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Abstract

Parameter sensitivities of prices for derivative contracts play an important role in model calibration as well as in quantification of model risk. In this paper a unified approach to the efficient numerical computation of all sensitivities for Markovian market models is presented. Variational approximations of the integro-differential equations corresponding to the infinitesimal generators of the market model differentiated with respect to the model parameters are employed. Superconvergent approximations to second and higher derivatives of prices w.r. to the price process' state variables are extracted from approximate, computed prices with low, C^0 regularity by postprocessing. The extracted numerical sensitivities are proved to converge with optimal rates as the mesh width tends to zero. Numerical experiments for uni- and multivariate models with sparse tensor product discretization confirm the theoretical results.

Keywords: Markov process, Greeks, sensitivity, sparse tensor finite elements

1 Introduction

A key task in financial engineering is the fast and accurate calculation of sensitivities of market models with respect to model parameters. This becomes necessary for example in model calibration, risk analysis and in the pricing and hedging of certain derivative contracts. Classical examples are variations of option prices with respect to the spot price or with respect to time-to-maturity, the so-called "Greeks" of the model. For classical, diffusion type models and plain vanilla type contracts, the Greeks can be obtained analytically (see [21]). With the trends to more general market models of jump-diffusion type and to more complicated contracts, closed form solutions are generally not available for pricing and calibration. Thus, prices and model sensitivities have to be approximated

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numerically. As model sensitivities are generally derivatives of the computed prices with respect to model input parameters, a naive approach consists in numerically differentiating computed prices by, e.g., Finite Difference formulas. This results in extra work (e.g. due to multiple “forward” pricing runs) and, due to the generally low regularity of Finite Difference or Finite Element approximations to prices of derivative contracts, in substantial loss of accuracy in the computed sensitivities.

To obtain stable numerical procedures yielding approximate, numerically computable sensitivities for general Markovian market models and for general contracts which converge at the same rate as the computed option prices, additional analytical considerations are necessary.

Most work in this direction has been devoted to Monte-Carlo methods (see [8, 13] and references therein) for diffusion and jump-diffusion models. This paper is focused on a more general class of Markov processes X, including stochastic volatility and multi-dimensional Lévy models. A mesh-based approach is used to solve the corresponding partial integro-differential equation (PIDE). A mesh-based approach is also described in [1] where automatic differentiation of a Finite Element code is used to approximate the Greeks.

In our approach, we distinguish between two classes of sensitivities. The sensitivity of the solution u to variation of a model parameter, like the Greek Vega ($\partial_\sigma u$) and the sensitivity of the solution u to a variation of state spaces such as the Greek Delta ($\partial_x u$). We show that an approximation for the first class can be obtained as a solution of the pricing PIDE with a right hand side depending on u. For the second class, a finite difference like differentiation procedure is presented which allows to obtain the sensitivities from the Finite Element forward price without additional forward solver.

The outline of the paper is as follows. We start by describing the problem setup. First we explain the abstract framework and the variational discretization of the forward Kolmogorov equation by Finite Element methods. Then, we derive for both classes of sensitivities an algorithm to compute these by postprocessing the Finite Element solution. It is shown that approximation of the sensitivities converge with the same rate as the approximation of the option price. Finally, we give numerical examples for different dimensions and models.

2 Variational option pricing

2.1 Parametric Markovian market models

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. We consider the process X to model the dynamics of a single underlying, a basket or a underlying and its ”background” volatility drivers in case of stochastic volatility models. For notational simplicity only we assume that the interest rate is zero. Let g be the payoff, $T > 0$ the maturity and $\mathbb{Q}$ an equivalent martingale measure (EMM) to $\mathbb{P}$, i.e. $\mathbb{Q} \sim \mathbb{P}$ such that the process $(X_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale. Since X is Markovian, the fair price of European style contingent claim with underlying X is given by

$$u(t,x) = \mathbb{E}^{\mathbb{Q}}\big[\, g(X_T) \mid X_t = x\big].$$

If the function value u is sufficiently smooth, it is known to solve the backward Kolmogorov equation

$$-\partial_t u + \mathcal{A}u \;=\; 0 \quad \text{in } (0,T)\times\mathbb{R}^d \tag{2.1}$$

$$u(T,x) \;=\; g(x) \;\text{ in } \mathbb{R}^d, \tag{2.2}$$

where $\mathcal{A}$ denotes the infinitesimal generator of X. We consider processes X where $\mathcal{A}$ splits into the diffusive part $\mathcal{A}_W$, the drift part $\mathcal{A}_\delta$ and the jump part $\mathcal{A}_J$ are given by

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_W + \mathcal{A}_\delta + \mathcal{A}_J, \quad \text{where}\\
\mathcal{A}_W[\varphi](x) &= -\frac{1}{2}\mathcal{Q}(x)D^2\varphi(x)\\
\mathcal{A}_\delta[\varphi](x) &= \langle b(x), D\varphi(x)\rangle\\
\mathcal{A}_J[\varphi](x) &= -\int_E \big(\varphi(x+\zeta(x,z)) - u(x) - \langle \zeta(x,z), D\varphi(x)\rangle 1_{|z|\leq 1}\big)\nu(\mathrm{d}z).
\end{aligned} \tag{2.3}$$

Here, $\mathcal{Q}:\mathbb{R}^d\to\mathbb{R}^{d\times d}$, $b:\mathbb{R}^d\to\mathbb{R}^d$ and $\zeta:\mathbb{R}^d\times E\to\mathbb{R}^d$ with the set of admissible jumps $E\subset\mathbb{R}^d\setminus\{0\}$. Furthermore, D and D^2 are the differential operators $D = (\partial_{x_i})_{1\leq i\leq d}$, and $D^2 = (\partial_{x_i x_j})_{1\leq i,j\leq d}$ and ν denotes the compensator of a Poisson random measure on E satisfying $\int_E \min\{1,|z|^2\}\nu(\mathrm{d}z) < \infty$.

Definition 2.1. We call a process X a parametric Markovian market model with admissible parameter set $\mathcal{S}_\eta$, if

(i) for all $\eta\in\mathcal{S}_\eta$ X is a strong Markov process w.r. to a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$

(ii) the infinitesimal generator $\mathcal{A}$ of the semigroup generated by X has the form (2.3), and the mapping $\mathcal{S}_\eta \ni \eta \to \{\mathcal{Q}, b, \nu, \zeta\}$ is infinitely differentiable.

In (2.3), we assume that the coefficients $\mathcal{Q}$, b, ζ and ν do not depend on time t. Recently, Carr et al. in [10] considered Sato processes (self-similar additive processes) as drivers for the underlying X. The authors introduce $\mathbb{R}$-valued pure jump processes with Lévy measure ν which has time-inhomogeneous Lévy density $k(z,t)$. Our approach to compute prices and sensitivities is not restricted to time independent coefficients but naturally extends to the case when the coefficients are time-inhomogeneous. We give some examples of Markov processes X and their infinitesimal generators covered by our approach.

Example 2.2 (Multidimensional Lévy model [16, 22]). The Markov process is given by the d dimensional Lévy process $X = (X_1, \ldots, X_d)$ with characteristic triplet $(\mathcal{Q}, \nu_{\mathbb{Q}}, \gamma)$ under the EMM $\mathbb{Q}$. We assume that the Lévy measure satisfies $\int_{|z|>1} e^{z_i} \nu_{\mathbb{Q}}(\mathrm{d}z) < \infty$, $i = 1, \ldots, d$. Then, the coefficients are given by

$$\mathcal{Q}(x) = (\mathcal{Q}_{ij})_{1\le i,j\le d}\,, \; b(x) = (\gamma_i)_{1\le i\le d}\,, \; \zeta(x,z) = z\,, \tag{2.4}$$

with $\gamma_i = \frac{1}{2}\mathcal{Q}_{ii} + \int_{\mathbb{R}^d} \left(e^{z_i} - 1 - z_i 1_{|z|\le 1}\right) \nu(\mathrm{d}z)$. The dependence structure of the Brownian motion part of X is characterized entirely by its covariance matrix $\mathcal{Q}$. The dependence structure of the purely discontinuous part of X can be described using Lévy copulas. These were introduced in Tankov [22] and developed in Kallsen and Tankov [16]. Analytic properties and wavelet discretization of the copula process' generator were discussed by Farkas et al. [11] . For the Clayton Lévy copula with tempered stable margins with CGMY [9] parameters the multidimensional Lévy density is given by

$$k(x_1, \ldots, x_d) = \partial_1 \ldots \partial_d F|_{\xi_1 = U_1(x_1), \ldots, \xi_d = U_d(x_d)} k_1(x_1) \ldots k_d(x_d)\,,$$

with marginal Lévy densities

$$k_i(z) = C_i \left(\frac{e^{G_i z}}{|z|^{1+Y_i}} 1_{\{z<0\}} + \frac{e^{-M_i z}}{|z|^{1+Y_i}} 1_{\{z>0\}} \right), \quad C_i, G_i > 0, M_i > 1, Y_i < 2\,, \tag{2.5}$$

marginal tail integrals

$$U_i(x) = C_i\, M_i^{Y_i}\, \Gamma(-Y_i, M_i x)\, 1_{\{x>0\}} - C_i\, G_i^{Y_i}\, \Gamma(-Y_i, -G_i x)\, 1_{\{x<0\}}\,,$$

and Lévy copula

$$F(x_1, \ldots, x_d) = 2^{2-d} \left(\sum_{i=1}^{d} |x_i|^{-\theta} \right)^{-\frac{1}{\theta}} \left(\eta 1_{\{x_1 \cdots x_d \ge 0\}} - (1-\eta) 1_{\{x_1 \cdots x_d \le 0\}} \right)\,,$$

where $i = 1, \ldots, d$, $\theta > 0$ and $\eta \in [0, 1]$. The Clayton copula density blends for $x_i \ge 0$, $i = 1, \ldots, d$ the complete independence density ($\theta = 0$) and the complete dependence density ($\theta \to \infty$).

