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ASYMPTOTIC MODELLING OF CONDUCTIVE THIN SHEETS

Kersten Schmidt[1] and Sébastien Tordeux[2]

Abstract. We derive and analyse models which reduce conducting sheets of a small thickness ε in two dimensions to an interface and approximate their shielding behaviour by conditions on this interface. For this we consider a model problem with a conductivity scaled reciprocal to the thickness ε, which leads a nontrivial limit solution for $\varepsilon \to 0$. The functions of the expansion are defined hierarchically, i.e. order by order. Our analysis shows that for smooth sheets the models are well defined for any order and have optimal convergence meaning that the H^1-modelling error for an expansion with N terms is bounded by $O(\varepsilon^{N+1})$ in the exterior of the sheet and by $O(\varepsilon^{N+1/2})$ in its interior. We explicitely specify the models of order zero, one and two. Numerical experiments for sheets with varying curvature validate the theoretical results.

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Introduction

Many electric devices contain very thin conducting parts either for electromagnetic shielding [13, 16], or as casings, tank walls [9, 23] or supply lines [5]. The large aspect ratio of these sheets of about few millimetres or centimetres to metres or hundreds of micrometres to centimetres and the high conductivity causes variations in thickness direction in much smaller scales than in the longitudinal directions. Their discretisation by the finite element method (FEM) is challenging when the thickness ε of the thin sheets is considerably smaller than the size of neighbouring parts for three reasons. First, domains with such thin sheets are difficult to mesh by most mesh generators. Secondly, a discretisation on meshes with cell sizes of different magnitudes can lead to ill-conditioned matrices, and thirdly, meshes of good quality may also contain cells around the sheet with sizes comparable to the sheet thickness which leads to a high number of additional degrees of freedom. By reducing the thin sheet to an interface and by approximating its effect by conditions on this interface a highly accurate modelling with standard discretisation schemes like the FEM is possible.

The so called impedance boundary conditions (IBCs), first proposed by Shchukin [27] and Leontovich [19], are traditionally used for replacing solid conductors, where the domain is artificially confined, by an approximate boundary condition [1–3, 8, 11, 15, 26]. This technique is proved to be accurate for smooth sheets and can be readily implemented.

However, in the context of thin conducting sheets this technique of Shchukin and Leontovich has been seldomly applied. Interface conditions for thin sheets are often based on a tensor product ansatz of a set of simple functions in thickness direction and functions defined on the interface. The simplest approaches assume

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no variation in thickness direction, which leads to a surface quantity [5, 22]. Using two functions in thickness direction Krähenbühl and Muller [18] derived a relation between the mean value of the tangential component of the electric or magnetic field on the interfaces of the sheet and the jump of the magnetic or electric field between the interfaces. This approach for time-harmonic Maxwell's equations is adopted by various authors [13, 17, 20] and is known as impedance boundary condition for thin layers. The functions in thickness direction depending on the frequency ω and the conductivity σ take the skin effect into account. In similar IBCs for simulations in time domain underlying functions are changed dynamically [6, 21]. Unfortunately, these interface conditions are of low order, and even with the use of a larger number of functions in thickness direction [14] this type of conditions do not achieve higher orders [25].

In this paper we derive a high order approximation technique to deal with thin sheets based on arguments similar to those that was used to derive IBCs. We attain these approximations in the context of a 2D scalar model problem with a smooth thin dissipative sheet.

The model problem defined in Section 1 includes the two major effects, the shielding and the skin effect. We investigate an asymptotics of constant shielding for $\varepsilon \to 0$ by scaling the conductivity c like $1/\varepsilon$. For this asymptotics we derive the problems defining together the expansion functions of the solution of arbitrary order inside and outside the sheet in Section 2. Then, in Section 3 we will rearrange the problems leading to hierarchical coupled problems defining the expansion functions for each order with the knowledge of the functions of previous orders only. We will decouple these problems, introduce their variational formulation and show existence and uniqueness of the internal and external expansion functions in Section 4. Then, in Section 5 we analyse the modelling error and give the models for the first three orders explicitely in Section 6. Finally, we describe in Section 7 the numerical discretisation of the asymptotic expansion models and the original model by means of high-order finite elements and show results for the modelling error in various indicators in dependence of the sheet thickness. These numerical simulations demonstrate the sharpness of the bounds for the modelling error.

1. Problem definition

Let Ω be a domain in $\mathbb{R}^2$ and $\Omega^\varepsilon_{\text{int}}$ be the sub-domain occupied by a sheet of thickness $\varepsilon > 0$ with conductivity c. The remaining sub-domain $\Omega^\varepsilon_{\text{ext}} := \Omega \backslash \overline{\Omega^\varepsilon_{\text{int}}}$ is non-conducting and we denote the conductivity function $c(\underline{x})$, where $c(\underline{x}) = c$ for $\underline{x} \in \Omega^\varepsilon_{\text{int}}$ and $c(\underline{x}) = 0$ otherwise. We call the sub-domain of the thin conducting sheet the interior and the non-conducting sub-domain the exterior.

Let u^ε be the solution of the problem

$$\begin{aligned} -\Delta u^\varepsilon(\underline{x}) + c(\underline{x}) u^\varepsilon(\underline{x}) &= f(\underline{x}) \quad \text{in } \Omega, \\ u^\varepsilon(\underline{x}) &= g(\underline{x}) \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with the source term $f(\underline{x})$ vanishing in $\Omega^\varepsilon_{\text{int}}$ and the Dirichlet data $g(\underline{x})$. This model problem borrows the eddy-current model in 2D and includes the skin and shielding effects. We use a bounded domain Ω and Dirichlet boundary conditions for sake of simplicity. However, the boundary condition is of no importance in the derivation of the thin sheet models and can be replaced easily, also by suitable radiation conditions for an unbounded domain.

We make the following assumptions on the sheet. The mid-line Γ_m is given as C^∞ continuous and C^∞ invertible map $\underline{x}_m(t)$ from a 1D torus (note that Γ_m is hence closed), identified with a reference interval $\hat{\Gamma} \subset \mathbb{R}$. Furthermore, we assume Γ_m to have a positive distance to the boundary of Ω, and, for simplicity, $|\underline{x}'_m(t)| = 1$, *i.e.* t is an arc length parameter. The left normed normal vector and the curvature of the sheet are denoted by $\underline{n}(t)$ and $\kappa(t)$, and the normal derivative by $\partial_n = \nabla \cdot \underline{n}$. Hence, we can define a parametrisation of the sheet

$$\underline{x}(t, s) = \underline{x}_m(t) + s\underline{n}(t)$$

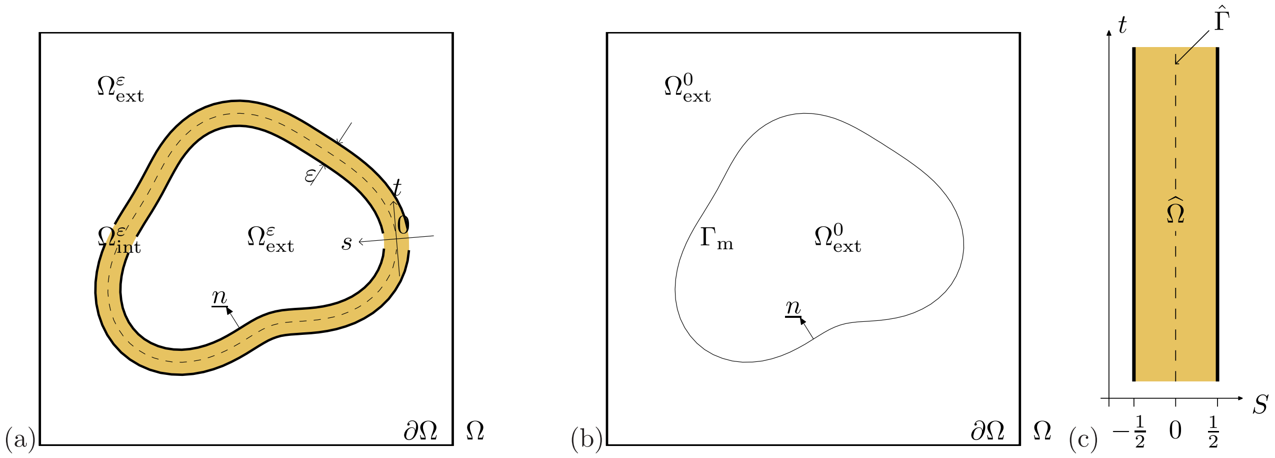


FIGURE 1. (a) Family of geometries for the family of problems for $u^\varepsilon(x)$. (b) Limit geometry for $\varepsilon \to 0$. (c) Normalised interior sub-domain.

over the parameter domain $\widehat\Omega^\varepsilon := \hat\Gamma \times \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, where $s \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$ (see Figure 1(a)). Due to the regularity of its midline Γ_{m} we can assert for the sheet that

$$C_\kappa^{-1} \le 1 + s\kappa(t) \le C_\kappa \quad \forall\,(t,s) \text{ in } \widehat\Omega^\varepsilon, \tag{2}$$

for ε small enough with a positive constant C_κ. Finally, we denote the interfaces of the sheet for $s = \pm\frac{\varepsilon}{2}$ by Γ^ε_+ and Γ^ε_- and its union by Γ^ε.

For an asymptotic analysis we embed the problem (1) for a sheet of a particular thickness into a family of problems with varying thicknesses and a conductivity depending on the respective thickness. There are several possible scalings of the conductivity with decreasing thickness ε, *e.g.* one can consider $c = \frac{c_0}{\varepsilon^\alpha}$ for different parameters α. The choice $c = c_0/\varepsilon$ is a borderline case between a perfect shielding ($\alpha > 1$) and no shielding ($\alpha < 1$) and corresponds asymptotically to a constant shielding [25]. Therefore, this choice is of practical interest.

Hence, we look for the solution $u^\varepsilon \in H^1(\Omega)$ satisfying

$$\begin{aligned}
-\Delta u^\varepsilon_{\mathrm{ext}} &= f && \text{in } \Omega^\varepsilon_{\mathrm{ext}},\\
-\Delta u^\varepsilon_{\mathrm{int}} + \frac{c_0}{\varepsilon}\, u^\varepsilon_{\mathrm{int}} &= 0 && \text{in } \Omega^\varepsilon_{\mathrm{int}}, \quad \Re\, c_0 \ge 0,\\
u^\varepsilon_{\mathrm{ext}} &= g && \text{on } \partial\Omega,\\
u^\varepsilon_{\mathrm{ext}}(t, \pm\tfrac{\varepsilon}{2}) &= u^\varepsilon_{\mathrm{int}}(t, \pm\tfrac{\varepsilon}{2}) && \text{on } \hat\Gamma,\\
\partial_n u^\varepsilon_{\mathrm{ext}}(t, \pm\tfrac{\varepsilon}{2}) &= \partial_n u^\varepsilon_{\mathrm{int}}(t, \pm\tfrac{\varepsilon}{2}) && \text{on } \hat\Gamma,
\end{aligned} \tag{3}$$

where we denote by $u^\varepsilon_{\mathrm{ext}}$ the solution restricted to $\Omega^\varepsilon_{\mathrm{ext}}$ and by $u^\varepsilon_{\mathrm{int}}$ its restriction to $\Omega^\varepsilon_{\mathrm{int}}$. We assume for a positive constant ε_0, that $f \in C^\infty(\Omega^{\varepsilon_0}_{\mathrm{ext}})$, and $g \in C^\infty(\partial\Omega)$ and $\partial\Omega$ to be C^∞.

2. Derivation of the coupled problems

In this section, we derive two asymptotic expansions of the exact solution, one in each of the exterior and interior sub-domains. These two expansions are defined by a coupled problem.

2.1. **The exterior and interior asymptotic expansions of the solution**

The *exterior asymptotic expansion* corresponds to the asymptotic expansion of u^ε restricted to $\Omega^\varepsilon_{\text{ext}}$. It consists in a formal power series

$$u^\varepsilon_{\text{ext}}(\underline{x}) \;=\; \sum_{i=0}^{\infty} \varepsilon^i u^i_{\text{ext}}(\underline{x}) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \quad \text{ in } \Omega^\varepsilon_{\text{ext}}, \tag{4}$$

in which the terms of the asymptotic expansion are independent of ε and defined on $\Omega^0_{\text{ext}} = \Omega \setminus \Gamma_{\text{m}}$ (see Figure 1(c)), the limit of $\Omega^\varepsilon_{\text{ext}}$ for $\varepsilon \to 0$.

The *interior asymptotic expansion* is an asymptotic expansion of u^ε restricted to $\Omega^\varepsilon_{\text{int}}$. In order to introduce the normalised domain $\widehat{\Omega} := \hat{\Gamma} \times [-\frac{1}{2}, \frac{1}{2}]$ (see Figure 1(c)), we consider the *stretched variable*

$$S = \varepsilon^{-1} s. \tag{5}$$

The normalised representation of a function v defined in $\Omega^\varepsilon_{\text{int}}$ is denoted by its capital letter V: $v(\underline{x}) = v(t,s) = V(t,S)$. The interior asymptotic expansion is postulated to be a formal power series in ε

$$U^\varepsilon_{\text{int}}(t,S) \;=\; \sum_{i=0}^{\infty} \varepsilon^i U^i_{\text{int}}(t,S) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \quad \text{ in } \widehat{\Omega}, \tag{6}$$

whose terms $U^\varepsilon_{\text{int}}$ are independent of ε and defined on $\widehat{\Omega}$.

