Anisotropic Stable Lévy Copula Processes – Analytical and Numerical Aspects

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Seminar für Angewandte Mathematik
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Abstract

We consider the valuation of derivative contracts on baskets of risky assets whose prices are Lévy like Feller processes of tempered stable type. The dependence among the marginals' jump structure is parametrized by a Lévy copula. For marginals of regular, exponential Lévy type in the sense of [6] we show that the infinitesimal generator $\mathcal{A}$ of the resulting Lévy copula process is a pseudo-differential operator whose principal symbol is a distribution of anisotropic homogeneity. We analyze the jump measure of the corresponding Lévy copula processes. We prove the domains of their infinitesimal generators $\mathcal{A}$ are certain anisotropic Sobolev spaces. In these spaces and for a large class of Lévy copula processes, we prove a Gårding inequality for $\mathcal{A}$.
We design a wavelet-based dimension-independent tensor product discretization for the efficient numerical solution of the parabolic Kolmogoroff equation $u_t + \mathcal{A}u = 0$ arising in valuation of derivative contracts under possibly stopped Lévy copula processes. In the wavelet basis diagonal preconditioning yields a bounded condition number of the resulting matrices.

Keywords: Lévy-copula, Lévy processes, Pseudo-differential Operators, Dirichlet Forms, Wavelet Finite Element Methods, Option Pricing

Subject Classification: 35K15, 45K05, 65N30

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1 Introduction

This paper addresses the pricing of derivative contracts on baskets of $d \geq 1$ assets whose prices are modelled by Lévy processes with particular attention on modelling the dependences in prices' jump structure.

Arbitrage-free values $v(x,T)$ of contingent claims on baskets of d assets whose log-returns are modelled by a Lévy process or, more generally, a strong Markov process X with state space $\mathbb{R}^d$ and $X_0 = x$, can be expressed as expected payoffs at maturity T over all price histories $(X_t)_{0 \leq t \leq T}$ conditional to $X_0 = x$ (see [10]),

$$v(x,T) = \mathbb{E}^x(g(X_T)), \tag{1.1}$$

where the expectation is taken with respect to a chosen martingale measure equivalent to the historical measure (for measure selection criteria we refer to [11, 12] and the references therein).

Deterministic methods to compute $v(x,T)$ are based on the semigroup $(T_t)_{t\geq 0}$ of X_t defined by

$$v(x,t) = (T_t g)(x) = \mathbb{E}^x(g(X_t)), \qquad t > 0 \tag{1.2}$$

or, more precisely, on the solution of the backward Kolmogoroff equation

$$v_t + \mathcal{A}v = 0, \qquad v|_{t=T} = g. \tag{1.3}$$

Here the *infinitesimal generator* $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ of the process X_t (resp. of the semigroup $(T_t)_{t\geq 0}$) is defined by the strong limit

$$\mathcal{A}u := \lim_{t\to 0^+} \frac{1}{t}\,(T_t u - u) \tag{1.4}$$

on all functions $u \in \mathcal{D}(\mathcal{A}) \subset C_0(\mathbb{R})$ for which the limit (1.4) exists w.r. to the sup-norm. $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is called a *Feller generator* of X.

In the classical setting of Black-Scholes, X is a geometric Brownian Motion and $\mathcal{A}$ is a diffusion operator so that closed form solution of (1.3) for plain vanilla contracts is possible in certain cases. For more general jump-diffusion or Lévy price processes X, $\mathcal{A}$ is in general a pseudo-differential operator with symbol ψ_X, i.e. (e.g. [21, 22])

$$(\mathcal{A}u)(x) = (\psi_X(D)u)(x) = \mathbf{F}^{-1}_{\xi\to x}\,(\psi_X(\xi)\widehat{u}(\xi)) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\xi\cdot x}\psi_X(\xi)\widehat{u}(\xi)d\xi \tag{1.5}$$

where $\widehat{u}(\xi) = (\mathbf{F}_{x\to\xi}u)(\xi)$ denotes the Fourier transform of $u(x)$. For exotic contracts, the Cauchy problem (1.3) is replaced by an initial boundary value problem (for barrier contracts) or by a variational inequality (arising from stopping X in American style contracts) or even a Hamilton-Jacobi-Bellmann Quasi-Variational inequality (for problems in portfolio optimization). Then numerical solutions are necessary which require efficient discretizations of the generator $\mathcal{A}$.

Numerical pricing of derivative contracts can be achieved either by stochastic, Monte-Carlo (MC) based methods or by so-called "PDE", or "mesh-based" methods. The latter deliver, if applicable, accurate solutions of derivative pricing problems not only in the Black-Scholes setting, but also for much more general stochastic processes modelling the dynamics of the risky asset. A general programme for the deterministic solution of the Kolmogoroff equation (1.3) involves time-stepping and space discretization of the infinitesimal generator $\mathcal{A}$ of X_t. We refer to [25] for wavelet discretization of (1.3) in the univariate Lévy case with the so-called θ time-stepping scheme, and to [27, 28] for numerical results and error analysis for an exponentially convergent time-stepping procedure which exploits the time-analyticity of $(T_t)_{t\geq 0}$ in (1.2). For optimal stopping problems arising in conjunction with American put style contracts in this setting, we refer to [26].

All these results are for contracts on single risky underlyings. For contracts on large baskets of risky assets in a Black-Scholes setting in [31] wavelet based deterministic solution methods for (1.3) have been analyzed and implemented, and in [20] stochastic volatility models with OU models have been treated with these methods. In particular, the wavelet based solution methods allow to reduce the computational complexity incurred due to the high-dimension of the computational domain for the pricing problem.

To employ these methods for pricing derivative contracts on baskets of risky assets in a Lévy setting, the correlation in the marginals' jump structure has to be modeled parametrically. One way to do this is by so-called Lévy copulas introduced in [37] and developed in [24]. For FE based solution methods of the pricing equation corresponding to these models, the generators $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ of $(T_t)_{t\geq 0}$ for the associated Lévy copula process X must be identified with appropriate function spaces, multilevel norm equivalences must be proved in these spaces and time-analyticity of the associated semigroup $(T_t)_{t\geq 0}$ in (1.2) must be established.

For example, for α-stable processes in $\mathbb{R}^d$, the jump measure $\nu(dx)$ of X is homogeneous of order $-d-\alpha$, i.e. $\nu(\lambda dx) = \lambda^{-d-\alpha}\nu(dx)$ for all $\lambda > 0$ and some $0 < \alpha < 2$, and the domain $\mathcal{D}(\mathcal{A})$ corresponds to certain fractional order Sobolev spaces $H^{\alpha}_{loc}(\mathbb{R}^d)$ (e.g. [21, 22]). Assuming the same jump intensity for each component process, however, is not realistic in practice [7], and copula constructions which blend univariate Lévy processes with different jump intensities should be considered [2].

The identification of the generators $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ for Lévy copula processes of tempered stable marginal processes and the establishment of the above results which form the basis of efficient, wavelet-based deterministic solution methods for the pricing equation (1.3) is the purpose of the present paper.

The outline of the paper is as follows.

To make this work as much as possible self contained in the next section we recall some basic facts on Lévy processes, define Lévy copulas and characterize all $\mathbb{R}^d$-valued Lévy processes X whose components are equal in law to d univariate Lévy processes $Y_1, ..., Y_d$. We conclude this preliminary section with fundamentals on generators and Dirichlet forms of Lévy copula processes.

Assuming that each marginal Lévy process Y_i is tempered stable with a 4-parameter density of tempered stable type, we show in Section 3 that the domains of generators of 1-homogeneous Lévy copula processes are certain classes of anisotropic Sobolev spaces. For an example of tempered stable processes that generalizes the variance gamma process by adding a new parameter in the Lévy density that allows the Lévy process to have both finite or infinite activity and finite or infinite variation we refer to [7] and the references therein.
The determination of these domains of generators (of 1-homogeneous Lévy copula processes) is done in two major steps.
Initially, we focus on 1-homogeneous Lévy copulas of $\underline{\alpha}$-stable marginals and determine the generators' domains, where $\underline{\alpha}$ is a vector $(\alpha_1, ..., \alpha_d)$.
Then, exponential tail decay of the marginal densities is introduced and it is shown that the domains of the generators and of the Dirichlet forms for these so-called tempered anisotropic stable Lévy processes coincide locally with those of the untempered versions.

Section 4 is devoted to aspects of the numerical solution of the Kolmogoroff equation (1.3) using the methods developed in [25, 26, 27, 28]. We propose a wavelet-based dimension independent tensor product discretization of the integro-differential generator of the anisotropic stable Lévy copula process. Unlike the Fourier transform, wavelets are well localized also in price space which allows to treat barrier and American contracts along the lines of [26, 28]. We conclude the paper with some remarks on possible extensions and generalizations.

2 Preliminaries

2.1 Lévy Processes

We start recalling definitions and basic properties of Lévy processes as presented e.g. in [3, 32].

A stochastic process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ with $X_0 = 0 \quad a.s.$ is a Lévy process if it has independent increments, is temporally homogeneous and stochastically continuous.

For $\xi \in \mathbb{R}^d$, define the characteristic function Φ_X and the characteristic exponent ψ_X of X by

$$\Phi_X(\xi) = \exp(-t\psi_X(\xi)) = \mathbb{E}(\exp(i\langle \xi, X_t\rangle)) \qquad \xi \in \mathbb{R}^d, \quad t > 0.$$

For a Lévy process X, the characteristic exponent $\psi_X(\xi)$ is also called *Lévy symbol*. It admits the *Lévy-Khinchin representation*

$$\psi_X(\xi) = i\langle \gamma, \xi\rangle + Q(\xi) + \int_{y\in\mathbb{R}^d} (1 - e^{i\langle \xi, y\rangle} + \frac{i\langle \xi, y\rangle}{1 + |y|^2})\nu(dy), \tag{2.1}$$

where $Q(\xi)$ denotes the quadratic form $\frac{1}{2}\xi^\top \mathbf{Q}\xi$ with a symmetric, nonnegative definite matrix $\mathbf{Q} = (q_{ij})_{1\le i,j\le d}$, a drift vector $\gamma \in \mathbb{R}^d$ and the Lévy measure $\nu(dy)$ which satisfies

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(dy) < \infty. \tag{2.2}$$

Functions ψ of the form (2.1) are called *negative definite* functions. The Lévy process X is completely determined by the *characteristic triple* $(\mathbf{Q}, \gamma, \nu)$ in (2.1). If $\mathbf{Q}$ in (2.1) vanishes, then X is called a *pure jump* process.

The Lévy-Khinchin formula (2.1) for the characteristic function ψ_X corresponds to the *Lévy-Itô decomposition* of X

$$\begin{aligned} X_t &= \Sigma B_t + t\mathbb{E}\left(X_t - \int_{|x|\ge 1} xN_1(\cdot, dx)\right) + \int_{|x|<1} x(N_t(\cdot, dx) - t\nu(dx)) + \int_{|x|\ge 1} xN_t(\cdot, dx) \\ &= \Sigma B_t + \gamma t + \int_{|x|<1} x(N_t(\cdot, dx) - t\nu(dx)) + \sum_{0<s\le t} \Delta X_s 1_{\{|\Delta X_s|\ge 1\}}. \end{aligned} \tag{2.3}$$

Remark 2.1. The Lévy-Khinchin formula (2.1) and the Lévy-Itô decomposition (2.3) for the Lévy process X_t decompose X_t into three pieces: a diffusion without drift $W_t = \Sigma B_t$ where $\Sigma\Sigma^\top = \mathbf{Q}$, a (deterministic) drift part $\delta_t = \gamma t$ and a quadratic, pure jump part

$$J_t = \int_{|x|<1} x(N_t(\cdot, dx) - t\nu(dx)) + \sum_{0<s\le t} \Delta X_s 1_{\{|\Delta X_s|\ge 1\}}.$$

These three pieces correspond to a decomposition of the characteristic exponent ψ_X and the infinitesimal generator $\mathcal{A}_X$ of X into three characteristic exponents and generators, resp., of the three component processes of X according to

$$\psi_X = \psi_W + \psi_\delta + \psi_J, \tag{2.4}$$

$$\mathcal{A}_X = \mathcal{A}_W + \mathcal{A}_\delta + \mathcal{A}_J. \tag{2.5}$$

For ψ_X as well as $\mathcal{A}_X$ each of the three elements in their representations is completely characterized by one component of the characteristic triple $(\mathbf{Q}, \gamma, \nu)$.