Example 2.3 (Stochastic volatility model of Heston [14]). The Markov process X is of the form $X = (S, Y)$, where the $\mathbb{R}$-valued process S describes the dynamics of the underlying and the $\mathbb{R}$-valued process Y its volatility. Under a EMM $\mathbb{Q}$, X satisfies the stochastic differential equation $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \Sigma(X_t)\mathrm{d}W_t$ where (W_t) denotes a two dimensional Brownian motion and the coefficients b, Σ are (see [14])

$$b = \begin{pmatrix} 0 \\ \alpha(m - Y_t) - \lambda(t, S_t, Y_t) \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sqrt{Y_t} S_t & 0 \\ \beta\rho\sqrt{Y_t} & \beta\sqrt{1-\rho^2}\sqrt{Y_t} \end{pmatrix},$$

with $\alpha > 0$ the rate of mean reversion, $m > 0$ the long-run mean level of volatility, $\beta \in \mathbb{R}$ and $\rho \in [-1, 1]$ the instantaneous correlation. The function $\lambda : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

appearing in the second component of the drift b represents the price of volatility and reflects the incompleteness of this market model. The infinitesimal generator $\mathcal{A}$ of X is as in (2.3) where for $x := (S,y) := (x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$b(x) = \begin{pmatrix} 0 \\ \alpha(m - x_2) - \lambda(t,x_1,x_2) \end{pmatrix}, \ \mathcal{Q}(x) = \Sigma\Sigma^\top = \begin{pmatrix} x_1^2 x_2 & \beta\rho x_1 x_2 \\ \beta\rho x_1 x_2 & \beta^2 x_2 \end{pmatrix}, \ \nu = 0.$$

Example 2.4 (Stochastic volatility model of BNS [3]). This stochastic volatility model specifies the volatility (of the underlying) as a Ornstein-Uhlenbeck process driven by a Lévy subordinator L_t. The Markov process $X = (S,Y) = (e^Z, \sigma^2)$ under an EMM $\mathbb{Q}$ satisfies the SDE (see [3] and [19])

$$\begin{aligned} \mathrm{d}Z_t &= \big(-\lambda\kappa - \frac{1}{2}\sigma_t^2\big)\mathrm{d}t + \sigma_t \mathrm{d}W_t + \rho\, \mathrm{d}L_{\lambda t} \\ \mathrm{d}\sigma_t^2 &= -\lambda\sigma_t^2 \mathrm{d}t + \mathrm{d}L_{\lambda t}, \end{aligned}$$

where (W_t) is a $\mathbb{Q}$-Brownian motion and $(L_{\lambda t})$ is a $\mathbb{Q}$-Lévy process. The parameters satisfy $\beta,\mu,\rho,\lambda \in \mathbb{R}$, $\lambda > 0$, $\rho \le 0$ and cumulant transform κ is $\kappa(\rho) = \int_{\mathbb{R}_+}(e^{\rho z} - 1)w(z)k(z)\mathrm{d}z$. Here, $w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\int_{\mathbb{R}_+}(\sqrt{w(z)} - 1)^2 k(z)\mathrm{d}z < \infty$, and k is density of the Lévy measure of L_t under the historical measure $\mathbb{P}$. The infinitesimal generator $\mathcal{A}$ of X has the form as in (2.3), where for $x := (x_1,x_2) \in \mathbb{R} \times \mathbb{R}_+$ and $z \in \mathbb{R}_+$ the coefficients are given by (with $c := \int_{z \le 1} z\nu(\mathrm{d}z)$)

$$\begin{aligned} b(x) &= \begin{pmatrix} -\lambda\kappa - \rho c - \frac{1}{2}x_2 \\ -\lambda x_2 - c \end{pmatrix}, \quad \mathcal{Q}(x) = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}, \\ \zeta(x,z) &= \begin{pmatrix} \rho \\ 1 \end{pmatrix} z, \quad \nu(\mathrm{d}z) = \lambda w(z)k(z)\mathrm{d}z, \quad E = \mathbb{R}_+ . \end{aligned}$$

Note that the term $\langle \zeta(x,z), D\varphi(x)\rangle 1_{|z|\le 1}$ appearing in $\mathcal{A}_J$ can be omitted here, since L_t is a subordinator and hence has sample paths of finite variation.

We calculate the sensitivities of the solution u of (2.1)–(2.2) w.r. to parameters in the infinitesimal generator $\mathcal{A}$ and w.r. to solution arguments x and t. We write $\mathcal{A}(\eta_0)$ for a fixed parameter $\eta_0 \in \mathcal{S}_\eta$ to emphasize the dependence of $\mathcal{A}$ on η_0 and change the time to time-to-maturity $t \to T - t$ in (2.1)–(2.2). For sensitivity computation (as well as for domain truncation, cf. [18]), it will be crucial below to admit a non-trivial right hand side. Accordingly, we consider from now on the forward parabolic problem

$$\begin{aligned} \partial_t u + \mathcal{A}(\eta_0)u &= f \ \text{ in } (0,T] \times \mathbb{R}^d && (2.6) \\ u(0,x) &= u_0 \ \text{ in } \mathbb{R}^d, && (2.7) \end{aligned}$$

with $u_0 = g$. For the numerical implementation we truncate the parabolic PIDE (2.6)–(2.7) to a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ and impose boundary conditions on $\partial\mathcal{D}$. Typically, $\mathcal{D}$ is d-dimensional hypercube, i.e. $\mathcal{D} = \prod_{k=1}^d (a_k, b_k)$ for some $a_k, b_k \in \mathbb{R}$, $b_k > a_k$, $k = 1,\dots,d$. We approximate the solution to (2.6)–(2.7) by the Finite Element method, which is based on the variational formulation of (2.6)–(2.7).

2.2 Variational setting

With a parametric Markovian market model X in the sense of Definition 2.1 with parameter set $\mathcal{S}_\eta$ and infinitesimal generator $\mathcal{A}(\eta_0)$ as in (2.3), $\eta_0 \in \mathcal{S}_\eta$, we associate to $\mathcal{A}(\eta_0)$ the Dirichlet form $a(\eta_0;\cdot,\cdot) : V \times V \to \mathbb{R}$ via

$$a(\eta_0;u,v) := \langle \mathcal{A}(\eta_0)u, v\rangle_{V^*\times V}, \quad u,v \in V,$$

with domain $V \overset{d}{\hookrightarrow} H$ (dense embedding). We identify H with its dual H^* and denote by V^* the dual of V so that $V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*$. We denote by $\|\cdot\|$, $\|\cdot\|_V$ the norms in H, V, by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle_{V^*\times V}$ the duality pairing between V and its dual V^*. $\mathcal{L}(V,W)$ is the vector space of linear and continuous operators $\mathcal{A} : V \to W$.

We assume $\mathcal{A}(\eta_0) \in \mathcal{L}(V,V^*)$ to be an elliptic, spatial operator given in weak form where the Dirichlet form $a(\cdot;\cdot,\cdot) : \mathcal{S}_\eta \times V \times V \to \mathbb{R}$ is continuous and satisfies a Gårding inequality: there exist non-negative constants $\alpha(\eta_0), \beta(\eta_0), \gamma(\eta_0)$ such that

$$|a(\eta_0;u,v)| \leq \alpha(\eta_0)\|u\|_V\|v\|_V, \quad \forall u,v \in V, \eta \in \mathcal{S}_\eta, \tag{2.8}$$

$$a(\eta_0;v,v) \geq \beta(\eta_0)\|v\|_V^2 - \gamma(\eta_0)\|v\|_H^2, \quad \forall v \in V, \eta_0 \in \mathcal{S}_\eta. \tag{2.9}$$

Remark 2.5. *(i) In general, the space V may depend on the parameter η_0 and we should write V_{η_0}. For notational simplicity, we drop the subscript η_0.*

(ii) We can assume without loss of generality that $\gamma(\eta_0) = 0$ in (2.9) since by the exponential shift $w := e^{-\gamma(\eta_0)\tau}u$ we obtain $\partial_t w + \mathcal{A}(\eta_0)w + \gamma(\eta_0)u = e^{-\gamma(\eta_0)t}f$ and the operator $\mathcal{A}(\eta_0) + \gamma(\eta_0)I$ is coercive on V.

For $f \in L^2(0,T;V^*)$ and $u_0 \in H$ the weak formulation to the problem (2.6)–(2.7) is given by:

$$\begin{aligned}&\text{Find } u \in L^2(0,T;V) \cap H^1(0,T;V^*) \text{ such that}\\ &(\partial_t u(t,\cdot), v) + a\big(\eta_0; u(t,\cdot), v\big) = \langle f(t), v\rangle_{V^*\times V}, \quad \forall v \in V\,,\\ &u(0,\cdot) = u_0\,.\end{aligned} \tag{2.10}$$

Under the assumption (2.8)–(2.9) the operator $\mathcal{A}(\eta_0) + \gamma(\eta_0)I \in \mathcal{L}(V,V^*)$ defines an isomorphism and (2.10) admits an unique solution.

We assume that V is Sobolev-type space with smoothness index r, i.e.

$$V = \widetilde{H}^r, \qquad \widetilde{H}^0 = H = L^2. \tag{2.11}$$

Note that r depends on the order of the operator $\mathcal{A}(\eta_0)$. We also assume that the solution $u(\eta_0)$ to (2.10) has higher regularity in space, $u(\eta_0)(t) \in \mathcal{H}^s \subset \widetilde{H}^r$ for $t \in (0,T]$, where $\mathcal{H}^s$ is again a Sobolev-type space with smoothness index s.

Example 2.6. Consider the multivariate Lévy copula model from Example 2.2.

(i) It can be shown similar to [18] that $V = \widetilde{H}^r(\mathcal{D})$ with $r = 1$, if $\mathcal{Q} > 0$. Here, for $r \geq 0$, the space $\widetilde{H}^r(\mathcal{D})$ is given by $\widetilde{H}^r(\mathcal{D}) = \{u|_{\mathcal{D}} \mid u \in H^r(\mathbb{R}),\ u|_{\mathbb{R}\setminus\mathcal{D}} = 0\}$.

(ii) Now let $\mathcal{Q}_{ij} = b_i = 0$, $1 \leq i,j \leq d$, and marginal Lévy densities k_i, $1 \leq i \leq d$, as in (2.5). In [11] it was proved that for multivariate barrier contracts V is the anisotropic Sobolev space $V = \widetilde{H}^r(\mathcal{D})$, with $r = (Y_1/2, \dots, Y_d/2)$. Here, for $r = (r_1, \dots, r_d)$, $r_i \geq 0$, $i = 1, \dots, d$, we denote the space $\widetilde{H}^r(\mathcal{D})$ by $\widetilde{H}^r(\mathcal{D}) = \overline{\{\overline{u} \mid u \in C_0^\infty(\mathcal{D})\}}$ where $\overline{u}$ is the zero extension of u to $\mathbb{R}^d$ and the closure is taken w.r. to the norm given by $\|u\|^2_{H^r(\mathbb{R}^d)} = \sum_{j=1}^d \|u\|^2_{H_j^{r_j}(\mathbb{R}^d)}$.

