Finite Element Valuation of Swing Options

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Abstract

In this paper an algorithm based on Finite Element Methods is presented to value American type of swing contracts with multiple exercise rights. Thereby the reduction of multiple stopping time problems to a cascade of single stopping time problems is utilized. The numerical results obtained with the proposed algorithm show a smooth and stable behavior. This allows an interpretation of the swing options' optimal exercise boundaries and an analysis of the dependence of swing option prices on the initial spot prices. A comparison of the Finite Element algorithm to Monte Carlo and lattice methods demonstrates the strengths of the proposed numerical algorithm.

Keywords: ffinite element methods, swing options, multiple optimal stopping time problems, power derivatives

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1 Introduction

Financial instruments enjoy an increasing popularity in risk management in energy markets by providing a custom-made protection against undesirable price movements. Especially swing types of derivatives are in favor as they offer the desired flexibility in delivery with respect to both timing and amount of energy. Due to different demands of flexibility, swing options are given in various forms, often including constraints from production processes. Therewith these derivatives partially mimic a portfolio of real assets, allowing the seller to hedge himself by owning production facilities.

Although swing options are given in various forms, many of them are mathematically of the same type, namely optimal multiple stopping time problems. Despite the simple and intuitive formulation, multiple stopping time problems do not seem to have attracted much attention in probability literature. Indeed, the first mathematical analysis was only recently proposed by [5] in which the authors proved the existence of optimal multiple exercise polices for continuous reward processes. In addition, they characterized the exercise boundary values for American put options with multiple exercise rights in the Black-Scholes model. Their work has been extended by [4] to general, linear diffusions and reward functions. Due to the lack of closed-form solutions for American type of contingent claims, various numerical schemes for approximating swing option prices are proposed in the literature.

To include different types of contract constraints, several authors modeled swing options as stochastic control problems. For numerically solving these problems, [26] and [15] proposed to use binomial and trinomial lattices, extending ideas of [11] where valuation of contingent claims with path-dependence was introduced. Both approaches made use of a special type of payoff function to make the computation more efficient. In addition, [15] considered the special case of gas swing contracts using a one factor, seasonal and mean reverting spot price model.

Regarding numerical pricing of derivatives, Monte Carlo methods are very popular. Especially, the Least Squares Monte Carlo method proposed by [19] has attracted much attention. Both, [23] and [25] combined this method with the dynamic programming principle and applied it to price swing derivatives. In contrast, [5] utilized an approach based on Malliavin integration by parts introduced in [3] which seems to be convenient for Gaussian spot price processes. Another method based on Monte Carlo simulation is presented in [12]. Here, the algorithm proposed by [13] which approximates the optimal exercise boundary has been extended. The insight of their method is that any point on an optimal exercise frontier can be computed as a fixed point of a simple algorithm.

Only recently an approach based on Hamilton-Jacobi-Bellman inequalities has been presented by [9] for the Black-Scholes model using Finite Difference methods. The work has been extended in [8] where the author has considered a fairly general setting in which technical constraints as well as different recovery times between exercise rights have been taken into account. Another approach based on a continuous time model is studied in [17]. Here, a replication strategy utilizing futures, call and put options has been derived. The problem is that the dynamic strategy appears to be difficult to compute and that needed derivatives are not traded at energy exchanges.

In contrast to the lately proposed numerical methods, we use the fact that multiple optimal stopping time problems can be reduced to a cascade of ordinary stopping time problems as proved in [5]. By applying this reduction to swing options an algorithm based on Finite Element methods is derived. These numerical schemes offer high flexibility with respect to payoff functions and spot price models as opposed to other numerical procedures where special structure properties are needed for efficient computation. In addition, the approximation of swing option prices as well as exercise regions is highly accurate permitting interpretations of these quantities.

The paper is organized as follows. First, we give an overview of multiple stopping time problems where the fundamental results needed for the numerical deviation are summarized. Then, in Section 3 the numerical algorithm for pricing swing options is described in detail. The derived procedure is applied to different spot price models as well as different types of swing derivatives in Section 4 and compared to lattice and Monte Carlo methods. Additionally, we extend the derived algorithm to Lévy processes and give a numerical example.