Currently, we do not give a mathematical sense to this expansion, even if the formal computation makes sense. The expansion of the exact solution by a power series in ε emerges as a proper choice, because all the expansions involve only polynomials in ε. This ansatz of a power series in ε will be ultimately validated by Theorem 5.1.

In the remainder of this section we derive a coupled problem defining the functions u^i_{ext} and U^i_{int}.

The coupled problem

Find the families of functions $(u^i_{\text{ext}})_{i\in\mathbb{N}_0}$ and $(U^i_{\text{int}})_{i\in\mathbb{N}_0}$ such that for all $i \in \mathbb{N}_0$

$$-\Delta u^i_{\text{ext}} = f\delta^i_0 \qquad \text{in } \Omega^0_{\text{ext}}, \tag{7a}$$

$$u^i_{\text{ext}} = g\delta^i_0 \qquad \text{on } \partial\Omega, \tag{7b}$$

$$\partial^2_S U^i_{\text{int}}(t,S) = c_0 U^{i-1}_{\text{int}}(t,S) - \sum_{\ell=1}^{i} \Delta_\ell U^{i-\ell}_{\text{int}}(t,S) \qquad \text{in } \widehat{\Omega}. \tag{7c}$$

$$U^i_{\text{int}}(t,\pm\tfrac{1}{2}) - u^i_{\text{ext}}(t,\pm 0) = \sum_{\ell=1}^{i} \left(\pm\frac{1}{2}\right)^{\ell} \frac{1}{\ell\,!} \partial^\ell_s u^{i-\ell}_{\text{ext}}(t,\pm 0) \qquad \text{on } \Gamma_{\text{m}}, \tag{7d}$$

$$\partial_S U^i_{\text{int}}(t,\pm\tfrac{1}{2}) = \sum_{\ell=1}^{i} \left(\pm\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!} \partial^\ell_s u^{i-\ell}_{\text{ext}}(t,\pm 0), \qquad \text{on } \Gamma_{\text{m}}. \tag{7e}$$

where we use the Kronecker symbol, $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$, and the differential operators Δ_ℓ for $\ell \in \mathbb{N}$ which are given by

$$\begin{aligned} \Delta_\ell(t,S) \;&=\; \widehat{\Delta}^0_\ell(t) S^{\ell-2} + \widehat{\Delta}^1_\ell(t) S^{\ell-1}\partial_S, \\ \widehat{\Delta}^0_\ell(t) &= (-\kappa(t))^{\ell-2}(\ell-1)\left(\partial^2_t + \frac{\ell-2}{2}\frac{\kappa'(t)}{\kappa(t)}\partial_t\right) \quad \text{and} \quad \widehat{\Delta}^1_\ell(t) = -(-\kappa(t))^{\ell}. \end{aligned} \tag{8}$$

Equations (7a) and (7b) are readily to derive by inserting (4) in (3) and identifying terms of the same order in ε. More steps, however, are needed to obtain the leading equation for $U^i_{\rm int}$. It relies on the asymptotic expansion

$$\Delta \;=\; \varepsilon^{-2}\partial_S^2 + \sum_{\ell=1}^{L-1} \varepsilon^{\ell-2}\Delta_\ell + \varepsilon^{L-2}\mathbf{R}^L_\varepsilon \quad \text{for all } L \geq 1 \tag{9}$$

of the Laplacian expressed in local coordinates [8, 25]

$$\begin{aligned}\Delta = \partial_s^2 + \frac{\kappa(t)}{1+s\kappa(t)}\partial_s + \frac{1}{1+s\kappa(t)}\,\partial_t\left(\frac{1}{1+s\kappa(t)}\partial_t\right)&\\ = \varepsilon^{-2}\partial_S^2 + \frac{\varepsilon^{-1}\kappa(t)}{1+\varepsilon S\kappa(t)}\partial_S + \frac{1}{1+\varepsilon S\kappa(t)}\,\partial_t\left(\frac{1}{1+\varepsilon S\kappa(t)}\partial_t\right) &\qquad \text{with } (t,s)=(t,S\varepsilon),\end{aligned} \tag{10}$$

where for its remainder it holds for any $L \in \mathbb{N}$

$$\|\mathbf{R}^L_\varepsilon U\|_{L^2(\widehat{\Omega})} \leq C_L\; \|U\|_{H^2(\widehat{\Omega})}$$

Inserting (4) and (9) in (3) leads to equation (7c). The coupling conditions (7d) and (7e) need a specific treatment that will be detailed in Section 2.2.

Remark. The first terms of the expansion of the Laplacian are required in the sequel

$$\Delta_0 = \partial_S^2, \qquad\qquad \Delta_1 = \kappa(t)\partial_S, \qquad\qquad \Delta_2 = \partial_t^2 - \kappa^2(t)S\partial_S. \tag{11}$$

2.2. The Dirichlet and Neumann coupling conditions

In this section we derive the transmission conditions (7d) and (7e). These relations result from the exact Dirichlet and Neumann transmission conditions on $\Gamma_{\rm m}$ written in local coordinates

$$u^\varepsilon_{\rm ext}(t,\pm\tfrac{\varepsilon}{2}) = U^\varepsilon_{\rm int}(t,\pm\tfrac{1}{2}), \tag{12}$$

$$\partial_s u^\varepsilon_{\rm ext}(t,\pm\frac{\varepsilon}{2}) = \frac{1}{\varepsilon}\partial_S U^\varepsilon_{\rm int}(t,\pm\tfrac{1}{2}). \tag{13}$$

Since these conditions are written at $s = \pm\varepsilon/2$, Taylor expansions of $u^i_{\rm ext}$ expressed on $\Gamma_{\rm m}$ will be used to obtain conditions on a single interface. They require regularity of $u^i_{\rm ext}$ that will be a posteriori validated in Theorem 4.4, by assuming smoothness of $\Gamma_{\rm m}$.

Remark. The interfaces between the thin sheet and the exterior domain consist of two parts ($s = \pm\varepsilon/2$). Even if one decides to shift the position of the mid-line $\Gamma_{\rm m}$ ($s = 0$), at least one of the interfaces is not fixed with respect to ε. This leads to rather more complicated coupling conditions than for thin coatings [2, 3, 8], where the interface consists of one part only and can be fixed independently of ε.

The Dirichlet transmission condition (7d).

The Taylor expansion of $u^i_{\rm ext}$ reads

$$u^i_{\rm ext}(t,\pm\frac{\varepsilon}{2}) = \sum_{j=0}^{\infty}\left(\pm\frac{\varepsilon}{2}\right)^j \frac{1}{j!}\,\partial_s^j u^i_{\rm ext}(t,\pm 0) + \underset{\varepsilon\to 0}{o}\left(\varepsilon^\infty\right)\;. \tag{14}$$

Inserting the expansion (4), (6) and (14) into (12), we obtain

$$
\begin{aligned}
0 &= u^{\varepsilon}_{\text{ext}}(t, \pm\tfrac{\varepsilon}{2}) - U^{\varepsilon}_{\text{int}}(t, \pm\tfrac{1}{2}) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{\infty} \left(\pm\frac{\varepsilon}{2}\right)^j \frac{1}{j!}\, \partial_s^j u^i_{\text{ext}}(t, \pm 0) - U^i_{\text{int}}(t, \pm\tfrac{1}{2}) \right) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{i} \left(\pm\frac{1}{2}\right)^j \frac{1}{j!}\, \partial_s^j u^{i-j}_{\text{ext}}(t, \pm 0) - U^i_{\text{int}}(t, \pm\tfrac{1}{2}) \right) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \ .
\end{aligned} \tag{15}
$$

Identifying terms of same orders leads to the Dirichlet transmission condition (7d).
*Remark.*The exterior expansion functions u^i_{ext} may be discontinuous across Γ_{m}.

The Neumann transmission condition (7e).

The Taylor expansion of $\partial_s u^i_{\text{ext}}$ reads

$$
\partial_s u^i_{\text{ext}}(t, \pm\frac{\varepsilon}{2}) = \sum_{j=0}^{\infty} \left(\pm\frac{\varepsilon}{2}\right)^j \frac{1}{j!}\, \partial_s^{j+1} u^i_{\text{ext}}(t, \pm 0) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \ . \tag{16}
$$

Inserting the expansions (4), (6) and (16) into (15), we get

$$
\begin{aligned}
0 &= \varepsilon \partial_s u^{\varepsilon}_{\text{ext}}(t, \pm\frac{\varepsilon}{2}) - \partial_S U^{\varepsilon}_{\text{int}}(t, \pm\tfrac{1}{2}) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \left(\varepsilon \sum_{j=0}^{\infty} \left(\pm\frac{\varepsilon}{2}\right)^j \frac{1}{j!}\, \partial_s^{j+1} u^i_{\text{ext}}(t, \pm 0) - \partial_S U^i_{\text{int}}(t, \pm\tfrac{1}{2}) \right) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{i-1} \left(\pm\frac{1}{2}\right)^j \frac{1}{j!}\, \partial_s^{j+1} u^{i-j-1}_{\text{ext}}(t, \pm 0) - \partial_S U^i_{\text{int}}(t, \pm\tfrac{1}{2}) \right).
\end{aligned} \tag{17}
$$

Identifying terms of the same order results in the Neumann transmission condition (7e).

3. The hierarchical coupled problem

In the last section, we derived a coupled problem (7) that defines the families of exterior and interior terms $(u^i_{\text{ext}})_{i\in\mathbb{N}}$ and $(U^i_{\text{int}})_{i\in\mathbb{N}}$. However, these equations do not define the family hierarchically. Indeed, given $(U^i_{\text{int}}, u^i_{\text{ext}})_{i<k}$, (7) written for $i = k$, and not for all $i \in \mathbb{N}$, does not uniquely define $(U^k_{\text{int}}, u^k_{\text{ext}})$. This is due to the fact that there is no condition for the normal derivative $\partial_s u^k_{\text{ext}}(\underline{x})$ on the mid-line Γ_{m}.[1]

Deriving a necessary condition for the existence U^{i+1}_{int} leads to a formulation of (7) which permits the computation of $(U^i_{\text{int}}, u^i_{\text{ext}})_{i\leq k}$ step by step.

[1]Remember, that for second order differential equations two transmission conditions are needed. We have with (7d) a Dirichlet transmission condition and (7e) defines the normal derivative from the interior. Thus, a condition for the normal derivative from the exterior of the sheet is missing.

Symbols for the mean and the jump

For the sake of brevity let us introduce the following symbols for the jumps and the mean values of the expansion functions

$$[V](t) := V(t, \tfrac{1}{2}) - V(t, -\tfrac{1}{2}), \quad \{V\}(t) := \frac{1}{2}\left(V(t, \tfrac{1}{2}) + V(t, -\tfrac{1}{2})\right),$$
$$[v](t) := v(t, 0^+) - v(t, 0^-), \qquad \{v\}(t) := \frac{1}{2}\left(v(t, 0^+) + v(t, 0^-)\right),$$

and a symbol for either the jump or the mean value of the external expansion function of both sides of the mid-line Γ_{m}

$$[v]^n(t) := \begin{cases} [v](t) & n \text{ even} \\ 2\,\{v\}(t) & n \text{ odd.} \end{cases}$$

The latter symbol is convenient for terms resulting from the Taylor expansions (14) and (16), in which the sign changes from term to term and, hence, the difference is the jump and the mean value, in turns.

Additional condition for the normal derivative $\partial_s u^i_{\mathrm{ext}}(\underline{x})$

The missing condition for the normal derivative $\partial_s u^i_{\mathrm{ext}}(\underline{x})$ is the compatibility condition for (7c) and (7d) which is necessary for the existence of the internal functions U^{i+1}_{int}. Inserting (7c) and (7e) into the following equality for U^i_{int}

$$0 = \partial_S U^i_{\mathrm{int}}(t, +\tfrac{1}{2}) - \partial_S U^i_{\mathrm{int}}(t, -\tfrac{1}{2}) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_S^2 U^i_{\mathrm{int}}(t, S)\,\mathrm{d}S,$$

we obtain

$$0 = \sum_{\ell=1}^{i} \left(\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!} \left[\partial_s^\ell u^{i-\ell}_{\mathrm{ext}}\right]^{\ell-1}(t) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(c_0 U^{i-1}_{\mathrm{int}}(t, S) - \sum_{\ell=1}^{i} \Delta_\ell U^{i-\ell}_{\mathrm{int}}(t, S)\right)\mathrm{d}S. \tag{18}$$

Furthermore, inserting the equality $\Delta_1 = \kappa(t)\partial_S$ (see (11)) we can rewrite (18) for $i = i + 1$ as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (c_0 - \kappa(t)\partial_S) U^i_{\mathrm{int}}(t, S)\,\mathrm{d}S - \left[\partial_s u^i_{\mathrm{ext}}\right](t) = \sum_{\ell=1}^{i} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_{\ell+1} U^{i-\ell}_{\mathrm{int}}(t, S)\,\mathrm{d}S + \frac{1}{2^\ell\,\ell\,!}[\partial_s^{\ell+1} u^{i-\ell}_{\mathrm{ext}}]^\ell(t)\right), \tag{19}$$

which is a condition for the normal derivative involving only terms of order i. Adding this condition to (7) yields a problem which defines the expansion functions hierarchically.