2.3 Variational discretization

Let V_h be a finite dimensional subspace $V_h \subset V$ consisting of continuous piecewise polynomials of degree $p \geq 1$ with $\dim V_h = N < \infty$. The FE semi-discretization in (log) price space of (2.10) reads:

Find $u_h \in L^2(J; V_h) \cap H^1(J; (V_h)^*)$ such that

$$(\partial_t u_h(t,\cdot), v_h) + a(\eta_0; u_h(t,\cdot), v_h) = \langle f(t), v_h \rangle_{V^* \times V} \quad \forall\, v_h \in V_h\,, \tag{2.12}$$

$$u_h(0,\cdot) = u_{0,h}\,, \tag{2.13}$$

where $u_{0,h}$ is an approximation of u_0 in V_h. The FE formulation (2.12)–(2.13) is equivalent to a large, but finite system of ODEs to be solved numerically on the time interval $(0,T)$. To this end, we fix a basis $\mathcal{B} := \{\Phi_j\}_{j=1}^N$ of V_h and let $\underline{u}$ denote the coefficient vector of u_h with respect to the basis $\mathcal{B}$. Then, (2.12)–(2.13) is equivalent to:

Find $\underline{u}(t) \in \mathbb{R}^N$ such that

$$\mathbf{M}\underline{\dot{u}} + \mathbf{A}\underline{u} = \underline{f}(t)\,,$$

where $\mathbf{M}$ and $\mathbf{A}$ are the so-called mass and stiffness matrix given by

$$\mathbf{M} = \Big((\Phi_i, \Phi_j) \Big)_{1 \leq i,j \leq N}, \qquad \mathbf{A} = \Big(a(\eta_0; \Phi_j, \Phi_i) \Big)_{1 \leq i,j \leq N}, \tag{2.14}$$

as well as $\underline{f}(t) = \langle f(t), \Phi_j \rangle_{1 \leq j \leq N}$.

For the convergence analysis of the Finite-Element based pricing algorithms, we assume the following approximation property of the space V_h: For all $u \in \mathcal{H}^s$ with $r \leq s \leq p+1$ there exists a $u_h \in V_h$ such that for $0 \leq \tau \leq r$ (with r as in (2.11))

$$\|u - u_h\|_{\widetilde{H}^\tau} \leq C h^{s-\tau} \|u\|_{\mathcal{H}^s} \tag{2.15}$$

We further assume the existence of a projector or an interpolant $P_h : V \to V_h$ which satisfies (2.15) with $u_h = P_h u$.

We give examples for the space V_h. In dimension $d = 1$, we consider V_h to be the wavelet Finite Element space on a uniform mesh with mesh width h on $\mathcal{D}$ as proposed e.g in

[18]. In this setting, the projector P_h is defined by truncating the wavelet expansion of $u \in V$. For problems in dimension $d \geq 2$, consider the sparse tensor space $\widehat{V}_h$ as defined e.g. in [26].

To discretize in time, we use the θ-scheme. For $M \in \mathbb{N}$ define the time step $\Delta t = \frac{T}{M}$ and $t^m = m\Delta t$, $m = 0, \ldots, M$. The fully discrete scheme reads: Find $u_h^{m+1} \in V_h$, $m = 0, 1, \ldots, M-1$ such that

$$\big(\Delta t^{-1}(u_h^{m+1} - u_h^m), v\big) + a(\eta_0; u_h^{m+\theta}, v_h) = (f^{m+\theta}, v), \quad \forall v \in V_h\,, \tag{2.16}$$

with $u_h^0 = u_{0,h}$. Here $u_h^{m+\theta} := \theta u_h^{m+1} + (1-\theta)u_h^m$ and $f^{m+\theta} := \theta f(t^{m+1}) + (1-\theta) f(t^m)$. In matrix form, (2.16) reads

$$\big(\Delta t^{-1}\mathbf{M} + \theta\mathbf{A}\big)\underline{u}^{m+1} = \big(\Delta t^{-1}\mathbf{M} - (1-\theta)\mathbf{A}\big)\underline{u}^m + \underline{f}^{m+\theta}, \quad m = 0, 1, \ldots, M-1,$$

where $\underline{u}^m$ is the coefficient vector of u_h^m with respect to the basis $\mathcal{B}$ of V_h.

3 Sensitivity analysis

For a parametric Markovian market model X in the sense of Definition 2.1 we distinguish two classes of sensitivities.

1. The sensitivity of the solution u to a variation $\mathcal{S}_\eta \ni \eta_s := \eta_0 + s\delta\eta$, $s > 0$, of an input parameter $\eta_0 \in \mathcal{S}_\eta$. Typical examples are the Greeks Vega ($\partial_\sigma u$), Rho ($\partial_r u$) and Vomma ($\partial_{\sigma\sigma} u$). Other sensitivities which are not so commonly used in the financial community are the sensitivity of the price w.r. to the jump intensity or the order of the process that models the underlying. We show that the Finite Element approximation to such sensitivities satisfies again the scheme (2.16) with a right hand side $f^{m+\theta}$ which depends on the approximation $u_h^{m+\theta}$ of the pricing function u. We also show that the approximation of these sensitivities converge with the same rate as u_h.

2. The sensitivity of the solution u to a variation of arguments t, x. Typical examples are the Greeks Theta ($\partial_\tau u$), Delta ($\partial_x u$) and Gamma ($\partial_{xx} u$). Higher derivatives like $\partial_{xxx} u$ are used in [12, Chapter 5] to approximate prices of European options under stochastic volatility models. We show that these sensitivities can directly be obtained by postprocessing the Finite Element solution u_h (2.12)–(2.13) without additional runs. Again our numerical approximations of these sensitivities converge with the same rate as u_h.

3.1 Sensitivity w.r. to model parameters

Let $\mathcal{C}$ be a Banach space over a domain $\mathcal{D} \subset \mathbb{R}^d$. $\mathcal{C}$ is the space of parameters or coefficients in the operator $\mathcal{A}$ and $\mathcal{S}_\eta \subseteq \mathcal{C}$ is the set of admissible coefficients. We denote by $u(\eta_0)$ the unique solution to (2.10) and introduce the derivative of $u(\eta_0)$ w.r. to $\eta_0 \in \mathcal{S}_\eta$ as the mapping $D_{\eta_0} u(\eta_0) : \mathcal{C} \to V$

$$\widetilde{u}(\delta\eta) := D_{\eta_0} u(\eta_0)(\delta\eta) := \lim_{s\to 0^+} \frac{1}{s}\big(u(\eta_0 + s\delta\eta) - u(\eta_0)\big), \quad \delta\eta \in \mathcal{C}.$$

We also introduce the derivative of $\mathcal{A}(\eta_0)$ w.r. to $\eta_0 \in \mathcal{S}_\eta$

$$\widetilde{\mathcal{A}}(\delta\eta)\varphi := D_{\eta_0}\mathcal{A}(\eta_0)(\delta\eta)\varphi := \lim_{s\to 0^+} \frac{1}{s}\big(\mathcal{A}(\eta_0 + s\delta\eta)\varphi - \mathcal{A}(\eta_0)\varphi\big), \quad \varphi \in V, \quad \delta\eta \in \mathcal{C}.$$

We assume that $\widetilde{\mathcal{A}}(\delta\eta) \in \mathcal{L}(\widetilde{V}, \widetilde{V}^*)$ with $\widetilde{V}$ a real and separable Hilbert space satisfying

$$\widetilde{V} \subseteq V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^* \subseteq \widetilde{V}^*.$$

We further assume that there exists a real and separable Hilbert space $\overline{V} \subseteq \widetilde{V}$ such that $\widetilde{\mathcal{A}}v \in V^*$, $\forall v \in \overline{V}$.

We have the following relation between $D_{\eta_0}u(\eta_0)(\delta\eta)$ and u.

Lemma 3.1. *Let $\widetilde{\mathcal{A}}(\delta\eta) \in \mathcal{L}(\widetilde{V}, \widetilde{V}^*)$, $\forall \delta\eta \in \mathcal{C}$ and $u(\eta_0) : (0,T] \to \overline{V}$, $\eta_0 \in \mathcal{S}_\eta$ be the unique solution to*

$$\partial_t u(\eta_0) + \mathcal{A}(\eta_0)u(\eta_0) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \tag{3.1}$$

$$u(\eta_0)(0,\cdot) = g(x) \quad \text{in } \mathbb{R}^d. \tag{3.2}$$

Then $\widetilde{u}(\delta\eta)$ solves

$$\partial_t \widetilde{u}(\delta\eta) + \mathcal{A}(\eta_0)\widetilde{u}(\delta\eta) = -\widetilde{\mathcal{A}}(\delta\eta)u(\eta_0) \quad \text{in } (0,T) \times \mathbb{R}^d \tag{3.3}$$

$$\widetilde{u}(\delta\eta)(0,\cdot) = 0 \quad \text{in } \mathbb{R}^d \tag{3.4}$$

Proof. Since $u(\eta_0)(0) = g$ does not depend on η_0 its derivative w.r. to η is 0. Now let $\eta_s := \eta_0 + s\delta\eta$, $s > 0$, $\delta\eta \in \mathcal{C}$. Subtract from the equation $\partial_t u(\eta_s)(t) + \mathcal{A}(\eta_s)u(\eta_s)(t) = 0$ equation (3.1) and divide by s to obtain

$$\partial_t \frac{1}{s}\big(u(\eta_s)(t) - u(\eta_0)(t)\big) + \frac{1}{s}\big(\mathcal{A}(\eta_s) - \mathcal{A}(\eta_0)\big)u(\eta_s)(t) + \frac{1}{s}\mathcal{A}(\eta_0)\big(u(\eta_s)(t) - u(\eta_0)(t)\big) = 0.$$

Taking $\lim_{s\to 0^+}$ gives equation (3.3). □

We associate to the operator $-\widetilde{\mathcal{A}}(\delta\eta)$ the Dirichlet form $\widetilde{a}(\delta\eta;\cdot,\cdot) : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ which is given by

$$-\widetilde{a}(\delta\eta; u, v) = -\big(\widetilde{\mathcal{A}}(\delta\eta)u, v\big).$$

The variational formulation to (3.3)-(3.4) reads: Find $\widetilde{u}(\delta\eta) \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that $\widetilde{u}(\delta\eta)(0,\cdot) = 0$ in H and such that

$$(\partial_t \widetilde{u}(\delta\eta)(t,\cdot), v) + a\big(\eta_0; \widetilde{u}(\delta\eta)(t,\cdot), v\big) = -\widetilde{a}\big(\delta\eta; u(\eta_0)(t,\cdot), v\big), \quad \forall v \in V. \tag{3.5}$$

Note that (3.5) has an unique solution $\widetilde{u}(\delta\eta) \in V$ due to the assumptions on $a(\eta_0;\cdot,\cdot)$, $\widetilde{\mathcal{A}}$ and $u(\eta_0) \in \overline{V}$.

