. Invoking the theory of pseudo-differential operators [41, Prop. 5.5] we obtain the following following continuity result, which is a special case of [16, Cor. 3.4]

THEOREM 2.3. *The mapping* R *can be extended to a* continuous *linear operator* $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$*, which is still denoted by* R*. It satisfies*

$$\mathbf{curl}\,\mathsf{R}\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \boldsymbol{H}(\operatorname{div} 0, D)\ . \tag{2.8}$$

The smoothed Poincaré lifting shares this continuity property with many other mappings, see [29, Sect. 2.4]. Yet, it enjoys another essential feature, which is immediate from its definition (2.2): R maps polynomials of degree p to other polynomials of degree $\leq p+1$. The next section will highlight the significance of this observation.

3. Tetrahedral edge elements. In [38] Nedéléc introduced a family of $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming, that is, tangentially continuous, finite element spaces. On a tetrahedral triangulation $\mathcal{M}$ of Ω, the corresponding finite element spaces of degree p are given by

$$\begin{aligned}\mathcal{W}^1_p(\mathcal{M}) &:= \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl},\Omega):\ \mathbf{v}_{|T} \in \mathcal{W}^1_p(T)\ \forall T \in \mathcal{M}\}\ ,\\ \mathcal{W}^1_p(T) &:= \{\mathbf{v} \in \boldsymbol{C}^\infty(T):\ \mathbf{v}(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + \mathbf{q}(\boldsymbol{x}) \times \boldsymbol{x},\ \mathbf{p},\mathbf{q} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3),\ \boldsymbol{x} \in T\}\ .\end{aligned}$$

We wrote $\mathcal{P}_p(\mathbb{R}^3)$ for the space of 3-variate polynomials of total degree $\leq p$, $p \in \mathbb{N}_0$, and the bold symbol $\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)$ for vectorfields with three components in $\mathcal{P}_p(\mathbb{R}^3)$. To emphasize that polynomials on a tetrahedron T are being considered, we may use the notations $\mathcal{P}_p(T)/\boldsymbol{\mathcal{P}}_p(T)$ instead of $\mathcal{P}_p(\mathbb{R}^3)/\boldsymbol{\mathcal{P}}(\mathbb{R}^3)$. We also adopt the convention that $\mathcal{P}_p(\mathbb{R}^3) = \{0\}$, if $p < 0$. Another relevant polynomial space is

$$\boldsymbol{\mathcal{P}}_p(\operatorname{div} 0,\mathbb{R}^3) := \{\mathbf{q} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3):\ \operatorname{div}\mathbf{q} = 0\}\ . \tag{3.1}$$

Deep insights can be gained by regarding edge elements as discrete 1-forms. This provides a very elegant construction of higher order edge element spaces and immediately reveals their relationships with standard Lagrangian finite elements and $\boldsymbol{H}(\operatorname{div},\Omega)$-conforming face elements (see below). In particular, the Poincaré lifting becomes a powerful tool for building discrete differential forms of high polynomial degree. This is explored in [27,28], [29, Sect. 3.4], and [4, Sect. 1.4] in arbitrary dimension, using the calculus of differential forms. In this article we prefer to stick to the classical calculus of vector analysis, because we are only concerned with 3D. We hope, that, thus, the presentation will be more accessible to an audience of numerical analysts. Yet, the differential forms background has inspired our notations: integer superscripts label spaces and operators related to differential forms. For instance, $\mathcal{W}_p^1(\mathcal{M})$ can be read as a space of discrete 1-forms.

According to [27, Sect. 3], for any $T \in \mathcal{M}$, $\boldsymbol{a} \in T$, we can obtain the local space as

$$\mathcal{W}_p^1(T) = \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) + \mathsf{R}_{\boldsymbol{a}}\big(\boldsymbol{\mathcal{P}}_p(\operatorname{div} 0, \mathbb{R}^3)\big) \; . \tag{3.2}$$

Independence of $\boldsymbol{a}$ is discussed in [27, Sect. 3]. The representation (3.2) can be established by dimensional arguments: from the formula (2.2) for the Poincaré lifting we immediately see that $\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)+\mathsf{R}_{\boldsymbol{a}}(\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)) \subset \mathcal{W}_p^1(T)$. In addition, from [38, Lemma 4] and [27, Thm. 6, case $l = 1$, $n = 3$] we learn that the dimensions of both spaces agree and are equal to

$$\dim \mathcal{W}_p^1(T) = \tfrac{1}{2}(1+p)(3+p)(4+p) \; . \tag{3.3}$$

As a consequence, the two finite dimensional spaces must agree.

For the remainder of this section, which focuses on local spaces, we single out a tetrahedron $T \in \mathcal{M}$. On T we can introduce a smoothed Poincaré lifting R_T according to (2.3) with $B = T$ and a suitable $\Phi \in C_0^\infty(T)$ complying with (2.4). An immediate consequence of (3.2) is that

$$\mathsf{R}_T\big(\boldsymbol{\mathcal{P}}_p(\operatorname{div} 0, \mathbb{R}^3)\big) \subset \mathcal{W}_p^1(T) \; . \tag{3.4}$$

We introduce the notation $\mathcal{F}_m(T)$ for the set of all m-dimensional facets of T, $m = 0, 1, 2, 3$. Hence, $\mathcal{F}_0(T)$ contains the vertices of T, $\mathcal{F}_1(T)$ the edges, $\mathcal{F}_2(T)$ the faces, and $\mathcal{F}_3(T) = \{T\}$. Moreover, for some $F \in \mathcal{F}_m(T)$, $m = 1, 2, 3$, $\mathcal{P}_p(F)$ denotes the space of m-variate polynomials of total degree $\leq p$ in a local coordinate system of the facet F, and $\boldsymbol{\mathcal{P}}_p(F)$ will designate corresponding tangential polynomial vectorfields. Further, we write

$$\mathcal{W}_p^1(e) = \mathcal{W}_p^1(T) \cdot \boldsymbol{t}_e \; , \quad \boldsymbol{t}_e \text{ the unit tangent vector of } e,\; e \in \mathcal{F}_1(T) \; , \tag{3.5}$$

$$\mathcal{W}_p^1(f) = \mathcal{W}_p^1(T) \times \boldsymbol{n}_f \; , \quad \boldsymbol{n}_f \text{ the unit normal vector of } f,\; f \in \mathcal{F}_2(T) \; , \tag{3.6}$$

for the tangential traces of local edge element vectorfields onto edges and faces. Simple vector analytic manipulations permit us to deduce from (3.2) that

$$\mathcal{W}_p^1(e) = \mathcal{P}_p(e) \; , \quad e \in \mathcal{F}_1(T) \; , \tag{3.7}$$

$$\mathcal{W}_p^1(f) = \boldsymbol{\mathcal{P}}_p(f) + \mathsf{R}_{\boldsymbol{a}}^{2D}(\mathcal{P}_p(f)) \; , \quad \boldsymbol{a} \in f \; , \quad f \in \mathcal{F}_2(T) \; , \tag{3.8}$$

where the projection $\mathsf{R}_{\boldsymbol{a}}^{2D}$ of the Poincaré lifting in the plane reads

$$\mathsf{R}_{\boldsymbol{a}}^{2D}(u)(\boldsymbol{x}) := \int_0^1 t u(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}))(\boldsymbol{x} - \boldsymbol{a}) \, \mathrm{d}t \; , \quad \boldsymbol{a} \in \mathbb{R}^2 \; . \tag{3.9}$$

It satisfies $\operatorname{div}_\Gamma \mathsf{R}_{\boldsymbol{a}}^{2D}(u) = u$ for all $u \in C^\infty(\mathbb{R}^2)$. We point out that, along with (3.2), the formulas (3.7) and (3.8) are special versions of the general representation formula for discrete 1-forms, see [27, Formula (16)]. Special facet tangential trace spaces will also be needed:

$$\mathring{\mathcal{W}}_p^1(e) := \{u \in \mathcal{W}_p^1(e) : \int_e u \, \mathrm{d}l = 0\} \; , \quad e \in \mathcal{F}_1(T) \; , \tag{3.10}$$

$$\mathring{\mathcal{W}}_p^1(f) := \{\mathbf{u} \in \mathcal{W}_p^1(f) : \mathbf{u} \cdot \boldsymbol{n}_{e,f} \equiv 0 \; \forall e \in \mathcal{F}_1(T),\; e \subset \partial f\} \; , \quad f \in \mathcal{F}_2(T) \; , \tag{3.11}$$

$$\mathring{\mathcal{W}}_p^1(T) := \{\mathbf{u} \in \mathcal{W}_p^1(T) : \mathbf{u} \times \boldsymbol{n}_f \equiv 0 \; \forall f \in \mathcal{F}_2(T)\} \; . \tag{3.12}$$

Here $\boldsymbol{n}_f$ represents an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ the in plane normal of a face w.r.t. an edge $e \subset \partial f$.

According to [38, Sect. 1.2] and [27, Sect. 4], the local degrees of freedom for $\mathcal{W}^1_p(T)$ are given by the first $p-2$ vectorial moments on the cells of $\mathcal{M}$, the first $p-1$ vectorial moments of the tangential components on the faces of $\mathcal{M}$ and the first p tangential moments along the edges of T, see (3.14) for concrete formulas. Then the set $\mathrm{dof}^1_p(T)$ can be partitioned as

$$\mathrm{dof}^1_p(T) = \bigcup_{e\in\mathcal{F}_1(T)} \mathrm{ldf}^1_p(e) \ \cup \bigcup_{f\in\mathcal{F}_2(T)} \mathrm{ldf}^1_p(f) \ \cup \ \mathrm{ldf}^1_p(T) \ , \tag{3.13}$$

where the functionals in $\mathrm{ldf}^1_p(e)$, $\mathrm{ldf}^1_p(f)$, and $\mathrm{ldf}^1_p(T)$ are supported on an edge, face, and T, respectively, and read

$$\begin{aligned} \kappa \in \mathrm{ldf}^1_p(e) \quad &\Rightarrow \quad \kappa(\mathbf{u}) = \textstyle\int_e p\boldsymbol{\xi}\cdot\boldsymbol{t}_e \,\mathrm{d}l && \text{for } e\in\mathcal{F}_1(T), \text{ suitable } p\in\mathcal{P}_p(e)\ , \\ \kappa \in \mathrm{ldf}^1_p(f) \quad &\Rightarrow \quad \kappa(\mathbf{u}) = \textstyle\int_f \mathbf{p}\cdot(\boldsymbol{\xi}\times\mathbf{n})\,\mathrm{d}S && \text{for } f\in\mathcal{F}_2(T), \text{ suitable } \mathbf{p}\in\boldsymbol{\mathcal{P}}_{p-1}(f)\ , \\ \kappa \in \mathrm{ldf}^1_p(T) \quad &\Rightarrow \quad \kappa(\mathbf{u}) := \textstyle\int_T \mathbf{p}\cdot\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x} && \text{for suitable } \mathbf{p}\in\boldsymbol{\mathcal{P}}_{p-2}(T)\ . \end{aligned} \tag{3.14}$$