Remark 2.2. A key issue in the construction of $\mathbb{R}^d$-valued Lévy processes is the parametrization of correlations between the d univariate Lévy "driving" processes taking values in $\mathbb{R}$. As the drift δ is deterministic, dependence modeling enters in the diffusion and the quadratic, pure jump part of the process X. In the diffusion part, dependence between the d univariate driving Brownian Motions is parametrized by the so-called "volatility correlation matrix", Σ. In the jump-part J of the process X, however, the dependence between the d Lévy measures $\nu_i(dy_i)$ on $\mathbb{R}$ of the driving Lévy processes enters into the construction of the jump-measure of J. One way to parametrize dependence in the jump structure of X are so-called *Lévy copulas*.

2.2 Lévy copulas

Here we recall the definition of Lévy copulas and present their main properties, following [37, 24]. We start with some notation. Denote $\overline{\mathbb{R}} := (-\infty, \infty]$, and, for $a, b \in \overline{\mathbb{R}}^d$ such that $a \leq b$ componentwise, we define the half-open intervals

$$(a, b] := (a_1, b_1] \times ... \times (a_d, b_d].$$

For a function $F : S \to \overline{\mathbb{R}}$ defined on some subset $S \subseteq \overline{\mathbb{R}}^d$, the F-volume of $(a, b]$ is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times ... \times \{a_d, b_d\}} (-1)^{N(u)} F(u)$$

where $N(u) = |\{k : u_k = a_k\}|$. Note that for $F(u) = u_1 u_2 ... u_d$, $V_F((a, b])$ equals the Lebesgue measure of $(a, b] \subseteq \overline{\mathbb{R}}^d$. The function $F : S \to \overline{\mathbb{R}}$ is called d-*increasing* if $V_F((a, b]) \geq 0$ for all $a, b \in S$ such that $a \leq b$ and $\overline{(a, b]} \subset S$. Examples of d-increasing functions relevant to us are furnished by distribution functions of random vectors $X \in \mathbb{R}^d$ via

$$F(x_1, ..., x_d) = P[X_1 \leq x_1, ..., X_d \leq x_d].$$

In dependence modelling, an important role is played by margins of multivariate distributions. To define them, let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be d-increasing and such that $F(u_1, ..., u_d) = 0$ if $u_i = 0$ for at least one $1 \leq i \leq d$. Let further $I \subset \{1, ..., d\}$ be a nonempty index set of cardinality $|I| \leq d$ and denote by $I^c := \{1, ..., d\} \backslash I$ its complement of cardinality $d - |I|$. For $u \in \overline{\mathbb{R}}^d$, define $u^I \in \overline{\mathbb{R}}^{|I|}$ to be the vector $(u_i)_{i \in I}$.

Then the I-*margin* of F is the function $F^I : \overline{\mathbb{R}}^I \to \overline{\mathbb{R}}$ defined by

$$F^I((u^I)_{i \in I}) := \lim_{c \to \infty} \sum_{u^{I^c} \in \{-c, \infty\}^{I^c}} \left(\Pi_{j \in I^c} \text{sgn} u_j\right) F(u_1, ..., u_d)$$

where sgn$x = 1$ for $x \geq 0$ and -1 otherwise.

After these preparations, we may define Lévy copulas.

Definition 2.3. $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ *is a Lévy-copula if*

1. $F(u_1, ..., u_d) \neq \infty$ *for* $(u_1, ..., u_d) \neq (\infty, ..., \infty)$,
2. $F(u_1, ..., u_d) = 0$ *if* $u_i = 0$ *for at least one* $i \in \{1, ..., d\}$,
3. F *is* d*-increasing,*
4. $F^{\{i\}}(u) = u$ *for any* $i \in \{1, ..., d\}$, $u \in \mathbb{R}$.

Lévy copulas are Lipschitz in the sense that

$$|F(u_1, ..., u_d) - F(v_1, ..., v_d)| \leq \sum_{i=1}^{d} |u_i - v_i|. \tag{2.6}$$

We also need tail integrals of Lévy processes.

Definition 2.4. *Let $X \in \mathbb{R}^d$ be a Lévy process with Lévy measure ν. The* tail integral of X *is the function $U : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ given by*

$$U(x_1, ..., x_d) := \Pi_{i=1}^d \text{sgn}(x_i) \nu(\Pi_{j=1}^d \mathcal{I}(x_j))$$

where $\mathcal{I}(x) := (x, \infty)$ if $x \geq 0$ and $\mathcal{I}(x) := (-\infty, x]$ otherwise.

Let further $I \subseteq \{1, 2, ..., d\}$ be an index set with $|I| > 0$ indices. Then the I - marginal tail integral U^I of X *is the tail integral of the process $X^I := (X_i)_{i \in I} \in \mathbb{R}^{|I|}$. If $I = \{i\}$, we write $U_i := U^I = U^{\{i\}}$.*

The next result due to Kallsen and Tankov [24] will play a key role in our considerations.

Theorem 2.5. *(Sklar's Theorem for Lévy copulas)*
1. For any Lévy process $X \in \mathbb{R}^d$ exists a Lévy copula F such that the tail integrals of X satisfy

$$U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I}) \tag{2.7}$$

for any $I \subseteq \{1, ..., d\}$ and any $\{x_i\}_{i \in I} \in \mathbb{R}^{|I|} \backslash \{0\}$. The Lévy copula is unique on $\Pi_{i=1}^d \overline{\text{Range} U_i}$.

2. Let F be a d-dimensional Lévy copula and U_i the tail integrals of univariate Levy processes. Then there exists a Lévy process $X \in \mathbb{R}^d$ such that its components have tail integrals U_i and its marginal tail integrals satisfy (2.7)

The purpose of copulas is parametric modelling of dependence in the jump structure of multivariate Lévy processes $X = (X^1, ..., X^d) \in \mathbb{R}^d$. By the Lévy-Itô decomposition (2.3), correlation in the diffusion part is accounted for in the volatility matrix Σ, so that the main objective of Lévy copulas is the parametric construction of the multivariate jump measure $\nu(dx)$ out of jump measures of the component processes.

The extreme cases complete dependence and complete independence among jumps of the components of X are characterized as follows.

Proposition 2.6. *(Independence Lévy copula)*
Let $X = (X^1, ..., X^d) \in \mathbb{R}^d$ be a Lévy process. Its components X^i are independent if and only if their Brownian parts are independent and if X has a Lévy copula of the form

$$F_\perp(x_1, ..., x_d) := \sum_{i=1}^{d} x_i \Pi_{j \neq i} 1_{\{\infty\}}(x_j)$$

To address the other extreme, namely complete dependence among the jumps of the components X^i, we observe that elements of a strictly ordered set $S \subset \mathbb{R}^d$ are completely determined by one coordinate only. Hence we have

Definition 2.7. *Let $X \in \mathbb{R}^d$ be a Lévy process. The jumps of X are* completely dependent *or* comonotonic *if there exists a strictly ordered subset $S \subset K := \{x \in \mathbb{R}^d : \text{sgn}(x_1) = ... = \text{sgn}(x_d)\}$ such that for all $t > 0$ holds $\Delta X_t := X_t - X_{t-} \in S$ almost sure. Equivalently, the jumps of X are completely dependent if there exists a strictly ordered subset $S \subset K$ such that $\nu(\mathbb{R}^d \backslash S) = 0$ where ν denotes the Lévy measure of X.*

Complete dependence among the components' jumps of X in terms of Lévy copulas is provided by

Proposition 2.8. *(Complete dependence Lévy copula)*
Let $X \in \mathbb{R}^d$ be a Lévy process with Lévy measure ν supported by an ordered set $S \subset K$. Then the complete dependence Lévy copula

$$F_{||}(x_1, ..., x_d) := \min\{|x_1|, ..., |x_d|\} 1_K(x_1, ..., x_d) \Pi_{i=1}^d \text{sgn}(x_i) \tag{2.8}$$

is a Lévy copula of X.

Vice versa, if $F_{||}$ in (2.8) *is a Lévy copula of X, then the Lévy measure of X is supported on an ordered subset of K. If, in addition, the tail integrals U_i of X^i are continuous and satisfy $\lim_{x \to 0} U_i(x) = \infty$ for $i = 1, ..., d$, then the jumps of X are completely dependent.*

2.3 Generators and Dirichlet forms of Lévy copula processes

Recall that a Dirichlet space (on $\mathbb{R}^d$ for simplicity) is a pair $(\mathcal{F}, \mathcal{E})$ consisting of a space of real-valued functions $\mathcal{F} \subset L^2(\mathbb{R}^d)$ and a symmetric quadratic form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ which is closed, densely defined, non-negative, and satisfies the following contraction condition:

$$\text{if} \quad u \in \mathcal{F} \quad \text{then} \quad v := (0 \vee u) \wedge 1 \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

All translation invariant (symmetric) Dirichlet forms (on $\mathbb{R}^d$) are given by

$$\mathcal{E}^\phi(u, v) = \int_{\mathbb{R}^d} \phi(\xi)\, \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi, \quad u, v \in \mathcal{S}(\mathbb{R}^d),$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a continuous negative definite function. The domain $\mathcal{F}^\phi$ of $\mathcal{E}^\phi$ is then given by

$$\mathcal{F}^\phi = H^{\phi,1}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} (1 + \phi(\xi)) |\widehat{u}(\xi)|^2\, d\xi < \infty \right\}. \tag{2.9}$$

It is well known that one can associate with ϕ (or with $(\mathcal{F}^\phi, \mathcal{E}^\phi)$) the operator semigroup $(T_t)_{t\geq 0}$ on $L^2(\mathbb{R}^d)$ defined by

$$T_t u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, e^{-t\phi(\xi)}\, \widehat{u}(\xi)\, d\xi = \int_{\mathbb{R}^d} u(x-y)\, \mu_t(dy),$$

where $(\mu_t)_{t\geq 0}$ is a vaguely continuous semigroup of sub-probability measures on $\mathbb{R}^d$ with Fourier transform $\widehat{\mu_t} = (2\pi)^{-d/2}\, e^{-t\phi(\xi)}$.

Note that the measures μ_t are also the transition probabilities for a Lévy process $(X_t)_{t\geq 0}$ and therefore we have

$$\mathbb{E}\left(e^{iX_t\xi}\right) = e^{-t\phi(\xi)}.$$

Thus ϕ is also a characteristic exponent of a Lévy process.

The generator $(\mathcal{A}, D(\mathcal{A}))$ of the semigroup $(T_t)_{t\geq 0}$ is given by

$$\mathcal{A}u(x) = -\phi(D)u(x) = -(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, \phi(\xi)\, \widehat{u}(\xi)\, d\xi$$

with domain

$$H^{\phi,2}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} (1+\phi(\xi))^2\, |\widehat{u}(\xi)|^2\, d\xi < \infty \right\}. \tag{2.10}$$

Remark 2.9. In fact the spaces $H^{\phi,2}(\mathbb{R}^d) =: H_2^{\phi,2}(\mathbb{R}^d)$ are part of the scale of Bessel potential spaces $H_p^{\phi,s}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$ for which $\mathcal{S}(\mathbb{R}^d)$ is a dense subset with respect to a norm defined generalizing (2.10). These spaces appear as generalizations of the classes $H_p^{\phi,2}$ which are domains of generators for L_p-sub-Markovian semigroups associated with the real valued continuous negative definite function ψ. These spaces were extensively investigated in [17] and [18] and we do not go into further details here since for our purposes it will be sufficient to consider the case $p = 2$.

The function spaces of type $H^{\phi,1}$ and $H^{\phi,2}$ we are interested in, appeared in their generality for the first time in the work of A. Beurling and J. Deny [4, 5], see also [13], on Dirichlet spaces. In general they are contained neither in the Besov- $B^s_{p,q}$ or Triebel - Lizorkin- $F^s_{p,q}$ scales nor in the anisotropic classes of function spaces considered so far. They are so-called function spaces of generalized smoothness, because the smoothness properties are related to the function ϕ. Function spaces of generalized smoothness have been introduced and considered by several authors, in particular since the middle of the seventies with different starting points and in different contexts. In [19] it was given an overview on the approaches known in the literature up to that moment.

Let $X \in \mathbb{R}^d$ be a Lévy process with characteristic triple $(\mathbf{Q}, \gamma, \nu) = (\Sigma^\top\Sigma, \gamma, \nu)$ as in (2.3) and with associated semigroup $(T_t)_{t\geq 0}$ as in (1.2) with infinitesimal generator $\mathcal{A}$. The Lévy symbol ψ_X of X given by (2.1) is a continuous, negative definite function and the symbol of the generator $\mathcal{A}$ of X as in (1.5).