The hierarchical coupled problem

For $i \in \mathbb{N}_0$, find u^i_{ext} and U^i_{int} such that

$$-\Delta u^i_{\text{ext}} = f\delta^i_0 \qquad \text{in } \Omega^0_{\text{ext}}, \tag{20a}$$

$$u^i_{\text{ext}} = g\delta^i_0 \qquad \text{on } \partial\Omega, \tag{20b}$$

$$\partial^2_S U^i_{\text{int}}(t,S) = c_0 U^{i-1}_{\text{int}}(t,S) - \sum_{\ell=1}^{i} \Delta_\ell U^{i-\ell}_{\text{int}}(t,S) \qquad \text{in } \widehat{\Omega}, \tag{20c}$$

$$U^i_{\text{int}}(t,\pm\tfrac{1}{2}) - u^i_{\text{ext}}(t,\pm 0) = \sum_{\ell=1}^{i} \left(\pm\frac{1}{2}\right)^{\ell} \frac{1}{\ell\,!}\,\partial^\ell_s u^{i-\ell}_{\text{ext}}(t,\pm 0) \qquad \text{on } \Gamma_{\text{m}}, \tag{20d}$$

$$\partial_S U^i_{\text{int}}(t,\pm\tfrac{1}{2}) = \sum_{\ell=1}^{i} \left(\pm\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!}\,\partial^\ell_s u^{i-\ell}_{\text{ext}}(t,\pm 0) \qquad \text{on } \Gamma_{\text{m}}, \tag{20e}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \big(c_0 - \kappa(t)\partial_S\big) U^i_{\text{int}}(t,S)\,\mathrm{d}S - \left[\partial_s u^i_{\text{ext}}\right](t) = \sum_{\ell=1}^{i} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_{\ell+1} U^{i-\ell}_{\text{int}}(t,S)\,\mathrm{d}S + \left(\frac{1}{2}\right)^{\ell} \frac{1}{\ell\,!} [\partial^{\ell+1}_s u^{i-\ell}_{\text{ext}}]^\ell(t) \right) \qquad \text{on } \Gamma_{\text{m}}. \tag{20f}$$

Remark. It can be easily proven that $U^i_{\text{int}}(t,S)$ is a polynomial of degree $2i$ in S for $i \in \mathbb{N}_0$. Thus, we define $U^0_{\text{int}}(t) := U^0_{\text{int}}(t,S)$.

In the next section, we prove the existence and uniqueness of the solution of problem (20).

4. Well-posedness of the hierarchical coupled problem

4.1. An algorithm to solve the hierarchical coupled problem

In this section we propose an algorithm to define successively the three functions

$$\widetilde{U}^i_{\text{int}}(t,S) := U^i_{\text{int}}(t,S) - \{U^i_{\text{int}}\}(t), \quad u^i_{\text{ext}}(\underline{x}) \quad \text{and} \quad \{U^i_{\text{int}}\}(t). \tag{21}$$

as the solutions of the following three problems which can be solved iteratively:

(i) Find $\widetilde{U}^i_{\text{int}}(t,S) : \widehat{\Omega} \longrightarrow \mathbb{C}$ such that

$$\begin{cases} \partial^2_S \widetilde{U}^i_{\text{int}}(t,S) = c_0 U^{i-1}_{\text{int}}(t,S) - \sum_{\ell=1}^{i} \Delta_\ell U^{i-\ell}_{\text{int}}(t,S) & \text{in } \widehat{\Omega}, \\ \partial_S \widetilde{U}^i_{\text{int}}(t,\pm\frac{1}{2}) = \sum_{\ell=1}^{i} \left(\pm\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!}\,\partial^\ell_s u^{i-\ell}_{\text{ext}}(t,\pm 0) & \text{on } \Gamma_{\text{m}}, \\ \{\widetilde{U}^i_{\text{int}}\}(t) = 0 & \text{on } \Gamma_{\text{m}}. \end{cases} \tag{22a}$$

(ii) Find $u^i_{\text{int}}(t,S) : \Omega^0_{\text{ext}} \longrightarrow \mathbb{C}$ such that

$$\left\{\begin{aligned} -\Delta u^i_{\text{ext}} &= f\delta^i_0 && \text{in } \Omega^0_{\text{ext}},\\ u^i_{\text{ext}} &= g\delta^i_0 && \text{on } \partial\Omega,\\ \left[u^i_{\text{ext}}\right](t) &= \left[\widetilde{U}^i_{\text{int}}\right](t) - \sum_{\ell=1}^{i}\left(\frac{1}{2}\right)^{\ell}\frac{1}{\ell\,!}\left[\partial^\ell_s u^{i-\ell}_{\text{ext}}\right]^{\ell}(t) && \text{on } \Gamma_{\text{m}},\\ \left[\partial_s u^i_{\text{ext}}\right](t) - c_0\left\{u^i_{\text{ext}}\right\}(t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} c_0\widetilde{U}^i_{\text{int}}(t,S)\,\mathrm{d}S - \kappa(t)\left[\widetilde{U}^i_{\text{int}}\right](t) + \\ &\quad \sum_{\ell=1}^{i}\Bigg(c_0\left(\frac{1}{2}\right)^{\ell}\frac{1}{\ell\,!}[\partial^\ell_s u^{i-l}_{\text{ext}}]^{\ell+1}(t) && \text{on } \Gamma_{\text{m}},\\ &\quad -\left(\frac{1}{2}\right)^{\ell}\frac{1}{\ell\,!}[\partial^{\ell+1}_s u^{i-\ell}_{\text{ext}}]^{\ell}(t)\\ &\quad -\int_{-\frac{1}{2}}^{\frac{1}{2}}\Delta_{\ell+1}U^{i-\ell}_{\text{int}}(t,S)\,\mathrm{d}S\Bigg). \end{aligned}\right. \tag{22b}$$

(iii) Find $\{U^i_{\text{int}}\} : \Gamma_{\text{m}} \longrightarrow \mathbb{C}$ such that

$$\{U^i_{\text{int}}\}(t) = \left\{u^i_{\text{ext}}\right\}(t) + \sum_{\ell=1}^{i}\left(\frac{1}{2}\right)^{\ell}\frac{1}{\ell\,!}[\partial^\ell_s u^{i-\ell}_{\text{ext}}]^{\ell+1}(t) \qquad \text{on } \Gamma_{\text{m}}. \tag{22c}$$

Lemma 4.1. *The problem* (22) *is equivalent to the problem* (20).

Proof. We first demonstrate that every solution of (20) is also a solution of (22). The equations (22a) are a direct consequence of (20c) and (20e) taking into account that $\{U^i_{\text{int}}\}(t)$ is a constant in S. The equation (22c) follows by applying the mean value operator to (20d). The third equation in (22b) follows by applying the jump operator to (20d). And, the fourth equation of (22b) is obtained, after calculation, by inserting $U^i_{\text{int}}(t,S) = \widetilde{U}^i_{\text{int}}(t,S) + \{U^i_{\text{int}}\}(t)$ and (22c) into (20f).

Applying the converse arguments, we can show that every solution of (22) is also solution of (20). □

4.2. **Variational framework**

The interior solution u^i_{ext} is defined by the system (22b). This section is devoted to the existence, uniqueness and regularity of the solution of such problems.

Given $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, $\gamma \in H^{\frac{1}{2}}(\Gamma_{\text{m}})$, $\delta \in H^{-\frac{1}{2}}(\Gamma_{\text{m}})$, we are looking for solution $u \in H^1(\Omega^0_{\text{ext}})$ of the problem

$$\left\{\begin{aligned} -\Delta u &= f, && \text{in } \Omega^0_{\text{ext}},\\ u &= g, && \text{on } \partial\Omega,\\ [u] &= \gamma, && \text{on } \Gamma_{\text{m}},\\ [\partial_n u] - c_0\{u\} &= \delta, && \text{on } \Gamma_{\text{m}}. \end{aligned}\right. \tag{23}$$

A classical route to deal with this non-homogeneous problem consists in introducing the harmonic offset function $\tilde{u} \in H^1(\Omega^0_{\mathrm{ext}})$ satisfying

$$\begin{cases} \Delta\tilde{u} = 0, & \text{in } \Omega^0_{\mathrm{ext}}, \\ \tilde{u}(\cdot, \pm 0) = \pm\frac{1}{2}\gamma(\cdot), & \text{on } \Gamma_{\mathrm{m}}, \\ \tilde{u} = g, & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Consequently, $\tilde{u}$ fulfils the jump condition $[\tilde{u}](t) = \gamma$ and has a vanishing mean $\{\tilde{u}\}(t) = 0$. Moreover, since $\Delta\tilde{u} = 0$ in Ω^0_{ext} and $\tilde{u} \in H^1(\Omega^0_{\mathrm{ext}})$, the jump of the normal trace $[\partial_n \tilde{u}]$ belongs to $H^{-\frac{1}{2}}(\Gamma_{\mathrm{m}})$.

Multiplying the first equation of (23) by a test function v, integrating over Ω and using the Green formula, we get the following weak formulation for $\hat{u} = u - \tilde{u}$:

$$\text{Find } \hat{u} \in H^1_0(\Omega) \text{ such that } \mathsf{a}(\hat{u}, v) = l(v) \qquad \forall v \in H^1_0(\Omega), \tag{25}$$

with the bilinear form $\mathsf{a}(\cdot,\cdot)$ and the linear form $l(\cdot)$ defined by

$$\mathsf{a}(\hat{u}, v) := \int_\Omega \nabla\hat{u}\cdot\nabla v \,\mathrm{d}\underline{x} + \int_{\Gamma_{\mathrm{m}}} c_0\, \hat{u} v \,\mathrm{d}t, \tag{26}$$

$$l(v) := \int_\Omega f v + \int_{\Gamma_{\mathrm{m}}} ([\partial_n \tilde{u}] - \delta)\, v \,\mathrm{d}t. \tag{27}$$

Using Poincaré-Friedrichs inequality [7], it is rather easy to prove the following lemma.

Lemma 4.2. *The system* (23)*, with data* $f \in L^2(\Omega)$*,* $g \in H^{1/2}(\partial\Omega), \gamma \in H^{1/2}(\Gamma_{\mathrm{m}})$ *and* $\delta \in H^{-1/2}(\Gamma_{\mathrm{m}})$ *admits a unique solution given that* $\Re\, c_0 \geq 0$.

Even if we seek the expansion functions $u \in H^1(\Omega^0_{\mathrm{ext}})$ they possess a higher regularity given that the mid-line of the sheet Γ_{m} and the source term f are smooth enough. This confirms the validity of the Taylor expansions (14) and (16).

Proposition 4.3. *For* $k_0 \in \mathbb{N}$*,* $f \in H^{k_0-2}(\Omega^0_{\mathrm{ext}})$*,* $g \in H^{k_0-1/2}(\partial\Omega)$*,* $\gamma \in H^{k_0-1/2}(\Gamma_{\mathrm{m}})$*,* $\delta \in H^{k_0-3/2}(\Gamma_{\mathrm{m}})$ *and* $\Gamma_{\mathrm{m}} \cup \partial\Omega$ C^{k_0}*-continuous, let* $u(\underline{x}) \in H^1(\Omega^0_{\mathrm{ext}})$ *be the solution of* (23)*.*

For any positive integer $k \leq k_0$*, there exists a constant* $C_k > 0$ *such that*

$$\|u\|_{H^k(\Omega^0_{\mathrm{ext}})} \leq C_k \left(\|f\|_{H^{k-2}(\Omega^0_{\mathrm{ext}})} + \|g\|_{H^{k-1/2}(\partial\Omega)} + \|\gamma\|_{H^{k-1/2}(\Gamma_{\mathrm{m}})} + \|\delta\|_{H^{k-3/2}(\Gamma_{\mathrm{m}})} \right).$$

Proof. Applying the techniques of Proposition 2.8 in [25] we get the statement of proposition. □

Remark. If the boundary of the domain is not smooth enough, the regularity statement of Proposition 4.3 has to be restricted to a sub-domain of Ω^0_{ext} excluding a neighbourhood of the boundary. A sub-domain of Ω^0_{ext} exluding the support of the source term f has to be taken, if this term is not smooth.

4.3. Existence and uniqueness of (u^i_{ext}) and (U^i_{int})

Theorem 4.4. *The sequences* (u^i_{ext}) *and* (U^i_{int}) *exist and are uniquely defined by* (22)*. For any* $k \in \mathbb{N}_0$ *and* $i \in \mathbb{N}_0$ *it holds* $u^i_{\mathrm{ext}} \in H^k(\Omega^0_{\mathrm{ext}})$*,* $\widetilde{U}^i_{\mathrm{int}} \in H^k(\widehat{\Omega})$*, and* $\{U^i_{\mathrm{int}}\} \in H^k(\widehat{\Omega})$ *and consequentely* $U^i_{\mathrm{int}} \in H^k(\widehat{\Omega})$ *as well.*

Proof. The proof is by induction in i.