These functionals are unisolvent on $\mathcal{W}^1_p(T)$ and locally fix the tangential trace of $\mathbf{u}\in\mathcal{W}^1_p(T)$. There is a splitting of $\mathcal{W}^1_p(T)$ dual to (3.13): Defining

$$\mathcal{Y}^1_p(F) := \{\mathbf{v}\in\mathcal{W}^1(T):\ \kappa(\mathbf{v}) = 0\ \forall\kappa\in\mathrm{dof}^1_p(T)\setminus\mathrm{ldf}^1_p(F)\} \tag{3.15}$$

for $F\in\mathcal{F}_m(T)$, $m=1,2,3$, we find the direct sum decomposition

$$\mathcal{W}^1_p(T) = \sum_{m=1}^{3}\sum_{F\in\mathcal{F}_m(T)} \mathcal{Y}^1_p(F)\ . \tag{3.16}$$

In addition, note that the tangential trace of $\mathbf{u}\in\mathcal{X}^1_p(F)$ vanishes on all facets $\neq F$, whose dimension is smaller or equal the dimension of F. By the unisolvence of $\mathrm{dof}^1_p(T)$, there are bijective linear *extension operators*

$$\mathsf{E}^1_{e,p} : \mathcal{W}^1_p(e) \mapsto \mathcal{Y}^1_p(e)\ , \quad e\in\mathcal{F}_1(T)\ , \tag{3.17}$$

$$\mathsf{E}^1_{f,p} : \overset{\circ}{\mathcal{W}}{}^1_p(f) \mapsto \mathcal{Y}^1_p(f)\ , \quad f\in\mathcal{F}_2(T)\ . \tag{3.18}$$

The **curl** connects the edge element spaces $\mathcal{W}^1_p(\mathcal{M})$ and the so-called face element spaces of discrete 2-forms [38, Sect. 1.3]

$$\begin{aligned} \mathcal{W}^2_p(\mathcal{M}) &:= \{\mathbf{v}\in\boldsymbol{H}(\mathrm{div},\Omega):\ \mathbf{v}_{|T}\in\mathcal{W}^2_p(T)\ \forall T\in\mathcal{M}\}\ , \\ \mathcal{W}^2_p(T) &:= \{\mathbf{v}\in\boldsymbol{C}^\infty(T):\ \mathbf{v}(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + q(\boldsymbol{x})\boldsymbol{x},\ \mathbf{p}\in\boldsymbol{\mathcal{P}}_p(T),\ q\in\mathcal{P}_p(T)\}\ . \end{aligned}$$

An alternative representation of the local face element space is [27, Formula (16) for $l=2$, $n=3$]

$$\mathcal{W}^2_p(T) = \boldsymbol{\mathcal{P}}_p(T) + \mathsf{D}_{\boldsymbol{a}}(\mathcal{P}_p(T))\ , \tag{3.19}$$

where the appropriate version of the Poincaré lifting reads

$$(\mathsf{D}_{\boldsymbol{a}}u)(\boldsymbol{x}) := \int_0^1 t^2 u(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))(\boldsymbol{x}-\boldsymbol{a})\,\mathrm{d}t\ , \quad \boldsymbol{a}\in T\ . \tag{3.20}$$

Like (3.2) this is a special incarnation of the general formula (16) in [27]. Again, dimensional arguments based on [38, Sect. 1.3] and [27, Thm. 6] confirm the representation (3.20). We remark that $\operatorname{div} \mathsf{D}_{\boldsymbol{a}} u = u$, see [25, Prop. 1.2].

The normal trace space of $\mathcal{W}_p^2(T)$ onto a face is

$$\mathcal{W}_p^2(f) := \mathcal{W}_p^2(T) \cdot \boldsymbol{n}_f = \mathcal{P}_p(f) \;, \quad f \in \mathcal{F}_2(T) \;, \tag{3.21}$$

and as relevant space "with zero trace" we are going to need

$$\overset{\circ}{\mathcal{W}}{}_p^2(f) := \{ u \in \mathcal{W}_p^2(f) : \int_f u \, \mathrm{d}S = 0 \} \;, \quad f \in \mathcal{F}_2(T) \;, \tag{3.22}$$

$$\overset{\circ}{\mathcal{W}}{}_p^2(T) := \{ \mathbf{u} \in \mathcal{W}_p^2(T) : \; \mathbf{u} \cdot \boldsymbol{n}_{\partial T} = 0 \} \;. \tag{3.23}$$

The connection between the local spaces $\mathcal{W}_p^1(T)$, $\mathcal{W}_p^2(T)$ and full polynomial spaces is established through a local discrete DeRham exact sequence: To elucidate the relationship between differential operators and various traces onto faces and edges, we also include those in the statement of the following theorem. There $\boldsymbol{n}_f$ stands for an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ for the in plane normal of a face w.r.t. an edge $e \subset \partial f$, and $\frac{d}{dl}$ is the differentiation w.r.t. arclength on an edge.

THEOREM 3.1. *For $f \in \mathcal{F}_2(T)$, $e \in \mathcal{F}_1(T)$, $e \subset \partial f$, all the sequences in*

$$\begin{tikzcd}
\text{const} \arrow[r, "\mathsf{Id}"] & \mathcal{P}_{p+1}(T) \arrow[r, "\mathbf{grad}"] \arrow[d, "\cdot_{|f}"'] & \mathcal{W}_p^1(T) \arrow[r, "\mathbf{curl}"] \arrow[d, "\cdot \times \boldsymbol{n}_{f\,|f}"'] & \mathcal{W}_p^2(T) \arrow[r, "\operatorname{div}"] \arrow[d, "\cdot \boldsymbol{n}_{f\,|f}"] & \mathcal{P}_p(T) \arrow[r, "\mathsf{Id}"] & \{0\} \\
\text{const} \arrow[r, "\mathsf{Id}"] & \mathcal{P}_{p+1}(f) \arrow[r, "\mathbf{curl}_\Gamma"] \arrow[d, "\cdot_{|e}"'] & \mathcal{W}_p^1(f) \arrow[r, "\operatorname{div}_\Gamma"] \arrow[d, "\cdot \boldsymbol{n}_{e,f\,|e}"'] & \mathcal{P}_p(f) \arrow[r, "\mathsf{Id}"] & \{0\} & \\
\text{const} \arrow[r, "\mathsf{Id}"] & \mathcal{P}_{p+1}(e) \arrow[r, "\frac{d}{dl}"] & \mathcal{P}_p(e) \arrow[r, "\mathsf{Id}"] & \{0\} & &
\end{tikzcd}$$

are exact and the diagram commutes.

Proof. The assertion about the top exact sequence is an immediate consequence of representations (3.2) and (3.19) and the relationships

$$\mathbf{curl}\, \mathsf{R}_{\boldsymbol{a}}(\mathbf{u}) = \mathbf{u} \quad \forall \mathbf{u} \in \boldsymbol{\mathcal{P}}_p(\operatorname{div} 0, T) \;, \quad \operatorname{div} \mathsf{D}_{\boldsymbol{a}}(u) = u \quad \forall u \in \mathcal{P}_p(T) \;.$$

For further discussions and the proof of the other exact sequence properties see [27, Sect. 5 for $n = 3$]. ☐

4. Projection based interpolation. The degrees of freedom introduced above define local finite element projectors onto $\mathcal{W}_p^1(T)$. In conjunction with suitably defined interpolation operators for degree p Lagrangian finite elements, they possess a very desirable commuting diagram property [27, Thm. 13], which will be explained below. However, they do not enjoy favorable continuity properties with increasing p. Thus, L. Demkowicz [19–21], taking the cue from the theory of p-version Lagrangian finite elements, invented an alternative in the form of local projection based interpolation.

4.1. Projections, liftings, and extensions. Again, consider a single tetrahedron $T \in \mathcal{M}$ and fix the polynomial degree $p \in \mathbb{N}$. Following the developments of [29, Sect. 3.5], projection based interpolation requires building blocks in the form of local *orthogonal* projections P^l_* and liftings L^l_*[4]. Some operators will depend on a regularity parameter $0 < \epsilon < \frac{1}{2}$, which is considered fixed below and will be specified in Sect. 5. To begin with, we define for every $e \in \mathcal{F}_1(T)$

$$\mathsf{P}^1_{e,p} : H^{-1+\epsilon}(e) \mapsto \frac{d}{dl}\mathring{\mathcal{P}}_{p+1}(e) = \mathring{\mathcal{W}}^1_p(e) \tag{4.1}$$

as the $H^{-1+\epsilon}(e)$-orthogonal projection. Here, $\mathring{\mathcal{P}}_p(F)$ denotes the space of degree p polynomials on a facet F that vanish on ∂F.

Similarly, for every face $f \in \mathcal{F}_2(T)$ introduce

$$\mathsf{P}^1_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathbf{curl}_\Gamma \mathring{\mathcal{P}}_{p+1}(f) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.2}$$
$$\mathsf{P}^2_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathrm{div}_\Gamma \mathring{\mathcal{W}}^1_p(f) = \mathring{\mathcal{W}}^2_p(f)\ , \tag{4.3}$$

as the corresponding $\boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f)$-orthogonal projections. Eventually, let

$$\mathsf{P}^1_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{grad}\, \mathring{\mathcal{P}}_{p+1}(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\}\ , \tag{4.4}$$
$$\mathsf{P}^2_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{curl}\, \mathring{\mathcal{W}}^1_p(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^2_p(T) :\ \mathrm{div}\, \mathbf{v} = 0\}\ , \tag{4.5}$$
$$\mathsf{P}^3_{T,p} : L^2(T) \mapsto \mathrm{div}\, \mathring{\mathcal{W}}^2_p(T) = \{v \in \mathcal{P}_p(T) :\ \int_T v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0\}\ , \tag{4.6}$$

stand for the respective $L^2(T)$-orthogonal projections.