2 Multiple stopping time problems

In this section the most fundamental results concerning the no-arbitrage valuation of finite horizon swing options are provided. Let $(\Omega, \mathcal{F}, P)$ be a complete, filtered probability space with natural filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by a standard Brownian motion $(W_t)_{t\geq 0}$ with values in $\mathbb{R}$. The risk neutral spot price $(S_t)_{t\geq 0}$ is assumed to be the solution of the stochastic differential equation

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \quad S_0 = s \tag{2.0.1}$$

where to ensure existence and uniqueness, it is supposed that the functions $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to (0, \infty)$ satisfy

$$\begin{aligned} &|\mu(t,x)| + |\sigma(t,x)| \leq c_1(1 + |x|)\,, \\ &|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq c_2|x - y|\,, \end{aligned}$$

for all $t \geq 0$, $x, y \in \mathbb{R}^+$ and some constants $c_1,\ c_2 > 0$. To emphasize the dependence of the process on the initial condition $S_0 = s$, we write $(S_t^{0,s})_{t\geq 0}$ instead of $(S_t)_{t\geq 0}$. It is assumed that the price of the risk-free asset is given as the solution of

$$dB_t = rB_t dt, \quad B_0 = 1\,,$$

where $r \in \mathbb{R}^+$ denotes the risk-free interest rate.

In energy markets the delivery of a commodity is limited by capacity constraints usually resulting in a pre-specified refracting time for contracts with several exercise rights. It can be agreed that the refraction period δ which is greater than the minimal delivery time is constant. This separation of two exercise times not only represents an important contract constraint, but also prevents the case of single optimal stopping time problems where all rights are exercised at once. The definition of admissible stopping times includes these important characteristics of commodity markets.

Let us denote by $\mathcal{T}_{t,T}$ the set of all $\mathbf{F}$-stopping times with values in $[t, T]$ and by $\mathcal{T}_{t,\infty}$ the set of all $\mathbf{F}$-stopping times with values greater or equal than t. For stopping time problems with $p \in \mathbb{N}$ exercise rights, constant refracting period δ and maturity T the following sets are defined.

Definition 2.1. The set of admissible stopping time vectors with length $p \in \mathbb{N}$ and refracting time $\delta > 0$ is defined by

$$\mathcal{T}_t^{(p)} := \{\tau^{(p)} = (\tau_1, \ldots, \tau_p) \mid \tau_i \in \mathcal{T}_{t,\infty} \text{ with } \tau_1 \leq T \text{ a.s. and } \tau_{i+1} - \tau_i \geq \delta \\ \text{for } i = 1, \ldots, p-1\}.$$

Note that the stopping times of a vector $\tau^{(p)} \in \mathcal{T}_t^{(p)}$ might not all have their values in the interval $[t, T]$. This is essential as it might be desirable not to exercise all rights of a swing option with maturity T. Then by letting an exercise right expire, one is not limited by the refraction period and so can fully benefit from potential better future prices.

According to American contingent claims, see e.g. [24], a continuous reward function $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies for all $t \in \mathbb{R}^+$ the linear growth condition $|\phi(t, s)| \leq k_1 + k_2 s$ for some constants k_1, k_2 is considered. In addition, it is assumed that $\phi(t, \cdot) = 0$ for $t > T$. The finite horizon multiple stopping time problem with maturity T and $p \in \mathbb{N}$ exercise rights is defined as

$$V^{(p)}(t, s) := \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} E\left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} \phi(\tau_i, S_{\tau_i}) \mid S_t = s\right], \tag{2.0.2}$$

for all $(t, s) \in [0, T] \times \mathbb{R}^+$.

The Theorem below states that the supremum in (2.0.2) is attained for continuous reward functions satisfying the linear growth condition above.