Example 3.2 (CGMY model). We consider an one-dimensional tempered stable Lévy process X with CGMY density k as in (2.5). According to (2.3)–(2.4), its infinitesimal generator $\mathcal{A}$ has the form

$$\mathcal{A}[\varphi] = -\frac{1}{2}\sigma^2\partial_{xx}\varphi + \left(\frac{1}{2}\sigma^2 + c\right)\partial_x\varphi - \int_{\mathbb{R}} \{\varphi(x+z) - \varphi(x) - z1_{\{|z|\le 1\}}\partial_x\varphi(x)\}k(z)\mathrm{d}z$$

where the constant c depends only on k via $c := \int_{\mathbb{R}} \left(e^z - 1 - z1_{\{|z|\le 1\}}\right)k(z)\mathrm{d}z$ to ensure that e^X is a martingale. The weak formulation for the price of European style contingent claim is as in (2.10) with $f = 0$.

For the sensitivity of the price w.r. to the volatility σ the set of admissible parameters $\mathcal{S}_\eta$ is $\mathcal{S}_\eta = \mathbb{R}_+$ with $\eta = \sigma$. We have

$$\widetilde{\mathcal{A}}(\delta\sigma)\varphi = -\delta\sigma\sigma_0\partial_{xx}\varphi + \delta\sigma\sigma_0\partial_x\varphi \in \mathcal{L}(V, V^*),$$

with $\delta\sigma \in \mathbb{R} = \mathcal{C}$. The Dirichlet form $\widetilde{a}(\delta\sigma;\cdot,\cdot)$ appearing in the weak formulation (3.5) of $\widetilde{u}(\delta\sigma)$ is given by

$$\widetilde{a}(\delta\sigma;\varphi,\psi) = \delta\sigma\sigma_0(\partial_x\varphi,\partial_x\psi) + \delta\sigma\sigma_0(\partial_x\varphi,\psi).$$

For the sensitivity of the price w.r. to the jump intensity parameter Y of the Lévy process X we let $0 < Y < 2$. Then, we have $\mathcal{S}_\eta = (0,2)$ with $\eta = Y$ and

$$\widetilde{\mathcal{A}}(\delta Y)\varphi = -\delta Y \int_{\mathbb{R}} \{\varphi(x+z) - \varphi(x) - z\partial_x\varphi(x)\}\widetilde{k}(z)\mathrm{d}z \in \mathcal{L}(\widetilde{V}, \widetilde{V}^*)$$

where the kernel $\widetilde{k}$ is given by

$$\widetilde{k}(z) := -\ln|z|k(z).$$

It is easy to check that due to $Y < 2$ in (2.5) $\int_{|z|\le 1} z^2\widetilde{k}(z)\mathrm{d}z < \infty$, $\int_{|z|>1}\widetilde{k}(z)\mathrm{d}z < \infty$. In this setting, $\widetilde{V} = V = \widetilde{H}^1(\mathcal{D})$, if $\sigma > 0$, and $\widetilde{V} = \widetilde{H}^{Y/2+\varepsilon}(\mathcal{D}) \subset \widetilde{H}^{Y/2}(\mathcal{D}) = V$, $\forall\varepsilon > 0$, if $\sigma = 0$ and if the drift has been removed by a change of variables as in [18].

The fully discrete scheme to find an approximation to $\widetilde{u}(\delta\eta)$ in (3.5) is:

Given $\widetilde{u}_h^0 = 0$, for $m = 0, 1, \ldots, M-1$ find $\widetilde{u}_h^{m+1} \in V_h$ such that

$$\left(\Delta t^{-1}(\widetilde{u}_h^{m+1} - \widetilde{u}_h^m), v\right) + a(\eta_0; \widetilde{u}_h^{m+\theta}, v_h) = -\widetilde{a}\left(\delta\eta; u_h^{m+\theta}, v\right), \quad \forall v \in V_h \tag{3.6}$$

or in matrix form

$$\left(\Delta t^{-1}\mathbf{M} + \theta\mathbf{A}\right)\underline{\widetilde{u}}^{m+1} = \left(\Delta t^{-1}\mathbf{M} - (1-\theta)\mathbf{A}\right)\underline{\widetilde{u}}^m - \widetilde{\mathbf{A}}\left(\theta\underline{u}^{m+1} + (1-\theta)\underline{u}^m\right),$$

where $\widetilde{\mathbf{A}}$ is matrix of the Dirichlet form $\widetilde{a}(\delta\eta;\cdot,\cdot)$ in the basis $\mathcal{B}$,

$$\widetilde{\mathbf{A}} = \left(\widetilde{a}(\delta\eta; \Phi_j, \Phi_i)\right)_{1\le i,j\le N}. \tag{3.7}$$

The resulting algorithm is illustrated as pseudo code in Table 1. Here, we denote by $\underline{y} \leftarrow \mathrm{solve}(\mathbf{B}, \underline{x})$ the output of a generic (exact or approximate) solver for a linear system $\mathbf{B}\underline{x} = \underline{y}$.

Choose $\eta_0 \in \mathcal{S}_\eta$, $\delta\eta \in \mathcal{C}$.
Calculate the matrices $\mathbf{M}$, $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ according to (2.14) and (3.7).
Let $\underline{u}^0$ be the coefficient vector of u_h^0 in the basis $\mathcal{B}$ of V_h.
Set $\underline{\widetilde{u}}^0 = \underline{0}$.
For $j = 0, 1, \ldots, M-1$
  $\underline{u}^1 \leftarrow \text{solve}\big(\Delta t^{-1}\mathbf{M} + \theta\mathbf{A}, (\Delta t^{-1}\mathbf{M} - (1-\theta)\mathbf{A})\underline{u}^0\big)$
  Set $\underline{f} := \widetilde{\mathbf{A}}(\theta\underline{u}^1 + (1-\theta)\underline{u}^0)$
  $\underline{\widetilde{u}}^1 \leftarrow \text{solve}\big(\Delta t^{-1}\mathbf{M} + \theta\mathbf{A}, (\Delta t^{-1}\mathbf{M} - (1-\theta)\mathbf{A})\underline{\widetilde{u}}^0 - \underline{f})\big)$
  Set $\underline{u}^0 := \underline{u}^1$, $\underline{\widetilde{u}}^0 := \underline{\widetilde{u}}^1$
Next j

Table 1: Algorithm to compute sensitivities w.r.t. to model parameters

3.2 Convergence rates for sensitivities w.r.t. to model parameters

In this section we establish convergence rates for the sequence $\{\underline{\widetilde{u}}^m\}_{m=0}^{M-1}$ of sensitivities w.r.t. to model parameters as the discretization parameter h in (2.12)–(2.13) tends to zero. We show that the computed sensitivities converge essentially at the same rate as the computed prices. For notational simplicity the subscript η_0 is omitted. We define the energy norm

$$\|u\|_a := \sqrt{a(u,u)} \sim \|u\|_V$$

which is, by (2.8) and (2.9), equivalent to the norm $\|\cdot\|_V$. For $f \in V_h^*$, we let

$$\|f\|_* := \sup_{0 \neq v_h \in V_h} \frac{(f, v_h)}{\|v_h\|_a}.$$

The main result of this section is the following Theorem. The proof is given in the Appendix A.1

Theorem 3.3. *Let the assumptions of Lemma A.5 and Lemma A.1 be fulfilled. Then there holds*

$$\begin{aligned}
&\|\widetilde{u}^M - \widetilde{u}_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\widetilde{u}^{m+\theta} - \widetilde{u}_h^{m+\theta}\|_V^2 \\
&\leq C \sum_{v \in \{u, \widetilde{u}\}} \left\{ \begin{array}{ll} (\Delta t)^2 \int_0^T \|\ddot{v}(\tau)\|_*^2 \mathrm{d}\tau & \theta \in [0,1] \\ (\Delta t)^4 \int_0^T \|\dddot{v}(\tau)\|_*^2 \mathrm{d}\tau & \theta = \frac{1}{2} \end{array} \right. + Ch^{2(s-r)} \sum_{v \in \{u, \widetilde{u}\}} \int_0^T \|\dot{v}(\tau)\|_{\mathcal{H}^{s-r}}^2 \mathrm{d}\tau \\
&\quad + Ch^{2(s-r)} \max_{0 \leq t \leq T} \|u(t)\|_{\mathcal{H}^s}^2.
\end{aligned}$$

Theorem 3.3 shows that if the error between the exact and the approximate price satisfies $\|u^m - u_h^m\| = O(h^{s-r}) + O((\Delta t)^\kappa)$, the error between the exact and approximate sensitivity preserves the same convergence rates both in space and time, i.e. $\|\widetilde{u}^m - \widetilde{u}_h^m\| = O(h^{s-r}) + O((\Delta t)^\kappa)$.