The lifting operators

$$\mathsf{L}^1_{e,p} : \mathring{\mathcal{W}}^1_p(e) \mapsto \mathring{\mathcal{P}}_{p+1}(e)\ , \quad e \in \mathcal{F}_1(T)\ , \tag{4.7}$$
$$\mathsf{L}^1_{f,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(f)\ , \quad f \in \mathcal{F}_2(T)\ , \tag{4.8}$$
$$\mathsf{L}^1_{T,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(T)\ , \tag{4.9}$$

are uniquely defined by requiring

$$\frac{d}{dl}\mathsf{L}^1_{e,p} u = u \quad \forall u \in \mathring{\mathcal{W}}^1_p(e)\ , \tag{4.10}$$
$$\mathbf{curl}_\Gamma \mathsf{L}^1_{f,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.11}$$
$$\mathbf{grad}\, \mathsf{L}^1_{T,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\}\ . \tag{4.12}$$

Another class of liftings provides right inverses for **curl** and div_Γ: Pick a face $f \in \mathcal{F}_2(T)$, and, without loss of generality, assume the vertex opposite to the edge $\widetilde{e}$ to coincide with 0. Then define

$$\mathsf{L}^2_{f,p} : \left\{ \begin{array}{ccl} \mathrm{div}_\Gamma \mathring{\mathcal{W}}^1_p(f) & \mapsto & \mathring{\mathcal{W}}^1_p(f) \\ u & \mapsto & \mathsf{R}^{2D}_0 u - \mathbf{curl}_\Gamma \mathsf{E}^0_{\widetilde{e},p} \mathsf{L}^1_{\widetilde{e},p}(\mathsf{R}^{2D}_0 u \cdot \boldsymbol{n}_{\widetilde{e},f})\ . \end{array} \right. \tag{4.13}$$

This is a valid definition, since, by virtue of definition (3.9), the normal components of $\mathsf{R}^{2D}_0 u$ will vanish on $\partial f \setminus \widetilde{e}$. Moreover, $\mathrm{div}_\Gamma \mathsf{R}^{2D}_0 u = u$ ensures that the normal component of $\mathsf{R}^{2D}_0 u$ has zero average on $\widetilde{e}$. We infer

$$\Big(\mathbf{curl}_\Gamma \mathsf{E}^0_{\widetilde{e},p} \mathsf{L}^1_{\widetilde{e},p}\big((\mathsf{R}^{2D}_0 \mathbf{u} \cdot \boldsymbol{n}_{\widetilde{e},f})_{|\widetilde{e}}\big) \cdot \boldsymbol{n}_{\widetilde{e},f}\Big)_{|\widetilde{e}} = \\ \frac{d}{dl}\mathsf{L}^1_{\widetilde{e},p}\big((\mathsf{R}^{2D}_0 \mathbf{u}) \cdot \boldsymbol{n}_{\widetilde{e},f}\big)_{|\widetilde{e}} = \mathsf{R}^{2D}_0 \mathbf{u} \cdot \boldsymbol{n}_{\widetilde{e},f} \quad \text{on } \widetilde{e}\ ,$$

[4]The parameter l in the notations for the extension operators E^l_*, the projections P^l_*, and the liftings L^l_* refers to the degree of the discrete differential form they operate on. This is explained more clearly in [29, Sect. 3.5].

and see that the zero trace condition on ∂f is satisfied. The same idea underlies the definition of

$$\mathsf{L}^2_{T,p} : \begin{cases} \mathbf{curl}\,\overset{\circ}{\mathcal{W}}{}^1_p(T) & \mapsto \quad \overset{\circ}{\mathcal{W}}{}^1_p(T) \\ \mathbf{u} & \mapsto \quad \mathsf{R}_0\mathbf{u} - \mathbf{grad}\,\mathsf{E}^0_{\widetilde{f},p}\mathsf{L}^1_{\widetilde{f},p}\big(((\mathsf{R}_0\mathbf{u}) \times \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}}\big) , \end{cases} \tag{4.14}$$

where $\widetilde{f}$ is the face opposite to vertex 0, and the definition of

$$\mathsf{L}^3_{T,p} : \begin{cases} \operatorname{div}\overset{\circ}{\mathcal{W}}{}^2_p(T) & \mapsto \quad \overset{\circ}{\mathcal{W}}{}^2_p(T) \\ u & \mapsto \quad \mathsf{D}_0 u - \mathbf{curl}\,\mathsf{E}^1_{\widetilde{f},p}\mathsf{L}^2_{\widetilde{f},p}\big((\mathsf{D}_0 u \cdot \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}}\big) . \end{cases} \tag{4.15}$$

The relationships between the various facet function spaces with vanishing traces can be summarized in the following exact sequences:

$$\begin{array}{l} \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(T) \underset{\mathsf{L}^1_{T,p}}{\xrightarrow{\mathbf{grad}}} \overset{\circ}{\mathcal{W}}{}^1_p(T) \underset{\mathsf{L}^2_{T,p}}{\xrightarrow{\mathbf{curl}}} \overset{\circ}{\mathcal{W}}{}^2_p(T) \underset{\mathsf{L}^3_{T,p}}{\xrightarrow{\operatorname{div}}} \overline{\mathcal{P}}_p(T) \xrightarrow{\mathsf{Id}} \{0\}, \\ \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(f) \underset{\mathsf{L}^1_{f,p}}{\xrightarrow{\mathbf{curl}_\Gamma}} \overset{\circ}{\mathcal{W}}{}^1_p(f) \underset{\mathsf{L}^2_{f,p}}{\xrightarrow{\operatorname{div}_\Gamma}} \overline{\mathcal{P}}_p(f) \xrightarrow{\mathsf{Id}} \{0\}, \\ \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(e) \underset{\mathsf{L}^1_{e,p}}{\xrightarrow{\frac{d}{dl}}} \overline{\mathcal{P}}_p(e) \xrightarrow{\mathsf{Id}} \{0\} , \end{array} \tag{4.16}$$

where $\overline{\mathcal{P}}_p(F)$ designates degree p polynomial spaces on F with vanishing mean. These relationships and the lifting mappings are studied in [29, Sect. 3.4].

Finally we need polynomial extension operators

$$\mathsf{E}^0_{e,p} : \overset{\circ}{\mathcal{P}}_{p+1}(e) \mapsto \mathcal{P}_{p+1}(T) , \tag{4.17}$$
$$\mathsf{E}^0_{f,p} : \overset{\circ}{\mathcal{P}}_{p+1}(f) \mapsto \mathcal{P}_{p+1}(T) \tag{4.18}$$

that satisfy

$$\mathsf{E}^0_{e,p} u_{|e'} = 0 \quad \forall e' \in \mathcal{F}_1(T) \setminus \{e\} , \tag{4.19}$$
$$\mathsf{E}^0_{f,p} u_{|f'} = 0 \quad \forall f' \in \mathcal{F}_2(T) \setminus \{f\} . \tag{4.20}$$

Such extension operators can be constructed relying on a representation of a polynomial on F, $F \in \mathcal{F}_m(T)$, $m = 1, 2$, as a homogeneous polynomial in the barycentric coordinates of F, see [29, Lemma 3.4]. As an alternative, one may use the polynomial preserving extension operators proposed in [22, 37] and [1]. We stress that continuity properties of the extensions E^l_F, $l = 0, 1$, $F \in \mathcal{F}_m(T)$, are immaterial.

4.2. Interpolation operators. Now, we are in a position to define the projection based interpolation operators locally on a generic tetrahedron T with vertices $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$.

First, we devise a suitable projection (depending on the regularity parameter $0 < \epsilon < \frac{1}{2}$, which is usually suppressed to keep notations manageable)

$$\Pi^0_{T,p}(= \Pi^0_{T,p}(\epsilon)) \; : \; C^\infty(\overline{T}) \mapsto \mathcal{P}_{p+1}(T) \tag{4.21}$$

for degree p Lagrangian $H^1(\Omega)$-conforming finite elements. For $u \in C^0(\overline{T})$ define (λ_i is the barycentric coordinate function belonging to vertex $\boldsymbol{a}_i$ of T)

$$u^{(0)} := u - \underbrace{\sum_{i=1}^{4} u(\boldsymbol{a}_i)\lambda_i}_{:=w^{(0)}} , \tag{4.22}$$

$$u^{(1)} := u^{(0)} - \underbrace{\sum_{e \in \mathcal{F}_1(T)} \mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}\frac{d}{ds}u^{(0)}{}_{|e}}_{:=w^{(1)}} , \tag{4.23}$$

$$u^{(2)} := u^{(1)} - \underbrace{\sum_{f \in \mathcal{F}_1(T)} \mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma(u^{(1)}{}_{|f})}_{:=w^{(2)}} , \tag{4.24}$$

$$\Pi^0_{T,p}u := \mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}\, u^{(2)} + w^{(2)} + w^{(1)} + w^{(0)} . \tag{4.25}$$

Observe that $w^{(i)}{}_{|F} = 0$ for all $F \in \mathcal{F}_m(T)$, $0 \le m < i \le 3$. We point out that $w^{(0)}$ is the standard linear interpolant of u.

LEMMA 4.1. *The linear mapping $\Pi^0_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $Cp_{p+1}(T)$*

Proof. Assume $u \in \mathcal{P}_{p+1}(T)$, which will carry over to all intermediate functions. Since $u^{(0)}(\boldsymbol{z}_i) = 0$, $i = 1, \dots, 4$, we conclude from the projection property of $\mathsf{P}^1_{e,p}$ that $\mathsf{L}^1_e\mathsf{P}^1_e\frac{d}{ds}u^{(0)}{}_{|e} = u^{(0)}{}_{|e}$ for any edge $e \in \mathcal{F}_1(T)$. As a consequence

$$u^{(1)} = u^{(0)} - \sum_{e \in \mathcal{F}_1(T)} \mathsf{E}^0_{e,p}u^{(0)}{}_{|e} \quad \Rightarrow \quad u^{(1)}_{|e} = 0 \quad \forall e \in \mathcal{F}_1(T) . \tag{4.26}$$

We infer $\mathsf{L}^1_{f,p}\mathsf{P}^1_f\mathbf{curl}_\Gamma(u^{(1)}{}_{|f}) = u^{(1)}{}_{|f}$ on each face $f \in \mathcal{F}_2(T)$, which implies

$$u^{(2)} = u^{(1)} - \sum_{f \in \mathcal{F}_1(T)} \mathsf{E}^0_{f,p}(u^{(1)}{}_{|f}) \quad \Rightarrow \quad u^{(2)}{}_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) . \tag{4.27}$$

This means that $\mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}\, u^{(2)} = u^{(2)}$ and finishes the proof. ☐