Theorem 2.2. *For any $p \in \mathbb{N}$ there exists $\tau^* = (\tau_1^*, \ldots, \tau_p^*) \in \mathcal{T}_t^{(p)}$ such that*

$$V^{(p)}(t, s) = E\left[\sum_{i=1}^{p} e^{-r(\tau_i^* - t)} \phi(\tau_i^*, S_{\tau_i^*}) \mid S_t = s\right].$$

for all $(t, s) \in [0, T] \times \mathbb{R}^+$.

Proof. See [5]. □

Using this fundamental result, the multiple stopping time problem can be reduced to a cascade of single stopping time problems.

Corollary 2.3. *For any $p \in \mathbb{N}$, $s \in \mathbb{R}^+$ and $t \in [0,T]$*

$$V^{(p)}(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau-t)}\Phi^{(p)}(\tau, S_\tau) \mid S_t = s\right],$$

with

$$\Phi^{(p)}(t,s) := \begin{cases} \phi(t,s) + e^{-r\delta}E\left[V^{(p-1)}(t+\delta, S_{t+\delta}) \mid S_t = s\right] & \text{if } t \leq T-\delta \\ \phi(t,s) & \text{if } t \in (T-\delta, T] \end{cases}$$

$V^{(0)}(t,s) := 0\,.$

Proof. See [5]. ☐

For a single stopping time problem, i.e. $p = 1$, Corollary 2.3 gives the standard formulation of American contingent claims. Additionally, it allows the intuitive interpretation of the first optimal stopping time τ_1^* as the time when the sum of the discounted instantaneous payoff and the value of the stopping time problem with $p-1$ rights is maximal. Due to the constant refracting period, the value of the stopping time problem with $p-1$ exercise rights is given as a discounted conditional expected value. Moreover, the refraction period limits the number of exercisable rights until T resulting in the following equation for $p \geq 2$

$$V^{(p)}(t,s) = V^{(p-1)}(t,s) \quad \text{for } t \in (T-(p-1)\delta, T],\ s \in \mathbb{R}^+. \tag{2.0.3}$$

which is seen from Corollary 2.3. Next we prove that the only price of a swing option with finite time horizon which does not create arbitrage is given by (2.0.2).

Corollary 2.4. *The only price of a swing option with $p \in \mathbb{N}$ exercise rights, payoff function ϕ and maturity T which does not create any arbitrage opportunities is given by*

$$V^{(p)}(t,s) = \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} E\left[\sum_{i=1}^{p} e^{-r(\tau_i - t)}\phi(\tau_i, S_{\tau_i}) \mid S_t = s\right] \tag{2.0.4}$$

for all $(t,s) \in [0,T] \times \mathbb{R}^+$.

Proof. For $p = 1$ the situation is equivalent to the standard American contingent claim. The process $(e^{-rt}\phi(t,S_t))_{t\in[0,T]}$ has continuous paths and ϕ satisfies the linear growth condition. Therefore, the only price which does not create arbitrage is given by (2.0.4) according to Theorem 5.3 in [16].
For $p > 1$ the proof is obtained from [10] by slight adaptations. To sketch the main ideas, the set of admissible strategies is defined by

$$\mathcal{U} := \left\{ u = \sum_{i=1}^{p} 1_{[\tau_i, \infty)} \mid \tau^{(p)} = (\tau_1, \ldots, \tau_p) \in \mathcal{T}_t^{(p)} \right\}$$

with which the multiple optimal stopping time problem can be written as

$$V^{(p)}(t,s) = \sup_{u \in \mathcal{U}} E\left[\int_t^T e^{-r(\tau-t)}\phi(\tau, S_\tau)du_\tau \mid S_t = s\right].$$