For $i = 0$, the Sturm-Liouville problem (22a) with homogeneous data uniquely defines $\widetilde{U}^0_{\mathrm{int}}(t) = 0$ (see [28] for a presentation of Sturm-Liouville problems). The source term and the mid-line of the sheet are C^∞ by assumption. Thus, by Proposition 4.3 there exists for any $k \in \mathbb{N}$ a constant $C_{0,k}$ such that $\|u^0_{\mathrm{ext}}(t)\|_{H^k(\Omega^0_{\mathrm{ext}})} \leq$

$C_{0,k}$. Since $H^1(\Omega^0_{\text{ext}}) \subset L^2(\Omega^0_{\text{ext}})$ the same holds for $k = 0$. By (22a) we can assert that $U^0_{\text{int},0}(t) = U^0_{\text{int}}(t) = \{U^0_{\text{int}}\}(t) = \{u^0_{\text{ext}}\}(t) \in H^{k-1/2}(\Gamma_{\text{m}})$ for any $k \in \mathbb{N}$. Hence, the statement of the theorem is proven for $i = 0$.

Assume that the assertion holds for all integer $j < i$. We divide the rest of the proof in three steps. In (i) we prove the existence, uniqueness and regularity of $\widetilde{U}^i_{\text{int}}$ (i), in (ii) those of u^i_{ext} and in (iii) the regularity of $\{U^i_{\text{int}}\}$.

(i) The function $\widetilde{U}^i_{\text{int}}$ is defined by the Sturm-Liouville problem (22a). This function exists and is unique if and only if the source terms satisfy the compatibility[2] condition (18). This condition is fulfilled since it is equivalent to (20f) written for $i = i-1$ which holds as $(u^{i-1}_{\text{ext}}, U^{i-1}_{\text{int}})$ is the solution of (22) and by Lemma 4.1 also of (20). The regularity of $\widetilde{U}^i_{\text{int}}$ follows from the regularity of $(u^j_{\text{ext}})_{j<i}$ and $(U^j_{\text{int}})_{j<i}$.

(ii) The function u^i_{ext} is defined by (22b). Since $(u^j_{\text{ext}})_{j<i}$, $(U^j_{\text{int}})_{j<i}$ and $\widetilde{U}^i_{\text{int}}$ are regular, the existence, uniqueness and regularity of u^i_{ext} result from Lemma 4.2 and Proposition 4.3.

(iii) The function $\{U^i_{\text{int}}\}$ is defined by (22c). The smoothness of $\{U^i_{\text{int}}\}$ follows from the regularity of $(u^j_{\text{ext}})_{j\le i}$.

□

Remark. Although we assume a smooth boundary $\partial\Omega$ and a smooth source term f, this assumption is not needed for the existence and uniqueness of the expansion functions (Theorem 4.4) since the former terms of the expansion $(u^j_{\text{ext}})_{j<i}$ appear only on the mid-line Γ_{m} and regularity is required for the traces to this mid-line only.

5. Estimates of the modelling error

To obtain an approximation $u^{\varepsilon,N}$ of order $N \in \mathbb{N}_0$ of the exact solution u^ε we truncate the expansions of $u^\varepsilon_{\text{ext}}$ and $U^\varepsilon_{\text{int}}$ to the first $N+1$ terms

$$u^{\varepsilon,N}_{\text{ext}}(\underline{x}) := \sum_{i=0}^{N} \varepsilon^i u^i_{\text{ext}}(\underline{x}), \qquad \text{and} \qquad U^{\varepsilon,N}_{\text{int}}(t,S) := \sum_{i=0}^{N} \varepsilon^i U^i_{\text{int}}(t,S), \tag{29}$$

and use the notation $u^{\varepsilon,N}_{\text{int}}(t,s) := U^{\varepsilon,N}_{\text{int}}(t, s/\varepsilon)$. Now, we formulate the main result about the modelling error in the following Theorem.

Theorem 5.1 (The modelling error in the H^1-norm)**.** *For any $N \in \mathbb{N}_0$, there exists a constant C_N independent of ε such that*

$$\|u^\varepsilon_{\text{ext}} - u^{\varepsilon,N}_{\text{ext}}\|_{H^1(\Omega^\varepsilon_{\text{ext}})} + \sqrt{\varepsilon}\, \|u^\varepsilon_{\text{int}} - u^{\varepsilon,N}_{\text{int}}\|_{H^1(\Omega^\varepsilon_{\text{int}})} \le C_N \varepsilon^{N+1}. \tag{30}$$

Proof. In order to prove Theorem 5.1 we need to estimate the remainder $r^{\varepsilon,N+1}$

$$r^{\varepsilon,N+1}_{\text{ext}} = u^\varepsilon_{\text{ext}} - u^{\varepsilon,N}_{\text{ext}} \quad \text{and} \quad r^{\varepsilon,N+1}_{\text{int}} = u^\varepsilon_{\text{int}} - u^{\varepsilon,N}_{\text{int}}. \tag{31}$$

In Section 5.1, we identify residuals by inserting $r^{\varepsilon,N+1}$ in the model problem (3). Then, these residuals are bounded in Section 5.2. Finally, we conclude using a stability argument in Section 5.3. □

[2]This compatibility condition corresponds to a necessary condition for the existence of $\widetilde{U}^i_{\text{int}}$:

$$\partial_S \widetilde{U}^i_{\text{int}}(t, +\tfrac{1}{2}) - \partial_S \widetilde{U}^i_{\text{int}}(t, -\tfrac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial^2_S \widetilde{U}^i_{\text{int}}(t,S)\, \mathrm{d}S. \tag{28}$$

5.1. The problem for the remainder

Contrary to u^ε, the approximation $u^{\varepsilon,N}$ given in (29) does not exactly fulfil our model problem (3). Indeed, the exact solution u^ε has continuous Dirichlet and Neumann data on Γ^ε, whereas the Dirichlet and Neumann traces of $u^{\varepsilon,N}$ have jumps. Moreover, the partial differential equation in the sheet is also not satisfied exactly. More precisely, the remainder $r^{\varepsilon,N+1}$ solves the following system of equations

$$\left\{\begin{aligned} -\Delta r_{\text{ext}}^{\varepsilon,N+1} &= 0 && \text{in } \Omega_{\text{ext}}^\varepsilon,\\ -\Delta r_{\text{int}}^{\varepsilon,N+1} + \frac{c_0}{\varepsilon} r_{\text{int}}^{\varepsilon,N+1} &= \delta_{\text{int}}^{\varepsilon,N+1} && \text{in } \Omega_{\text{int}}^\varepsilon\\ r_{\text{ext}}^{\varepsilon,N+1}(t,\pm\frac{\varepsilon}{2}) - r_{\text{int}}^{\varepsilon,N+1}(t,\pm\frac{\varepsilon}{2}) &= \delta_{D,\pm}^{\varepsilon,N+1} && \text{on } \Gamma^\varepsilon,\\ \partial_s r_{\text{ext}}^{\varepsilon,N+1}(t,\pm\frac{\varepsilon}{2}) - \partial_s r_{\text{int}}^{\varepsilon,N+1}(t,\pm\frac{\varepsilon}{2}) &= \delta_{N,\pm}^{\varepsilon,N+1} && \text{on } \Gamma^\varepsilon,\\ r_{\text{ext}}^{\varepsilon,N+1} &= 0 && \text{on } \partial\Omega, \end{aligned}\right. \tag{32}$$

with the internal residual

$$\delta_{\text{int}}^{\varepsilon,N+1}(\underline{x}) := \left(-\Delta + \frac{c_0}{\varepsilon}\right)\left(u_{\text{int}}^\varepsilon(\underline{x}) - u_{\text{int}}^{\varepsilon,N}(\underline{x})\right) \overset{(3)}{=} -\left(-\Delta + \frac{c_0}{\varepsilon}\right) u_{\text{int}}^{\varepsilon,N}(\underline{x}) \overset{(29)}{=} -\sum_{i=0}^N \varepsilon^i \left(-\Delta + \frac{c_0}{\varepsilon}\right) u_{\text{int}}^i(\underline{x}), \tag{33a}$$

the residual of the Dirichlet jump

$$\delta_{D,\pm}^{\varepsilon,N+1}(t) := \underbrace{\left(u_{\text{ext}}^\varepsilon(t,\pm\frac{\varepsilon}{2}) - u_{\text{int}}^\varepsilon(t,\pm\frac{\varepsilon}{2})\right)}_{0 \text{ by } (3)} - \left(u_{\text{ext}}^{\varepsilon,N}(t,\pm\frac{\varepsilon}{2}) - u_{\text{int}}^{\varepsilon,N}(t,\pm\frac{\varepsilon}{2})\right) \overset{(29)}{=} \sum_{i=0}^N \varepsilon^i \left(U_{\text{int}}^i(t,\pm\frac{1}{2}) - u_{\text{ext}}^i(t,\pm\frac{\varepsilon}{2})\right) \tag{33b}$$

and the residual of the Neumann jump

$$\begin{aligned}\delta_{N,\pm}^{\varepsilon,N+1}(t) &:= \underbrace{\left(\partial_n u_{\text{ext}}^\varepsilon(t,\pm\frac{\varepsilon}{2}) - \partial_n u_{\text{int}}^\varepsilon(t,\pm\frac{\varepsilon}{2})\right)}_{0 \text{ by } (3)} - \left(\partial_n u_{\text{ext}}^{\varepsilon,N}(t,\pm\frac{\varepsilon}{2}) - \partial_n u_{\text{int}}^{\varepsilon,N}(t,\pm\frac{\varepsilon}{2})\right)\\ &\overset{(29)}{=} \sum_{i=0}^N \varepsilon^i \left(\frac{1}{\varepsilon}\partial_S U_{\text{int}}^i(t,\pm\frac{1}{2}) - \partial_n u_{\text{ext}}^i(t,\pm\frac{\varepsilon}{2})\right).\end{aligned} \tag{33c}$$

5.2. Consistency estimates

In this section, we estimate the residuals $\delta_{\text{int}}^{\varepsilon,N+1}$, $\delta_{D,\pm}^{\varepsilon,N+1}$ and $\delta_{N,\pm}^{\varepsilon,N+1}$ defined in (33).

5.2.1. *The internal residual*

Proposition 5.2 (Consistency error in the sheet)**.** *There exists $C_N > 0$, independent of ε, such that*

$$\|\delta_{\text{int}}^{\varepsilon,N+1}\|_{L^2(\Omega_{\text{int}}^\varepsilon)} \le C_N\, \varepsilon^{N-1/2}.$$

Proof. We write the interior residual given by (33a) in local coordinates, $\mathcal{D}_{\text{int}}^{\varepsilon,N+1}(t,S) := \delta_{\text{int}}^{\varepsilon,N+1}(t,s)$, with $s = S\varepsilon$. Inserting the expansion of the Laplace operator (9) we have

$$\mathcal{D}_{\text{int}}^{\varepsilon,N+1}(t,S) \;=\; -\sum_{i=0}^N \varepsilon^i \left(-\varepsilon^{-2}\left(\partial_S^2 + \sum_{\ell=1}^{N-i}\left(\varepsilon^\ell \Delta_\ell\right) + \varepsilon^{N-i+1}\mathbf{R}_\varepsilon^{N-i+1}\right) U_{\text{int}}^i(t,S) + \frac{c_0}{\varepsilon} U_{\text{int}}^i(t,S)\right). \tag{34}$$

With the convention $U_{\text{int}}^{-1} \equiv 0$, we collect the terms of same powers of ε

$$\mathcal{D}_{\text{int}}^{\varepsilon,N+1}(t,S) \;=\; \varepsilon^{N-1}\left(\sum_{i=0}^{N} \mathbf{R}_{\varepsilon}^{N-i+1} U_{\text{int}}^{i}(t,S) - c_0 U_{\text{int}}^{N}(t,S)\right) + \sum_{i=0}^{N} \varepsilon^{i-2} \underbrace{\left(\partial_S^2 U_{\text{int}}^{i}(t,S) - c_0 U_{\text{int}}^{i-1}(t,S) + \sum_{\ell=1}^{i} \Delta_\ell u_{\text{int}}^{i-\ell}(t,S)\right)}_{0 \text{ by } (22a)}. \quad (35)$$

Since $U_{\text{int}}^{i}(t,S)$ is independent of ε for all i by Theorem 4.4, we obtain using (9)

$$\left\|\mathcal{D}_{\text{int}}^{\varepsilon,N+1}\right\|_{L^2(\widehat{\Omega})} \le \varepsilon^{N-1}\left(\sum_{i=0}^{N} C\|U_{\text{int}}^{i}\|_{H^2(\widehat{\Omega})} + c_0\|U_{\text{int}}^{N}\|_{L^2(\widehat{\Omega})}\right) \le C_N \varepsilon^{N-1}. \quad (36)$$

Considering the curved geometry, see (2), we can write the integral in the original coordinates

$$\|\delta_{\text{int}}^{\varepsilon,N+1}\|^2_{L^2(\Omega_{\text{int}}^{\varepsilon})} \le C_\kappa \int_{\hat{\Gamma}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\delta_{\text{int}}^{\varepsilon,N+1}(t,s))^2 \,\mathrm{d}s\,\mathrm{d}t = C_\kappa \varepsilon \left\|\mathcal{D}_{\text{int}}^{\varepsilon,N+1}\right\|^2_{L^2(\widehat{\Omega})} \le C_N\, \varepsilon^{2N-1}.$$

The proof is complete. □

5.2.2. *A preliminary result on the Taylor expansion remainder*

In Sections 5.2.3 and 5.2.3, the estimates of the Dirichlet and the Neumann jump residuals will require the following proposition. We give the proof of this classical result for the sake of completeness.