Given the generator $\mathcal{A} = \psi_X(D)$, its *Dirichlet bilinear form* $\mathcal{E}(\cdot,\cdot)$ of $\mathcal{A}$ given by

$$\mathcal{E}(u,v) = -(u, \mathcal{A}v), \qquad u,v \in \mathcal{S}(\mathbb{R}^d)$$

is crucial for variational formulation and the numerical solution of the pricing problem (1.3).

We identify generators $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and Dirichlet forms $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ of certain Lévy copula processes X modelling dynamics of baskets of risky assets whose prices are univariate, so-called *regular Lévy processes of exponential type* which have been found useful in financial modelling (e.g. [14, 15, 6, 7]).

A first example is furnished by α-stable processes $X \in \mathbb{R}^d$ as proved in [24, Theorem 4.8].

Proposition 2.10. *Let $X \in \mathbb{R}^d$ be a Lévy process and $\alpha \in (0,2)$. Then X is α-stable if and only if its components $X_i \in \mathbb{R}$ are α-stable and if X has a Lévy copula that is homogeneous of order 1, i.e.*

$$F(r\xi_1, ..., r\xi_d) = r\,F(\xi_1, ..., \xi_d) \quad \textit{for all} \quad \xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d, \quad \textit{and all} \quad r > 0. \tag{2.11}$$

The next example is taken from [24, Example 5.2] .

Example 2.11. *Let $d \geq 2$. The function F defined as*

$$F(u_1, ..., u_d) = 2^{2-d} \left(\sum_{j=1}^{d} |u_j|^{-\theta} \right)^{-1/\theta} \left(\eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_d < 0\}} \right)$$

defines a two parameter family of Lévy copulas which resembles the Clayton family of ordinary copulas. It is in fact a Lévy copula homogeneous of order 1, for any $\theta > 0$ and any $\eta \in [0,1]$.

3 Domains of generators of Lévy copula processes

As we indicated in Remark 2.1, the infinitesimal generator $\mathcal{A}_X$ and the Dirichlet form $\mathcal{E}_X(\cdot,\cdot)$ of the Lévy process X each are the sum (2.5) of diffusion W, drift δ and quadratic, pure jump part J in the Lévy-Itô decomposition (2.3) of X. Therefore, the domains $D(\mathcal{A}_X)$ and $D(\mathcal{E}_X)$ are intersections of the corresponding domains:

$$D(\mathcal{A}) = D(\mathcal{A}_W) \cap D(\mathcal{A}_\delta) \cap D(\mathcal{A}_J), \quad D(\mathcal{E}) = D(\mathcal{E}_W) \cap D(\mathcal{E}_\delta) \cap D(\mathcal{E}_J). \tag{3.1}$$

The structure of the domains $D(\mathcal{E}_W)$, $D(\mathcal{E}_\delta)$ is known, see e.g. [18, Remark 2.1.7], therefore we focus in the present section on the structure of the domains $D(\mathcal{A}_J)$ and $D(\mathcal{E}_J)$, or, equivalently, of $D(\mathcal{A}_X)$ *under the assumption that* in the Lévy-Itô decomposition (2.3) of X it holds

$$W = \delta = 0. \tag{3.2}$$

By [37, Remark 3.2] we know that if the tail integrals given by (2.7) are absolutely continuous, we can compute the Lévy density of the Lévy copula process by differentiation as follows:

$$\nu(dx_1, ..., dx_d) = \partial_1 ... \partial_d F \mid_{\xi_1 = U_1(x_1), ..., \xi_d = U_d(x_d)} \nu_1(dx_1) ... \nu_d(dx_d)$$

where $\nu_1(dx_1)$, ..., $\nu_d(dx_d)$ are marginal Lévy densities.

Let $0 < \alpha_1, ..., \alpha_d < 2$ and let real numbers $\beta_1, ..., \beta_d \geq 0$ governing the Lévy densities' tail behavior be given.

Writing for each $j = 1, ..., d$: $\nu_j(dx_j) = k_j^{\beta_j}(x_j)\, dx_j$ with densities $k_j^{\beta_j} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ of the jump measures' in the coordinates given by

$$k_j^{\beta_j}(x_j) = C_j \frac{e^{-\beta_j |x_j|}}{|x_j|^{1+\alpha_j}} \tag{3.3}$$

for some constant $C_j > 0$, we get

$$\nu(dx_1, ..., dx_d) = \partial_1 ... \partial_d F \mid_{\xi_1 = U_1^{\beta_1}(x_1), ..., \xi_d = U_d^{\beta_d}(\xi_d)} k_1^{\beta_1}(x_1) ... k_d^{\beta_d}(x_d)\, dx_1 ... dx_d \tag{3.4}$$

and this can be written as

$$\nu(dx_1, ..., dx_d) = k^{\beta}(x_1, ..., x_d)\, dx_1 ... dx_d \tag{3.5}$$

for some $\beta = (\beta_1, ..., \beta_d)$.

3.1 Homogeneous marginal densities and anisotropic stable Lévy copula processes

3.1.1 Definition and preliminaries

This subsection is dedicated to the treatment of the case in which the marginal densities in (3.3) are given with parameters $\beta_j = 0$ for all $j = 1, ..., d$. Then each of the functions k_j^0 ($j = 1, ..., d$) (corresponding to $\beta_j = 0$) is homogeneous of degree $-1 - \alpha_j$.
More specifically, we will work under the following assumption.

Assumption 3.1. *The function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is a homogeneous Lévy copula of order 1 such that $\partial_1 ... \partial_d F : \mathbb{R}^d \to \mathbb{R}$ exists.*
The numbers $\alpha_1, ..., \alpha_d$ are in $(0, 2)$ and the functions $k_1^0, ..., k_d^0$, as marginal densities of univariate Lévy processes, are homogeneous of order $-1 - \alpha_1$,..., $-1 - \alpha_d$, respectively, i.e.

$$k_j^0(rx_j) = r^{-1-\alpha_j}\, k_j^0(x_j) \quad \textit{for all} \quad r > 0, \quad \textit{and all} \quad x_j \in \mathbb{R} \setminus \{0\},$$

for any $j = 1, ..., d$.

Note that under the above assumption, the tail integrals $U_1^0, ..., U_d^0 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, of the univariate Lévy processes, with marginal densities $k_1^0, ..., k_d^0$, are homogeneous of order $-\alpha_1$,...,$-\alpha_d$, respectively, i.e.

$$U_j^0(rx_j) = r^{-\alpha_j} U_j^0(x_j) \quad \text{for all} \quad r > 0, \quad \text{and all} \quad x_j \in \mathbb{R} \setminus \{0\},$$

for any $j = 1, ..., d$. In order to see this, one has only to recall Definition 2.4 of the tail integrals: $U_j^0(x_j) = \nu_j(x_j, \infty)$ if $x_j \geq 0$ and $U_j^0(x_j) = \nu_j((-\infty, x])$ if $x_j < 0$ where here $\nu_j(dx_j) = k_j^0(x_j)dx_j$.

We now introduce the anisotropic stable Lévy copula processes.

Definition 3.2. *Let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a homogeneous Lévy copula of order 1 such that $\partial_1 ... \partial_d F : \mathbb{R}^d \to \mathbb{R}$ exists and let $k_1^0, ..., k_d^0$ be marginal densities of univariate Lévy processes which are homogeneous of order $-1-\alpha_1$,..., $-1-\alpha_d$, respectively, where $\alpha_1, ..., \alpha_d \in (0, 2)$; let $U_1^0, ..., U_d^0$ be the corresponding tail integrals.*

By Sklar's theorem for Lévy copulas, Theorem 2.5*, there exists a Lévy process $X = (X_1, ..., X_d) \in \mathbb{R}^d$ such that its components have tail integrals U_i^0, $i = 1, ..., d$.*
We call this process an $\underline{\alpha}$-stable Lévy copula process, where $\underline{\alpha} = (\alpha_1, ..., \alpha_d)$, or simply an anisotropic stable Lévy copula process if it will be clear from the context to which anisotropy parameters α_1, ..., α_d we will refer.

As already announced we are able to determine the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form associated to the generator of the $\underline{\alpha}$-stable Lévy copula process associated to F and to prescribed marginals $U_1^0, ..., U_d^0$.

3.1.2 Anisotropic homogeneity of the Lévy symbol

We will start proving an anisotropic homogeneity property of the Lévy density k^0, corresponding to the case when each $\beta_j = 0$, $j = 1, ..., d$.

Theorem 3.3. *Let the copula F, the numbers $\alpha_1, ..., \alpha_d$, and let the marginal densities $k_1^0, ..., k_d^0$ be as in Assumption* 3.1*. Let $U_1^0, ..., U_d^0$ denote the corresponding marginal tail integrals. Then the function k^0, defined by*

$$k^0(x_1, ..., x_d) = \partial_1 ... \partial_d F \mid_{\xi_1 = U_1^0(x_1), ..., \xi_d = U_d^0(x_d)} k_1^0(x_1) ... k_d^0(x_d),$$

compare (3.4) *and* (3.5)*, satisfies*

$$k^0 \left(t^{-\frac{1}{\alpha_1}} x_1, ..., t^{-\frac{1}{\alpha_d}} x_d \right) = t^{1 + \frac{1}{\alpha_1} + ... + \frac{1}{\alpha_d}} k^0(x_1, ..., x_d)$$

for all $t > 0$ and all $x = (x_1, ..., x_d) \in \mathbb{R}^d$ with $x_1, ..., x_d \neq 0$.

Proof. *Step* 1. The function $\partial_1 ... \partial_d F$ is homogeneous of order $1-d$, i.e.

$$\partial_1 ... \partial_d F(r\xi_1, ..., r\xi_d) = r^{1-d}\, F(\xi_1, ..., \xi_d),$$

for all $r > 0$ and all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, since one has only to take partial derivatives in (2.11) and to use definition of homogeneity.

Step 2. Let F, $0 < \alpha_1, ..., \alpha_d < 2$, and $U_1^0, ..., U_d^0$ as above. Using step 1, we get by a direct computation that $\partial_1 ... \partial_d F \circ (U_1^0, ..., U_d^0) : \mathbb{R}^d \setminus \bigcup_{j=1}^{d} \{x_j = 0\} \to \mathbb{R}$ satisfies

$$\left[\partial_1 ... \partial_d F \circ (U_1^0, ..., U_d^0)\right] \left(t^{-\frac{1}{\alpha_1}} x_1, ..., t^{-\frac{1}{\alpha_d}} x_d\right) = t^{1-d} \left[\partial_1 ... \partial_d F \circ (U_1^0, ..., U_d^0)\right] (x_1, ..., x_d)$$

for all $t > 0$, and all $x_j \in \mathbb{R} \setminus \{0\}$, $j = 1, ..., d$.

Step 3. Let F, $0 < \alpha_1, ..., \alpha_d < 2$, $U_1^0, ..., U_d^0$, and $k_1^0, ..., k_d^0$ as above. Then for $x_1, ..., x_d \neq 0$ we have

$$\begin{aligned}
& k^0 \left(t^{-\frac{1}{\alpha_1}} x_1, ..., t^{-\frac{1}{\alpha_d}} x_d\right) \\
= \;& \partial_1 \cdots \partial_d F \mid_{\xi_1 = U_1^0(t^{-\frac{1}{\alpha_1}} x_1), ..., \xi_d = U_d^0(t^{-\frac{1}{\alpha_d}} x_d)} \; k_1^0 \left(t^{-\frac{1}{\alpha_1}} x_1\right) ... k_d^0 \left(t^{-\frac{1}{\alpha_d}} x_d\right) \\
= \;& t^{1-d}\, \partial_1 \cdots \partial_d F \mid_{\xi_1 = U_1^0(x_1), ..., \xi_d = U_d^0(x_d)} \left(t^{-\frac{1}{\alpha_1}}\right)^{-1-\alpha_1} k_1(x_1) ... \left(t^{-\frac{1}{\alpha_d}}\right)^{-1-\alpha_d} k_d(x_d) \\
= \;& t^{1 + \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_d}}\, k^0(x_1, \cdots, x_d)
\end{aligned}$$

and this completes the proof. ■

By Definition 3.2, the so-called $\underline{\alpha}$-stable Lévy copula process, denote it by X, is itself a Lévy process. Its Lévy density is given by k^0, as in Theorem 3.3. The corresponding Lévy symbol $\psi_X : \mathbb{R}^d \to \mathbb{C}$ of X obtained from (2.1) is a continuous negative definite function.