We have to prove, that if the initial price of the swing option differs from $V^{(p)}(t,s)$, there is an arbitrage opportunity.
First suppose that the option is offered at a price $V' < V^{(p)}(t,s)$. Then, there is a *long arbitrage* which can be seen by the following considerations. The agent enters a long position at V' and exercises the contract by the optimal policy $u^* \in \mathcal{U}$ which according to Theorem 2.2 exists. Simultaneously, he writes a contingent claim promising the cash flow of the strategy u^*. Due to completeness, the market pays $V^{(p)}(t,s)$ for this claim. Thus, the agent takes the arbitrage $V^{(p)}(t,s) - V' > 0$.
If the contract is asked at a price $V' > V^{(p)}(t,s)$, there is a *short arbitrage*. The crucial point shown in [10] is that for each strategy $u \in \mathcal{U}$ there exists a centered martingale M^u on $[t,T]$ for which the mapping $u \mapsto M^u$ is non-anticipating. In addition, M^u fulfills for each $u \in \mathcal{U}$ and all $\tau \in [t,T]$ the inequality

$$\int_t^\tau e^{-r(q-t)}\phi(q, S_q^{t,s})du_q \leq V^{(p)}(t,s) + M_\tau^u \ . \tag{2.0.5}$$

Using this result, an agent entering a short position to receive V' is considered. The part $V^{(p)}(t,s)$ of his capital is used to start a trading strategy $\pi(M^u)$ whose discounted wealth equals to the right-hand side of (2.0.5). According to (2.0.5) the wealth of this strategy covers the agent's liabilities of the short position for every $u \in \mathcal{U}$. As a result, the agent takes arbitrage $V' - V^{(p)}(t,s) > 0$. □

From the existence of multiple optimal stopping times it follows that the only swing option price which does not create arbitrage is determined by a series of single stopping time problems. This reduction forms the basis of the numerical algorithm presented below.

Remark 2.5. In [5] a result similar to Corollary 2.3 is also derived for the perpetual multiple stopping time problem.

3 Finite Element pricing algorithm

In the previous section it is shown that the value of a swing derivative is given by a multiple stopping time problem which can be reduced to a cascade of single stopping time problems. The fair price of a swing option is thus given as an American contingent claim with reward function equal to the discounted instantaneous payoff plus a European option value. The latter represents the swing option price with one exercise right less after the refracting period δ. Therefore, a numerical algorithm which inductively calculates prices of swing derivatives can be constructed in the following way. First, the p^{th} reward function $\Phi^{(p)}$ including the price of a European contingent claim is calculated. Next, an American contingent claim is evaluated using $\Phi^{(p)}$ as payoff function. According to Corollary 2.3, the obtained results are the fair prices of a swing option with p exercise rights.

Any algorithm could be used for approximating the involved standard European and American contingent claim prices. However, since the option price curves are reused for the p^{th} reward function it is essential for these to be highly accurate. Using Finite Element methods the whole price curve is obtained fast and precise.

Before presenting the algorithm for swing type of derivatives, the Finite Element method for pricing European and American options is briefly reviewed (for more details see [1] and the references therein). To simplify the notation the Black-Scholes framework is considered where the risk neutral spot price process $(S_t)_{t\geq 0}$ has the dynamics

$$dS_t = S_t(rdt + \sigma dW_t), \quad S_0 = s\,,$$

with positive, constant interest rate r and volatility σ. Note that this assumption is not necessary for the numerical procedure and is just done for illustrating purposes. For $\Omega \subset \mathbb{R}$ we denote with $L^2(\Omega)$, $H^1(\Omega)$ the usual Lebesgue, respectively Sobolev space on Ω and with $(\cdot,\cdot)$ the L^2 inner product.

3.1 European options

The price of a European contingent claim with payoff ϕ is defined as

$$h(t,s) := E\left[e^{-r(T-t)}\phi(S_T)|S_t = s\right] \qquad \forall (t,s) \in [0,T] \times \mathbb{R}^+.$$