Proposition 5.3 (Estimate of the remainder of the Taylor expansion)**.** *Let $L \in \mathbb{N}$.*

$$\exists C_L > 0 \quad \forall \varepsilon > 0 \quad \forall u \in H^L([-\frac{\varepsilon}{2}; \frac{\varepsilon}{2}] \setminus \{0\}) \quad \left|\mathbf{r}_{\varepsilon,\pm}^{L}(u)\right| \le\; C_L\; \varepsilon^{L-1/2}\; |u|_{H^L([0,\pm\frac{\varepsilon}{2}])} \quad (37)$$

with $\mathbf{r}_{\varepsilon,+}^{L}(u)$ and $\mathbf{r}_{\varepsilon,+}^{L}(u)$ the two reals defined by

$$\mathbf{r}_{\varepsilon,\pm}^{L}(u) := u(\pm\frac{\varepsilon}{2}) - \sum_{\ell=0}^{L-1}\left(\pm\frac{\varepsilon}{2}\right)^{\ell} u^{(\ell)}(\pm 0). \quad (38)$$

Proof. We use the well-known expression of the remainder term of Taylor polynomials

$$|\mathbf{r}_{\varepsilon,\pm}^{L}(u)| = \frac{1}{(L-1)!}\left|\int_0^{\pm\frac{\varepsilon}{2}} \left(\pm\frac{\varepsilon}{2} - s\right)^{L-1} \partial_s^L u(s)\,\mathrm{d}s\right|.$$

Bounding $\left|\pm\frac{\varepsilon}{2} - s\right|^{L-1}$ by its maximal value $\left(\frac{\varepsilon}{2}\right)^{L-1}$ and applying the Cauchy-Schwarz inequality, we obtain

$$\left|\int_0^{\pm\frac{\varepsilon}{2}} \left(\pm\frac{\varepsilon}{2} - s\right)^{L-1} \partial_s^L u(s)\,\mathrm{d}s\right| \le \left(\frac{\varepsilon}{2}\right)^{L-\frac12} \|\partial_s^L u\|_{L^2([0,\pm\frac{\varepsilon}{2}])} = \left(\frac{\varepsilon}{2}\right)^{L-\frac12} |u|_{H^L([0,\pm\frac{\varepsilon}{2}])}.$$

The composition of the estimates completes the proof. □

5.2.3. *The Dirichlet jump residual*

The functions $\delta^{\varepsilon,N+1}_{D,\pm}(t)$ for the Dirichlet jumps are defined on Γ^ε_+ or Γ^ε_-, respectively. However, we can regard them as functions on the mid-line Γ_{m}. In the following proposition we bound the L^2-norm of the error of the Dirichlet jumps evaluated on the mid-line. In Proposition 5.5, we will then define and estimate an extension function of the Dirichlet jump into the sheet.

Proposition 5.4 (Estimate of the Dirichlet jump residual). *There exists a constant $C_N > 0$, independent of ε, such that for $j = 0, 1$*

$$\|\partial_t^j \delta^{\varepsilon,N+1}_{D,\pm}\|_{L^2(\Gamma_{\mathrm{m}})} \le C_N \varepsilon^{N+1/2}. \tag{39}$$

Proof. The Dirichlet jump residual is given by (33b). Replacing $u^i_{\mathrm{ext}}(t, \pm\frac{\varepsilon}{2})$ by its Taylor expansion, see Proposition 5.3, we get

$$\delta^{\varepsilon,N+1}_{D,\pm}(t) \;=\; \sum_{i=0}^{N} \varepsilon^i \left(U^i_{\mathrm{int}}(t, \pm\frac{1}{2}) - \sum_{j=0}^{i} \left(\pm\frac{1}{2}\right)^j \frac{1}{j!} \partial_s^j u^{i-j}_{\mathrm{ext}}(t, \pm 0) \right) - \sum_{i=0}^{N} \varepsilon^i \mathbf{r}^{N-i+1}_{\varepsilon,\pm}(u^i_{\mathrm{ext}})(t). \tag{40}$$

Due to (20d), this simplifies to

$$\delta^{\varepsilon,N+1}_{D,\pm}(t) \;=\; -\sum_{i=0}^{N} \varepsilon^i \mathbf{r}^{N-i+1}_{\varepsilon,\pm}(u^i_{\mathrm{ext}})(t). \tag{41}$$

Applying (37), we get the estimate with C_N a generic constant depending on N

$$|\delta^{\varepsilon,N+1}_{D,\pm}(t)| \le \sum_{i=0}^{N} \varepsilon^i \left(C_{N-i}\, \varepsilon^{N-i+1/2}\, \|\partial_s^{N-i+1} u^i_{\mathrm{ext}}\|_{L^2([0,\pm\frac{\varepsilon}{2}])} \right) \le\; C_N\, \varepsilon^{N+1/2} \sum_{i=0}^{N} \|\partial_s^{N-i+1} u^i_{\mathrm{ext}}\|_{L([0,\pm\frac{\varepsilon}{2}])}. \tag{42}$$

Thus, we can bound the $L^2(\Gamma_m)$-norm of $\delta^{\varepsilon,N+1}_{D,\pm}$ by a triangular inequality

$$\|\delta^{\varepsilon,N+1}_{D,\pm}(t)\|_{L^2(\Gamma_{\mathrm{m}})} \;\le\; C_N\, \varepsilon^{N+\frac{1}{2}} \sum_{i=0}^{N} \|\partial_s^{N-i+1} u^i_{\mathrm{ext}}\|_{L^2(\Gamma_{\mathrm{m}}\times[0,\pm\frac{\varepsilon}{2}])}.$$

Considering the curvature, $C_\kappa^{-1} \le 1 + s\kappa(t)$ by (2), and since $\Gamma_{\mathrm{m}} \times (0, \pm\frac{\varepsilon}{2}] \subset \Omega^0_{\mathrm{ext}}$ we can write

$$\|\partial_s^{N-i+1} u^i_{\mathrm{ext}}\|^2_{L^2(\Gamma_{\mathrm{m}}\times[0,\pm\frac{\varepsilon}{2}])} \le C_\kappa \int_{\Gamma_{\mathrm{m}}} \int_0^{\pm\frac{\varepsilon}{2}} (\partial_s^{N-i+1} u^i_{\mathrm{ext}}(s,t))^2 (1 + s\kappa(t))\, \mathrm{d}s\, \mathrm{d}t \le C_\kappa |u^i_{\mathrm{ext}}|^2_{H^{N-i+1}(\Omega^0_{\mathrm{ext}})}.$$

Thus, we obtain $\|\partial_s^{N-i+1} u^i_{\mathrm{ext}}\|_{L^2(\Gamma_{\mathrm{m}}\times[0,\pm\frac{\varepsilon}{2}])} \le \|u^i_{\mathrm{ext}}\|_{H^{N+1}(\Omega^0_{\mathrm{ext}})}$. It follows that

$$\|\delta^{\varepsilon,N+1}_{D,\pm}(t)\|_{L^2(\Gamma_{\mathrm{m}})} \le C_N \varepsilon^{N+\frac{1}{2}} \sum_{i=0}^{N} |u^i_{\mathrm{ext}}|_{H^{N-i+1}(\Omega^0_{\mathrm{ext}})} \le C_N \varepsilon^{N+\frac{1}{2}} \|u^i_{\mathrm{ext}}\|_{H^{N+1}(\Omega^0_{\mathrm{ext}})}.$$

By inserting the regularity bound for the expansion functions u^i_{ext} (see Theorem 4.4) we obtain

$$\|\delta^{\varepsilon,N+1}_{D,\pm}(t)\|_{L^2(\Gamma_{\mathrm{m}})} \le C_{N,0}\, \varepsilon^{N+1/2},$$

which is our claim for $j = 0$. With similar arguments we find

$$\|\partial_t \delta^{\varepsilon,N+1}_{D,\pm}(t)\|_{L^2(\Gamma_{\mathrm{m}})} \;\le\; C_N\, \varepsilon^{N+\frac{1}{2}}\, \|\partial_t u^i_{\mathrm{ext}}\|_{H^{N+1}(\Omega^0_{\mathrm{ext}})} \le\; C_N\, \varepsilon^{N+\frac{1}{2}} \|u^i_{\mathrm{ext}}\|^2_{H^{N+2}(\Omega^0_{\mathrm{ext}})} \;\le\; C_N\, \varepsilon^{N+\frac{1}{2}}.$$

This completes the proof. □

Proposition 5.5 (An extension function of the Dirichlet jump residual)**.** *There exists an extension* $\delta_D^{\varepsilon,N+1}(t,s)$ *of* $\delta_{D,\pm}^{\varepsilon,N+1}(t)$ *defined in* (33b) *into* $\Omega_{\rm int}^\varepsilon$ *with*

$$\partial_s \delta_D^{\varepsilon,N+1}\left(t,\pm\frac{\varepsilon}{2}\right) = 0, \ \text{and}\ \exists\, C_N > 0\ \forall \varepsilon > 0\ :\ \|\delta_D^{\varepsilon,N+1}\|_{H^1(\Omega_{\rm int}^\varepsilon)} \leq C_N \varepsilon^N. \tag{43}$$

Proof. Let us define the piecewise linear, continuous function (see Figure 2)

$$\chi_\varepsilon(s) := \begin{cases} 0 & : -\varepsilon/2 < s < -\varepsilon/4, \\ \dfrac{1}{2} + \dfrac{2s}{\varepsilon} & : -\varepsilon/4 \leq s \leq \ \ \varepsilon/4\,, \\ 1 & : \ \ \varepsilon/4 < s < \ \ \varepsilon/2, \end{cases} \tag{44}$$

for which it holds

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \chi_\varepsilon^2(s)\,\mathrm{d}s = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (1-\chi_\varepsilon(s))^2\ \mathrm{d}s = \frac{5}{12}\,\varepsilon, \qquad \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\chi_\varepsilon'(s))^2\,\mathrm{d}s = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} ((1-\chi_\varepsilon)'(s))^2\,\mathrm{d}s = \frac{2}{\varepsilon}. \tag{45}$$

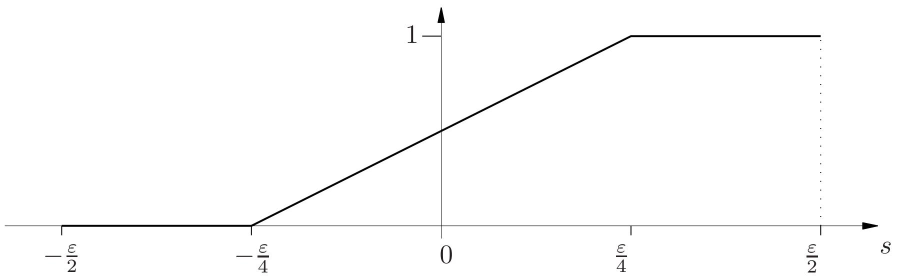


FIGURE 2. The extension function $\chi_\varepsilon(s)$.