Below we will see that the domain of the Dirichlet form associated to X can be completely characterized by the real part $\Re\psi_X$ of ψ_X. Therefore, we will complement now Theorem 3.3 with a result concerning the anisotropic homogeneity of the real part of the jump part $\Re\psi_J$ of the Lévy symbol ψ_X. As in (2.4), by the Lévy-Khinchin formula (2.1) we have

$$\psi_J(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i\langle \xi, x\rangle} + \frac{i\langle \xi, x\rangle}{1 + |x|^2}) k^0(x) dx. \tag{3.6}$$

Theorem 3.4. *Under the assumptions of Theorem 3.3, let the Lévy kernel $k^0(x_1, ..., x_d)$ be as above. Then real part of the Lévy symbol $\psi_J : \mathbb{R}^d \to \mathbb{C}$ of the pure jump part of the $\underline{\alpha}$-Lévy copula process X with density k^0 is an anisotropic homogeneous function of type $(1/\alpha_1, ..., 1/\alpha_d)$ and order 1, i.e. it satisfies*

$$\Re\psi_J(t^{1/\alpha_1}\xi_1, ..., t^{1/\alpha_d}\xi_d) = t\, \Re\psi_J(\xi_1, ..., \xi_d) \quad \textit{for all} \quad t > 0 \quad \textit{and all} \quad \xi = (\xi_1, .., \xi_d) \in \mathbb{R}^d.$$

Proof. Since the Lévy symbol ψ_X of X is a continuous negative definite function, also ψ_J and hence $\Re\psi_J$ are continuous negative definite. Hence, $\Re\psi_J : \mathbb{R}^d \to \mathbb{R}$ can be regarded as the symbol of a pure jump process with Lévy density k^0. Thus, by [21, Corollary 3.7.9] the Lévy-Khinchin formula (3.6) can be simplified to

$$\Re\psi_J(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos\langle\xi, x\rangle) k^0(x) dx. \tag{3.7}$$

With this connection of ψ_J and k^0, Theorem 3.3 yields for any $t > 0$ and $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$,

$$\begin{aligned}
\Re\psi_J(t^{1/\alpha_1}\xi_1, ..., t^{1/\alpha_d}\xi_d) &= \int_{\mathbb{R}^d\setminus\{0\}} \big(1 - \cos\big(\sum_{i=1}^d x_i t^{1/\alpha_i}\xi_i\big)\big) k^0(x) dx \\
&= \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos\langle\xi, z\rangle) k^0(t^{-1/\alpha_1} z_1, ..., t^{-1/\alpha_d} z_d) t^{-(\alpha_1+...+\alpha_d)} dz \\
&= \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos\langle\xi, z\rangle) t\, k^0(z_1, ..., z_d) dz \\
&= t\, \Re\psi_J(\xi),
\end{aligned}$$

where we have used the change of variables $x_i = t^{-1/\alpha_i} z_i$, $i = 1, ..., d$. ■

3.1.3 The domain of the generator as an anisotropic Sobolev space

We will now identify the domains of generators of anisotropic, $\underline{\alpha}$-stable Lévy processes. As indicated before, we focus on $D(\mathcal{E}_J)$, i.e. on the pure jump part of the process and assume (3.2) throughout. As we will see, $\mathcal{E}_J$ coincides with Bessel potential spaces resp. with Sobelev spaces of mixed smoothness. We start by recapitulating basic facts on anisotropic Bessel potential spaces.

If $(s_1, ..., s_d)$ is a d- tuple of natural numbers then

$$W^{(s_1,...,s_d)}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{L^2(\mathbb{R}^d)} + \sum_{j=1}^d \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \right\|_{L^2(\mathbb{R}^d)} < \infty \right\}$$

is the classical anisotropic Sobolev space on $\mathbb{R}^d$. In contrast to the usual (isotropic) Sobolev space ($s_1 = \cdots = s_d$) the smoothness properties of an element from $W^{(s_1,...,s_d)}(\mathbb{R}^d)$ depend on the chosen direction in $\mathbb{R}^d$.

These spaces are generalized in a natural way: given $s_1, ..., s_d \in \mathbb{R}$ one can define anisotropic Bessel potential spaces, or fractional Sobolev spaces,

$$H^{(s_1,...,s_d)}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{H^{(s_1,...,s_d)}(\mathbb{R}^d)} = \left\| \sum_{j=1}^d (1 + \xi_j^2)^{s_j/2} \widehat{f} \right\|_{L^2(\mathbb{R}^d)} < \infty \right\}. \tag{3.8}$$

Similar to the isotropic case, the study of anisotropic Bessel potential spaces is a part of the more general theory of anisotropic spaces of Besov and Triebel-Lizorkin type. However for our purposes it will be enough to restrict ourselves to the above classes of anisotropic Bessel potential spaces.

It turns out that it is very useful to remark that given $s \in \mathbb{R}$ and $j = 1, ..., d$, denoting

$$H_j^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|f\|_{H_j^s(\mathbb{R}^d)} = \left\| (1 + \xi_j^2)^{s/2} \widehat{f} \right\|_{L^2(\mathbb{R}^d)} < \infty \right\} \tag{3.9}$$

by [29, Section 9.2] we have (in the sense of equivalent norms):

$$H^{(s_1,...,s_d)}(\mathbb{R}^d) = \bigcap_{j=1}^{d} H_j^{s_j}(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H^{(s_1,...,s_d)}(\mathbb{R}^d)} = \left(\sum_{j=1}^{d} \|f\|^2_{H_j^{s_j}(\mathbb{R}^d)} \right)^{\frac{1}{2}}. \tag{3.10}$$

Henceforth, we will use the notation $\underline{s} = (s_1, ...s_d)$.

In the analysis of anisotropic spaces, following [36] an important tool are so-called anisotropic distance functions.

Let $d \geq 2$ and let $a = (a_1, ..., a_d)$ a given anisotropy, that is a fixed d- tuple of positive numbers. The case $a = (1, ..., 1)$ is usually known as the ”isotropic case”.

Definition 3.5. *Let $a = (a_1, ..., a_d)$ a given anisotropy. An anisotropic distance function (with respect to a) is a continuous function $\varrho : \mathbb{R}^d \to \mathbb{R}$ with the properties $\varrho(x) > 0$ if $x \neq 0$ and*

$$\varrho(t^{a_1} x_1, ..., t^{a_d} x_d) = t\, \varrho(x) \quad \textit{for all} \quad t > 0 \quad \textit{and all} \quad x \in \mathbb{R}^d.$$

It is easy to see that $\varrho_\lambda : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\varrho_\lambda(x) = \left(\sum_{i=1}^{d} |x_i|^{\lambda/a_i} \right)^{1/\lambda} \tag{3.11}$$

is an anisotropic distance function for every $0 < \lambda < \infty$. Remark that for appropriate values of λ we can obtain arbitrary (finite) smoothness of the function ϱ_λ introduced above, cf. [9, Lemma 2.2].

The next result essentially goes back to E. M. Stein and S. Wainger, see [36]. A nice and very detailed exposition can be found in the thesis of H. Dappa, see [9]. Together with the example from (3.11) it will play a key role in our further considerations so that we state it separately.

Lemma 3.6. *Let $a = (a_1, ..., a_d)$ a given anisotropy and ϱ and ϱ' two anisotropic distance functions. Then they are equivalent in the sense that there exist constants $c, c' > 0$ such that $c\, \varrho(x) \leq \varrho'(x) \leq c'\, \varrho(x)$ for all $x \in \mathbb{R}^d$.*

Now we are ready to identify the domain of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ of an anisotropic stable Lévy copula process as an anisotropic Bessel potential space.

Theorem 3.7. *Let the copula F, the numbers $\alpha_1, ..., \alpha_d$, the marginal densities $k_1^0, ..., k_d^0$, and the tail integrals $U_1^0, ..., U_d^0$, as in Assumption* 3.1. *Then the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form associated to the generator of the $\underline{\alpha}$-stable Lévy copula process associated to F and $U_1^0, ..., U_d^0$, is the anisotropic Bessel potential space*

$$H^{\underline{\alpha}/2}(\mathbb{R}^d) = H^{(\alpha_1/2,\cdots,\alpha_d/2)}(\mathbb{R}^d),$$

compare (3.8).

Proof. As in Theorem 3.4, we denote by X the $\underline{\alpha}$-Lévy copula process. Its characteristic triple is denoted by $(A,\gamma,\nu) = (\Sigma^\top\Sigma,\gamma,\nu)$ as in (2.3) and denote its associated semigroup $(T_t)_{t\geq 0}$ as in (1.2) with infinitesimal generator $\mathcal{A}$. Also as in Theorem 3.4, the Lévy symbol $\psi_X : \mathbb{R}^d \to \mathbb{C}$ of X is a continuous, negative definite function.

Since X is a Lévy process, $\mathcal{A}$ is translation invariant and its Dirichlet bilinear form $\mathcal{E}(\cdot,\cdot)$ is given by

$$\mathcal{E}(u,v) = -(u,\mathcal{A}v), \qquad u,v \in \mathcal{S}(\mathbb{R}^d),$$

Since it is a translation invariant Dirichlet form on $L^2(\mathbb{R}^d;\mathbb{R})$, it can be expressed in terms of the characteristic function $\psi_X(\xi)$ of X by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \psi_X(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi. \tag{3.12}$$

We know by [21, Example 4.7.32] that for such forms, there exists $c > 0$ such that

$$|\Im\psi_X(\xi)| \leq c\,(1 + \Re\psi_X(\xi)) \qquad \text{for all} \quad \xi \in \mathbb{R}^d, \tag{3.13}$$

and that the domain of $\mathcal{E}$, denoted $\mathcal{D}(\mathcal{E})$, is a Sobolev space of generalized smoothness

$$\mathcal{D}(\mathcal{E}) = H^{\Re\psi_X,1}(\mathbb{R}^d;\mathbb{R}).$$

Let us specify this in the context of $\underline{\alpha}$-stable processes. By hypothesis (3.2) there holds $\Re\psi_X = \Re\psi_J$, where ψ_J denotes the Lévy symbol of the jump part of X as in formula (3.6). By Theorem 3.4 the function $\Re\psi_X = \Re\psi_J$ is an anisotropic distance function with respect to the anisotropy

$$a_1 = \frac{1}{\alpha_1}, \cdots, a_d = \frac{1}{\alpha_d}.$$

Consequently $\Re\psi_X$ is equivalent to any of the anisotropic distance functions ϱ_λ defined in (3.11), in particular for $\lambda = 1$ or $\lambda = 2$ we get

$$\Re\psi_X(\xi) \sim |\xi_1|^{\alpha_1} + \cdots + |\xi_d|^{\alpha_d} \sim \left(|\xi_1|^{2\alpha_1} + \cdots + |\xi_d|^{2\alpha_d}\right)^{1/2}.$$

From the last characterization, using (3.9) and (3.10), we now get that the space $H^{\Re\psi_X,1}(\mathbb{R}^d)$ coincides with the anisotropic Bessel potential space $H^{(\alpha_1/2,\ldots,\alpha_d/2)}(\mathbb{R}^d)$ and this completes the proof. ∎

3.1.4 Analyticity of the associated semigroup of operators

We start recalling some basic facts on analytic semigroups of operators and on Gårding inequalities. Recall a semigroup $(T_t)_{t\geq 0}$ of operators between Banach spaces is called *analytic* if $t \mapsto T_t u$ admits an analytic extension $z \mapsto T_z u$ to some sector $S_{\theta,d_0} := \{z \in \mathbb{C} : \arg(z - d_0) < \theta\}$. A sufficient condition for the semigroup $(T_t)_{t\geq 0}$ to be analytic is a Gårding inequality for its infinitesimal generator $\mathcal{A}$ in (1.4).

The Gårding inequality can be stated also in terms of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ associated with $\mathcal{A}$ (compare [35]): let $\mathcal{A} : V \to V^*$ and $\mathcal{E}(\cdot,\cdot) : V \times V \to \mathbb{R}$ be continuous, where V^* is the dual space of V with respect to a pivot space H so that

$$V \hookrightarrow H \simeq H^* \hookrightarrow V^* \quad \text{with dense injections;}$$

the semigroup $(T_t)_{t\geq 0}$ with generator $\mathcal{A}$ is analytic, if $\mathcal{E}(\cdot,\cdot)$ satisfies the Gårding inequality: there are $\gamma > 0$ and $C \geq 0$ such that

$$\forall v \in V : \quad \Re\,\mathcal{E}(v,v) \geq \gamma \|v\|_V^2 - C\|v\|_H^2. \tag{3.14}$$

If (3.14) holds, then the Kolmogoroff equation (1.3) is, for $g \in H$, well-posed in $L^2([0,T];V) \cap C^0([0,T];H)$ (e.g., [1]).

We note that (3.14) implies apart from analyticity also the exponential convergence of a suitable high order time-stepping scheme of discontinuous Galerkin type for the numerical integration of the Kolmogoroff equation (1.3) (e,g. [31] and the references there).