3.3 Sensitivity w.r. to solution arguments

Let u be the solution of the variational problem (2.10). We discuss the computation of $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}u$ for arbitrary multi index $\alpha \in \mathbb{N}_0^d$. The approximation of derivatives of solutions to elliptic and parabolic partial differential equations in the context of Finite Elements is well studied [2, 6, 5, 7, 20, 23, 24]. In these papers, derivatives are approximated by applying difference operators or local averaging operators to the Finite Element solution of the problem. We follow this approach. For $\mu \in \mathbb{Z}^d$ a multi-integer and $h \in \mathbb{R}_+$ we define the translation operator $T_h^\mu \varphi(x) = \varphi(x + \mu h)$ and the forward difference quotient $\partial_{h,j}\varphi(x) = h^{-1}(T_h^{e_j}\varphi(x) - \varphi(x))$, where e_j, $j = 1, \dots, d$, denotes the j-th standard basis vector in $\mathbb{R}^d$. For $\alpha \in \mathbb{N}_0^d$ we denote by $\partial_h^\alpha \varphi = \partial_{h,1}^{\alpha_1}\cdots\partial_{h,d}^{\alpha_d}\varphi$ and by D_h^α the difference operator of order $n \geq 0$

$$D_h^\alpha \varphi := \sum_{\gamma, |\alpha| = n} C_{\gamma,\alpha} T_h^\gamma \partial_h^\alpha \varphi.$$

Definition 3.4. The difference operator D_h^α of order $|\alpha| = n$ and mesh width h is called an approximation to the derivative D^α of order $s \in \mathbb{N}_0$ if for any $\mathcal{D}_0 \subset \mathcal{D}$ there holds

$$\|D^\alpha \varphi - D_h^\alpha \varphi\|_{\widetilde{H}^r(\mathcal{D}_0)} \leq C h^s \|\varphi\|_{\mathcal{H}^{s+r+n}(\mathcal{D})}, \forall \varphi \in \mathcal{H}^{s+r+n}(\mathcal{D}). \tag{3.8}$$

Given a basis $\mathcal{B} := \{\Phi_j\}_{j=1}^N$ of V_h, the action of D_h^α to $v_h \in V_h$ can be realized as matrix-vector multiplication $\underline{v}_h \mapsto \mathbf{D}_h^\alpha \underline{v}_h$, where

$$\mathbf{D}_h^\alpha = \left(D_h^\alpha \Phi_1, \cdots, D_h^\alpha \Phi_N\right) \in \mathbb{R}^{N\times N} \tag{3.9}$$

and $\underline{v}_h$ is the coefficient vector of v_h w.r. to basis $\mathcal{B}$, respectively.

Example 3.5. Let $\mathcal{V}_h = \operatorname{span}\{\varphi_j(x) \mid 1 \leq j \leq N\}$, $\varphi_j(x) = \max\{0, 1 - h^{-1}|x - jh)|\}$, $h = \frac{1}{N+1}$, be the space of piecewise linear continuous functions on $[0, 1]$ vanishing at the end points 0, 1. For $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu \in \mathbb{N}_0$ we denote by $\operatorname{diag}_{\pm\mu}(\alpha, \beta, \gamma)$ the matrices

$$\operatorname{diag}_{-\mu}(\alpha,\beta,\gamma) = \begin{pmatrix} \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots \\ & & & \alpha & \beta & \gamma & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\uparrow$$
$$\mu + 1 - \text{th column}$$

and $\operatorname{diag}_\mu(\alpha, \beta, \gamma) = (\operatorname{diag}_{-\mu}(\alpha, \beta, \gamma))^\top$. Then the matrices $\mathbf{Q}_h$ of the forward difference quotient ∂_h and $\mathbf{T}_\mu$ of the translation operator T_h^μ respectively are given by

$$\mathbf{Q}_h = h^{-1}\operatorname{diag}_0(0, -1, 1), \quad \mathbf{T}_\mu = \operatorname{diag}_\mu(0, 1, 0).$$

Hence, for example, we have for the centered finite difference quotient

$$D_h^2 \varphi(x) = h^{-2}(\varphi(x + h) - 2\varphi(x) + \varphi(x - h))$$

Choose $\eta_0 \in \mathcal{S}_\eta$.
Calculate the matrices $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{D}_h^\alpha$ according to (2.14) and (3.9).
Let $\underline{u}^0$ be the coefficient vector of u_h^0 in the basis $\mathcal{B}$ of V_h.
For $j = 0, 1, \ldots, M-1$
    $\underline{u}^1 \leftarrow \text{solve}\big(\Delta t^{-1}\mathbf{M} + \theta\mathbf{A}, (\Delta t^{-1}\mathbf{M} - (1-\theta)\mathbf{A})\underline{u}^0\big)$
    Set $\underline{u}^0 := \underline{u}^1$
Next j
Set $\underline{v} := \mathbf{D}_h^\alpha \underline{u}^1$

Table 2: Algorithm to compute w.r. to arguments of solution

of order 2 in one dimension $\mathbf{D}_h^2 = \mathbf{T}_{-1}\mathbf{Q}_h^2 = h^{-2}\text{diag}_0(1, -2, 1)$.

Now let $V_h = \mathcal{V}_h \otimes \cdots \otimes \mathcal{V}_h$ be the d-fold tensor product of $\mathcal{V}_h$. Then the matrix $\mathbf{D}_h^\alpha$ is given by

$$\mathbf{D}_h^\alpha = \sum_{\gamma, |\alpha|=n} C_{\gamma,\alpha} \mathbf{T}_{\gamma_1} \otimes \cdots \otimes \mathbf{T}_{\gamma_d} \mathbf{Q}_h^{\alpha_1} \otimes \cdots \otimes \mathbf{Q}_h^{\alpha_d}.$$

In Table 2 the algorithm how to obtain an approximation to the derivative $D^\alpha u(T, x)$ at maturity T is illustrated. The vector $\underline{v} \in \mathbb{R}^N$ is the coefficient vector of $D_h^\alpha u_h^M$ in the basis $\mathcal{B}$ of V_h.

3.4 Convergence rates for sensitivities w.r. to solution arguments

For simplicity, we shall assume in this section that the function $\zeta : \mathbb{R}^d \times E \to \mathbb{R}^d$ appearing in (2.3) depends only on z, i.e. $\zeta : E \to \mathbb{R}^d$.

We have the following convergence result for the approximation of sensitivities w.r. to solution arguments. Its proof can be found in the Appendix A.2.

Theorem 3.6. *Let the assumptions of Lemma A.1 be fulfilled and assume that $u(x, t)$ is sufficiently smooth in $[0, T] \times \overline{\mathcal{D}}$. Assume that the approximation $\partial_h^\beta u_h^0$ is quasi-optimal in $L^2(\mathcal{D})$ for all $\beta \le \alpha$. Assume further that D_h^α approximates D^α in the sense of Definition 3.4. Then there holds*

$$\begin{aligned}
&\|D^\alpha u^M - D_h^\alpha u_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|D^\alpha u^{m+\theta} - D_h^\alpha u_h^{m+\theta}\|_V^2 \\
&\le C \begin{cases} (\Delta t)^2 \int_0^T \|\ddot{u}(\tau)\|_*^2 \mathrm{d}\tau & \theta \in [0,1] \\ (\Delta t)^4 \int_0^T \|\dddot{u}(\tau)\|_*^2 \mathrm{d}\tau & \theta = \frac{1}{2} \end{cases} + Ch^{2(s-r)} \int_0^T \|\dot{u}(\tau)\|_{\mathcal{H}^{s-r}}^2 \mathrm{d}\tau \\
&\quad + Ch^{2(s-r)} \max_{0 \le t \le T} \|u(t)\|_{\mathcal{H}^s}^2.
\end{aligned}$$

Remark 3.7. *(i) Note that we can not get higher convergence rates than $s - r$, even if u has higher regularity ($u(t) \in \mathcal{H}^{s+r+n}$).*

(ii) Theorem 3.6 shows that arbitrary derivatives of u can be approximated with the same rate as u itself, provided u is sufficiently smooth.

4 Numerical examples

In this section we compute various sensitivities for different models. We mainly choose models where the price is known in closed form such that we are able to compute the errors between the exact price/sensitivities and their Finite Element approximations. In Theorems 3.3 and 3.6 these discretization errors are estimated in the energy norm. In the numerical examples, however, we measure the errors in the L^∞ norm. This choice stems from the fact that the above mentioned closed form solutions are not given explicitly as functions of x and t, which turns the computation of $\|u(T,x) - u_h^M\|_V$ very expensive and only approximative.

We measure the L^∞ norm of the error on a subset $\mathcal{D}_0$ of the computational domain $\mathcal{D}$ at time to maturity $t = T$. In all computations, we choose wavelet Finite Element spaces spanned by continuous wavelets of polynomial degree $p = 1$. For problems in dimension $d \geq 2$, we choose the sparse grid spaces $\widehat{V}_h$ to reduce the computational complexity of the approximations. In the θ-scheme, we let $\theta = \frac{1}{2}$ and choose the time step Δt sufficiently small.

The experimental convergence rates are obtained by the least square method applied to the data $(\log h, \log e_h)$, where $e_h := \|u(T,x) - u_h^M\|_{L^\infty(\mathcal{D}_0)}$.

4.1 One-dimensional models

We consider Example 2.2 with $d = 1$ for two models: (i) the Black-Scholes model [4] and (ii) Variance Gamma model [17] with parameters (σ, ν, ϑ), i.e one has in (2.5) $Y = 0$, $C = \nu^{-1}$, $M = (\nu\mu_+)^{-1}$ and $G = (\nu\mu_+)^{-1}$ where $\mu_+ = \frac{1}{2}\sqrt{\vartheta^2 + 2\sigma^2\nu^{-1}} + \frac{\vartheta}{2}$, $\mu_- = \mu_+ - \vartheta$. For both models, we consider a European put with strike $K = 1$ and maturity $T = 0.1$, and we calculate the Greeks Delta, $\Delta = Du = \frac{\partial u}{\partial S}$, and Gamma, $\Gamma = D^2u = \frac{\partial^2 u}{\partial S^2}$. For the Black-Scholes model we additionally compute the Vega, $\mathcal{V} = \widetilde{u}(\delta\sigma) = \frac{\partial u}{\partial \sigma}$. We choose for both models the parameter $\sigma = 0.4$ as well as $\nu = 0.04$, $\vartheta = -0.2$ for (ii). Using the analytic solution for the Black-Scholes model [4] and the Variance Gamma model [17], we can compute the error of the FEM solution. The convergence rates in $\|\cdot\|_{L^\infty([0.5,2])}$ are shown in Figure 1.

As predicted in Theorem 3.3 and 3.6, all Greeks convergence with the same rate as the price u itself. Here, since the degree of polynomials in V_h is $p = 1$, the rate is $s = p+1 = 2$, even when the error is measured in the L^∞-norm.