A similar stage by stage construction applies to edge elements and gives a projection

$$\Pi^1_{T,p}(= \Pi^1_{T,p}(\epsilon)) \; : \; \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^1(T) \; : \tag{4.28}$$

for a directed edge $e := [\mathbf{a}_i, \mathbf{a}_j]$ we introduce the Whitney-1-form basis function

$$\mathbf{b}_e = \lambda_i\,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_i . \tag{4.29}$$

These functions span $\mathcal{W}^1_0(T)$. Next, for $\mathbf{u} \in \boldsymbol{C}^0(\overline{T})$ define

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\Big(\sum_{e\in\mathcal{F}_1(T)} \int_e \mathbf{u}\cdot \mathrm{d}\vec{s}\Big)\,\mathbf{b}_e}_{:=\mathbf{w}^{(0)}} \;, \tag{4.30}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{e\in\mathcal{F}_1(T)} \mathbf{grad}\,\mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}((\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e})}_{:=\mathbf{w}^{(1)}} \;, \tag{4.31}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathsf{E}^1_{f,p}\mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)}\times \boldsymbol{n}_f)_{|f})}_{:=\mathbf{w}^{(2)}} \;, \tag{4.32}$$

$$\mathbf{u}^{(3)} := \mathbf{u}^{(2)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathbf{grad}\,\mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}((\mathbf{u}^{(2)}\times \boldsymbol{n}_f)_{|f})}_{:=\mathbf{w}^{(3)}} \;, \tag{4.33}$$

$$\mathbf{u}^{(4)} := \mathbf{u}^{(3)} - \underbrace{\mathsf{L}^2_{T,p}\mathsf{P}^2_{T,p}\,\mathbf{curl}\,\mathbf{u}^{(3)}}_{:=\mathbf{w}^{(4)}} \;, \tag{4.34}$$

$$\Pi^1_{T,p}\mathbf{u} := \mathbf{grad}\,\mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\mathbf{u}^{(4)} + \mathbf{w}^{(4)} + \mathbf{w}^{(3)} + \mathbf{w}^{(2)} + \mathbf{w}^{(1)} + \mathbf{w}^{(0)} \;. \tag{4.35}$$

The contribution $\mathbf{w}^{(0)}$ is the standard interpolant $\Pi^1_{T,0}$ of $\mathbf{u}$ onto the local space of Whitney-1-forms (lowest order edge elements, see [35, Sect. 5.5.1]). The extension operators were chosen in a way that guarantees that $\mathbf{w}^{(2)}\cdot \boldsymbol{t}_e = 0$ and $\mathbf{w}^{(3)}\cdot \boldsymbol{t}_e = 0$ for all $e \in \mathcal{F}_1(T)$.

LEMMA 4.2. *The linear mapping $\Pi^1_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{W}^1_p(T)$ and satisfies the* commuting diagram property

$$\Pi^1_{T,p}\circ \mathbf{grad} = \mathbf{grad} \circ \Pi^0_{T,p} \quad \text{on } C^\infty(\overline{T}) \;. \tag{4.36}$$

Proof. The proof of the projection property runs parallel to that of Lemma 4.1. Assuming $\mathbf{u} \in \mathcal{W}^1_p(T)$ it is obvious that the same will hold for all $\mathbf{u}^{(i)}$ and $\mathbf{w}^{(i)}$ from (4.30)-(4.35). In order to confirm that all projections can be discarded, we have to check that their arguments satisfy conditions of zero trace on the facet boundaries and, in some cases, belong to the kernel of differential operators.

First, recalling the properties of the interpolation operator Π^1_0 for Whitney-1-forms, we find $(\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e} \in \mathcal{W}^1_p(e)$. This implies

$$\mathbf{grad}\,\mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}((\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e}) = (\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) \;, \tag{4.37}$$

and

$$(\mathbf{u}^{(1)}\cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T) \;. \tag{4.38}$$

We see that $(\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f} \in \overset{\circ}{\mathcal{W}}{}^1_p(f)$ for any $f \in \mathcal{F}_2(T)$, so that

$$\begin{aligned}
& \mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) && (4.39)\\
\Rightarrow\ & \mathrm{div}_\Gamma\mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) && (4.40)\\
\Rightarrow\ & \mathrm{div}_\Gamma((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T)\ , \quad (\mathbf{u}^{(2)} \cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T) && (4.41)\\
\Rightarrow\ & \mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) && (4.42)\\
\Rightarrow\ & \mathbf{grad}\,\mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) \times \boldsymbol{n}_f = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) && (4.43)\\
\Rightarrow\ & (\mathbf{u}^{(3)} \times \boldsymbol{n}_f)_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) && (4.44)\\
\Rightarrow\ & \mathsf{P}^2_{T,p}\,\mathbf{curl}\,\mathbf{u}^{(3)} = \mathbf{curl}\,\mathbf{u}^{(3)} && (4.45)\\
\Rightarrow\ & \mathbf{curl}\,\mathsf{L}^2_{T,p}\mathsf{P}^2_{T,p}\,\mathbf{curl}\,\mathbf{u}^{(3)} = \mathbf{curl}\,\mathbf{u}^{(3)} && (4.46)\\
\Rightarrow\ & \mathbf{curl}\,\mathbf{u}^{(4)} = 0 \quad \Rightarrow \quad \mathsf{P}^1_T\mathbf{u}^{(4)} = \mathbf{u}^{(4)} && (4.47)\\
\Rightarrow\ & \mathbf{grad}\,\mathsf{L}^1_T\mathsf{P}^1_T\mathbf{u}^{(4)} = \mathbf{u}^{(4)}\ , && (4.48)
\end{aligned}$$

which confirms the projector property.

Now assume $\mathbf{u} = \mathbf{grad}\,u$ for some $u \in C^\infty(\overline{T})$. The commuting diagram property will follow, if we manage to show $\mathbf{grad}\,u^{(0)} = \mathbf{u}^{(0)}$, $\mathbf{grad}\,u^{(1)} = \mathbf{u}^{(1)}$, $\mathbf{grad}\,u^{(2)} = \mathbf{u}^{(3)}$, etc., for the intermediate functions in (4.22)-(4.25) and (4.30)-(4.35), respectively.

By the commuting diagram property for the standard local interpolation operators onto the spaces of Whitney-0-forms (linear polynomials) and Whitney-1-forms, we conclude

$$\begin{aligned}
& \mathbf{grad}\,u^{(0)} = \mathbf{u}^{(0)} \quad \Rightarrow \quad \frac{d}{ds}u^{(0)}{}_{|e} = (\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) && (4.49)\\
\Rightarrow\ & \mathbf{u}^{(1)} = \mathbf{grad}\,u^{(1)} \quad \Rightarrow \quad \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T) && (4.50)\\
\Rightarrow\ & \mathbf{u}^{(2)} = \mathbf{u}^{(1)} && (4.51)\\
\Rightarrow\ & (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} = \mathbf{curl}_\Gamma u^{(1)}{}_f \quad \forall f \in \mathcal{F}_2(T) \quad \Rightarrow \quad \mathbf{u}^{(3)} = \mathbf{grad}\,u^{(2)} && (4.52)\\
\Rightarrow\ & \mathbf{u}^{(4)} = \mathbf{u}^{(3)}\ . && (4.53)
\end{aligned}$$

Of course, analogous relationships for the functions $w^{(i)}$ and $\mathbf{w}^{(i)}$ hold, which yields $\Pi^1_{T,p}\mathbf{u} = \mathbf{grad}\,\Pi^0_{T,p}u$. □

Following [29, Sect. 3.5], a projection based interpolation onto $\mathcal{W}^2_p(T)$, the operator $\Pi^2_{T,p}(= \Pi^2_{T,p}(\epsilon)) : \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^2_p(T)$, involves the stages

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\Big(\sum_{f\in\mathcal{F}_2(T)} \int_f \mathbf{u} \cdot \boldsymbol{n}_f \,\mathrm{d}S\Big)\,\mathbf{b}_f}_{:=\mathbf{w}^{(0)}}\ , \tag{4.54}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathbf{curl}\,\mathsf{E}^1_{f,p}\mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}\big((\mathbf{u}^{(0)} \cdot \boldsymbol{n}_f)_{|f}\big)}_{:=\mathbf{w}^{(1)}} \tag{4.55}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\mathsf{L}^3_{T,p}\mathsf{P}^3_{T,p}\,\mathrm{div}\,\mathbf{u}^{(1)}}_{:=\mathbf{w}^{(2)}} \tag{4.56}$$

$$\Pi^2_{T,p}\mathbf{u} := \mathbf{curl}\,\mathsf{L}^2_{T,p}\mathsf{P}_{T,p}\mathbf{u}^{(2)} + \mathbf{w}^{(0)} + \mathbf{w}^{(1)} + \mathbf{w}^{(2)}\ . \tag{4.57}$$

Here, $\mathbf{b}_f$ refers to the local basis functions for Whitney-2-forms [29, Sect. 3.2]:

$$\mathbf{b}_f = \lambda_i \,\mathbf{grad}\, \lambda_j \times \mathbf{grad}\, \lambda_k + \lambda_j \,\mathbf{grad}\, \lambda_k \times \lambda_i + \lambda_k \,\mathbf{grad}\, \lambda_i \times \lambda_j \ . \tag{4.58}$$

Analogous to Lemma 4.2 one proves the following result.

LEMMA 4.3. *The linear operator $\Pi^2_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{W}^2_p(T)$ and satisfies the* commuting diagram property

$$\Pi^2_{T,p} \circ \mathbf{curl} = \mathbf{curl} \circ \Pi^1_{T,p} \quad \text{on } \boldsymbol{C}^\infty(\overline{T}) \ . \tag{4.59}$$

The next lemma makes it possible to patch together the local projection based interpolation operator to obtain global interpolation operators

$$\Pi^l_p : \boldsymbol{C}^\infty(\overline{\Omega}) \mapsto \mathcal{W}^l_p(\mathcal{M}) \ , \quad l = 1, 2 \ . \tag{4.60}$$

LEMMA 4.4. *For any $F \in \mathcal{F}_m(T)$, $m = 0, 1, 2$, and $u \in C^\infty(\overline{T})$ the restriction $\Pi^0_{T,p} u_{|F}$ depends only on $u_{|F}$.*

For any $F \in \mathcal{F}_m(T)$, $m = 1, 2$, and $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ the tangential trace of $\Pi^1_{T,p}\mathbf{u}$ onto F depends only on the tangential trace of $\mathbf{u}$ on F.