Remark 3.8. With the substitution $v = \exp(\lambda t)u$, for sufficiently large $\lambda > 0$, we can change the backward Kolmogoroff equation (1.3) so that a stronger form of (3.14) holds:

$$\forall v \in V : \quad \Re\,\mathcal{E}(v,v) \geq \gamma_1 \|v\|_V^2$$

with a positive constant $\gamma_1 > 0$.

We next establish Gårding inequalities for generators of $\underline{\alpha}$-stable processes. The case of tempered $\underline{\alpha}$-stable processes will be treated later.

Theorem 3.9. *The Dirichlet form induced by the generator $\mathcal{A}$ of the copula of $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d)$-stable marginals satisfying Assumption* 3.1 *and (3.2) satisfies a Gårding inequality in the anisotropic space $H^{\underline{\alpha}/2}(\mathbb{R}^d)$: there exist constants $\gamma > 0$ and $C \geq 0$ such that*

$$\Re\,\mathcal{E}(u,u) \geq \gamma\, \|u\|^2_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} - C\, \|u\|^2_{L^2(\mathbb{R}^d)} \quad \textit{for all} \quad u \in C_0^\infty(\mathbb{R}^d). \tag{3.15}$$

Moreover, the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ is continuous in $H^{\underline{\alpha}/2}(\mathbb{R}^d)$, i.e.

$$|\mathcal{E}(u,v)| \leq c_2 \|u\|_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} \|v\|_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} \quad \textit{for all} \quad u,v \in H^{\underline{\alpha}/2}(\mathbb{R}^d). \tag{3.16}$$

Proof. Based on (2.9) and (3.2), and using Theorem 3.7 we have

$$\begin{aligned} H^{\underline{\alpha}/2}(\mathbb{R}^d) &= H^{\Re\Psi_X,1}(\mathbb{R}^d;\mathbb{R}) \\ &= \left\{ u \in L^2(\mathbb{R}^d) \,:\, \|u\|^2_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+\Re\Psi_X(\xi))\,|\widehat{u}(\xi)|^2\,d\xi < \infty \right\} \end{aligned} \tag{3.17}$$

Based on the definition (3.12) of the Dirichlet form induced by the generator $\mathcal{A}$ of an $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d)$-stable Lévy copula process, with marginals satisfying Assumption 3.1, we know

$$\Re\,\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \Re\Psi_X(\xi))\,|\widehat{u}(\xi)|^2\,d\xi$$

so that the Gårding inequality (3.15) holds in this case with $\gamma = C = 1$.
To prove the continuity of $\mathcal{E}$ in the anisotropic space $H^{\underline{\alpha}/2}(\mathbb{R}^d)$, we use the characterization (3.17) and the fact that the imaginary part of ψ_X is controlled by its real part, i.e. (3.13), to infer (3.16). ■

3.2 Tempered stable Lévy copula processes

In this subsection we introduce tempered $\underline{\alpha}$-stable Lévy copula processes and obtain a counterpart of the Gårding inequality Theorem 3.7 for these processes.

Definition 3.10. *Assume that the copula function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is a homogeneous Lévy copula of order 1 such that $\partial_1 ... \partial_d F : \mathbb{R}^d \to \mathbb{R}$ exists.*
Let $\alpha_1, ..., \alpha_d \in (0,2)$ and the marginal densities be of tempered stable (also known as 'CGMY', [6] *and the references there) type, i.e. (3.3) holds for some $C_j \geq 0$, $\beta_j \geq 0$.*
Then the corresponding Lévy copula process $(X_t^{\beta})_{t\geq 0} \in \mathbb{R}^d$, which exists according to Theorem 2.5*, is called a tempered, $\underline{\alpha}$-stable Lévy copula process.*

Obviously, tempered stable Lévy copula processes include the $\underline{\alpha}$-stable ones (compare Definition 3.2) as special cases ($\beta_1 = ... = \beta_d = 0$). We show next that the infinitesimal generator $\mathcal{A}^{\beta}$ of the quadratic, pure jump tempered, $\underline{\alpha}$-stable process X_t^{β} satisfies, for any choice of $\beta_1, ..., \beta_d \geq 0$, a Gårding inequality (3.14) in the anisotropic Sobolev spaces $V = H^{\underline{\alpha}/2}(\mathbb{R}^d)$ and $H = L^2(\mathbb{R}^d)$.

We start with a lemma which will be useful later.

Lemma 3.11. *Under the above assumptions for any $j = 1, ..., d$ there exist constants $c_j, c_j' > 0$ such that*

$$|k_j^{\beta_j}(x_j) - k_j^0(x_j)| \leq c_j\,|x_j|^{-\alpha_j}.$$

and

$$|U_j^{\beta_j}(x_j) - U_j^0(x_j)| \leq c_j'\,|x_j|^{-\alpha_j+1} \tag{3.18}$$

for any $x_j \in \mathbb{R}$ with $0 < |x_j| \leq 1$.

Proof. The first inequality follows easy using the Taylor expansion of $\exp(-\zeta)$ around $\zeta = 0$. The second inequality follows immediately. ■

Remark 3.12. As a direct consequence of (3.3) it follows that for every $j = 1, ..., d$ there are constants $0 < c_1 \le c_2 < \infty$ such that

$$0 < c_1 \le |x_j|^{\alpha_j+1}|k_j^{\beta_j}(x_j)| \le c_2 < \infty \qquad 0 < c_1 \le |x_j|^{\alpha_j}|U_j^{\beta_j}(x_j)| \le c_2 < \infty \tag{3.19}$$

for any $x_j \in \mathbb{R}$ with $0 < |x_j| \le 1$.

Let us denote by $U_j^{\beta_j}(x) = \int_{-\infty}^{x} k_j^{\beta_j}(\xi)d\xi$, $j = 1, ..., d$ and write $\underline{U}^\beta(x) = (U_1^{\beta_1}, ..., U_d^{\beta_d})$. For a 1-homogeneous copula Lévy copula function F, we denote

$$(F \circ \underline{U}^\beta)(x_1, ..., x_d) = F(U_1^{\beta_1}(x_1), ..., U_d^{\beta_d}(x_d)).$$

Applying the chain rule we immediately get

$$\partial_1...\partial_d(F \circ \underline{U}^\beta)(x_1, ..., x_d) = ((\partial_1...\partial_d F) \circ \underline{U}^\beta)(x_1, ..., x_d)\, k_1^{\beta_1}(x_1)...k_d^{\beta_d}(x_d)$$

which implies that the density of the tempered, $\underline{\alpha}$-stable, Lévy copula process from Definition 3.10 is given by

$$k^\beta(x_1, ..., x_d) = \partial_1...\partial_d(F \circ \underline{U}^\beta)(x_1, ..., x_d) = ((\partial_1...\partial_d F) \circ \underline{U}^\beta)(x_1, ..., x_d)\Pi_{j=1}^d k_j^{\beta_j}(x_j).$$

In the special case when all $\beta_j = 0$, $j = 1, ..., d$, we recover the density of the $\underline{\alpha}$-stable Lévy copula process, i.e.

$$k^0(x_1, ..., x_d) = \partial_1...\partial_d(F \circ \underline{U}^\beta)(x_1, ..., x_d) = ((\partial_1...\partial_d F) \circ \underline{U}^0)(x_1, ...x_d)\Pi_{j=1}^d k_j^0(x_j).$$

Denote by $\mathcal{E}^\beta(\cdot, \cdot)$ and by $\mathcal{E}^0(\cdot, \cdot)$ the corresponding Dirichlet forms which are given by, cf. (3.12)

$$\mathcal{E}^\beta(u, v) = \int_{\mathbb{R}^d} \psi^\beta(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi.$$

Analogously, $\mathcal{E}^0(\cdot, \cdot)$ is defined. Here ψ^β and ψ^0 are the characteristic functions of the tempered $\underline{\alpha}$-stable, resp. $\underline{\alpha}$-stable processes. They are given by the Lévy-Khinchin formula (2.1) with $\nu(dx) = k^\beta(x)dx$, resp. $\nu(dx) = k^0(x)dx$.

We estimate, for $u \in C_0^\infty(\mathbb{R}^d)$,

$$\Re\mathcal{E}^\beta(u, u) \ge \Re\mathcal{E}^0(u, u) - |\mathcal{C}(u, u)|$$

where the form $\mathcal{C}(\cdot, \cdot) := (\mathcal{E}^\beta - \mathcal{E}^0)(\cdot, \cdot)$ is expressed in terms of the characteristic functions by

$$\mathcal{C}(u, v) := \int_{\mathbb{R}^d} (\psi^\beta(\xi) - \psi^0(\xi))\widehat{u}(\xi)\overline{\widehat{v}}(\xi)d\xi, \qquad u, v \in C_0^\infty(\mathbb{R}^d).$$

To establish the Gårding inequality (3.14) for $\mathcal{E}^\beta(\cdot, \cdot)$, we use for $\mathcal{E}^0(\cdot, \cdot)$ the Theorem 3.9 for the $\underline{\alpha}$-homogeneous case. For the tempered stable case (i.e. for $\underline{\beta} \neq 0$), we have:

Proposition 3.13. *For any $K \subset\subset \mathbb{R}^d$ and any choice of $\beta_1, ..., \beta_d \geq 0$, the form $\mathcal{C}(\cdot,\cdot)$ is compact on $V = H^{\alpha/2}_{comp}(K)$.*

Proof. By Theorem 3.10.5 in [21] in order to prove the compactness of $\mathcal{C}(\cdot,\cdot)$ it suffices to verify that

$$\frac{|\psi^\beta(\xi) - \psi^0(\xi)|}{|\psi^0(\xi)|} \to 0 \quad \text{as} \quad |\xi| \to \infty. \tag{3.20}$$

Since the Lévy densities k^β, k^0 of the $\underline{\alpha}$-stable copula processes are smooth for $x_i \neq 0$, the decay for $|\xi| \to \infty$ of the characteristic functions $\psi^\beta(\xi)$ and $\psi^0(\xi)$ as $|\xi| \to \infty$ is determined by the singularities of the densities $k^\beta(x_1, ..., x_d)$, $k^0(x_1, ..., x_d)$ at zero.

By substituting $k^\beta = \partial_1 ... \partial_d(F \circ \underline{U}^\beta)$ and $k^0 = \partial_1 ... \partial_d(F \circ \underline{U}^0)$ in the Lévy-Khinchin formula (2.1) one obtains,

$$\psi^0(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{i\langle\xi,y\rangle} + \frac{i\langle\xi,y\rangle}{1+|y|^2})\partial_{y_1}...\partial_{y_d}(F \circ \underline{U}^0)(y)dy.$$

and

$$(\psi^\beta - \psi^0)(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{i\langle\xi,y\rangle} + \frac{i\langle\xi,y\rangle}{1+|y|^2})\partial_{y_1}...\partial_{y_d}((F \circ \underline{U}^\beta) - (F \circ \underline{U}^0))(y)dy.$$

Using integration by parts we have

$$\psi^0(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(\partial_{y_1}...\partial_{y_d}(1 - e^{i\langle\xi,y\rangle} + \frac{i\langle\xi,y\rangle}{1+|y|^2})\right)(F \circ \underline{U}^0)(y)dy,$$

and

$$(\psi^\beta - \psi^0)(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(\partial_{y_1}...\partial_{y_d}(1 - e^{i\langle\xi,y\rangle} + \frac{i\langle\xi,y\rangle}{1+|y|^2})\right)((F \circ \underline{U}^\beta) - (F \circ \underline{U}^0))(y)dy.$$

To establish (3.20) it therefore suffices to investigate the singularities of $[(F\circ\underline{U}^\beta) - (F\circ\underline{U}^0)](x)$ and of $(F \circ \underline{U}^0)(x)$ at $x = 0$.

Since the copula function F in Definition 3.10 is 1-homogeneous and Lipschitz (recall (2.6)), we estimate using (3.18)

$$\left|[F \circ \underline{U}^\beta - F \circ \underline{U}^0](x)\right| \leq \sum_{j=1}^d |U_j^{\beta_j}(x_j) - U_j^0(x_j)| \leq C \sum_{j=1}^d |x_j|^{-\alpha_j+1}.$$

On the other hand, by the 1-homogeneity of the copula function F we can write for $\xi \neq 0$

$$F(\xi) = F\left(|\xi|(\frac{\xi_1}{|\xi|}, ..., \frac{\xi_d}{|\xi|})\right) = |\xi|\, F\left(\frac{\xi_1}{|\xi|}, ..., \frac{\xi_d}{|\xi|}\right)$$

and using (3.19) we find that there is a $c > 0$ such that

$$|(F \circ \underline{U}^0)(x)| \geq c \sum_{j=1}^{d} |x_j|^{-\alpha_j}.$$

Consequently we obtain that the leading singularity of $F \circ \underline{U}^0$ at $x = 0$ is cancelled in the difference $[F \circ \underline{U}^\beta - F \circ \underline{U}^0](x)$ whence the growth of $\psi^\beta(\xi) - \psi^0(\xi)$ as $|\xi| \to \infty$ is slower than that of $\psi^0(\xi)$ which proves (3.20).