As a further illustration of the method, we compute the sensitivity w.r. to the order Y $\widetilde{u}(\delta Y) = \frac{\partial u}{\partial Y}$ as explained in Example 3.2. We consider a European call with strike $K = 1$

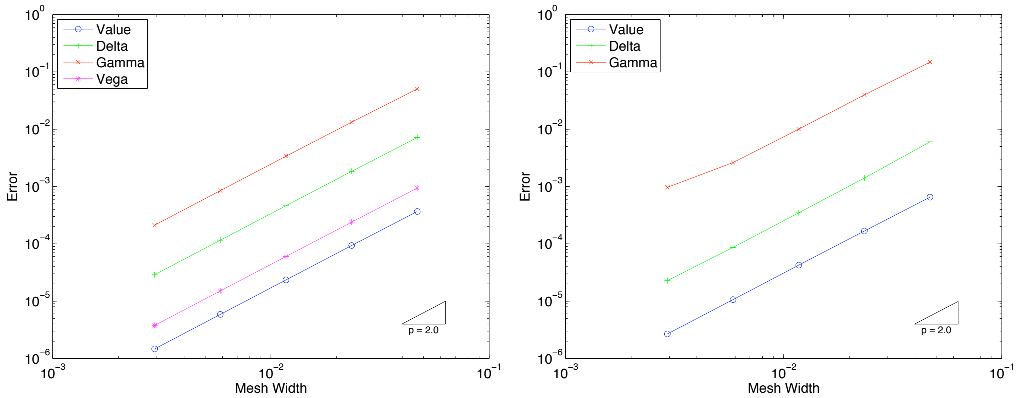

Figure 1: Convergence rates of Greeks for a European put in the Black-Scholes (left) and Variance Gamma (right) model.

and maturity $T = 0.5$ and choose the model parameters $C = 1$, $G = 12$, $M = 10$ and $\eta_0 = Y = 1$. The functions $u(T, S)$ and $\widetilde{u}(\delta Y)(T, S)$ for $\delta Y = 1$ are shown in Figure 2.

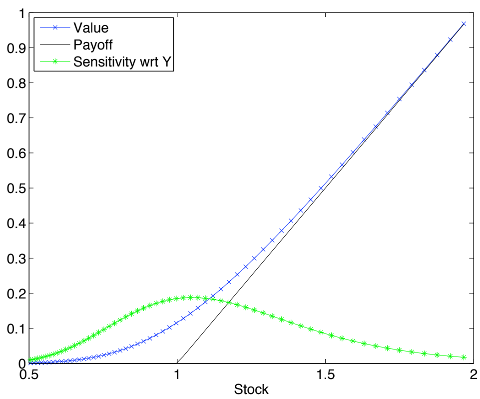

Figure 2: Computed sensitivity of a European call w.r.t. to the jump intensity parameter Y in the CGMY model.

4.2 Multi-variate models

We consider the Heston stochastic volatility model and a three dimensional basket option.

4.2.1 Heston model

We calculate the sensitivities $\widetilde{u}(\delta\rho)$ and $\widetilde{u}(\delta\alpha)$ w.r.t. to correlation ρ of the Brownian motions that drive the underlying and the volatility and the rate of mean reversion α (see Example 2.3). To this end, we consider a European call with strike $K = 1$ and maturity $T = \frac{1}{2}$. The model parameters for both sensitivity runs are $\lambda = 0$, $\sigma = 0.5$ and $m = 0.06$. Additionally, for the sensitivity w.r.t. to ρ we let $\rho_0 = -0.5$, $\delta\rho = 1$ and $\alpha = 2.5$. For the sensitivity w.r.t. to α we set $\alpha_0 = 2.5$, $\delta\alpha = 1$ and $\rho = -0.5$. We compare the FEM solution with the closed form solution given in [14]. The convergence rates in $\|\cdot\|_{L^\infty([e^{-0.25},e^{0.75}]\times[0.24,1.2])}$ are shown in Figure 3.

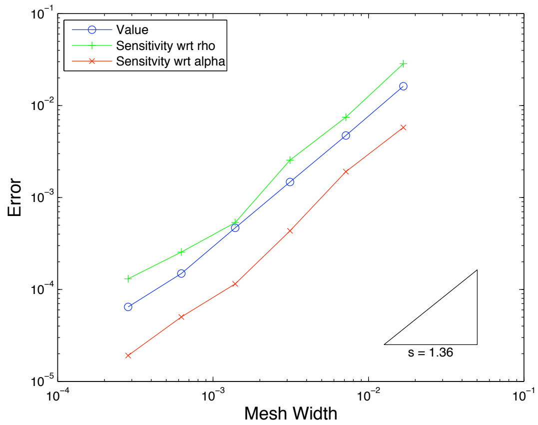

Figure 3: Convergence rates of sensitivities $\widetilde{u}(\delta\rho)$, $\widetilde{u}(\delta\alpha)$ for a European call in the Heston stochastic volatility model.

The experimental convergence rate s is $s \approx 1.36$ for u and $\widetilde{u}(\delta\rho)$ and $s \approx 1.43$ for $\widetilde{u}(\delta\alpha)$. This confirms the theoretical finding of Theorem 3.3 that computed prices and sensitivities convergence with the same rate. In Figure 4 the sensitivities w.r.t. to ρ and α are shown.

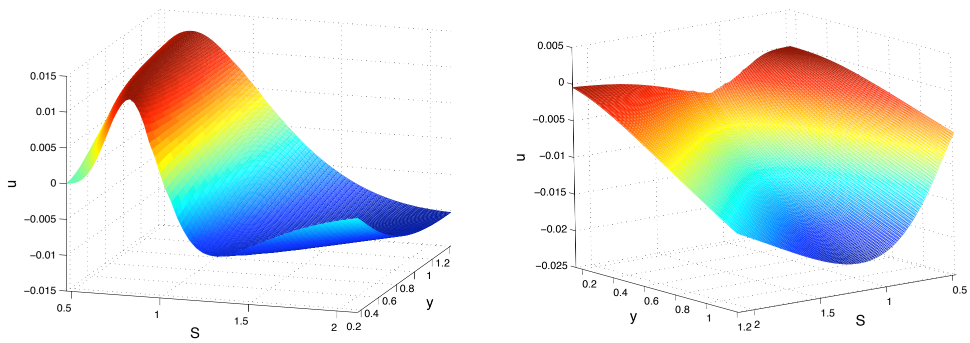

Figure 4: Computed sensitivities for a European call w.r.t. to model parameters ρ and α: $\widetilde{u}(\delta\rho)$ (left) and $\widetilde{u}(\delta\alpha)$ (right) in the Heston stochastic volatility model.

4.2.2 Basket option

We again need an analytic solution to compare our FEM solution with. Therefore, we choose $g(S) = \left(\prod_{i=1}^{d} S_i - K\right)^+$ where for the multidimensional Black-Scholes model analytic solutions can be found by reducing the problem to an one-dimensional problem [15, Section 7.5]. We consider the dimension $d = 3$, strike $K = 1$ and maturity $T = 0.1$, and we calculate the Greeks Delta, $\Delta_1 = \frac{\partial u}{\partial S_1}$, and Gamma, $\Gamma_{11} = \frac{\partial^2 u}{\partial S_1^2}$. The parameters are $\sigma = 0.4$ and $\rho = 0$. The convergence rates in $\|\cdot\|_{L^\infty([0.5,2]^3)}$ are shown in Figure 5.

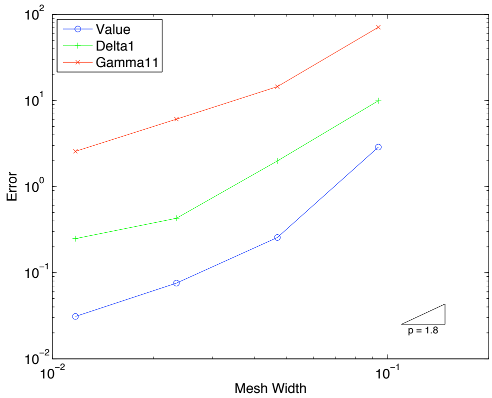

Figure 5: Convergence rates of Greeks of a basket option in a three dimensional Black-Scholes model.

The experimental convergence rate s is $s \approx 1.82$ for Δ_1 and $s \approx 1.56$ for Γ_{11}.

A Appendix

A.1 Proof of Theorem 3.3

We first recall the following stability result for θ-scheme from [25, Proposition 4.1]. We denote by λ the constant

$$\lambda := \sup_{0 \neq v_h \in V_h} \frac{\|v_h\|^2}{\|v_h\|_*^2}.$$

Lemma A.1. *If* $\frac{1}{2} \leq \theta \leq 1$, *assume that* $0 < C_1 < 2$, $C_2 \geq (2 - C_1)^{-1}$. *If* $0 \leq \theta < \frac{1}{2}$ *assume that*

$$\sigma := \Delta t(1 - 2\theta)\lambda < 2, \quad 0 < C_1 < 2 - \sigma, \quad C_2 \geq \frac{1 + (4 - C_1)\sigma}{2 - \sigma - C_1}.$$

Then the sequence $\{u_h^m\}_{m=0}^{M}$ *generated by the* θ*-scheme (2.16) satisfies the stability esti-*

mate

$$\|u_h^M\|^2 + C_1\Delta t \sum_{m=0}^{M-1} \|u_h^{m+\theta}\|_a^2 \le \|u_h^0\|^2 + C_2\Delta t \sum_{m=0}^{M-1} \|f^{m+\theta}\|_*^2. \tag{A.1}$$

We will also need the following convergence result proved in [25, Theorem 5.4].

Lemma A.2. *Let the assumptions of Lemma A.1 be fulfilled. Assume that $u(x,t)$ is sufficiently smooth in $[0,T]\times\overline{\mathcal{D}}$ and assume that the approximation property (2.15) holds. Assume further that the approximation $u_{0,h} \in V_h$ of u^0 is quasi-optimal in $L^2(\mathcal{D})$. Then the sequence $\{u_h^m\}_{m=0}^{M-1}$ in (2.16) satisfies*

$$\begin{aligned}
&\|u^M - u_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|u^{m+\theta} - u_h^{m+\theta}\|_V^2 \\
&\le C \begin{cases} (\Delta t)^2 \int_0^T \|\ddot{u}(\tau)\|_*^2 \mathrm{d}\tau & \theta \in [0,1] \\ (\Delta t)^4 \int_0^T \|\dddot{u}(\tau)\|_*^2 \mathrm{d}\tau & \theta = \frac{1}{2} \end{cases} + Ch^{2(s-r)} \max_{0\le t\le T} \|u(t)\|_{\mathcal{H}^s}^2 \\
&\quad + Ch^{2(s-r)} \int_0^T \|\dot{u}(\tau)\|_{\mathcal{H}^{s-r}}^2 \mathrm{d}\tau\,.
\end{aligned}$$

We now estimate the error $e_h^m := \widetilde{u}^m - \widetilde{u}_h^m$, where we set $\widetilde{u}^m := \widetilde{u}(t^m)$. We write

$$e_h^m = \underbrace{\widetilde{u}^m - P_h\widetilde{u}^m}_{\eta^m} + \underbrace{P_h\widetilde{u}^m - \widetilde{u}_h^m}_{\xi_h^m} = \eta^m + \xi_h^m.$$

Since η^m can be estimated by the approximation property (2.15), we now focus on ξ_h^m.