According to the Feynman-Kac formula (see e.g. [24]) the function h satisfies for $t \in (0,T)$ and $s \in \mathbb{R}^+$ the partial differential equation

$$\frac{\partial h}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 h}{\partial s^2} + rs\frac{\partial h}{\partial s} - rh = 0\,,$$

with terminal condition

$$h(T,s) = \phi(s) \qquad \forall s \in \mathbb{R}^+\,.$$

To remove the degeneracy at $s = 0$, we change to logarithmic price $x := \log(s) \in \mathbb{R}$ and denote $u(t,x) := h(t,e^x)$, $\psi(x) := \phi(e^x)$ for all $t \in [0,T]$, $x \in \mathbb{R}$. Furthermore, by changing to time to maturity $\tau := T - t \in [0,T]$ the partial differential equation can be written as

$$\begin{aligned} \frac{\partial u}{\partial \tau} + A_{\mathrm{BS}}[u] &= 0 \qquad \text{in } (0,T) \times \mathbb{R}\,, \\ u(0,\cdot) &= \psi \qquad \text{in } \mathbb{R}\,, \end{aligned} \tag{3.1.1}$$

where A_{BS} is the Black-Scholes operator

$$A_{\mathrm{BS}} := -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r\,. \tag{3.1.2}$$

For the numerical implementation the unbounded domain of the variable x needs to be truncated to a bounded domain $\Omega_R = [-R,R]$. Using the excess to payoff function

$$U(\tau,x) := u(\tau,x) - e^{-r\tau}\psi(x + r\tau) \qquad \forall (\tau,x) \in [0,T] \times \mathbb{R}\,.$$

and weighted Sobolev spaces, it is shown in [22] that the localization error decays exponentially with R. This is illustrated in Figure 1 for a European put option ($T = 1$, $K = 1$, $\sigma = 0.5$, $r = 0.05$).

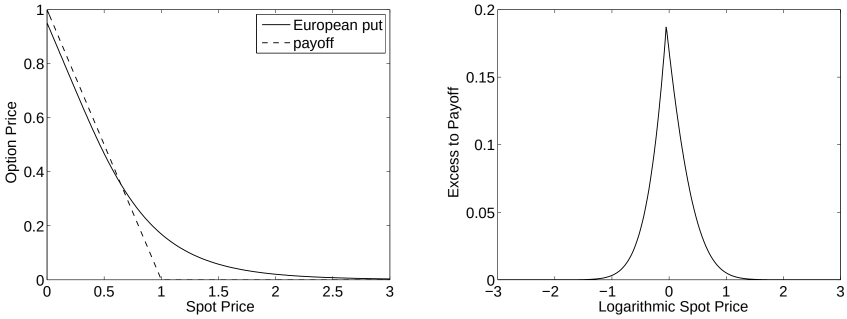

Figure 1: European put option price u (left) and the corresponding excess to payoff function U (right) in logarithmic spot price.

Therefore, zero Dirichlet boundary conditions can be imposed on the function

$$U_{\mathrm{R}}(\tau,x) := u(\tau,x) - e^{-r\tau}\psi(x+r\tau) \qquad \forall(\tau,x)\in[0,T]\times\Omega_R\,,$$

which only defined on the bounded domain. The problem is to determine for all $\tau\in(0,T)$ the function $U_{\mathrm{R}}(\tau,\cdot)\in H_0^1(\Omega_R)$ with

$$H_0^1(\Omega_R) := \left\{\varphi\in H^1(\Omega_R)\mid \varphi(R)=\varphi(-R)=0\right\}\,.$$

Note that since only continuous spot price models of the form (2.0.1) are considered u could be calculated directly by imposing artificial time-dependent non-zero boundary conditions. This time dependence would make the algorithm more technical. Furthermore, for an extension to Lévy models the excess to payoff is important.

Proposition 3.1. *The excess to payoff function $U_R(\tau,\cdot)$ is the variational solution in $H_0^1(\Omega_R)$ of*

$$\begin{aligned}\frac{\partial U_R}{\partial\tau} + A_{BS}[U_R] &= g \qquad \text{in } (0,T)\times\Omega_R\,,\\ U_R(0,\cdot) &= 0 \qquad \text{in } \Omega_R\,,\end{aligned} \tag{3.1.3}$$

with

$$g(\tau,\cdot) := -\frac{\partial\left(e^{-r\tau}\psi(\cdot+r\tau)\right)}{\partial\tau} - A_{BS}[e^{-r\tau}\psi(\cdot+r\tau)]\,, \tag{3.1.4}$$