For any face $f \in \mathcal{F}_2(T)$ and $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ the normal trace of $\Pi^2_{T,p}\mathbf{u}$ onto f depends only on the normal component of $\mathbf{u}$ on f.

Proof. The assertion is immediate from the construction, in particular, the properties of the extension operators used therein. ∎

It goes without saying that density arguments permit us to extend Π^l_p, $l = 0, 1, 2$, to Sobolev spaces, as long as they are continuous in the respective norms. (Repeated) application of trace theorems [26, Sect. 1.5] reveals that it is possible to obtain continuous projectors

$$\Pi^0_p : H^{1+s}(\Omega) \mapsto \{v \in H^1(\Omega) : \ v_{|T} \in \mathcal{P}_{p+1}(T) \ \forall T \in \mathcal{M}\} \ , \tag{4.61}$$

$$\Pi^1_p : \boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \mapsto \mathcal{W}^1_p(\mathcal{M}) \ , \tag{4.62}$$

$$\Pi^2_p : \boldsymbol{H}^{s}(\Omega) \mapsto \mathcal{W}^2_p(\mathcal{M}) \ , \tag{4.63}$$

for any $s > \frac{1}{2}$. In addition, by virtue of Lemma 4.4, zero pointwise/tangential/normal trace on $\partial\Omega$ of the argument function will be preserved by Π^l_p, $l = 0, 1, 2$, for instance,

$$\Pi^1_p(\boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)) = \mathcal{W}^1_p(\mathcal{M}) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega) \ . \tag{4.64}$$

5. Interpolation error estimates. Closely following the ingenious approach in [21, Section 6] we first examine the interpolation error for $\Pi^0_{T,p}$. Please notice that $\Pi^0_{T,p}$ still depends on the fixed regularity parameter $0 < \epsilon < \frac{1}{2}$. The argument function of $\Pi^0_{T,p}$ is assumed to lie in $H^{1+s}(T)$ for some $s > \frac{1}{2}$, *cf.* (4.61). The continuous embedding $H^{1+s}(T) \hookrightarrow C^0(\overline{T})$ plus trace theorems for Sobolev spaces render all operators well defined in this case.

We start with an observation related to the local best approximation properties of the projection based interpolant.

LEMMA 5.1. *For any $u \in H^{1+s}(T)$ holds*

$$\big(\mathbf{grad}(u - \Pi^0_{T,p}u), \mathbf{grad}\, v\big)_{L^2(T)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(T) \ , \tag{5.1}$$

$$\Big(\mathbf{curl}_\Gamma(u - \Pi^0_{T,p}u)_{|f}, \mathbf{curl}_\Gamma v\Big)_{H^{-\frac{1}{2}+\epsilon}(f)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(f), \ f \in \mathcal{F}_2(T) \ , \tag{5.2}$$

$$\left(\frac{d}{dl}(u - \Pi^0_{T,p}u)_{|e}, \frac{d}{dl}v\right)_{H^{-1+\epsilon}(e)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(e), \ e \in \mathcal{F}_1(T) \ . \tag{5.3}$$

Proof. We use the notations of (4.22)-(4.25). Setting $w := w^{(0)} + w^{(1)} + w^{(2)}$, we find

$$\Pi^0_{T,p} u = \mathsf{L}^1_{T,p} \mathsf{P}^1_{T,p} \,\mathbf{grad}(u - w) + w \;, \tag{5.4}$$

which implies, because $\mathsf{L}^1_{T,p}$ is a right inverse of $\frac{d}{dl}$,

$$\mathbf{grad}\, \Pi^0_{T,p} u = \mathsf{P}^1_{T,p} \,\mathbf{grad}\, u + (\mathsf{Id} - \mathsf{P}^1_{T,p}) \,\mathbf{grad}\, w \;. \tag{5.5}$$

This means that $u - \mathbf{grad}\, \Pi^0_{T,p} u$ belongs to the range of $\mathsf{Id} - \mathsf{P}^1_{T,p}$ and (5.1) follows from (4.4) and the properties of orthogonal projections. Similar manipulations establish (5.2):

$$\begin{aligned}
\mathbf{curl}_\Gamma \Pi^0_{T,p} u_{|f} &= \mathbf{curl}_\Gamma w_{|f} && (5.6)\\
&= \underbrace{\mathbf{curl}_\Gamma \mathsf{L}^1_{f,p}}_{=\mathsf{Id}} \mathsf{P}^1_{f,p} \mathbf{curl}_\Gamma u^{(1)} + \mathbf{curl}_\Gamma (w^{(0)} + w^{(1)})_{|f} && (5.7)\\
&= \mathsf{P}^1_{f,p} \mathbf{curl}_\Gamma u_{|f} + (\mathsf{Id} - \mathsf{P}^1_{f,p}) \mathbf{curl}_\Gamma (w^{(0)} + w^{(1)}) \quad \forall f \in \mathcal{F}_2(T) \;. && (5.8)
\end{aligned}$$

The same arguments as above verify (5.3). □

From this we can conclude the result of [21, Section 6, Corollary 1]. To state it we now assume a dependence

$$0 < \epsilon = \epsilon(p) := \frac{1}{10 \log(p+1)} < \frac{1}{4} \;, \quad p \in \mathbb{N} \;, \tag{5.9}$$

of the parameter ϵ in the definition of the local projection based interpolation operators. Below, all parameters ϵ are linked to p via (5.9). Please note that we retain the notation $\left(\Pi^l_{T,p}\right)_{p \in \mathbb{N}}$, $l = 0, 1, 2$, for these new families of operators.

THEOREM 5.2 (Spectral interpolation error estimate for $\Pi^0_{T,p}$). *There is a constant $C_T > 0$ depending only on T and $\frac{1}{2} < s \leq 1$, and, in particular, independent of p, such that*

$$\left| (\mathsf{Id} - \Pi^0_{T,p}) v \right|_{H^1(T)} \leq C_T \frac{\log^{3/2} p}{p^s} |v|_{H^{1+s}(T)} \quad \forall v \in H^{1+s}(T) \;, \quad p \geq 1 \;. \tag{5.10}$$

Stable polynomial extensions are instrumental for the proof, which will be postponed until Page 16. First, we recall the results of [37, Thm. 1] and [1, Thm. 1]:

THEOREM 5.3 (Stable polynomial extension for tetrahedra). *For a tetrahedron T there is linear operator $\mathsf{S}_T : H^{\frac{1}{2}}(\partial T) \mapsto H^1(T)$ such that*

$$\mathsf{S}_T u_{|\partial T} = u \quad \forall u \in H^{\frac{1}{2}}(\partial T) \;, \tag{5.11}$$

$$|\mathsf{S}_T u|_{H^1(T)} \leq C \, |u|_{H^{\frac{1}{2}}(\partial T)} \quad \forall u \in H^{\frac{1}{2}}(\partial T) \;, \tag{5.12}$$

$$\mathsf{S}_T w \in \mathcal{P}_{p+1}(T) \quad \forall w \in \mathcal{P}_{p+1}(T)_{|\partial T} \;, \tag{5.13}$$

where $C > 0$ only depends on the shape regularity measure[5] *of T.*

THEOREM 5.4 (Stable polynomial extension for triangles). *Given a triangle F,*

[5] The shape regularity measure of a tetrahedron is the ratio of the radii of its circumscribed sphere and the largest inscribed sphere.

there is a continuous *linear mapping* $\mathsf{S}_F : L^2(\partial F) \mapsto H^{\frac{1}{2}}(T)$ *such that*

$$|\mathsf{S}_F u|_{H^1(F)} \leq C\, |u|_{H^{\frac{1}{2}}(\partial F)} \quad \forall u \in H^{\frac{1}{2}}(\partial F)\;, \tag{5.14}$$

$$\mathsf{S}_F w \in \mathcal{P}_{p+1}(F) \quad \forall w \in \mathcal{P}_{p+1}(F)_{|\partial F}\;, \tag{5.15}$$

where $C > 0$ *depends only on the shape regularity measure of* T.

By interpolation in Sobolev scale from the last theorem we can conclude

$$\exists C > 0 : \quad |\mathsf{S}_F u|_{H^s(F)} \leq C |u|_{H^{s-\frac{1}{2}}(\partial F)} \quad \forall u \in H^{s-\frac{1}{2}}(\partial F),\ \frac{1}{2} \leq s \leq 1\;. \tag{5.16}$$

We also need to deal with the awkward property of the $H^{\frac{1}{2}}(\partial T)$-norm that it cannot be localized to faces. To that end we resort to a result from [34, Proof of Lemma 3.31], see also [21, Lemma 13].

Lemma 5.5 (Splitting of $H^{\frac{1}{2}}(\partial T)$-norm). *The exists* $C > 0$ *depending only on the shape regularity of the tetrahedron* T *such that*

$$|u|_{H^{\frac{1}{2}+\epsilon}(\partial T)} \leq \frac{C}{\epsilon} \sum_{f \in \mathcal{F}_2(T)} |u|_{H^{\frac{1}{2}+s}(f)} \quad \forall u \in H^{\frac{1}{2}+\epsilon}(\partial T),\ 0 < \epsilon \leq \frac{1}{2}\;. \tag{5.17}$$

Another natural ingredient for the proof are polynomial best approximation estimates, see [40] or [37, Sect. 3].

Lemma 5.6. *Let* F *be either a tetrahedron or a triangle. Then, there is a constant* $C > 0$ *depending only on* F *such that for all* $p \geq 1$

$$\inf_{v_p \in \mathcal{P}_{p+1}(F)} |u - v_p|_{H^r(F)} \leq C p^{r-1-s} |u|_{H^{1+s}(F)} \quad \forall u \in H^{1+s}(F),\ 0 \leq r \leq 1\;. \tag{5.18}$$

Define a semi-norm projection $\mathsf{Q}_{T,p} : H^1(T) \mapsto \mathcal{P}_{p+1}(T)$ on the tetrahedron T by

$$\begin{aligned} &\int_T \mathbf{grad}(u - \mathsf{Q}_{T,p} u) \cdot \mathbf{grad}\, v_p \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T)\;, \\ &\int_T u - \mathsf{Q}_{T,p} u \,\mathrm{d}\boldsymbol{x} = 0\;, \end{aligned} \tag{5.19}$$

and semi-norm projections $\mathsf{Q}_{f,p} : H^{s+\frac{1}{2}}(f) \mapsto \mathcal{P}_{p+1}(f)$, $f \in \mathcal{F}_2(T)$, by

$$\begin{aligned} &(\mathbf{curl}_\Gamma(u - \mathsf{Q}_{f,p} u), \mathbf{curl}_\Gamma v_p)_{H^{\epsilon-\frac{1}{2}}(f)} = 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T)\;, \\ &\int_f u - \mathsf{Q}_{f,p} u \,\mathrm{d}\boldsymbol{x} = 0\;. \end{aligned} \tag{5.20}$$

These definitions involve best approximation properties of $\mathsf{Q}_{T,p} u$ and $\mathsf{Q}_{f,p} u$. Thus, we learn from Lemma 5.6 that with constants independent of $0 < \epsilon < \frac{1}{2} < s \leq 1$

$$|u - \mathsf{Q}_{T,p} u|_{H^1(T)} \leq C(p+1)^{-s} |u|_{H^{1+s}(T)} \quad \forall u \in H^s(T)\;, \tag{5.21}$$

$$|u - \mathsf{Q}_{f,p} u|_{H^{\frac{1}{2}+\epsilon}(f)} \leq C(p+1)^{\epsilon-s} |u|_{H^{\frac{1}{2}+s}(T)} \quad \forall u \in H^{\frac{1}{2}+s}(f)\;. \tag{5.22}$$

The latter estimate follows from the fact that $|\cdot|_{H^{\frac{1}{2}+\epsilon}(f)}$ and $\|\mathbf{curl}_\Gamma \cdot\|_{H^{-\frac{1}{2}+\epsilon}(f)}$ are equivalent semi-norms, uniformly in ϵ.