Using this extension function $\chi_\varepsilon(s)$ we define an extension of the error in the Dirichlet jumps

$$\delta_D^{\varepsilon,N+1}(t,s) := \chi_\varepsilon(s)\,\delta_{D,+}^{\varepsilon,N+1}(t) + (1-\chi_\varepsilon(s))\,\delta_{D,-}^{\varepsilon,N+1}(t).$$

Applying the triangle estimate we can assert that

$$\left\|\delta_D^{\varepsilon,N+1}(t,s)\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} \ \leq\ \left\|\chi_\varepsilon(s)\,\delta_{D,+}^{\varepsilon,N+1}(t)\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} + \left\|(1-\chi_\varepsilon(s))\,\delta_{D,-}^{\varepsilon,N+1}(t)\right\|_{H^1(\Omega_{\rm int}^\varepsilon)}. \tag{46}$$

Due to (2), it holds for any $u \in H^1(\Omega_{\rm int}^\varepsilon)$

$$\begin{aligned} \left\|u\right\|_{H^1(\Omega_{\rm int}^\varepsilon)}^2 &= \int\limits_{\Gamma_{\rm m}} \int\limits_{-\varepsilon/2}^{\varepsilon/2} \left(\big(u(t,s)\big)^2 + \left(\frac{\partial_t u(t,s)}{1+s\kappa(t)}\right)^2 + (\partial_s u(t,s))^2 \right) (1+s\kappa(t))\,\mathrm{d}s\,\mathrm{d}t \\ &\leq C \int\limits_{\Gamma_{\rm m}} \int\limits_{-\varepsilon/2}^{\varepsilon/2} \left(\big(u(t,s)\big)^2 + \big(\partial_t u(t,s)\big)^2 + \big(\partial_s u(t,s)\big)^2 \right) \mathrm{d}s\,\mathrm{d}t \ \leq\ C\,\left\|u\right\|_{H^1(\Gamma_{\rm m}\times[-\varepsilon/2,\varepsilon/2])}^2. \end{aligned}$$

Consequently, it is sufficient to estimate the functions in $H^1(\Gamma_{\rm m}\times[-\varepsilon/2,\varepsilon/2])$

$$\left\|\delta_D^{\varepsilon,N+1}\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} \ \leq\ C\Big(\left\|\chi_\varepsilon(s)\,\delta_{D,+}^{\varepsilon,N+1}(t)\right\|_{H^1(\Gamma_{\rm m}\times[-\varepsilon/2,\varepsilon/2])} + \left\|(1-\chi_\varepsilon(s))\,\delta_{D,-}^{\varepsilon,N+1}(t)\right\|_{H^1(\Gamma_{\rm m}\times[-\varepsilon/2,\varepsilon/2])}\Big).$$

Due to the tensorial nature of the two terms on the right hand side, we can roughly bound

$$\|\delta_D^{\varepsilon,N+1}\|_{H^1(\Omega^\varepsilon_{\rm int})} \le C\Big(\|\chi_\varepsilon\|_{H^1(\Gamma_{\rm m})}\|\delta_{D,+}^{\varepsilon,N+1}\|_{H^1([-\varepsilon/2,\varepsilon/2])} + \|(1-\chi_\varepsilon)\|_{H^1(\Gamma_{\rm m})}\,\|\delta_{D,-}^{\varepsilon,N+1}\|_{H^1([-\varepsilon/2,\varepsilon/2])}\Big).$$

Inserting the estimates (39) and (45) we finally obtain (43). □

5.2.4. *The Neumann jump residual*

Proposition 5.6 (Estimate of the Neumann jump residual). *There exists a constant $C_N > 0$, independent of ε, such that*

$$\|\delta_{N,\pm}^{\varepsilon,N+1}\|_{L^2(\Gamma^\varepsilon)} \le C_N\,\varepsilon^{N-1/2}.$$

Proof. The error in the Neumann jump is given by

$$\begin{aligned}
\delta_{N,\pm}^{\varepsilon,N+1}(\underline{x}) &\overset{(33c)}{=} \sum_{i=0}^{N}\varepsilon^i\left(\frac{1}{\varepsilon}\partial_S U^i_{\rm int}(t,\pm\frac{1}{2}) - \partial_s u^i_{\rm ext}(t,\pm\frac{\varepsilon}{2})\right)\\
&\overset{(38)}{=} \sum_{i=0}^{N}\varepsilon^{i-1}\underbrace{\left(\partial_S U^i_{\rm int}(t,\pm\frac{1}{2}) - \sum_{j=0}^{i-1}\left(\pm\frac{1}{2}\right)^j\frac{1}{j!}\partial_s^{j+1}u^{i-j-1}_{\rm ext}(t,\pm 0)\right)}_{0\text{ by (7e)}} - \sum_{i=0}^{N}\varepsilon^i\mathbf{r}^{N-i}_{\varepsilon,\pm}(\partial_s u^i_{\rm ext})(t)\\
&= \quad -\sum_{i=0}^{N}\varepsilon^i\mathbf{r}^{N-i}_{\varepsilon,\pm}(\partial_s u^i_{\rm ext})(t),
\end{aligned}$$

where we inserted the Taylor polynomial of $\partial_s u^i_{\rm ext}(t,\pm\frac{\varepsilon}{2})$ with their remainder terms in the second step. Note, that $\mathbf{r}^L_{\varepsilon,\pm}(\partial_s u^i_{\rm ext})(t)$ depends on $t\in\hat\Gamma$ since $\partial_s u^i_{\rm ext}$ is a function of t. The terms in the first sum cancel due to the approximation of the Neumann continuity in (7e). Now, we use (37) to estimate the remainders of the truncated Taylor expansion:

$$\begin{aligned}
\Rightarrow |\delta_{N,\pm}^{\varepsilon,N+1}(t)| &\overset{(37)}{\le} \sum_{i=0}^{N}\varepsilon^i\left(C_{N-i}\,\varepsilon^{N-i+1/2}\,\|\partial_s^{N-i+2}u^i_{\rm ext}\|_{L^2([0,\pm\frac{\varepsilon}{2}])}\right)\\
&\le\; C_N\,\varepsilon^{N+1/2}\sum_{i=0}^{N}\|\partial_s^{N-i+2}u^i_{\rm ext}\|_{L^2([0,\pm\frac{\varepsilon}{2}])}.
\end{aligned}$$

The proof of the bound in the L^2-norms is then similar to the one of Proposition 5.5. □

5.3. **Proof of Theorem 5.1**

Let $\hat r_{\rm ext} := r^{\varepsilon,N+1}_{\rm ext}$, $\hat r_{\rm int} := r^{\varepsilon,N+1}_{\rm int} - \delta_D^{\varepsilon,N+1}$, with $\delta_D^{\varepsilon,N+1}$ the extension function of $\delta_{D,\pm}^{\varepsilon,N+1}$ of Proposition 5.5. Then, the function $\hat r$ is continuous over the interfaces Γ^ε of the sheet and inherits the vanishing trace on the boundary from $r^{\varepsilon,N+1}$. It lies consequently in $H^1_0(\Omega)$.

Multiplying (32) with a test function $v\in H^1_0(\Omega)$ and integrating by parts in $\Omega^\varepsilon_{\rm ext}$ and in $\Omega^\varepsilon_{\rm int}$ we get the variational formulation: Seek $\hat r\in H^1_0(\Omega)$, such that

$$\begin{aligned}
\int_{\Omega^\varepsilon_{\rm ext}}\nabla\hat r_{\rm ext}\cdot\nabla v_{\rm ext}\,{\rm d}\underline{x} + \int_{\Omega^\varepsilon_{\rm int}}\left(\nabla\hat r_{\rm int}\cdot\nabla v_{\rm int} + \frac{c_0}{\varepsilon}\hat r_{\rm int}v_{\rm int}\right){\rm d}\underline{x} &= \int_{\Gamma^\varepsilon_+} -\delta^{\varepsilon,N+1}_{N,+}v\,{\rm d}t + \int_{\Gamma^\varepsilon_-}\delta^{\varepsilon,N+1}_{N,-}v\,{\rm d}t\\
&+ \int_{\Omega^\varepsilon_{\rm int}}\left(\nabla\delta_D^{\varepsilon,N+1}\cdot\nabla v_{\rm int} + \frac{c_0}{\varepsilon}\delta_D^{\varepsilon,N+1}v_{\rm int}\right){\rm d}\underline{x} + \int_{\Omega^\varepsilon_{\rm int}}\delta^{\varepsilon,N+1}_{\rm int}v\,{\rm d}\underline{x}. \qquad (47)
\end{aligned}$$

For $\Re c_0 \geq 0$, the left hand side defines a $H_0^1(\Omega)$-elliptic continuous bilinear form. By the estimates of the Propositions 5.2, 5.5 and 5.6 the right hand side defines a $H^1(\Omega)$-continuous linear form. The Lax-Milgram lemma [24] ensures stability. Inserting the results of the Propositions 5.2, 5.5 and 5.6 yields

$$\begin{aligned}\|r^{\varepsilon,N+1}\|_{H^1(\Omega)} &\leq \|\hat{r}\|_{H^1(\Omega)} + \|\delta_D^{\varepsilon,N}\|_{H^1(\Omega_{\rm int}^\varepsilon)} \\ &\leq \tilde{C}\Bigg(\Big(2+\sqrt{\frac{c_0}{\varepsilon}}\Big)\underbrace{\|\delta_D^{\varepsilon,N+1}\|_{H^1(\Omega_{\rm int}^\varepsilon)}}_{O(\varepsilon^N)} + \sum_{\sigma=\{+,-\}}\underbrace{\|\delta_{N,\sigma}^{\varepsilon,N+1}\|_{L^2(\Gamma_\sigma^\varepsilon)}}_{O(\varepsilon^{N-\frac{1}{2}})} + \underbrace{\|\delta_{\rm int}^{\varepsilon,N+1}\|_{L^2(\Omega_{\rm int}^\varepsilon)}}_{O(\varepsilon^{N-\frac{1}{2}})}\Bigg) \leq C\,\varepsilon^{N-\frac{1}{2}},\end{aligned} \quad (48)$$

with $C > 0$ a constant independent of ε. Moreover, by definition (31)

$$r^{\varepsilon,N+1} = \varepsilon^{N+1}u^{N+1} + \varepsilon^{N+2}u^{N+2} + r^{\varepsilon,N+3}. \quad (49)$$

Using the fact that for every integer i, $\|u^i\|_{H^1(\Omega_{\rm ext}^\varepsilon)} = O(1)$ and $\|u^i\|_{H^1(\Omega_{\rm int}^\varepsilon)} = O(\varepsilon^{-1/2})$, inserting (48) into (49) and applying the triangle inequality we conclude that

$$\begin{aligned}\left\|r_{\rm ext}^{\varepsilon,N+1}\right\|_{H^1(\Omega_{\rm ext}^\varepsilon)} &\leq \varepsilon^{N+1}\left\|u_{\rm ext}^{N+1}\right\|_{H^1(\Omega_{\rm ext}^\varepsilon)} + \varepsilon^{N+2}\left\|u_{\rm ext}^{N+2}\right\|_{H^1(\Omega_{\rm ext}^\varepsilon)} + \left\|r_{\rm ext}^{\varepsilon,N+3}\right\|_{H^1(\Omega_{\rm ext}^\varepsilon)} \\ &\leq C_1\varepsilon^{N+1} + C_2\varepsilon^{N+2} + C_3\varepsilon^{N+3/2} \leq C\varepsilon^{N+1}, \\ \left\|r_{\rm int}^{\varepsilon,N+1}\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} &\leq \varepsilon^{N+1}\left\|u_{\rm int}^{N+1}\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} + \varepsilon^{N+2}\left\|u_{\rm int}^{N+1}\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} + \left\|r_{\rm int}^{\varepsilon,N+3}\right\|_{H^1(\Omega_{\rm int}^\varepsilon)} \\ &\leq C_1\varepsilon^{N+1/2} + C_2\varepsilon^{N+3/2} + C_3\varepsilon^{N+3/2} \leq C\varepsilon^{N+1/2}.\end{aligned}$$

6. The three first orders

In Section 4, the external function $u_{\rm ext}^i$ and the internal function $U_{\rm int}^i$ were defined by a coupled problem, see (22). We could use a finite element method for the approximation on two meshes – a first one for $\Omega_{\rm ext}^0$ and a second one for $\widehat{\Omega}$. Since this formulation is not common, we propose an equivalent definition of the internal and external functions by uncoupled problems, whose solutions will be much easier to approximate numerically.

More precisely, we elaborate a procedure that allows to compute the exterior functions of order 0, 1, and 2, with no need of the interior functions. This factorisation leads to three problems defining $u_{\rm ext}^0$, $u_{\rm ext}^1$ and $u_{\rm ext}^2$ involving only exterior fields of lower order, see (56), (62) and (64). The details for the second order will not be given.

6.1. **Preliminary results: replacing higher normal derivatives on the mid-line**

The asymptotic expansion models (22) involve derivatives of high order with respect to the normal direction. Because it is from a practical point of view easier to handle tangential derivatives than normal derivatives of the same order we intend to replace these higher normal derivatives. Due to the absence of a source term f in $\Omega_{\rm int}^\varepsilon$ for all ε smaller than ε_0, *i.e.*

$$-\Delta u_{\rm ext}^i(t,s) = 0, \quad s \in \left[-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right], \quad (50)$$

it is possible to replace the normal derivatives by derivatives in t.