or in variational form

$$\begin{aligned}\left(\frac{\partial U_R}{\partial\tau},v\right) + a_{BS}\left(U_R,v\right) &= \langle g,v\rangle \quad \forall v\in H_0^1(\Omega_R)\,,\\ U_R(0,\cdot) &= 0 \qquad \text{in } \Omega_R\,,\end{aligned} \tag{3.1.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the $\left(H^1(\Omega_R)\right)^*\times H^1(\Omega_R)$ duality pairing and

$$a_{BS}\left(\varphi,\eta\right) := \frac{\sigma^2}{2}\left(\frac{\partial\varphi}{\partial x},\frac{\partial\eta}{\partial x}\right) + \left(\frac{\sigma^2}{2}-r\right)\left(\frac{\partial\varphi}{\partial x},\eta\right) + r\left(\varphi,\eta\right) \tag{3.1.6}$$

for $\varphi,\ \eta\in H^1(\Omega_R)$.

Proof. See [22]. □

Example 3.2. For the standard European put option with $\psi(x) = (K - e^x)^+$ the function g in (3.1.4) is

$$g(\tau, \cdot) = e^{-r\tau} \frac{\sigma^2}{2} K \delta_{\log(K) - r\tau} \,. \tag{3.1.7}$$

The partial differential equation (3.1.3) cannot be discretized using Finite Differences due to the presence of the Dirac functional on the right hand side. Nevertheless, using the variational formulation (3.1.5) gives the framework for Finite Element discretizations. Before discretizing the variational form we need to show that (3.1.5) has a unique solution.

Proposition 3.3. *The bilinear form (3.1.6) is continuous and satisfies a Gårding inequality, i.e. there exist constants* $c_0, c_1, c_2 > 0$ *such that*

$$\begin{aligned}
|a_{BS}(\varphi, \eta)| &\leq c_0 \, \|\varphi\|_{H^1(\Omega_R)} \, \|\eta\|_{H^1(\Omega_R)} \\
|a_{BS}(\varphi, \varphi)| &\geq c_1 \, \|\varphi\|^2_{H^1(\Omega_R)} - c_2 \, \|\varphi\|^2_{L^2(\Omega_R)}
\end{aligned}$$

for $\varphi, \eta \in H^1(\Omega_R)$. *This implies existence and uniqueness of the solution of (3.1.5).*

Proof. See [22]. □

To approximate the unique solution $U_{\mathrm{R}}(\tau, \cdot) \in H^1_0(\Omega_R)$, $\tau \in (0, T)$ of (3.1.5) we next discretize the problem by Finite Elements in Ω_R and by the θ-scheme in time.

Discretization. The truncated domain Ω_R is partitioned into an equidistant mesh of size $h = \frac{2R}{N+1}$ for which a finite dimensional subspace $\mathcal{V}^q_h \subset H^1_0(\Omega_R)$ is defined as

$$\begin{aligned}
\mathcal{V}^q_h := \big\{ & v_h \in C^0(\Omega_R) : v_h|_{[x_{i-1},\, x_i]} \in \mathbb{P}_q \quad \text{for } i = 1, \ldots, N+1 \\
& \text{and } v_h(-R) = v_h(R) = 0 \big\}
\end{aligned}$$

where $\mathbb{P}_q$ is the space of polynomials with degree $q \in \mathbb{N}$. Let $\{\varphi_i, \ i = 1, \ldots, N\}$ be the Lagrange basis of $\mathcal{V}^q_h$. Using the Galerkin method, the function $U_{\mathrm{R}}(\tau, \cdot) \in H^1_0(\Omega_R)$ is approximated by

$$U_{\mathrm{R}}(\tau, \cdot) \approx U_h(\tau, \cdot) = \sum_{i=1}^{N} U^i_h(\tau) \varphi_i(\cdot) \,.$$