Lemma A.3. *Assume $\widetilde{u} \in C^1([0,T];H)$. Then $\{\xi_h^m\}_{m=0}^{M-1}$ satisfy the θ-scheme: Given $\xi_h^0 = 0$, for $m = 0,1,\dots,M-1$*

$$(\Delta t)^{-1}(\xi_h^{m+1} - \xi_h^m, v_h) + a(\xi_h^{m+\theta}, v_h) = (r^m, v_h), \quad \forall v_h \in V_h, \tag{A.2}$$

with weak residual $r^m : V_h \to \mathbb{R}$ given by $r^m = \sum_{j=1}^4 r_j^m$ where

$$\begin{aligned}
(r_1^m, v_h) &:= \big((\Delta t)^{-1}(\widetilde{u}^{m+1} - \widetilde{u}^m) - \dot{\widetilde{u}}^{m+\theta}, v_h\big), \\
(r_2^m, v_h) &:= \big((\Delta t)^{-1}(P_h\widetilde{u}^{m+1} - P_h\widetilde{u}^m) - (\Delta t)^{-1}(\widetilde{u}^{m+1} - \widetilde{u}^m), v_h\big), \\
(r_3^m, v_h) &:= a\big(P_h\widetilde{u}^{m+\theta} - \widetilde{u}^{m+\theta}, v_h\big), \\
(r_4^m, v_h) &:= \widetilde{a}(\delta\eta; u^{m+\theta} - u_h^{m+\theta}, v_h).
\end{aligned}$$

Proof. We proceed as in the proof of Lemma 5.1 in [25]. We first recall that $\widetilde{u} \in C^1([0,T];H)$ implies

$$(\dot{\widetilde{u}}^{m+\theta}, v) + a(\widetilde{u}^{m+\theta}, v) = -\widetilde{a}(\delta\eta; u^{m+\theta}, v), \quad \forall v \in V. \tag{A.3}$$

Inserting the definition of ξ_h^m into the left hand side of equation (A.2) yields

$$
\begin{aligned}
&(\Delta t)^{-1}(\xi_h^{m+1} - \xi_h^m, v_h) + a(\xi_h^{m+\theta}, v_h) \\
&\quad = \big((\Delta t)^{-1}[(P_h\widetilde{u}^{m+1} - \widetilde{u}_h^{m+1}) - (P_h\widetilde{u}^m - \widetilde{u}_h^m)], v_h\big) + a(P_h\widetilde{u}^{m+\theta}, v_h) - a(\widetilde{u}_h^{m+\theta}, v_h) \\
&\quad = \big((\Delta t)^{-1}(P_h\widetilde{u}^{m+1} - P_h\widetilde{u}^m), v_h\big) + a(P_h\widetilde{u}^{m+\theta}, v_h) - \big((\Delta t)^{-1}(\widetilde{u}_h^{m+1} - \widetilde{u}_h^m), v_h\big) \\
&\quad\quad - a(\widetilde{u}_h^{m+\theta}, v_h) \\
&\quad \overset{(3.6)}{=} \big((\Delta t)^{-1}(P_h\widetilde{u}^{m+1} - P_h\widetilde{u}^m), v_h\big) + a(P_h\widetilde{u}^{m+\theta}, v_h) \\
&\quad\quad + \widetilde{a}(\delta\eta; u^{m+\theta} - u^{m+\theta} + u_h^{m+\theta}, v_h) \\
&\quad \overset{(A.3)}{=} \big((\Delta t)^{-1}(P_h\widetilde{u}^{m+1} - P_h\widetilde{u}^m) - \dot{\widetilde{u}}^{m+\theta}, v_h\big) + a(P_h\widetilde{u}^{m+\theta} - \widetilde{u}^{m+\theta}, v_h) \\
&\quad\quad - \widetilde{a}(\delta\eta; u^{m+\theta} - u_h^{m+\theta}, v_h)
\end{aligned}
$$

□

By stability (A.1) we have

Corollary A.4. *Let $\widetilde{u} \in C^1([0,T]; H)$. Then, under the assumptions of Lemma A.1, there holds*

$$\|\xi_h^M\|^2 + C_1 \Delta t \sum_{m=0}^{M-1} \|\xi_h^{m+\theta}\|_a^2 \le \|\xi_h^0\|^2 + C_2 \Delta t \sum_{m=0}^{M-1} \|r^{m+\theta}\|_*^2. \tag{A.4}$$

We estimate the quantities $\|r_j^m\|_*$, $j = 1, \dots, 4$. For $j = 1, 2, 3$ the estimates can be found in [25, Section 5]. For clarity, we restate them.

$$\|r_1^m\|_* \le C \begin{cases} (\Delta t)^{\frac{1}{2}} \Big(\int_{t_m}^{t_{m+1}} \|\ddot{\widetilde{u}}(\tau)\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} & \theta \in [0,1] \\ (\Delta t)^{\frac{3}{2}} \Big(\int_{t_m}^{t_{m+1}} \|\dddot{\widetilde{u}}(\tau)\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} & \theta = \frac{1}{2} \end{cases} \tag{A.5}$$

$$\|r_2^m\|_* \le C(\Delta t)^{-\frac{1}{2}} h^{s-r} \Big(\int_{t_m}^{t_{m+1}} \|\dot{\widetilde{u}}(\tau)\|_{\mathcal{H}^{s-r}}^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} \tag{A.6}$$

$$\|r_3^m\|_* \le C h^{s-r} \|\widetilde{u}^{m+\theta}\|_{\mathcal{H}^s} \tag{A.7}$$

To estimate r_4^m, we assume that the bilinear form $\widetilde{a}(\delta\eta, \cdot, \cdot)$ is continuous on $V \times V$. Hence

$$|(r_4^m, v_h)| \le \widetilde{\alpha} \|u^{m+\theta} - u_h^{m+\theta}\|_a \|v_h\|_a$$

We obtain

Lemma A.5. *Assume $\widetilde{u}(x,t)$ is sufficiently smooth in $[0,T] \times \overline{\mathcal{D}}$ and assume that $\widetilde{a}(\delta\eta, \cdot, \cdot)$ is continuous on $V \times V$. Then there holds*

$$\|r^m\|_* \le \sum_{j=1}^{3} \|r_j^m\|_* + C\|u^{m+\theta} - u_h^{m+\theta}\|_a,$$

with $\|r_j^m\|_*$ given by (A.5)–(A.7).

We are able to prove Theorem 3.3.

Proof (of Theorem 3.3). By definition $e_h^m = \eta^m + \xi_h^m$, $m = 0, 1, \dots, M-1$. Hence

$$\|e_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|e_h^{m+\theta}\|_a^2 \le 2\Big(\|\eta^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\eta^{m+\theta}\|_a^2\Big) + 2\Big(\|\xi_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\xi_h^{m+\theta}\|_a^2\Big)$$

The first term can be estimated by the approximation property (2.15). For the second term we have by the stability (A.4)

$$\|\xi_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\xi_h^{m+\theta}\|_a^2 \quad \le \quad \|\xi_h^0\|^2 + C_2 \Delta t \sum_{m=0}^{M-1} \|r^{m+\theta}\|_*^2.$$

Using the estimates for $\|r^{m+\theta}\|_*$ of Lemma A.5 and the convergence result Lemma A.2 for the sequence $\{u_h^m\}$ finishes the proof. □

A.2 Proof of Theorem 3.6

We estimate the error $e_h^m := D^\alpha u^m - D_h^\alpha u_h^m$. We consider the splitting

$$e_h^m = \underbrace{(D^\alpha u^m - P_h D^\alpha u^m)}_{\eta^m} + \underbrace{(P_h D^\alpha u^m - P_h D_h^\alpha u^m)}_{\nu_h^m} + \underbrace{(P_h D_h^\alpha u^m - D_h^\alpha u_h^m)}_{\xi_{h,\alpha}^m}.$$

In order for $D_h^\alpha u^m$ and $D_h^\alpha u_h^m$ being well defined, we extend u^m and u_h^m by zero to all of $\mathbb{R}^d$.

If u^m is sufficiently smooth such that $D^\alpha u^m \in \mathcal{H}^s$, η^m can be estimated using the approximation property (2.15). If we further assume that the projector $P_h : V \to V_h$ is uniformly stable (i.e. there exists a constant C independent of h such that $\|P_h v\|_V \le C\|v\|_V$), the term ν_h^m can be estimated using the approximation property (3.8) of D_h^α. It therefore remains to estimate $\xi_{h,\alpha}^m$.

We shall need the following subspace of V_h. For $\mathcal{D}_0 \subset\subset \mathcal{D}$ we denote by $V_h(\mathcal{D}_0)$ the space

$$V_h(\mathcal{D}_0) = \{v \in V_h \mid \operatorname{supp} v \subsetneqq \overline{\mathcal{D}}_0\} \subset V_h.$$

We may assume that $\mathcal{D}_0$ is such that $\varphi \in V_h(\mathcal{D}_0)$ implies $D_h^\alpha \varphi \in V_h$. It is obviously sufficient to consider $\xi_{h,\alpha}^m := P_h \partial_h^\alpha u^m - \partial_h^\alpha u_h^m$.