Taking the two limits of the expression (10) of the Laplace operator for $s \to \pm 0$ we obtain

$$\Delta = \partial_n^2 + \kappa(t)\partial_n + \partial_t^2$$

and inserting the above expression into (50) yields

$$\partial_n^2 u_{\rm ext}^i(t,\pm 0) = -\kappa(t)\,\partial_n u_{\rm ext}^i(t,\pm 0) - \partial_t^2 u_{\rm ext}^i(t,\pm 0). \quad (51)$$

Applying the normal derivative ∂_s to (10) we get the expression

$$0 = \partial_s \Delta = \partial_s^3 - \frac{\kappa^2(t)}{(1+s\kappa(t))^2}\partial_s + \frac{\kappa(t)}{1+s\kappa(t)}\partial_s^2 - \frac{\kappa(t)}{(1+s\kappa(t))^2}\partial_t\left(\frac{\kappa(t)}{1+s\kappa(t)}\partial_t\right) + \frac{\kappa(t)}{1+s\kappa(t)}\partial_t\left(-\frac{\kappa(t)}{(1+s\kappa(t))^2}\partial_t + \frac{\kappa(t)}{1+s\kappa(t)}\partial_s\partial_t\right).$$

Taking the two limits for $s \to \pm 0$ we have

$$\begin{aligned}\partial_n^3 u_{\mathrm{ext}}^i(t,\pm 0) &= -\kappa(t)\partial_n^2 u_{\mathrm{ext}}^i(t,\pm 0) + \left(\kappa^2(t) - \partial_t^2\right)\partial_n u_{\mathrm{ext}}^i(t,\pm 0) + \left(2\kappa(t)\partial_t^2 + \kappa'(t)\partial_t\right) u_{\mathrm{ext}}^i(t,\pm 0)\\ &\overset{(51)}{=} \left(2\kappa^2(t) - \partial_t^2\right)\partial_n u_{\mathrm{ext}}^i(t,\pm 0) + \left(3\kappa(t)\partial_t^2 + \kappa'(t)\partial_t\right) u_{\mathrm{ext}}^i(t,\pm 0). \end{aligned} \tag{52}$$

Such expressions hold also for the jump and for the mean value of higher order derivatives

$$\left.\begin{array}{l}\left[\partial_n^2 u_{\mathrm{ext}}^i\right](t)\\ \left\{\partial_n^2 u_{\mathrm{ext}}^i\right\}(t)\end{array}\right\} = -\kappa(t)\left\{\begin{array}{l}\left[\partial_n u_{\mathrm{ext}}^i\right](t)\\ \left\{\partial_n u_{\mathrm{ext}}^i\right\}(t)\end{array}\right\} - \partial_t^2\left\{\begin{array}{l}\left[u_{\mathrm{ext}}^i\right](t)\\ \left\{u_{\mathrm{ext}}^i\right\}(t),\end{array}\right. \tag{53}$$

$$\left.\begin{array}{l}\left[\partial_n^3 u_{\mathrm{ext}}^i\right](t)\\ \left\{\partial_n^3 u_{\mathrm{ext}}^i\right\}(t)\end{array}\right\} = \left(2\kappa^2(t) - \partial_t^2\right)\left\{\begin{array}{l}\left[\partial_n u_{\mathrm{ext}}^i\right](t)\\ \left\{\partial_n u_{\mathrm{ext}}^i\right\}(t)\end{array}\right\} + \left(3\kappa(t)\partial_t^2 - \kappa'(t)\partial_t\right)\left\{\begin{array}{l}\left[u_{\mathrm{ext}}^i\right](t)\\ \left\{u_{\mathrm{ext}}^i\right\}(t).\end{array}\right. \tag{54}$$

6.2. **Order 0**

First, we express the internal function U_{int}^0 as expression of u_{ext}^0. Then, inserting this expression into (22b) leads to an uncoupled problem for u_{ext}^0.

6.2.1. *Internal function*

The internal function is given as the sum of the mean value $\{U_{\mathrm{int}}^0\}(t)$ and of the function $\widetilde{U}_{\mathrm{int}}^0(t,S)$, which are defined in (22a) and (22c), respectively. By evaluating these equations we find

$$\left.\begin{array}{r}\partial_S^2 \widetilde{U}_{\mathrm{int}}^0(t,S) = 0\\ \partial_S \widetilde{U}_{\mathrm{int}}^0(t,\pm\frac{1}{2}) = 0\\ \{\widetilde{U}_{\mathrm{int}}^0\}(t) = 0\end{array}\right\} \Rightarrow \widetilde{U}_{\mathrm{int}}^0(t,S) = 0 \quad \text{and} \quad \{U_{\mathrm{int}}^0\}(t) = \{u_{\mathrm{ext}}^0\}(t).$$

Consequently, the internal function is given by

$$U_{\mathrm{int}}^0(t,S) = U_{\mathrm{int}}^0(t) = \{u_{\mathrm{ext}}^0\}(t). \tag{55}$$

6.2.2. *External function*

Inserting $\widetilde{U}_{\mathrm{int}}^0 = 0$ into (22b) yields the completely uncoupled problem for the external function u_{ext}^0

$$\left\{\begin{array}{rll}-\Delta u_{\mathrm{ext}}^0(\underline{x}) = f(\underline{x}) & \text{in } \Omega_{\mathrm{ext}}^0,\\ u_{\mathrm{ext}}^0(\underline{x}) = g(\underline{x}) & \text{on } \partial\Omega,\\ \left[u_{\mathrm{ext}}^0\right](t) = 0 & \text{on } \Gamma_{\mathrm{m}},\\ \left[\partial_s u_{\mathrm{ext}}^0\right](t) - c_0\left\{u_{\mathrm{ext}}^0\right\}(t) = 0 & \text{on } \Gamma_{\mathrm{m}}.\end{array}\right. \tag{56}$$

Note that u^0_{ext} is uniquely defined by Lemma 4.2. As u^0_{ext} has no jump over Γ_{m} we denote $u^0_{\mathrm{ext}}(t) := u^0_{\mathrm{ext}}(t,\pm 0) = \{u^0_{\mathrm{ext}}\}(t)$. Thus, we can write the last equation of (56) as

$$\left[\partial_s u^0_{\mathrm{ext}}\right](t) - c_0\, u^0_{\mathrm{ext}}(t) = 0 \quad \text{on } \Gamma_{\mathrm{m}}. \tag{57}$$

6.3. **Order 1**

In the same way as for order 0 we express $\widetilde{U}^1_{\mathrm{int}}$ in terms of u^0_{ext} and u^1_{ext} and derive the uncoupled problem defining u^1_{ext}. Then, we replace a second normal derivative by a second tangential and a simple normal derivative. The resulting model for u^1_{ext} depends only on the external function of order 0.

6.3.1. *Internal function*

The internal function $U^1_{\mathrm{int}}(t,S)$ is given as the sum of the mean value $\{U^1_{\mathrm{int}}\}(t)$ and the function $\widetilde{U}^1_{\mathrm{int}}(t,S)$, which are defined in (22a) and (22c), respectively. For $i=1$, the problem (22a) takes the form

$$\begin{cases} \partial^2_S \widetilde{U}^1_{\mathrm{int}}(t,S) = c_0 U^0_{\mathrm{int}}(t) - \kappa(t)\, \underbrace{\partial_S U^0_{\mathrm{int}}(t)}_{0} = c_0 u^0_{\mathrm{ext}}(t) \\ \partial_S \widetilde{U}^1_{\mathrm{int}}(t,\pm\frac{1}{2}) = \partial_s u^0_{\mathrm{ext}}(t,\pm 0) \\ \{\widetilde{U}^1_{\mathrm{int}}\}(t) = 0. \end{cases} \tag{58}$$

Consequently, we can assert that

$$\widetilde{U}^1_{\mathrm{int}}(t,S) = \frac{c_0}{2} u^0_{\mathrm{ext}}(t)\, \left(S^2 - \tfrac{1}{4}\right) + \left\{\partial_s u^0_{\mathrm{ext}}\right\}(t)\, S. \tag{59}$$

From (22c) the mean value of the internal function is given by

$$\{U^1_{\mathrm{int}}\}(t) = \{u^1_{\mathrm{ext}}\}(t) + \tfrac{1}{4}[\partial_s u^0_{\mathrm{ext}}](t) \overset{(56)}{=} \{u^1_{\mathrm{ext}}\}(t) + \frac{c_0}{4} u^0_{\mathrm{ext}}(t), \tag{60}$$

and we can re-compose the internal function to

$$U^1_{\mathrm{int}}(t,S) = \frac{c_0}{2} u^0_{\mathrm{ext}}(t)\, \left(S^2 + \tfrac{1}{4}\right) + \left\{\partial_s u^0_{\mathrm{ext}}\right\}(t)\, S + \{u^1_{\mathrm{ext}}\}(t). \tag{61}$$

6.3.2. *External function*

Inserting (59) into (22b) we obtain a vanishing Dirichlet jump

$$\left[u^1_{\mathrm{ext}}\right](t) = \left[\widetilde{U}^1_{\mathrm{int}}\right](t) - \left\{\partial_s u^0_{\mathrm{ext}}\right\}(t) = \left\{\partial_s u^0_{\mathrm{ext}}\right\}(t) - \left\{\partial_s u^0_{\mathrm{ext}}\right\}(t) = 0,$$

and for the Neumann jump

$$\begin{aligned} \left[\partial_s u^1_{\mathrm{ext}}\right](t) - c_0\left\{u^1_{\mathrm{ext}}\right\}(t) = & \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} c_0 \widetilde{U}^1_{\mathrm{int}}(t,S)\,\mathrm{d}S}_{-\frac{c_0^2}{12}u^0_{\mathrm{ext}}(t) \text{ by } (59)} - \kappa(t) \underbrace{\left[\widetilde{U}^1_{\mathrm{int}}\right](t)}_{\{\partial_s u^0_{\mathrm{ext}}\}(t) \text{ by } (59)} + \frac{c_0}{4} \underbrace{[\partial_s u^0_{\mathrm{ext}}](t)}_{c_0 u^0_{\mathrm{ext}}(t) \text{ by } (57)} \\ & - \{\partial_s^2 u^0_{\mathrm{ext}}\}(t) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_t^2 \underbrace{U^0_{\mathrm{int}}(t,S)}_{u^0_{\mathrm{ext}}(t) \text{ by } (55)} - \kappa^2(t) S \underbrace{\partial_S U^0_{\mathrm{int}}(t,S)}_{0 \text{ by } (55)}\,\mathrm{d}S. \end{aligned}$$

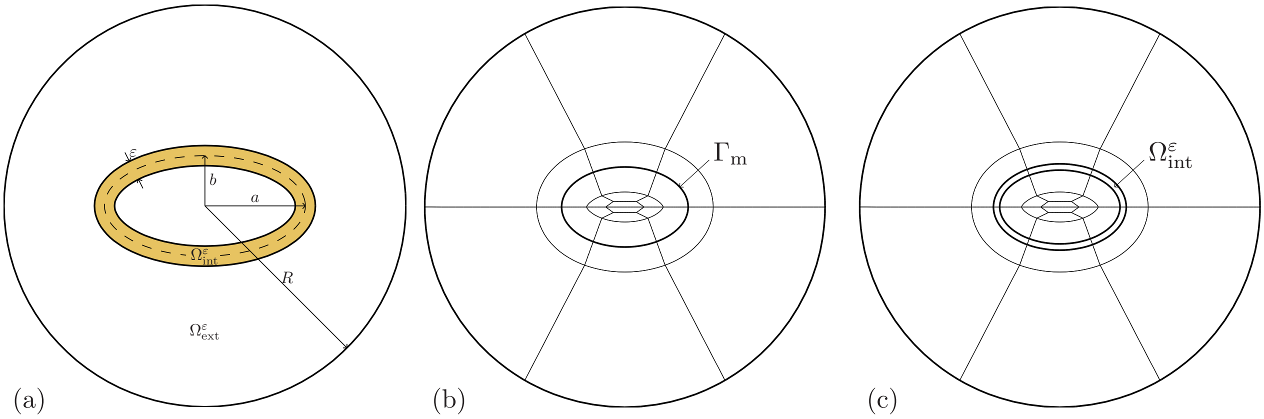


FIGURE 3. (a) Geometrical setting with elliptic mid-line (dashed line) with the semi-major axis a and semi-minor axis b. The boundary is a circle of radius R ($R = 2$, $a = \sqrt{0.4}$, $b = 0.4$). (b) Mesh $\mathcal{M}^0$ for the finite element solution of the asymptotic expansion models. The mid-line $\Gamma_{\rm m}$ is labelled. (c) Associate mesh $\mathcal{M}^\varepsilon$ for the finite element solution of the exact model with the cells in the sheet, here of thickness $\varepsilon = 1/16$.