We also need error estimates for the $L^2(e)$-orthogonal projections,

$$\mathsf{Q}^*_{e,p} : L^2(e) \mapsto \overset{\circ}{\mathcal{P}}_{p+1}(e)\ , \quad e \in \mathcal{F}_1(T)\ . \tag{5.23}$$

LEMMA 5.7 (see [21, Lemma 18]). *With a constant $C > 0$ independent of p, $0 \leq \epsilon \leq \frac{1}{2}$, and $2\epsilon \leq r \leq 1+\epsilon$*

$$\left|e - \mathsf{Q}^*_{e,p} u\right|_{H^\epsilon(e)} \leq C(p+1)^{2\epsilon - r} |u|_{H^r(e)} \quad \forall u \in H^r(e) \cap H^1_0(e)\ .$$

Proof. Write $\mathsf{I}_{e,p} : H^1_0(e) \mapsto \overset{\circ}{\mathcal{P}}_{p+1}$ for the interpolation operator

$$(\mathsf{I}_{e,p} u)(\xi) = u(0) + \int_0^\xi \big(\mathsf{Q}_{e,p} \frac{du}{d\xi}\big)(\tau)\,\mathrm{d}\tau\ , \quad 0 \leq \xi \leq |e|\ ,$$

where ξ is the arclength parameter for the edge e and $\mathsf{Q}_{e,p} : L^2(\Omega) \mapsto \mathcal{P}_p(e)$ is the $L^2(e)$-orthogonal projection. From [40, Sect. 3.3.1, Thm. 3.17] we learn that

$$|u - \mathsf{I}_{e,p} u|_{H^1(e)} \leq C(p+1)^{-1} |u|_{H^2(e)} \quad \forall u \in H^2(e)\ , \tag{5.24}$$

$$\|u - \mathsf{I}_{e,p} u\|_{L^2(e)} \leq C(p+1)^{-m} |u|_{H^m(e)} \quad \forall u \in H^m(e)\ , \quad m = 1, 2\ . \tag{5.25}$$

Here and the in the remainder of the proof, all constants may depend only on the length of e. As $\mathsf{I}_{e,p} u \in \overset{\circ}{\mathcal{P}}_{p+1}(e)$ for $u \in H^1_0(e)$, this permits us to conclude

$$\left\|u - \mathsf{Q}^*_{e,p} u\right\|_{L^2(e)} \leq \|u - \mathsf{I}_{e,p} u\|_{L^2(e)} \leq C(p+1)^{-1} \|u\|_{H^1(e)}\ , \tag{5.26}$$

which yields, by interpolation between $H^1(e)$ and $L^2(e)$,

$$\left\|u - \mathsf{Q}^*_{e,p} u\right\|_{L^2(e)} \leq C(p+1)^{-q} \|u\|_{H^q(e)}\ , \quad 0 \leq q \leq 1\ , \tag{5.27}$$

where $C > 0$ is independent of q. On the other hand, using the inverse inequality [6, Lemma 1]

$$\|u\|_{H^1(e)} \leq C(p+1)^2\, \|u\|_{L^2(e)} \quad \forall u \in \mathcal{P}_{p+1}(e) \tag{5.28}$$

and (5.24), (5.25) we find the estimate

$$\begin{aligned} \left|u - \mathsf{Q}^*_{e,p} u\right|_{H^1(e)} &\leq |u - \mathsf{I}_{e,p} u|_{H^1(e)} + \left|\mathsf{Q}^*_{e,p} u - \mathsf{I}_{e,p} u\right|_{H^1(e)} \\ &\leq |u - \mathsf{I}_{e,p} u|_{H^1(e)} + (p+1)^2 \left\|\mathsf{Q}^*_{e,p} u - \mathsf{I}_{e,p} u\right\|_{L^2(e)} \\ &\leq |u - \mathsf{I}_{e,p} u|_{H^1(e)} + C(p+1)^2 \left\|u - \mathsf{I}_{e,p} u\right\|_{L^2(e)} \\ &\leq C \|u\|_{H^2(e)}\ . \end{aligned} \tag{5.29}$$

Interpolation between (5.27) with $q = \frac{r-2\epsilon}{1-\epsilon}$ and (5.29) finishes the proof. □

Proof. [of Thm. 5.2, borrowed from [21, Sect. 6]] Orthogonality (5.1) of Lemma 5.1 combined with the definition of $\mathsf{Q}_{T,p}$ involves

$$\int_T \mathbf{grad}((\Pi^0_{T,p} - \mathsf{Q}_{T,p})u) \cdot \mathbf{grad}\, v_p\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall v_p \in \overset{\circ}{\mathcal{P}}_{p+1}(T)\ . \tag{5.30}$$

Hence, $(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u$ turns out to be the $|\cdot|_{H^1(T)}$-minimal degree $p+1$ polynomial extension of $(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u_{|\partial T}$, which,thanks to Thm. 5.3, implies

$$\begin{aligned} \left|(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u\right|_{H^1(T)} &\leq \left|\mathsf{S}_T((\Pi^0_{T,p}u - \mathsf{Q}_{T,p}u)_{|\partial T})\right|_{H^1(T)} \\ &\leq C\left|(\Pi^0_{T,p}u - \mathsf{Q}_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)} . \end{aligned} \tag{5.31}$$

Thus, by the continuity of the trace operator $H^1(T) \mapsto H^{\frac{1}{2}}(\partial T)$,

$$\begin{aligned} \left|u - \Pi^0_{T,p}u\right|_{H^1(T)} \leq& \left|u - \mathsf{Q}_{T,p}u\right|_{H^1(T)} \\ &+ C\left(\left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)} + \left|(u - \mathsf{Q}_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)}\right) \\ \leq& C\left(\left|u - \mathsf{Q}_{T,p}u\right|_{H^1(T)} + \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)}\right) . \end{aligned} \tag{5.32}$$

To estimate $\left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)}$ we appeal to Lemma 5.5 and get

$$\begin{aligned} \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}}(\partial T)} &\leq \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{H^{\frac{1}{2}+\epsilon}(\partial T)} \\ &\leq \frac{C}{\epsilon} \sum_{f \in \mathcal{F}_2(T)} \left|(u - \Pi^0_{T,p}u)_{|f}\right|_{H^{\frac{1}{2}+\epsilon}(f)} . \end{aligned} \tag{5.33}$$

Next, we use (5.2) from Lemma 5.1 together with (5.20), which confirms that $(\Pi^0_{T,p}u)_{|f} - \mathsf{Q}_{f,p}u$ is the minimum $|\cdot|_{H^{\frac{1}{2}+\epsilon}(f)}$-seminorm polynomial extension of $(\Pi^0_{T,p}u)_{|\partial f} - \mathsf{Q}_{f,p}(u)_{|\partial f}$. Hence, based on arguments parallel to the derivation of (5.32), this time using Thm. 5.4, we can bound

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{H^{\frac{1}{2}+\epsilon}(f)} \leq \left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{H^{\frac{1}{2}+\epsilon}(f)} + C\left|(\Pi^0_{T,p}u - \mathsf{Q}_{f,p}u)_{|\partial f}\right|_{H^{\epsilon}(\partial f)} , \tag{5.34}$$

where the (ϵ-independent !) continuity constant of the trace mapping S_f enters the constant $C > 0$. Also recall the continuity of the trace mapping $H^{\frac{1}{2}+\epsilon}(f) \mapsto H^{\epsilon}(\partial f)$ [34, Proof of Lemma 3.35]: with $C > 0$ independent of ϵ,

$$\left\|u_{|\partial f}\right\|_{H^{\epsilon}(\partial f)} \leq \frac{C}{\sqrt{\epsilon}} \|u\|_{H^{\frac{1}{2}+\epsilon}(f)} \quad \forall u \in H^{\frac{1}{2}+\epsilon}(f) . \tag{5.35}$$

Use this to continue the estimate (5.34)

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{H^{\frac{1}{2}+\epsilon}(f)} \leq C\left(\frac{1}{\sqrt{\epsilon}}\left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{H^{\frac{1}{2}+\epsilon}(f)} + \left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{H^{\epsilon}(\partial f)}\right) . \tag{5.36}$$

As $\epsilon < \frac{1}{2}$, we can localize the norm $\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{H^{\epsilon}(\partial f)}$ to the edges of f:

$$\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{H^{\epsilon}(\partial f)} \leq \frac{C}{\frac{1}{2} - \epsilon} \sum_{e \in \mathcal{F}_1(T), e \subset \partial f} \left|(u - \Pi^0_{T,p}u)_{|e}\right|_{H^{\epsilon}(e)} . \tag{5.37}$$

Recall the ϵ-uniform equivalence of the norms $|\cdot|_{H^\epsilon(e)}$ and $\left\|\frac{d}{dl}\cdot\right\|_{H^{-1+\epsilon}(e)}$. Hence, owing to (5.3), we have from Lemma 5.7 with $r = s$:

$$\begin{aligned}\left|(u - \Pi^0_{T,p}u)_{|e}\right|_{H^\epsilon(e)} &\leq C \inf_{v_p \in \mathring{\mathcal{P}}_{p+1}} \left|(u - \Pi^0_{T,0}u)_{|e} - v_p\right|_{H^\epsilon(e)} \\ &\leq C\left|(u - \Pi^0_{T,0}u)_{|e} - \mathsf{Q}^*_{e,p}((u - \Pi^0_{T,0}u)_{|e})\right|_{H^\epsilon(e)} \\ &\leq C(p+1)^{2\epsilon-s}\left|(u - \Pi^0_{T,0}u)_{|e}\right|_{H^s(e)} .\end{aligned} \tag{5.38}$$