This completes the proof of Proposition 3.13. ■

4 Numerical Solution of the Kolmogoroff Equation

Here, we address the evaluation of the expectation (1.1) by the numerical solution of the Kolmogoroff equation (1.3) for tempered, $\underline{\alpha}$-stable Lévy copula processes X^β.

Throughout, we continue to work under the hypothesis (3.2), i.e. that the Lévy process X^β is quadratic, pure jump. Note that if the driftless diffusion $W \neq 0$ and, more precisely, Σ in the Lévy-Itô decomposition (2.3) of X is non-singular, the domain of $\mathcal{A}_W$ in the decomposition (2.5) is $H^1(\mathbb{R}^d)$. Hence, because of the intersection structure of $D(\mathcal{A}_X)$ described in (3.1), the domain of the generator of X is given by $D(\mathcal{A}_X) = D(\mathcal{A}_W) = H^1(\mathbb{R}^d)$ as in the classical Black-Scholes setting.

Remark 4.1. To obtain arbitrage free prices of derivative contracts in mathematical models one requires the stochastic process driving the underlying assets to be a martingale under some suitable equivalent martingale measure. As described in e.g. [8], requiring this martingale property determines the drift part δ of the driving process X. In one dimension this drift part can easily be calculated in closed form (see e.g. [8, Proposition 8.20] for the case of exponential Lévy processes). Since for a Lévy measure ν of a Copula process there holds for any $i \in \{1, ..., d\}$ and $f \in C(\mathbb{R})$,

$$\int_{\mathbb{R}^d \setminus \{0\}} f(x_i)\nu(dx) = \int_{\mathbb{R} \setminus \{0\}} f(x_i)\nu_i(dx_i), \tag{4.1}$$

where ν_i denotes the i-th marginal measure (cf. [24]), the required drift vector $\gamma \in \mathbb{R}^d$ of the multivariate copula process comprises of the one-dimensional drifts $\gamma_i \in \mathbb{R}$, $i = 1, ..., d$, corresponding to each marginal process being a martingale.

Nevertheless, one may see that hypothesis (3.2) does not contradict these considerations as follows: In case the driftless diffusion part W of X does not vanish (and Σ is non-singular) the domain of the generator of X is independent of γ as indicated above. Therefore one may assume $\gamma = 0$ without loss of generality.
If W vanishes and the drift vector $\gamma \in \mathbb{R}^d$ is fixed by the martingale condition one may perform a

"removal of drift", i.e. a suitable transformation, to obtain vanishing drift again. Thus, one may assume $\gamma = 0$ as in (3.2). In the one dimensional case the "removal of drift" is described in [25, Section 4.4]. This procedure can be adapted to multivariate copula processes using the marginal preserving property of the Lévy copula.

4.1 Localization

For the numerical solution of (1.3), we localize this equation from $\mathbb{R}^d$ to the bounded computational domain $D = (0,1)^d$; this setting arises, for example, when pricing barrier contracts or in the case of first passage times from D. We start by defining anisotropic Bessel spaces in D.

For $u \in C_0^\infty(D)$, define $\bar{u}$ to be the zero extension of u to all of $\mathbb{R}^d$. Then we define for $s_i \geq 0$, $i = 1, ..., d$ the vector of smoothness indices $\underline{s} = (s_1, ..., s_d)$ and

$$\tilde{H}^{\underline{s}}(D) := \overline{\{\bar{u} | u \in C_0^\infty(D)\}}$$

where the closure is taken w.r. to the norm in $H^{\underline{s}}(\mathbb{R}^d)$, defined in (3.10).

By definition, for all $u \in \tilde{H}^{\underline{s}}(D)$ it holds that $\bar{u} \in H^{\underline{s}}_{comp}(\mathbb{R}^d)$. For the parabolic setting of the Kolmogoroff equation (1.3), we replace t by $-t$ and obtain for $u(t) = v(-t)$

$$u_t = \mathcal{A}u \quad \text{in} \quad (0,T), \qquad u|_{t=0} = g. \tag{4.2}$$

$$V = \tilde{H}^{\underline{\alpha}/2}(D), \quad H = L^2(D), \quad V_\theta := (V^*, V)_{\theta,2}, \quad 0 < \theta < 1$$

where we used the real method of interpolation. Finally, we localize the Dirichlet form $\mathcal{E}^\beta$ to D by

$$\mathcal{E}^\beta_D(u,v) := \mathcal{E}^\beta(\bar{u},\bar{v}), \qquad u,v \in \tilde{H}^{\underline{\alpha}/2}(D).$$

The compactness result Proposition 3.13 allows to state the analog of Theorem 3.9 for tempered stable Lévy copula processes.

Theorem 4.2. *For any bounded domain $D \subset \mathbb{R}^d$ and any vector $\underline{\beta}$ in $\mathbb{R}^d_+$, there exists a constant $\gamma > 0$ (depending on D and $\underline{\beta}$) and a compact bilinear form $\mathcal{C}(\cdot,\cdot) : \tilde{H}^{\underline{\alpha}/2}(D) \times \tilde{H}^{\underline{\alpha}/2}(D) \to \mathbb{R}$ such that*

$$\forall u \in \tilde{H}^{\underline{\alpha}/2}(D): \qquad \mathcal{E}^\beta_D(u,u) \geq \gamma \, \|u\|^2_{\tilde{H}^{\underline{\alpha}/2}(D)} - \mathcal{C}(u,u) \tag{4.3}$$

and that for all $u, v \in \tilde{H}^{\underline{\alpha}/2}(D)$ it holds

$$|\mathcal{E}^\beta_D(u,v)| = |\mathcal{E}^\beta(\bar{u},\bar{v})| \leq c_3 \|\bar{u}\|_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} \|\bar{v}\|_{H^{\underline{\alpha}/2}(\mathbb{R}^d)} = c_3 \|u\|_{\tilde{H}^{\underline{\alpha}/2}(D)} \|v\|_{\tilde{H}^{\underline{\alpha}/2}(D)} \tag{4.4}$$

Therefore, the form $\mathcal{E}_D(\cdot,\cdot)$ is continuous and coercive on $V \times V \to \mathbb{C}$. Henceforth, we omit the subscript D on $\mathcal{E}^\beta_D$ with the understanding that it will be considered only on D.

As already indicated above, in case of non-singular Σ (and thus non-vanishing diffusion part W of X^β) the domain of the generator $\mathcal{A}_D$ that canonically corresponds to $\mathcal{E}_D(\cdot,\cdot)$ is given by $\tilde{H}^1(D) = H^1_0(D)$ and in particular Theorem 4.2 also holds in this case.

By (4.3) and (4.4) the Kolmogoroff Equation (1.3) for the tempered, $\underline{\alpha}$-stable Lévy copula process X_t^β, i.e. (4.2), is well-posed and admits a unique solution $u \in L^2(0,T;V) \cap C^0([0,T];H)$.

4.2 Space Discretization

We address the space discretization of (1.3) in $D = (0,1)^d$. We consider only the case when $V = \tilde{H}^{\alpha_1/2,\dots,\alpha_d/2}(D)$ where $0 \le \alpha_i \le 2$, i.e. the case of 1-homogeneous copulas of α_i-stable (viz. tempered α_i-stable) marginals satisfying Assumption 3.1.

We consider discretization of the Dirichlet form $\mathcal{E}_D^\beta(\cdot,\cdot)$ by Galerkin projection onto finite dimensional subspaces $V_N = \operatorname{span}\{b_j : j = 1, ..., N\}$ of $V = \tilde{H}^{\underline{\alpha}/2}(D)$.

To realize the Galerkin Finite Element discretization of the infinitesimal generator $\mathcal{A}$, its Dirichlet form $\mathcal{E}(\cdot,\cdot)$ given in (3.12) must be evaluated on the basis functions of V_N, resulting in the generator's moment (or stiffness) matrix $\mathbf{A}$ given by

$$\mathbf{A}_{i,j} = \mathcal{E}(b_j, b_i), \qquad i,j = 1, ..., N.$$

More specifically, based on the Lévy-Khinchin formula (2.1), on the pseudodifferential representation (1.5) of $\mathcal{A}$ and on (3.12), we find for the representation

$$\mathcal{E}(u,v) = D(u,v) + J(u,v)$$

where the *jump-diffusion part* $D(u,v)$ of $\mathcal{E}(\cdot,\cdot)$ is given by

$$D(u,v) = \int_{\mathbb{R}^d} \sum_{i=1}^d v(x)\gamma_i \frac{\partial u}{\partial x_i} dx + \int_{\mathbb{R}^d} \sum_{i,j=1}^d q_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx \quad \forall \quad u,v \in \mathcal{S}(\mathbb{R}^d)$$

and the jump-part $J(u,v)$ of $\mathcal{E}(\cdot,\cdot)$ is given by

$$J(u,v) = \frac{1}{2}\int_{x\in\mathbb{R}^d}\int_{y\in\mathbb{R}^d} (u(x+y)-u(x))(v(x+y)-v(x))\nu(dy)dx \quad \forall \quad u,v \in \mathcal{S}(\mathbb{R}^d). \tag{4.5}$$

Using hypothesis (3.2) we set $D(u,v) = 0$ in the remainder of this paper. By the considerations we made at the beginning of Section 4 and since numerically the form $D(\cdot,\cdot)$ can be treated separately from $J(\cdot,\cdot)$ this is no real restriction of the following considerations.

Remark 4.3. *If the marginals' densities $k_j(x_j)$ are symmetric about $x_j = 0$, as, e.g. the densities in (3.3), the density $k^\beta(y_1, ..., y_d)$ is symmetric with respect to each coordinate axis and also the copula's Lévy measure $\nu(dy)$. We emphasize, however, that Lévy densities which are successful in financial modelling as, e.g. the tempered stable densities [6, 7] have densities $k_j(x_j)$ of the form*

$$k_j^{\beta_j}(x_j) = C_j \begin{cases} \frac{e^{-\beta_j^-|x_j|}}{|x_j|^{1+\alpha_j}} & x_j < 0, \\ \frac{e^{-\beta_j^+ x_j}}{|x_j|^{1+\alpha_j}} & x_j > 0, \end{cases} \tag{4.6}$$

for some $C_j > 0$, $0 < \alpha_j < 2$, $\beta_j^{\pm} \geq 0$. Here, the singular part of the density near $x_j = 0$ is symmetric and, hence, the principal part of the generator A_J is self-adjoint, whereas the tails (describing the probability of large jumps) are asymmetric, if $\beta_j^- \neq \beta_j^+$.

For Lévy processes X with infinite intensity of small jumps, the Lévy measure $\nu(dy) = k(y)dy$ has a density $k(\cdot)$ which is nonintegrable with respect to the Lebesgue measure near $y = 0$. This causes difficulties in various discretizations of the jump part (4.5) of $\mathcal{A}$ due to the appearance of divergent integrals. The variational representation (4.5), however, ensures that in Galerkin discretizations, no divergent integrals will appear *provided* the basis functions $b_i(x)$ of the subspace V_N satisfy some minimal smoothness.

Proposition 4.4. *Assume that u, v in* (4.5) *have compact support and satisfy a Lipschitz condition. Then $|J(u,v)| < \infty$.*

Proof. The assertion follows from representation (4.5), since the Lipschitz condition for u and v implies in (4.5) with (2.2)

$$|J(u,v)| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x+y) - u(x)||v(x+y) - v(x)|\nu(dy)dx \leq C \int_{y \in \mathbb{R}^d} |y|^2 \nu(dy) < \infty,$$

where C depends on the support of u, v and on the respective Lipschitz constants. ■

It remains to choose a particular basis $b_j(x)$ of V_N. Based on the tensor product nature of the domain D, the Lipschitz condition required to cancel the singularity of the jump measure near zero, and the fact that the order of the generator $\mathcal{A}$ never exceeds two, we build subspaces V_N out of univariate spaces of continuous piecewise polynomials of degree $p \geq 1$.