Applying (53) we can replace the mean value of the second normal derivative by

$$\{\partial_s^2 u^0_{\rm ext}\}(t) = -\kappa(t)\,\{\partial_s u^0_{\rm ext}\}(t) - \partial_t^2\,\{u^0_{\rm ext}\}(t).$$

Summarising, we have, after mutual cancellation of most of the terms,

$$\begin{aligned}\left[\partial_s u^1_{\rm ext}\right](t) - c_0\,\{u^1_{\rm ext}\}(t) &= -\frac{c_0^2}{12} u^0_{\rm ext}(t) - \kappa(t)\,\{\partial_s u^0_{\rm ext}\}(t) + \frac{c_0^2}{4} u^0_{\rm ext}(t) \\ &\quad + \kappa(t)\,\{\partial_s u^0_{\rm ext}\}(t) + \partial_t^2\,\{u^0_{\rm ext}\}(t) - \partial_t^2\,\{u^0_{\rm ext}\}(t) = \frac{c_0^2}{6} u^0_{\rm ext}(t).\end{aligned}$$

Hence, $u^1_{\rm ext}$ is uniquely defined, see Lemma 4.2, as solution of

$$\left\{\begin{aligned} -\Delta u^1_{\rm ext}(\underline{x}) &= 0, && \text{in } \Omega^0_{\rm ext}, \\ u^1_{\rm ext}(\underline{x}) &= 0, && \text{on } \partial\Omega, \\ \left[u^1_{\rm ext}\right](t) &= 0, && \text{on } \Gamma_{\rm m}, \\ \left[\partial_s u^1_{\rm ext}\right](t) - c_0\,\{u^1_{\rm ext}\}(t) &= \frac{c_0^2}{6} u^0_{\rm ext}(t), && \text{on } \Gamma_{\rm m}. \end{aligned}\right. \tag{62}$$

As $u^1_{\rm ext}(\underline{x})$ has no jump over $\Gamma_{\rm m}$ we denote $u^1_{\rm ext}(t) := u^1_{\rm ext}(t, \pm 0) = \{u^1_{\rm ext}\}(t)$. Thus, we can write the last equation of (62) as

$$\left[\partial_s u^1_{\rm ext}\right](t) - c_0\, u^1_{\rm ext}(t) = \frac{c_0^2}{6} u^0_{\rm ext}(t) \quad \text{on } \Gamma_{\rm m}. \tag{63}$$

6.4. **Order 2**

6.4.1. *External function*

In the same way, one can obtain that the second order term u^2_{ext} is uniquely defined by (see Lemma 4.2)

$$\begin{cases} \Delta u^2_{\mathrm{ext}}(\underline{x}) = 0 & \text{in } \Omega^0_{\mathrm{ext}}, \\ \left[u^2_{\mathrm{ext}}\right](t) = -\dfrac{c_0}{24}\kappa(t) u^0_{\mathrm{ext}}(t,0) - \dfrac{c_0}{12}\left\{\partial_n u^0_{\mathrm{ext}}\right\}(t) & \text{on } \Gamma_{\mathrm{m}}, \\ \left[\partial_n u^2_{\mathrm{ext}}\right](t) - c_0\left\{u^2_{\mathrm{ext}}\right\}(t) = \dfrac{c_0^2}{6} u^1_{\mathrm{ext}}(t) + \dfrac{c_0}{24}\kappa(t)\left\{\partial_n u^0_{\mathrm{ext}}\right\}(t) & \\ \qquad\qquad + c_0\left(\dfrac{7}{240}c_0^2 - \dfrac{\partial_t^2}{12}\right) u^0_{\mathrm{ext}}(t,0) & \text{on } \Gamma_{\mathrm{m}}, \\ u^2_{\mathrm{ext}}(\underline{x}) = 0 & \text{on } \partial\Omega, \end{cases} \tag{64}$$

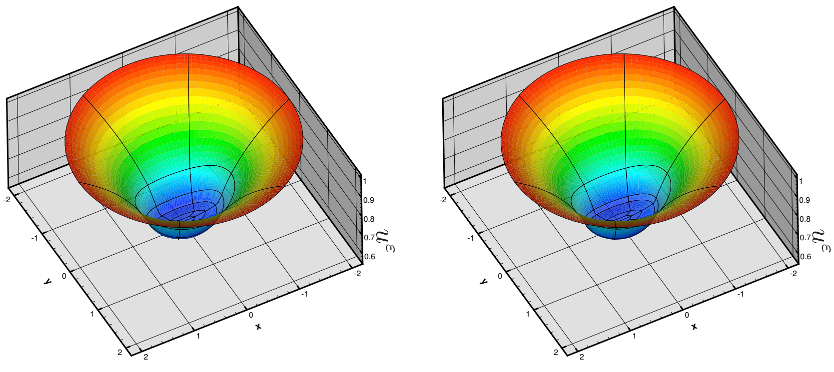
whose Dirichlet and Neumann traces are both discontinuous over the mid-line of the sheet in general. The transmission conditions depend on the solutions of order 0 and 1 and include even a second tangential derivative of u^0_{ext}. Once again no boundary data or source term is involved.

6.4.2. *Internal function*

The internal expansion function of order 2 is the fourth order polynomial

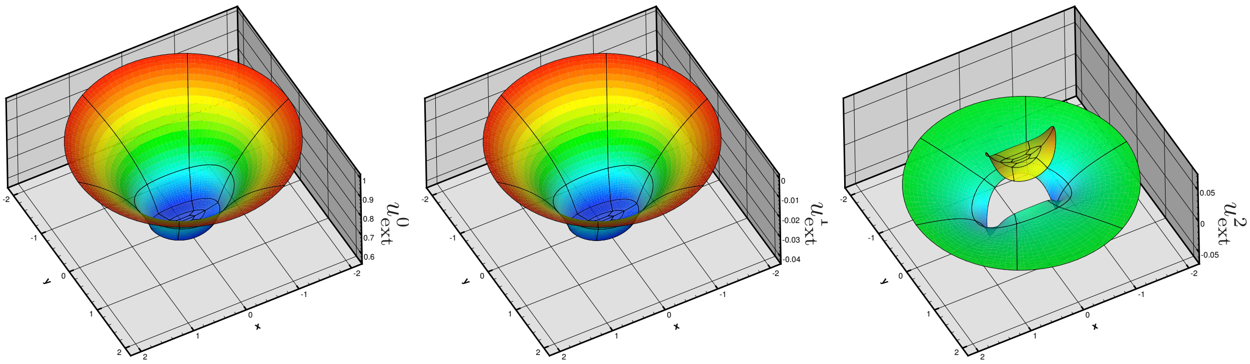
$$\begin{aligned} U^2_{\mathrm{int}}(t,S) = {} & \frac{c_0^2}{24} u^0_{\mathrm{ext}}(t)\left(S^2 + \frac{3}{4}\right)^2 + \frac{c_0}{6}\left\{\partial_s u^0_{\mathrm{ext}}\right\}(t)\left(S^3 - \frac{3}{4}S\right) - \frac{c_0}{6}\kappa(t) u^0_{\mathrm{ext}}(t)\left(S^3 + \frac{3}{4}S\right) \\ & + \frac{c_0}{2} u^1_{\mathrm{ext}}(t)\left(S^2 + \frac{1}{4}\right) - \frac{1}{2}\left(\kappa(t)\left\{\partial_s u^0_{\mathrm{ext}}\right\}(t) + \partial_t^2 u^0_{\mathrm{ext}}(t)\right) S^2 + \left\{\partial_s u^1_{\mathrm{ext}}\right\}(t)\, S + \left\{u^2_{\mathrm{ext}}\right\}(t), \end{aligned} \tag{65}$$

which involves the curvature of the sheet and a second tangential derivative of the external function of order 0.



(a) $\varepsilon = 1/8$ (b) $\varepsilon = 1/16$

FIGURE 4. High order Finite element approximation of the solution u^ε of the exact model for two values of ε for Dirichlet data $g = 1$, source term $f = 0$ and relative conductivity $c_0 = 1$.



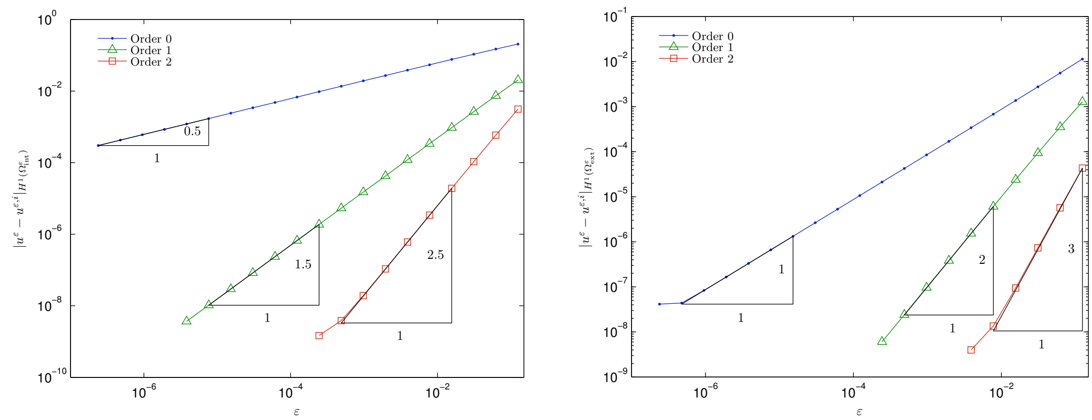
(a) Order 0 (b) Order 1 (c) Order 2

Figure 5. Numerical approximation of the asymptotic expansion coefficient $u^i_{\text{ext}}(\underline{x})$ for the geometry with an ellipsoidal thin sheet ($a = \sqrt{0.4}$, $b = 0.4$), $c_0 = 1$, $g = 1$, and $f = 0$, computed by high order finite elements.

7. Numerical examples

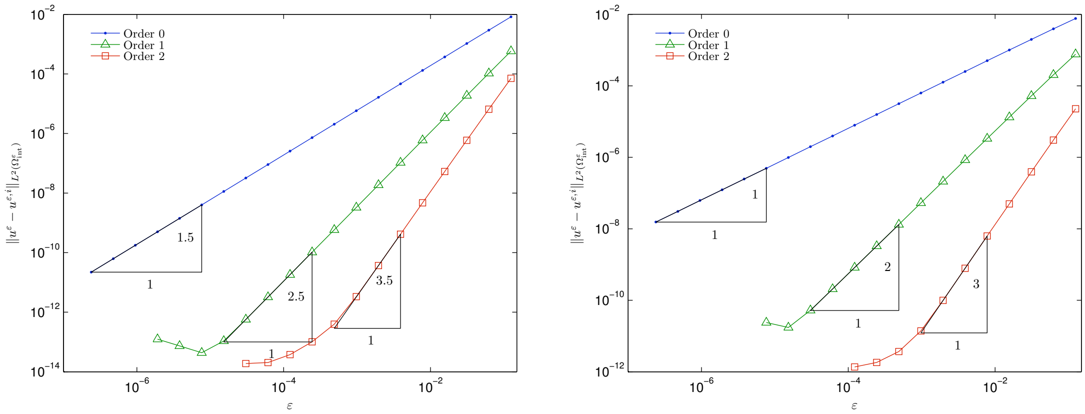
In this section, we numerically investigate the rate of convergence of the approximate asymptotic models with the numerical C++ library *Concepts* [10, 12] . We consider a domain with an ellipsoidal sheet as an example for varying curvature (see Figure 3(a)).

We discretise both, the exact model and the asymptotic expansion models, by means of high-order finite elements. The smooth shape is taken into account by curved elements of high-order so that the discretisation error does not dominate the modelling error. The meshes for the exact model are denoted by $\mathcal{M}^\varepsilon$ (see Figure 3(b)) whereas $\mathcal{M}^0$ denotes the mesh for asymptotic expansion models (see Figure 3(c)). For the computation of the modelling errors in the L^2-norm and the H^1-seminorm we represent the asymptotic expansion functions $u^i(\underline{x})$ and $U^i(t, S)$ after their computation on the meshes $\mathcal{M}^\varepsilon$.



(a) In the sheet $\Omega^\varepsilon_{\text{int}}$. (b) In the exterior sub-domain $\Omega^\varepsilon_{\text{ext}}$.

Figure 6. The modelling error in the H^1-seminorm for ellipsoid sheets of varying thicknesses ε and a constant relative conductivity $c_0 = 1$, computed by high-order FEM.



(a) In the sheet $\Omega^{\varepsilon}_{\text{int}}$. (b) In the exterior sub-domain $\Omega^{\varepsilon}_{\text{ext}}$.

FIGURE 7. The modelling error in the L^2-norm for ellipsoid sheets of varying thicknesses ε and a constant relative conductivity $c_0 = 1$, computed by high-order FEM.

The numerical simulations are performed with Dirichlet boundary data $g = 1$, a vanishing source term $f = 0$ and a relative conductivity $c_0 = 1$. We use linear trunk spaces with a uniform polynomial degree $p = 15$ and at least 17^2 Gauß-Jacobi-quadrature points per cell to highly resolve the solution of the exact (3380 degrees of freedom) and the asymptotic model (2738 degrees of freedom). For the exact model these high polynomial degrees are also applied in the cells inside the sheet.

The solutions $u^{\varepsilon}(\underline{x})$ of the exact model are shown in Figure 4 for two values of ε. The area, which is enclosed by the sheet, is apparently shielded. The according expansion functions $u^0_{\text{ext}}(\underline{x})$, $u^1_{\text{ext}}(\underline{x})$ and $u^2_{\text{ext}}(\underline{x})$ are shown in Figure 5.

In Figure 6 the modelling error in the H^1-seminorm evaluated first inside the sheet and secondly in the exterior is shown, both in dependence of ε. The convergence rate is 0.5, 1.5 and 2.5 in the sheet and 1, 2 and 3 in the exterior area for the asymptotic expansion models of order 0, 1 or 2, respectively. This validates the sharpness of the a-priori estimates of Lemma 5.1. The corresponding L^2-errors are shown in Figure 7. We observe rates of convergence in the thin sheet of 1.5, 2.5 and 3.5 and of 1, 2 and 3 in the exterior for the three models. The improved rates inside the sheet in comparison to the H^1-seminorm results due to the different scaling with changing thickness ε.

Conclusion

In the context of eddy current modelling, we derived the asymptotic expansion at any order of the solution of a model problem with a dissipative thin sheet, see (3). For the three first orders, we obtained formulations, see Section 6, that are easy to implement, do not require to mesh the sheet and do not lead to ill-conditioned matrices. This asymtptotic expansion is not only formal but justified by error estimates. The theoretical results have been validated through numerical simulations which also demonstrate the numerical feasibility.

Like it was achieved for IBCs, this approach can be generalised to 3D (where one has to take care of the geometry of the sheet) and to other systems of equation including non exclusively the Helmholtz equation, the Maxwell equations, or the wave equation in time domain (for IBCs see respectively [3], [15] and [4] for example).

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