Moreover, $H^{1+s}(T)$ is continuously embedded into $C^0(\overline{T})$. Consequently, applying trace theorems twice and appealing to the equivalence of all norms on the finite dimensional space $\mathcal{P}_1(T)$,

$$\left|(u - \Pi^0_{T,0}u)_{|e}\right|_{H^s(e)} \leq \left|u_{|e}\right|_{H^s(e)} + \left|(\Pi^0_{T,0}u)_{|e}\right|_{H^s(e)} \leq C|u|_{H^{1+s}(T)} , \tag{5.39}$$

where $C > 0$ depends on s and T, but not on p. Combining the estimates (5.32), (5.33), (5.36), and (5.37), (5.38) with (5.39), we find

$$\begin{aligned}\left|u - \Pi^0_{T,p}u\right|_{H^1(T)} \leq C\Big(&|u - \mathsf{Q}_{T,p}u|_{H^1(T)} + \frac{1}{\epsilon^{3/2}} \sum_{f \in \mathcal{F}_2(T)} \left|u_{|f} - \mathsf{Q}_{f,p}(u_{|f})\right|_{H^{\frac{1}{2}+\epsilon}(f)} + \\ &\frac{(p+1)^{2\epsilon-s}}{\epsilon(\frac{1}{2}-\epsilon)} \sum_{e \in \mathcal{F}_1(T)} |u|_{H^{1+s}(T)}\Big) ,\end{aligned} \tag{5.40}$$

with $C > 0$ independent of p. Finally, we plug in the projection error estimates (5.21), (5.22), and arrive at ($C > 0$ independent of u, ϵ, p, s)

$$\begin{aligned}\left|u - \Pi^0_{T,p}(\epsilon)u\right|_{H^1(T)} \leq C\Big(&(p+1)^{-s}|u|_{H^{1+s}(T)} + \frac{(p+1)^{-s+\epsilon}}{\epsilon^{3/2}} \sum_{f \in \mathcal{F}_2(T)} |u|_{H^{\frac{1}{2}+s}(f)} + \\ &\frac{(p+1)^{-s+2\epsilon}}{\epsilon(\frac{1}{2}-\epsilon)} \sum_{e \in \mathcal{F}_1(T)} |u|_{H^s(e)}\Big) .\end{aligned} \tag{5.41}$$

The choice (5.9) of ϵ together with an application of trace theorems then finishes the proof. □

The next lemma plays the role of [8, Lemma 9] and makes it possible to adapt the approach of [8, Sect. 4.4] to 3D edge elements.

Lemma 5.8. *If $\frac{1}{2} < s \leq 1$ and $\mathbf{u} \in \boldsymbol{H}^s(\Omega)$ satisfies $\mathbf{curl}\,\mathbf{u}_{|T} \in \boldsymbol{\mathcal{P}}_p(T)$ for all $T \in \mathcal{M}$, then*

$$\left\|(\mathsf{Id} - \Pi^1_p)\mathbf{u}\right\|_{L^2(\Omega)} \leq C\, \frac{\log^{3/2} p}{p^s} \left(\|\mathbf{u}\|_{H^s(\Omega)} + \|\mathbf{curl}\,\mathbf{u}\|_{L^2(T)}\right) , \tag{5.42}$$

with a constant $C > 0$ depending only on $\mathcal{M}$ and s.

Proof. Pick any $\mathbf{u}$ complying with the assumptions of the lemma. The locality of the projector allows purely local considerations. Single out one tetrahedron $T \in \mathcal{M}$, still write $\mathbf{u} = \mathbf{u}_{|T}$, and split on T

$$\mathbf{u} = (\mathbf{u} - \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}) + \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} , \tag{5.43}$$

Note that the properties of the smoothed Poincaré lifting R_T stated in Thm. 2.3 imply

1. $\mathbf{curl}(\mathbf{u} - \mathsf{R}_T \,\mathbf{curl}\,\mathbf{u}) = 0$ on T, as a consequence of (2.8), and
2. $\mathsf{R}_T \,\mathbf{curl}\,\mathbf{u} \in \boldsymbol{H}^1(T)$ and the bound

$$\|\mathsf{R}_T \,\mathbf{curl}\,\mathbf{u}\|_{H^1(T)} \leq C \,\|\mathbf{curl}\,\mathbf{u}\|_{L^2(\Omega)} \ , \tag{5.44}$$

where here and below no constant may depend on $\mathbf{u}$ or p.

Hence, as $\mathbf{u} \in \boldsymbol{H}^s(T)$, there exists $v \in H^{1+s}(T)$ such that

$$\mathbf{u} = \mathbf{grad}\, v + \mathsf{R}_T \,\mathbf{curl}\,\mathbf{u} \ . \tag{5.45}$$

The continuity of R_T reveals that, with a constant $C > 0$ depending only on T,

$$|v|_{H^{1+s}(T)} \leq \|\mathbf{u}\|_{H^s(T)} + |\mathsf{R}_T \,\mathbf{curl}\,\mathbf{u}|_{H^1(T)} \leq \|\mathbf{u}\|_{H^s(T)} + C\,\|\mathbf{curl}\,\mathbf{u}\|_{L^2(T)} \ . \tag{5.46}$$

By the assumptions of the lemma and (3.4) we know that

$$\mathsf{R}_T \,\mathbf{curl}\,\mathbf{u} \in \mathcal{W}^1_p(T) \ . \tag{5.47}$$

By the commuting diagram property from Lemma 4.2 and the projector property of $\Pi^1_{T,p}$ the task is reduced to an interpolation estimate for $\Pi^0_{T,p}$:

$$(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u} \overset{(5.45)}{=} \mathbf{grad}(\mathsf{Id} - \Pi^0_{T,p})v + \underbrace{(\mathsf{Id} - \Pi^1_{T,p})\mathsf{R}_T \,\mathbf{curl}\,\mathbf{u}}_{=0} \tag{5.48}$$

As a consequence, invoking Thm. 5.2,

$$\begin{aligned} \left\|(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u}\right\|_{L^2(T)} &\overset{(5.48)}{=} \left|(\mathsf{Id} - \Pi^0_{T,p})v\right|_{H^1(T)} \\ &\leq C\frac{\log^{2/3} p}{p^s}|v|_{H^{1+s}(T)} \overset{(5.46)}{\leq} C\frac{\log^{3/2} p}{p^s}\big(\|\mathbf{u}\|_{H^s(T)} + \|\mathbf{curl}\,\mathbf{u}\|_{L^2(T)}\big) \ , \end{aligned} \tag{5.49}$$

which furnishes a local version of the estimate. Squaring (5.49) and summing over all tetrahedra finishes the proof. ▯

6. Discrete compactness. Smoothness of the solenoidal part of the Helmholtz decomposition of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$, respectively, plays a pivotal role. It can be deduced from elliptic lifting theorems for 2nd-order elliptic boundary value problems [18, Ch. 6]. Proofs of the following lemma can be found in [29, Sect. 4.1] and [2, Sect. 3].

Lemma 6.1. *For any Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ there is a $\frac{1}{2} < s \leq 1$ such that $\mathcal{X} := \boldsymbol{H}_0(\mathrm{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\mathcal{X} := \boldsymbol{H}(\mathrm{div}, \Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ are continuously embedded into $\boldsymbol{H}^s(\Omega)$, that is,*

$$\exists C = C(s, \Omega) > 0 : \quad \|\mathbf{u}\|_{H^s(\Omega)} \leq C\left(\|\mathbf{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\mathbf{u}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}\right) \quad \forall \mathbf{u} \in \mathcal{X} \ . \tag{6.1}$$

We first verify the discrete compactness property of Def. 1.1 for $\boldsymbol{\epsilon} \equiv 1$: consider a sequence $(\mathbf{u}_p)_{p\in\mathbb{N}}$, which satisfies

$$\text{(i)}\ \mathbf{u}_p \in \mathcal{W}^1_p(\mathcal{M}) \ , \tag{6.2}$$

$$\text{(ii)}\ (\mathbf{u}_p, \mathbf{z}_p)_{L^2(\Omega)} = 0 \quad \forall \mathbf{z}_p \in \{\mathbf{v} \in \mathcal{W}^1_p(\mathcal{M}) :\ \mathbf{curl}\,\mathbf{v} = 0\} \ , \tag{6.3}$$

$$\text{(iii)}\ \|\mathbf{u}_p\|_{L^2(\Omega)} + \|\mathbf{curl}\,\mathbf{u}_p\|_{L^2(\Omega)} \leq 1 \quad \forall p \in \mathbb{N} \ . \tag{6.4}$$

THEOREM 6.2. *A sequence $(\mathbf{u}_p)_{p\in\mathbb{N}}$ compliant with* (6.2)–(6.4) *possesses a subsequence that converges in $\boldsymbol{L}^2(\Omega)$.*

Proof. The proof resorts to the "standard policy" for tackling the problem of discrete compactness, introduced by Kikuchi [32,33] for analyzing the h-version of edge elements. It forms the core of most papers tackling the issue of discrete compactness, see [9, Thm. 2], [8, Thm. 11], [29, Thm. 4.9], [23, Thm. 2], [8, Thm. 11], etc.

We start with the continuous Helmholtz decomposition of $\mathbf{u}_p$: let $\widetilde{\mathbf{u}}_p$ be the unique vector field in $\boldsymbol{H}(\mathbf{curl},\Omega)$ with

$$\mathbf{curl}\,\widetilde{\mathbf{u}}_p = \mathbf{curl}\,\mathbf{u}_p\ , \tag{6.5}$$

$$(\widetilde{\mathbf{u}}_p,\mathbf{z})_{L^2(\Omega)} = 0 \quad \forall \mathbf{z}\in \mathrm{Ker}(\mathbf{curl})\cap \boldsymbol{H}(\mathbf{curl},\Omega)\ . \tag{6.6}$$

The inclusion $\mathbf{grad}\,H^1(\Omega)\subset \mathrm{Ker}(\mathbf{curl})$ enforces

$$\operatorname{div}\widetilde{\mathbf{u}}_p = 0 \quad \text{in } \Omega \quad , \quad \widetilde{\mathbf{u}}_p\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\ . \tag{6.7}$$

Hence, by virtue of Lemma 6.1, $\widetilde{\mathbf{u}}_p$ satisfies

$$\widetilde{\mathbf{u}}_p \in \boldsymbol{H}^s(\Omega) \quad , \quad \left\|\widetilde{\mathbf{u}}_p\right\|_{H^s(\Omega)} \leq C \left\|\mathbf{u}_p\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\ , \tag{6.8}$$

where $C>0$ depends only on Ω and $\frac{1}{2} < s \leq 1$.