4.2.1 Spline Wavelets in $\mathbb{R}$

In the interval $I = (0, 1)$, we define the mesh $\mathcal{T}^\ell$ given by the nodes $j2^{-\ell}$, $j = 0, \ldots, 2^\ell$, with the meshwidth $h_\ell = 2^{-\ell}$. We define $\mathcal{V}^\ell$ as the space of piecewise polynomials of degree $p \geq 1$ on the mesh $\mathcal{T}^\ell$ which are in $C^{p'-1}([0,1])$ with $1 \leq p' \leq p$ and vanish at the endpoints $0, 1$. We write $N^\ell = \dim\, \mathcal{V}^\ell$, $M^\ell := N^\ell - N^{\ell-1}$, $N^{-1} := 0$; then $N^\ell = O(2^\ell)$, $\ell = 0, 1, 2, \ldots$. We employ a wavelet basis ψ_j^ℓ, $j = 1, \ldots, M^\ell$, $\ell = 0, 1, 2, \ldots$ of $\mathcal{V}^\ell$ with the properties:

$$\mathcal{V}^\ell = \mathrm{span}\{\psi_j^\ell \,|\, 0 \leq \ell \leq L;\, 1 \leq j \leq M^\ell\}, \quad \mathrm{diam}\,(\mathrm{supp}\,\psi_j^\ell) \leq C\, 2^{-\ell}\,.$$

Any function $v \in \mathcal{V}^L$ has the representation

$$v = \sum_{\ell=0}^{L} \sum_{j=1}^{M^\ell} v_j^\ell\, \psi_j^\ell$$

with $v_j^\ell = (v, \widetilde{\psi}_j^\ell)$ where $\widetilde{\psi}_j^\ell$ are the so-called dual wavelets. For $v \in V$ one obtains the series

$$v = \sum_{\ell=0}^{\infty} \sum_{j=1}^{M^\ell} v_j^\ell \, \psi_j^\ell \tag{4.7}$$

which converges in $L^2(I)$ and in $H_0^1(I)$. Moreover, there holds the norm equivalence

$$c_1 \|v\|_{H_\theta}^2 \le \sum_{\ell=0}^{\infty} \sum_{j=1}^{M^\ell} |v_j^\ell|^2 \, 2^{2\ell\theta} \le c_2 \|v\|_{H_\theta}^2, \quad 0 \le \theta \le 1 \tag{4.8}$$

where $H_\theta = (L^2(I), H_0^1(I))_{\theta,2} = \tilde{H}^\theta(I)$ for $0 \le \theta \le 1$.

For $v \in L^2(I)$ we can define a projection $P_L \colon L^2(I) \to \mathcal{V}^L$ by truncating (4.7):

$$P_L v := \sum_{\ell=0}^{L} \sum_{j=1}^{M^\ell} v_j^\ell \, \psi_j^\ell \,, \; P_{-1} := 0 \,.$$

This projection satisfies the approximation property

$$\|u - P_L u\|_{H_\theta} \le c \, 2^{-(t-\theta)L} \|u\|_{H^t(I)}, \; 0 \le \theta \le 1, \; \theta \le t \le p+1 \,. \tag{4.9}$$

The Increment or detail spaces $\mathcal{W}^\ell$ are defined by

$$\begin{cases} \mathcal{W}^\ell := & \operatorname{span}\{\psi_j^\ell : 1 \le j \le M^\ell\}, \; \ell = 1, 2, 3, \ldots \\ \mathcal{W}^0 := & \mathcal{V}^0 \,. \end{cases}$$

Then

$$\mathcal{V}^\ell = \mathcal{V}^{\ell-1} \oplus \mathcal{W}^\ell \text{ for } \ell \ge 1, \quad \text{and } \; \mathcal{V}^\ell = \mathcal{W}^0 \oplus \cdots \oplus \mathcal{W}^\ell, \ell \ge 0 \,. \tag{4.10}$$

and $Q_\ell := P_\ell - P_{\ell-1}$ is a projection from $L^2(I)$ to $\mathcal{W}^\ell$.

4.2.2 Examples of wavelets

We give an example for $p = p' = 1$, i.e., for piecewise linear continuous functions on $[0, 1]$ vanishing at the endpoints $0, 1$. Since there is no nonzero function which vanishes at the endpoints and which is linear on the whole interval $[0, 1]$ we now define the mesh $\mathcal{T}^\ell$ for $\ell \ge 0$ by the nodes $x_j^\ell := j 2^{-\ell-1}$ with $j = 0, \ldots, 2^{\ell+1}$. We have $N_\ell = 2^{\ell+1} - 1$ and $M_\ell = 2^\ell$.

We define the wavelets ψ_j^ℓ for level $\ell = 0, 1, 2, \ldots$, $j = 1, \ldots, M_\ell$: for $\ell = 0$ we have $N_0 = M_0 = 1$ and ψ_1^0 is the function with value c_0 at $x_1^0 = \frac{1}{2}$.

For $\ell \ge 1$ we have $N_\ell = 2^{\ell+1} - 1$. and we let $c_\ell := 2^{-\ell/2}, \ell = 0, 1, \ldots$. Then the wavelet ψ_1^ℓ has values $\psi_1^\ell(x_1^\ell) = 2c_\ell$, $\psi_1^\ell(x_2^\ell) = -c_\ell$ and zero at all other nodes. The wavelet $\psi_{M_\ell}^\ell$ has values $\psi_{M_\ell}^\ell(x_{N_\ell}^\ell) = 2c_\ell$, $\psi_{M_\ell}^\ell(x_{N_\ell - 1}^\ell) = -c_\ell$ and zero at all other nodes. The wavelet ψ_j^ℓ with $1 < j < M_\ell$ has values $\psi_j^\ell(x_{2j-2}^\ell) = -c_\ell$, $\psi_j^\ell(x_{2j-1}^\ell) = 2c_\ell$, $\psi_j^\ell(x_{2j}^\ell) = -c_\ell$ and zero at all other nodes.

4.2.3 Sparse tensor product spaces and approximation rates

In $\Omega = I^d = (0,1)^d$, $d > 1$ we define the subspace V^L as the tensor product of the one-dimensional spaces:

$$V^L := \mathcal{V}^L \otimes \cdots \otimes \mathcal{V}^L \tag{4.11}$$

which can be written using (4.10) as

$$V^L = \sum_{0 \le \ell_i \le L} \mathcal{W}^{\ell_1} \otimes \cdots \otimes \mathcal{W}^{\ell_d}.$$

The space V^L has $O(2^{\ell d})$ degrees of freedom and is too costly if d is large. We shall use the sparse tensor product space

$$\begin{aligned} \widehat{V}^L &:= \operatorname{span}\left\{ \psi^{\ell_1}_{j_1}(x_1) \dots \psi^{\ell_d}_{j_d}(x_d) \mid 1 \le j_i \le M^{\ell_i}, \ell_1 + \cdots + \ell_d \le L \right\} \\ &= \sum_{0 \le \ell_1 + \cdots + \ell_d \le L} \mathcal{W}^{\ell_1} \otimes \cdots \otimes \mathcal{W}^{\ell_d}. \end{aligned} \tag{4.12}$$

As $L \to \infty$, we have $N_L := \dim(V^L) = O(2^{dL})$, and $\widehat{N}_L := \dim(\widehat{V}^L) = O(L^{d-1}\, 2^L)$, i.e. the spaces $\widehat{V}^L$ have considerably smaller dimension than V^L. On the other hand, they do have similar approximation properties as V^L, provided the function to be approximated is sufficiently smooth: To characterize the smoothness we introduce the spaces $\mathcal{H}^k$ with square integrable mixed k-th derivatives: Let $\mathcal{H}^0 := L^2(\Omega)$, and define for integer $k \ge 1$

$$\mathcal{H}^k := \{\, u \in H^1_0(\Omega) \mid \mathcal{D}^\alpha u \in L^2(\Omega),\ 0 \le \alpha_i \le k \,\}$$

equipped with the norm

$$\|u\|_{\mathcal{H}^k} := \Big(\sum_{\substack{0 \le \alpha_i \le k \\ 1 \le i \le d}} \|\mathcal{D}^\alpha u\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}.$$

We then define $\mathcal{H}^s$ for arbitrary $s \ge 0$ by interpolation.

For a function $v \in L^2(\Omega)$ we have as a consequence of (4.7), (4.11)

$$v(x) = \sum_{\ell_1, \dots, \ell_d \ge 0} \ \sum_{1 \le j_k \le n_{\ell_k}} v^{\ell_1 \dots \ell_d}_{j_1 \dots j_d}\ \psi^{\ell_1}_{j_1}(x_1) \dots \psi^{\ell_d}_{j_d}(x_d), \tag{4.13}$$

where $v^{\ell_1 \dots \ell_d}_{j_1 \dots j_d} = \langle v, \widetilde{\psi}^{\ell_1}_{j_1} \dots \widetilde{\psi}^{\ell_d}_{j_d} \rangle$.

We then define the sparse projection operator $\widehat{P}^L \colon L^2(\Omega) \to \widehat{V}^L$ by truncating the wavelet expansion:

$$(\widehat{P}_L v)(x) := \sum_{\substack{0 \le \ell_1 + \cdots + \ell_d \le L \\ 1 \le j_k \le n_{\ell_k}, k = 1, \dots, d}} v^{\ell_1 \dots \ell_d}_{j_1 \dots j_d}\ \psi^{\ell_1}_{j_1}(x_1) \dots \psi^{\ell_d}_{j_d}(x_d). \tag{4.14}$$

The projection $\widehat{P}_L$ can be represented in terms of the projections Q_ℓ as follows,

$$\widehat{P}_L = \sum_{0\le \ell_1+\cdots+\ell_d\le L} Q_{\ell_1}\otimes\cdots\otimes Q_{\ell_d}. \tag{4.15}$$

The projection $\widehat{P}_L : V \to \widehat{V}^L$ is stable in the anisotropic, fractional order Sobolev spaces.

Proposition 4.5. *(Stability of $\widehat{P}_L$) For $0\le\theta\le 1$, $0\le\alpha_i\le 2$, and $v\in H_\theta := (L^2(D), \tilde{H}^{\underline{\alpha}/2}(D))_{\theta,2}$ we have*

$$\|\widehat{P}_L v\|_{H_\theta} \le C\,\|v\|_{H_\theta}\,. \tag{4.16}$$

Proof. For $\theta = 0$, we have with

$$|||v|||_0^2 := \sum_{\substack{\ell_k=0\\ k=1,\dots,d}}^{\infty}\ \sum_{1\le j_k\le n_{\ell_k}} \left|v^{\ell_1\dots\ell_d}_{j_1\dots j_d}\right|^2$$

that

$$\|\widehat{P}_L v\|_{H_0} \le C_1 \left|\left|\left|\widehat{P}_L v\right|\right|\right|_0 \le C_1\,|||v|||_0 \le C_2\,\|v\|_{L^2(\Omega)}\,.$$

We also have from the norm equivalence (4.8) and the characterization (3.10) that for every $v\in\tilde{H}^{\underline{\alpha}/2}(D)$:

$$\|v\|_{\mathcal{E}}^2 \sim \|v\|^2_{\tilde{H}^{\underline{\alpha}/2}(D)} \sim \sum_{\substack{\ell_k=0\\ k=1,\dots,d}}^{\infty}\ \sum_{1\le j_k\le n_{\ell_k}} \left|v^{\ell_1\dots\ell_d}_{j_1\dots j_d}\right|^2 (1+2^{\alpha_1\ell_1}+\cdots+2^{\alpha_d\ell_d}) =: |||v|||_1^2\,. \tag{4.17}$$

It follows

$$\|\widehat{P}_L v\|_{\tilde{H}^{\underline{\alpha}/2}(D)} \le C_3 \left|\left|\left|\widehat{P}_L v\right|\right|\right|_1 \le C_3\,|||v|||_1 \le C_4\|v\|_{\tilde{H}^{\underline{\alpha}/2}(D)}\,.$$

Interpolation gives (4.16). ■

Let us denote in what follows by $\bar{\alpha} = \max\{\alpha_i : i = 1,...,d\}\in[0,2]$.