Lemma A.6. *Assume $u \in C^1([0,T]; H)$. Assume that the operators P_h and ∂_h^α commute. Then $\{\xi_{h,\alpha}^m\}_{m=0}^{M-1}$ satisfy for any $\alpha \in \mathbb{N}_0^d$, $|\alpha| = n \ge 0$ the θ-scheme: Given $\xi_{h,\alpha}^0 = P_h \partial_h^\alpha u_0 - \partial_h^\alpha u_h^0$, for $m = 0, 1, \dots, M-1$*

$$(\Delta t)^{-1}(\xi_{h,\alpha}^{m+1} - \xi_{h,\alpha}^m, v_h) + a(\xi_{h,\alpha}^{m+\theta}, v_h) = (r^m, v_h), \quad \forall v_h \in V_h(\mathcal{D}_0), \qquad \text{(A.8)}$$

with weak residual $r^m : V_h \to \mathbb{R}$ *given by*

$$r^m = \sum_{j=1}^{4} r_j^m$$

where

$$\begin{aligned}
(r_1^m, v_h) &:= (-1)^{|\alpha|}\big((\Delta t)^{-1}(u^{m+1} - u^m) - \dot{u}^{m+\theta}, \partial_h^\alpha v_h\big),\\
(r_2^m, v_h) &:= (-1)^{|\alpha|}\big((\Delta t)^{-1}(P_h u^{m+1} - P_h u^m) - (\Delta t)^{-1}(u^{m+1} - u^m), \partial_h^\alpha v_h\big),\\
(r_3^m, v_h) &:= (-1)^{|\alpha|} a\big(P_h u^{m+\theta} - u^{m+\theta}, \partial_h^\alpha v_h\big),\\
(r_4^m, v_h) &:= -\int_{\mathcal{D}} \sum_{\beta<\alpha} C_{\alpha,\beta}\Big\{\sum_{i,j=1}^{d}(\partial_h^{\alpha-\beta}\mathcal{Q}_{ij})\big(T_h^{\alpha-\beta}\partial_h^\beta D_i(P_h u^{m+\theta} - u_h^{m+\theta})\big)D_j v_h\\
&\qquad + \sum_{i=1}^{d}(\partial_h^{\alpha-\beta} b_i)\big(T_h^{\alpha-\beta}\partial_h^\beta D_i(P_h u^{m+\theta} - u_h^{m+\theta})\big)v_h\Big\}\mathrm{d}x
\end{aligned}$$

with $C_{\alpha,\beta} := \begin{pmatrix} \alpha \\ \alpha-\beta \end{pmatrix}$.

Proof. Recall that $u \in C^1([0,T],H)$ implies

$$(\dot{u}^{m+\theta}, v) + a(u^{m+\theta}, v) = (f^{m+\theta}, v), \quad \forall v \in V. \tag{A.9}$$

Let $v_h \in V_h(\mathcal{D}_0)$. Inserting $\xi_{h,\alpha}^m$ in the θ-scheme yields

$$\begin{aligned}
&(\Delta t)^{-1}(\xi_{h,\alpha}^{m+1} - \xi_{h,\alpha}^m, v_h) + a(\xi_{h,\alpha}^{m+\theta}, v_h)\\
&\quad = \big((\Delta t)^{-1}(P_h\partial_h^\alpha u^{m+1} - P_h\partial_h^\alpha u^m), v_h\big) + a(P_h\partial_h^\alpha u^{m+\theta}, v_h)\\
&\qquad - \big\{(\Delta t)^{-1}((\partial_h^\alpha u_h^{m+1} - \partial_h^\alpha u_h^m), v_h) + a(\partial_h^\alpha u_h^{m+\theta}, v_h)\big\}.
\end{aligned}$$

By the discrete Leibniz rule, we have

$$\begin{aligned}
&(\Delta t)^{-1}((\partial_h^\alpha u_h^{m+1} - \partial_h^\alpha u_h^m), v_h) + a(\partial_h^\alpha u_h^{m+\theta}, v_h)\\
&\quad = (-1)^{|\alpha|}\big((\Delta t)^{-1}(u_h^{m+1} - u_h^m), \partial_h^\alpha v_h\big) + (-1)^{|\alpha|}a(u_h^{m+\theta}, \partial_h^\alpha v_h) + R_h^\alpha(u_h^{m+\theta})\\
&\quad \overset{(2.16)}{=} (-1)^{|\alpha|}(f^{m+\theta}, \partial_h^\alpha v_h) + R_h^\alpha(u_h^{m+\theta})\\
&\quad \overset{(A.9)}{=} (-1)^{|\alpha|}\big(\dot{u}^{m+\theta}, \partial_h^\alpha v_h\big) + (-1)^{|\alpha|}a\big(u^{m+\theta}, \partial_h^\alpha v_h\big) + R_h^\alpha(u_h^{m+\theta}).
\end{aligned}$$

Here, we denote by $R_h^\alpha(u_h^{m+\theta})$ the residual term

$$\begin{aligned}
R_h^\alpha(u_h^{m+\theta}) &:= -\int_{\mathcal{D}} \sum_{\beta<\alpha} C_{\alpha,\beta}\Big\{\sum_{i,j=1}^{d}(\partial_h^{\alpha-\beta}\mathcal{Q}_{ij})\big(T_h^{\alpha-\beta}\partial_h^\beta D_i u_h^{m+\theta}\big)D_j v_h\\
&\qquad + \sum_{i=1}^{d}(\partial_h^{\alpha-\beta} b_i)\big(T_h^{\alpha-\beta}\partial_h^\beta D_i u_h^{m+\theta})\big)v_h\Big\}\mathrm{d}x
\end{aligned}$$

Utilizing once more the discrete Leibniz rule and using that the operators P_h and ∂_h^α commute yields

$$\begin{aligned}
&\big((\Delta t)^{-1}(P_h\partial_h^\alpha u^{m+1} - P_h\partial_h^\alpha u^m), v_h\big) + a(P_h\partial_h^\alpha u^{m+\theta}, v_h) \\
&\quad = (-1)^{|\alpha|}\big(P_h u^{m+1} - P_h u^m, \partial_h^\alpha v_h\big) + (-1)^{|\alpha|} a(P_h u^{m+\theta}, \partial_h^\alpha v_h) + R_h^\alpha(P_h u^{m+\theta}).
\end{aligned}$$

The representation of r^m in (A.8) is now obvious. □

Remark A.7. *Note that the residual r_4^m in the Lemma A.6 satisfies $r_4^m = 0$ if the coefficients $\mathcal{Q}, b$ in (2.3) are constant, as it is the case for the Multidimensional Lévy model in Example 2.2.*

By the stability result Lemma A.1 we obtain the stability estimate for $\xi_{h,\alpha}^m$ in (A.8)

$$\|\xi_{h,\alpha}^M\|^2 + C_1\Delta t\sum_{m=0}^{M-1}\|\xi_{h,\alpha}^{m+\theta}\|_a^2 \le \|\xi_{h,\alpha}^0\|^2 + C_2\Delta t\sum_{m=0}^{M-1}\|r^{m+\theta}\|_*^2. \tag{A.10}$$

We estimate the residuals $\|r_j^m\|_*$. For $j = 1, 2, 3$ these are the same as Lemma A.5 (with u replacing $\widetilde{u}$, see also [25, Section 5]. To estimate $\|r_4^m\|$, we use again that the operators P_h and ∂_h^α commute. We find

$$\|r_4^m\|_* \le C\sum_{\beta<\alpha}\|\partial_h^\beta(P_h u^{m+\theta} - u_h^{m+\theta})\|_a = C\sum_{\beta<\alpha}\|\xi_{h,\beta}^{m+\theta}\|_a.$$

Rewriting the estimate (A.10) shows that for any $\alpha \in \mathbb{N}_0^d$ there holds

$$\begin{aligned}
&\|\xi_{h,\alpha}^M\|^2 + C_1\Delta t\sum_{m=0}^{M-1}\|\xi_{h,\alpha}^{m+\theta}\|_a^2 \\
&\quad \le \|\xi_{h,\alpha}^0\|^2 + 4C_2\Delta t\sum_{m=0}^{M-1}\sum_{j=1}^{3}\|r_j^{m+\theta}\|_a^2 + C\Delta t\sum_{m=0}^{M-1}\sum_{\beta<\alpha}\|\xi_{h,\beta}^{m+\theta}\|_a^2.
\end{aligned} \tag{A.11}$$

Since (A.11) holds for an arbitrary (but fixed) $\alpha \in \mathbb{N}_0^d$, we may iterate the inequality until $\beta = 0$ to obtain

$$\begin{aligned}
\|\xi_{h,\alpha}^M\|^2 + C_1\Delta t\sum_{m=0}^{M-1}\|\xi_{h,\alpha}^{m+\theta}\|_a^2 &\le C(\alpha)\Delta t\sum_{m=0}^{M-1}\|r_1^{m+\theta}\|_a^2 + \|r_2^{m+\theta}\|_a^2 + \|r_3^{m+\theta}\|_a^2 + \|\xi_{h,0}^{m+\theta}\|_a^2 \\
&\quad + C(\alpha)\sum_{\beta\le\alpha}\|\xi_{h,\beta}^0\|^2.
\end{aligned} \tag{A.12}$$

Proof (of Theorem 3.6). We have for $e_h^m = D^\alpha u^m - D_h^\alpha u_h^m$ and $M \geq 1$

$$\begin{aligned}
&\|e_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|e_h^{m+\theta}\|_a^2 \\
&\quad \leq 3\Big\{ \|\eta^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\eta^{m+\theta}\|_a^2 + \|\nu_h^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\nu_h^{m+\theta}\|_a^2 + \|\xi_{h,\alpha}^M\|^2 \\
&\qquad + \Delta t \sum_{m=0}^{M-1} \|\xi_{h,\alpha}^{m+\theta}\|_a^2 \Big\}.
\end{aligned}$$

If $D^\alpha u(t) \in \mathcal{H}^s$ for $t \in [0,T]$, the first term can be estimated with the approximation property (2.15). The second term is estimated using the uniform stability of the projector P_h and the approximation property (3.8) of D_h^α (provided $u(t) \in \mathcal{H}^{s+r+n}$ for $t \in [0,T]$). For the last term we have by (A.12) and the fact that $\Delta t \|\xi_{h,0}^{m+\theta}\|_a$ can be estimated also by the quantities $\|r_j^{m+\theta}\|_*$, $j = 1,2,3$

$$\|\xi_{h,\alpha}^M\|^2 + \Delta t \sum_{m=0}^{M-1} \|\xi_{h,\alpha}^{m+\theta}\|_a^2 \leq C \sum_{\beta \leq \alpha} \|\xi_{h,\beta}^0\|^2 + C\Delta t \sum_{m=0}^{M-1} \sum_{j=1}^{3} \|r_j^{m+\theta}\|_a^2.$$

Now conclude by using (A.5)–(A.7) to bound $\|r_j^{m+\theta}\|_*$ (replacing $\widetilde{u}$ by u), the quasi-optimality of $\partial_h^\beta u_h^0$ and the approximation property (2.15) to estimate $\|\xi_{h,\beta}^0\|$. □

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