In addition, $\widetilde{\mathbf{u}}_p$ is $L^2(\Omega)$-orthogonal to $\mathrm{Ker}(\mathbf{curl})\cap\boldsymbol{H}(\mathbf{curl},\Omega)$, see (6.6). Thus, using Nedélec's trick [38], we have

$$\begin{aligned}\left\|\widetilde{\mathbf{u}}_p-\mathbf{u}_p\right\|^2_{L^2(\Omega)} &= \left(\widetilde{\mathbf{u}}_p-\mathbf{u}_p, \widetilde{\mathbf{u}}_p - \Pi^1_p\widetilde{\mathbf{u}}_p + \Pi^1_p\widetilde{\mathbf{u}}_p - \mathbf{u}_p\right)_{L^2(\Omega)} \\ &= \left(\widetilde{\mathbf{u}}_p-\mathbf{u}_p, \widetilde{\mathbf{u}}_p - \Pi^1_p\widetilde{\mathbf{u}}_p\right)_{L^2(\Omega)}\ ,\end{aligned} \tag{6.9}$$

because, thanks to Lemma 4.3 and (6.5),

$$\mathbf{curl}(\Pi^1_p\widetilde{\mathbf{u}}_p - \mathbf{u}_p) = (\Pi^2_p - \mathsf{Id})\underbrace{\mathbf{curl}\,\mathbf{u}_p}_{\in\mathcal{W}^2_p(\mathcal{M})} = 0\ . \tag{6.10}$$

$$\Rightarrow \quad \Pi^1_p\widetilde{\mathbf{u}}_p - \mathbf{u}_p \in \{\mathbf{v}\in\mathcal{W}^1_p(\mathcal{M}):\ \mathbf{curl}\,\mathbf{v} = 0\}\ . \tag{6.11}$$

Hence, appealing to Lemma 5.8, with $C>0$ independent of p,

$$\begin{aligned}\left\|\widetilde{\mathbf{u}}_p-\mathbf{u}_p\right\|_{L^2(\Omega)} &\leq \left\|\widetilde{\mathbf{u}}_p - \Pi^1_p\widetilde{\mathbf{u}}_p\right\|_{L^2(\Omega)} \leq C\frac{\log^{3/2}p}{p^s}\left(\left|\widetilde{\mathbf{u}}_p\right|_{H^s(\Omega)} + \left\|\mathbf{curl}\,\widetilde{\mathbf{u}}_p\right\|_{L^2(\Omega)}\right) \\ &\leq C\frac{\log^{3/2}p}{p^s}\ \left\|\mathbf{u}_p\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \to 0 \quad \text{for } p\to\infty\ .\end{aligned} \tag{6.12}$$

Since bounded in $\boldsymbol{H}^s(\Omega)$, by Rellich's theorem $(\widetilde{\mathbf{u}}_p)_{p\in\mathbb{N}}$ has a convergent subsequence in $\boldsymbol{L}^2(\Omega)$: owing to (6.12), the same subsequence of $(\mathbf{u}_p)_{p\in\mathbb{N}}$ will converge in $\boldsymbol{L}^2(\Omega)$. ◻

THEOREM 6.3. *Replacing $\mathcal{W}^1_p(\mathcal{M})$ with $\mathcal{W}^1_p(\mathcal{M})\cap\boldsymbol{H}_0(\mathbf{curl},\Omega)$ in* (6.2)–(6.4)*, the assertion of Thm. 6.2 remains true.*

Proof. Since the projection based interpolation operators respect homogeneous Dirichlet boundary conditions, *cf.* (4.64), the proof runs parallel to that of Thm. 6.2. We point out that now $\widetilde{\mathbf{u}}_p\times\boldsymbol{n} = 0$ on $\partial\Omega$ instead of $\widetilde{\mathbf{u}}_p\cdot\boldsymbol{n}=0$, but Lemma 6.1 can still be applied. ◻

Now we are able to switch from $\boldsymbol{\epsilon} \equiv 1$ to general dielectric tensor, thus completing the proof of the main theorem Thm. 1.2.

Proof. [of Thm. 1.2 in the Introduction] We adapt the proof of [29, Thm. 4.9]. Consider a $\boldsymbol{H}(\mathbf{curl}, \Omega)$-bounded sequence $(\mathbf{w}_p)_{p\in\mathbb{N}}$, $\mathbf{w}_p \in \mathcal{W}_p^1(\mathcal{M})$, in the $L^2_{\boldsymbol{\epsilon}}(\Omega)$-orthogonal complement $\mathcal{X}_p^1(\mathcal{M})$ (see (1.2)) of the discrete kernel of $\mathbf{curl}$, *i.e.*,

$$(\boldsymbol{\epsilon}\mathbf{w}_p, \mathbf{z}_p)_{L^2(\Omega)} = 0 \quad \forall \mathbf{z}_p \in \mathrm{Ker}(\mathbf{curl}) \cap \mathcal{W}_p^1(\mathcal{M}) \ . \tag{6.13}$$

We continue with the $L^2(\Omega)$-orthogonal *discrete* Helmholtz decomposition

$$\mathbf{w}_p = \mathbf{u}_p \oplus_{L^2} \mathbf{w}_p^0 \ , \quad \mathbf{u}_p \in \mathcal{W}_p^1(\mathcal{M}), \ \mathbf{w}_p^0 \in \mathrm{Ker}(\mathbf{curl}) \cap \mathcal{W}_p^1(\mathcal{M}) \ . \tag{6.14}$$

As $(\mathbf{u}_p, \mathbf{z}_p)_{L^2(\Omega)} = 0$ for all $\mathbf{z}_p \in \mathrm{Ker}(\mathbf{curl}) \cap \mathcal{W}_p^1(\mathcal{M})$, by Thm. 6.2 we can find a subsequence, again denoted by $(\mathbf{u}_p)_{p\in\mathbb{N}}$, with

$$\mathbf{u}_p \xrightarrow{p\to\infty} \mathbf{q} \quad \text{in } L^2(\Omega) \ . \tag{6.15}$$

Since $\mathbf{w}_p$ satisfies (5.41), we conclude

$$\left(\boldsymbol{\epsilon}\mathbf{w}_p^0, \mathbf{z}_p\right)_{L^2(\Omega)} = -\left(\boldsymbol{\epsilon}\mathbf{u}_p, \mathbf{z}_p\right)_{L^2(\Omega)} \quad \forall \mathbf{z}_p \in \mathrm{Ker}(\mathbf{curl}) \cap \mathcal{W}_p^1(\mathcal{M}) \ . \tag{6.16}$$

This can be regarded as a perturbed spectral Galerkin approximation of the following continuous variational problem: seek $\mathbf{y} \in \boldsymbol{H}(\mathbf{curl}\, 0, \Omega) := \mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$ such that

$$(\boldsymbol{\epsilon}\mathbf{y}, \mathbf{z})_{L^2(\Omega)} = -\left(\boldsymbol{\epsilon}\mathbf{q}, \mathbf{z}\right)_{L^2(\Omega)} \quad \forall \mathbf{z} \in \boldsymbol{H}(\mathbf{curl}\, 0, \Omega) \ , \tag{6.17}$$

which, obviously, has unique solution $\mathbf{y}$. From Strang's lemma [14, Thm. 4.4.1] we infer

$$\left\|\mathbf{y} - \mathbf{w}_p^0\right\|_{L^2(\Omega)} \le C\Big(\inf_{\mathbf{z}_p \in \mathcal{W}_p^1(\mathcal{M})\cap\mathrm{Ker}(\mathbf{curl})} \left\|\mathbf{y} - \mathbf{z}_p\right\|_{L^2(\Omega)} + \underbrace{\left\|\mathbf{u}_p - \mathbf{q}\right\|_{L^2(\Omega)}}_{\to 0 \text{ for } p\to\infty}\Big) \ , \tag{6.18}$$

where $C > 0$ depends only on $\boldsymbol{\epsilon}$. Next recall, that there is a representation

$$\mathbf{y} = \mathbf{grad}\, v + \mathbf{h} \ , \quad v \in H^1(\Omega) \ , \quad \mathbf{h} \in \boldsymbol{\mathcal{H}}^1(\Omega) \ , \tag{6.19}$$

with $\mathcal{H}^1(\Omega)$ standing for the *finite dimensional* first co-homology space $\boldsymbol{\mathcal{H}}^1(\Omega)$ of harmonic vector fields, which is contained in $\boldsymbol{H}(\mathbf{curl}\, 0, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}\, 0, \Omega)$, [29, Lemma 2.2] and [2, Prop. 3.14, Prop. 3.18]. Owing to Lemma 6.1 it belongs to $\boldsymbol{H}^s(\Omega)$ for some $s > \frac{1}{2}$, which implies, thanks to Lemma 5.8,

$$\left\|\mathbf{h} - \Pi_p^1 \mathbf{h}\right\|_{L^2(\Omega)} \le C \frac{\log^{3/2} p}{p^s} |\mathbf{h}|_{H^s(\Omega)} \to 0 \quad \text{for } p \to \infty \ . \tag{6.20}$$

Further, the commuting diagram property of Lemma 4.3 confirms that $\mathbf{curl}\, \Pi_p^1 \mathbf{h} = 0$. Besides, asymptotic density of the spectral family of Lagrangian finite element spaces in $H^1(\Omega)$ means that also the first term on the right hand side of (6.18) tends to zero as $p \to \infty$.

Thus, selecting the same subsequence of $(\mathbf{w}_p)_{p\in\mathbb{N}}$ (and keeping the notation), it is immediate that

$$\mathbf{w}_p \xrightarrow{p\to\infty} \mathbf{q} + \mathbf{y} \quad \text{in } L^2(\Omega) \ . \tag{6.21}$$

The case of $\mathbf{w}_p \in \mathcal{W}_p^1(\mathcal{M}) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is amenable to almost the same proof: the boundary conditions are imposed on all fields and in the counterpart of (6.19) the second co-homology space $\boldsymbol{\mathcal{H}}^2(\Omega)$ has to be considered [29, Lemma 2.2]. $\square$

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