Proposition 4.6. *(Approximation property of $\widehat{P}_L$)*

Assume that the component spaces $\mathcal{V}^\ell$ of $\widehat{V}^L$ have the approximation property (4.9)*. Then for $0\le\bar{\alpha}/2 < p' + \frac{1}{2}$ and $\bar{\alpha}/2 < t \le p+1$*

$$\|u-\widehat{P}_L u\|_{H^{\bar{\alpha}/2}(\Omega)} \le \begin{cases} Ch^{p+1}\,|\log h|^{(d-1)/2}\,\|u\|_{\mathcal{H}^{p+1}} \textit{if } \bar{\alpha}=0 \textit{ and } t=p+1\\ Ch^{t-\bar{\alpha}/2}\|u\|_{\mathcal{H}^t} \textit{otherwise.}\end{cases}$$

4.3 Time-stepping

4.3.1 Time Discretization

Under hypothesis (3.2) the Kolmogoroff equation (1.3) is parabolic, a stable discretization scheme for its numerical solution is the implicit Euler Scheme: to define it, subdivide the interval $(0,T)$ into M subintervals of length $k = T/M$. We approximate the time derivative u_t at $t_m = mk$ by the difference quotient $(u(t_{m+1}) - u(t_m))/k$ and define a sequence of functions $u_L^m \in \hat{V}^L$, $m = 0, 1, ..., M$ approximating $u(t_m)$ as solutions of a sequence of stationary elliptic problems. We initialize them by projection of the data g into the sparse tensor product space $\hat{V}^L$

$$u_L^0 = \hat{P}_L g \in \hat{V}^L$$

and obtain the approximations $u^m \in \widehat{V}^L$ for $m = 1, 2, 3, ...$ as solutions of the stationary elliptic problems

$$\forall v \in \hat{V}^L: \quad (v, u_L^{m+1}) + k\mathcal{E}(v, u_L^{m+1}) = (v, u_L^m), \qquad m = 0, 1, ..., M-1 \tag{4.18}$$

Note that we use the sparse tensor product space (4.12). In the wavelet basis, each u_L^m has the representation (4.13), and we denote the coefficient vector by $\mathbf{u}^m$. Then the sequence of problems (4.18) is equivalent to the matrix equations

$$\mathbf{B}\mathbf{u}^{m+1} = (\mathbf{M} + k\mathbf{A})\mathbf{u}^{m+1} = \mathbf{u}^m \tag{4.19}$$

where $\mathbf{M}$ denotes the mass-matrix and $\mathbf{A}$ the stiffness matrix with respect to the wavelet basis of $\hat{V}^L$.

4.3.2 Multilevel Preconditioning

For iterative solution of the linear systems (4.19), we use multilevel preconditioning. In order to obtain robust algorithms, a unifying preconditioning approach which is effective in the anisotropic spaces $H^{\underline{\alpha}/2}(D)$ will be developed. Here, the wavelet basis is essential.

The norm equivalence (4.17) with $\theta = 0$ implies for every $v \in \hat{V}^L$ of the form (4.13) with coefficient vector $\mathbf{v} = (v_j^\ell)$

$$C_1 \|v\|^2 \leq \mathbf{v}^H \mathbf{M} \mathbf{v} \leq C_2 \|v\|^2 \tag{4.20}$$

with constants C_1, C_2 independent of L. Let $\mathbf{D}_A$ denote the diagonal matrix with entries $2^{\alpha_1 l_1} + \cdots + 2^{\alpha_d l_d}$ for an index corresponding to level $(l_1, \ldots, l_d)$. Then (3.15), (3.16) and (4.8) with $\theta = 1$ imply that

$$C_1 \mathbf{v}^H \mathbf{D}_A \mathbf{v} \leq \mathbf{v}^H \mathbf{A} \mathbf{v} \leq C_2 \mathbf{v}^H \mathbf{D}_A \mathbf{v} \tag{4.21}$$

with constants C_1, C_2 independent of L.

Let $\|\mathbf{w}\|_{\mathbf{D}_A} := (\mathbf{w}^H\mathbf{D}_A\mathbf{w})^{1/2}$, $\|\mathbf{w}\|_{\mathbf{D}_A^{-1}} := (\mathbf{w}^H\mathbf{D}_A^{-1}\mathbf{w})^{1/2}$. For $v \in \hat{V}^L$ with coefficient vector $\mathbf{v}$ and $f \in (\hat{V}^L)^*$ with coefficient vector $\mathbf{f}$ we then have

$$\|v\|_V \sim \|\mathbf{v}\|_{\mathbf{D}_A}, \qquad \|f\|_{V^*} \sim \|\mathbf{f}\|_{\mathbf{D}_A^{-1}}$$

where the norm equivalences hold with constants independent of L.

We now define for preconditioning the diagonal matrix $\mathbf{S}$ and the scaled matrix $\hat{\mathbf{B}}$ as

$$\mathbf{S} := \Big(\mathbf{I} + k\mathbf{D}_A\Big)^{1/2}, \qquad \hat{\mathbf{B}} := \mathbf{S}^{-1}\mathbf{B}\mathbf{S}^{-1}. \tag{4.22}$$

The next result shows that the linear system with the preconditioned matrix $\hat{\mathbf{B}}$ can be solved with GMRES in a number of steps which is independent of the meshwidth.

Lemma 4.7. *For the linear system $\hat{\mathbf{B}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ let $\hat{\mathbf{x}}_j$ denote the iterates obtained by the restarted GMRES(m_0) method with initial guess $\hat{\mathbf{x}}_0$. Then there is $0 < q < 1$ independent of L and k such that*

$$\left\|\hat{\mathbf{b}} - \hat{\mathbf{B}}\mathbf{x}_j\right\| \le Cq^j \left\|\hat{\mathbf{b}} - \hat{\mathbf{B}}\mathbf{x}_0\right\| \tag{4.23}$$

Proof. Since $\Re(\mathbf{x}^H i\Im(\lambda)\mathbf{M}\mathbf{x}) = 0$ we obtain from (4.20), (4.21) that

$$\Re(\mathbf{x}^H\mathbf{B}\mathbf{x}) \ge C\mathbf{x}^H\mathbf{S}^2\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^{\hat{N}_L}$$

implying with $\mathbf{y} = \mathbf{S}\mathbf{x}$

$$\Re(\mathbf{y}^H\hat{\mathbf{B}}\mathbf{y}) \ge C_3\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathbb{R}^{\hat{N}_L}. \tag{4.24}$$

We have

$$\left|\mathbf{x}^H\mathbf{B}\mathbf{y}\right| = \left|\mathbf{x}^H\mathbf{M}\mathbf{y}\right| + k\left|\mathbf{x}^H\mathbf{A}^L\mathbf{y}\right| \le C\|\mathbf{x}\|\|\mathbf{y}\| + Ck\left\|\mathbf{D}_A^{1/2}\mathbf{x}\right\|\left\|\mathbf{D}_A^{1/2}\mathbf{y}\right\|$$

With $\mathbf{D} := \mathbf{I} + k\mathbf{D}_A$ we get

$$\left|\mathbf{x}^H\mathbf{B}\mathbf{y}\right| \le C(\mathbf{x}^H\mathbf{D}\mathbf{x})^{1/2}(\mathbf{y}^H\mathbf{D}\mathbf{y})^{1/2}$$

Using the definition of $\mathbf{S}$, we obtain

$$\left|\mathbf{x}^H\mathbf{B}\mathbf{y}\right| \le C(\mathbf{x}^H\mathbf{S}^2\mathbf{x})^{1/2}(\mathbf{y}^H\mathbf{S}^2\mathbf{y})^{1/2}$$

or

$$\left|\mathbf{x}^H\hat{\mathbf{B}}\mathbf{y}\right| \le C_4\|\mathbf{x}\|\|\mathbf{y}\|.. \tag{4.25}$$

Inequalities (4.24) and (4.25) can be stated as

$$\lambda_{\min}\Big((\hat{\mathbf{B}} + \hat{\mathbf{B}}^H)/2\Big) \ge C_3, \qquad \left\|\hat{\mathbf{B}}\right\| \le C_4$$

According to [16] the non-restarted GMRES method for the matrix $\hat{\mathbf{B}}$ yields iterates $\mathbf{x}_m$ and residuals $\mathbf{r}_m$ satisfying

$$\|\mathbf{r}_m\| \le \Big(1 - \frac{C_3^2}{C_4^2}\Big)^{m/2}\|\mathbf{r}_0\|$$

which shows (4.23). ■

4.3.3 Convergence and Complexity

The implicit time stepping scheme (4.18) yields a sequence $\{u_L^m\}_{m=0}^M \subset \hat{V}^L$ of approximate solutions which approximates the exact solution $u(x,t)$ for $0 \le t \le T$ and $x \in D$.

Using the standard error analysis for the backward Euler scheme, as e.g. in Theorem 5.4 of [30] (assuming that the *exact* stiffness matrix $\mathbf{A}$ in (4.19) is available, i.e. *without* the effect of matrix compression or numerical integration considered in [30]), and assuming full regularity of the exact solution $u(x,t)$, we get the following error bound.

Proposition 4.8. *The sequence* $\{u_L^m\}_{m=0}^M \subset \hat{V}^L$ *of approximate solutions obtained from the time stepping scheme* (4.18) *with exact solutions of the linear systems* (4.19) *in each timestep satisfy the error estimate*

$$k\sum_{m=1}^{M} \|u^m - u_L^m\|_{\mathcal{E}}^2 \le Ch^{2(p+1-\bar{\alpha}/2)}\left\{\max_{0<t<T}\|u(t)\|_{p+1}^2 + \int_0^T \|\dot{u}(s)\|_{p+1-\rho/2}^2 ds\right\} \quad (4.26)$$

$$+ \; Ck^2\int_{t=0}^{T}\|\ddot{u}(s)\|_*^2 ds \quad (4.27)$$

$$= \; O(k^2) + O(h^{2(p+1-\bar{\alpha}/2)}) \quad (4.28)$$

where $u^m := u(\cdot, t_m) \in \tilde{H}^{\underline{\alpha}/2}(D)$ *denotes the exact solution at time* $t_m = mk$, $m = 1,...,M$, $k = T/M$ *and* $\bar{\alpha} = \max\{\alpha_i : i = 1,...,d\} \in [0,2]$. *Here,* $\| \circ \|_{\mathcal{E}}$ *denotes the natural norm associated with the bilinear form* $\mathcal{E}(\cdot,\cdot)$. *It is defined by* $\|u\|_{\mathcal{E}}^2 = \mathcal{E}(u,u)$.

Moreover, if the linear systems (4.19) *are solved in each time step approximately by* $O(\log(\widehat{N}_L))$ *steps of GMRES iteration with preconditioner* $\mathbf{S}$ *defined in* (4.22)*, the resulting approximate solutions* $\tilde{u}_L^m \in\subset \hat{V}^L$ *satisfy* (4.26) *and the overall work for generating the sequence* $\{\tilde{u}_L^m\}_{m=0}^M \subset \hat{V}^L$ *still satisfying the error bound* (4.26) *is bounded by* $O(M\widehat{N}_L^2(\log(\widehat{N}_L)^c))$ *operations for some* $c > 0$.

We conclude by several remarks on possible extensions and generalizations.

Remark 4.9. In Proposition 4.8 we considered only the backward Euler timestepping procedure (4.18) with uniform timestep. Likewise, the θ-scheme could be analyzed, along the lines of [30]. All these results assume maximal time regularity and uniform time steps k.

The time analyticity of the semigroup $(T_t)_{t\ge 0}$ can be exploited also for an *exponentially convergent timestepping scheme* of discontinuous Galerkin type. It would retain the error bounds in Proposition 4.8 with only $O(\log \widehat{N}_L)$ many timesteps; see [35] for analytic time regularity as well as for details on the dG time stepping scheme. A matrix formulation of this scheme is available in [31], for example; it applies also in the present case, once the stiffness matrix $\mathbf{A}$ for the diffusion operator in [31] is exchanged for the matrix $\mathbf{A}$ of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ of the Lévy copula's generator $\mathcal{A}$.

Remark 4.10. The complexity bound $O(M\widehat{N}_L^2(\log(\widehat{N}_L)^c))$ operations in Proposition 4.8 is sub-optimal for two reasons: first, as mentioned before, the time-analyticity of the copula's semi-group T_t is not exploited by the backward Euler timestepping scheme (4.18); this could be remedied by the hp dG timestepping procedure from [35, 31]. A second, more severe source of nonoptimality is the fact that the stiffness matrix $\mathbf{A}$ of $\mathcal{A}$ in the *sparse tensor product* wavelet basis of $\widehat{V}^L$ in (4.12) is fully populated, so that each matrix vector multiplication requires $O(\widehat{N}_L^2)$ many operations. As is well known (e.g [33]), generators of *isotropic* α-stable processes have Calderón Zygmund kernels whose Galerkin discretization matrices in *full* tensor product wavelet bases can be compressed from N_L^2 to $O(N_L \log N_L)$ many nonzero entries without compromising accuracy.

Remark 4.11. So far, we discussed only European style contracts without early exercise features. For American style contracts, arbitrage-free prices are characterized as (viscosity) solutions of parabolic variational inequalities. Their numerical solution, however, can be achieved along the lines of [26], based on the norm equivalence (4.17) which implies also optimal preconditioning of the linear complementarity solver in [26] applied to $\mathbf{M}$ and $\mathbf{A}$ in (4.19), using (4.20) and (4.21).

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