Substituting U_h into the variational form (3.1.5) we obtain for all $\tau \in (0, T)$

$$\begin{aligned}
\sum_{i=1}^{N} (\varphi_i, \varphi_j) \frac{\partial U^i_h}{\partial \tau} + \sum_{i=1}^{N} a_{\mathrm{BS}}(\varphi_i, \varphi_j) \, U^i_h &= \langle g, \varphi_j \rangle \qquad \text{for } j = 1, \ldots, N \,, \\
U^i_h(0) &= 0 \qquad \text{for } i = 1, \ldots, N \,.
\end{aligned} \tag{3.1.8}$$

Next, let $\underline{U}_h := (U_h^1, \ldots, U_h^N)^T$, $\underline{U}_h' := (\frac{\partial U_h^1}{\partial \tau}, \ldots, \frac{\partial U_h^N}{\partial \tau})^T$ and define the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{A}$ and the load vector $\underline{g}$ as

$$\begin{aligned} &\mathbf{M}_{j,i} := (\varphi_i, \varphi_j) \,, \quad \mathbf{A}_{j,i} := a_{\mathrm{BS}}(\varphi_i, \varphi_j) \qquad i,j = 1, \ldots, N, \\ &g_j := \langle g, \varphi_j \rangle \qquad j = 1, \ldots, N \,. \end{aligned} \tag{3.1.9}$$

With these definitions (3.1.8) can be rewritten in a matrix-vector notation as

$$\begin{aligned} \mathbf{M}\underline{U}_h'(\tau) + \mathbf{A}\underline{U}_h(\tau) &= \underline{g}(\tau) \qquad \forall \tau \in (0,T) \\ \underline{U}_h(0) &= \underline{0} \,. \end{aligned} \tag{3.1.10}$$

Finally, the time interval $[0,T]$ is partitioned in an uniform fashion $\tau_0 = 0 < \tau_1 < \ldots < \tau_M = T$ with $\Delta t = \tau_1$. Applying the θ-time stepping scheme we find

$$\begin{aligned} (\mathbf{M} + \theta \Delta t \mathbf{A})\,\underline{U}_h(\tau_m) &= \Delta t \underline{g}^\theta(\tau_m) + (\mathbf{M} - (1-\theta)\Delta t \mathbf{A})\,\underline{U}_h(\tau_{m-1}) \quad m = 1, \ldots, M \\ \underline{U}_h(\tau_0) &= \underline{0} \,, \end{aligned} \tag{3.1.11}$$

with $\underline{g}^\theta(\tau_m) = \theta \underline{g}(\tau_m) + (1-\theta)\underline{g}(\tau_{m-1})$. Hence, computing the value of a European contingent claim at time $t = 0$, i.e. calculating $\underline{U}_h(\tau_M)$, the linear system (3.1.11) has to be solved at each time step.

3.2 American options

The price of an American contingent claim with time-dependent reward function ϕ is given by

$$h(t,s) := \sup_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau - t)} \phi(\tau, S_\tau) \mid S_t = s \right] \qquad \forall (t,s) \in [0,T] \times \mathbb{R}^+ \,.$$

The value h satisfies (see e.g. [16]) the variational inequality

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 h}{\partial s^2} + rs \frac{\partial h}{\partial s} - rh &\leq 0 &&\text{in } (0,T) \times \mathbb{R}^+ \\ h &\geq \phi &&\text{in } (0,T) \times \mathbb{R}^+ \\ (h - \phi)\left(\frac{\partial h}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 h}{\partial s^2} + rs \frac{\partial h}{\partial s} - rh \right) &= 0 &&\text{in } (0,T) \times \mathbb{R}^+ \\ h(T, \cdot) &= \phi(T, \cdot) &&\text{in } \mathbb{R}^+ \,. \end{aligned}$$

Like in the case of European options, we first change to logarithmic prices $x := \log(s) \in \mathbb{R}$ and to time to maturity $\tau := T - t \in [0,T]$ for which the functions $u(\tau, x) := h(T - \tau, e^x)$, $\psi(\tau, x) := \phi(T - \tau, e^x)$ are defined. Then, the function u satisfies

$$\begin{aligned} \frac{\partial u}{\partial \tau} + A_{\mathrm{BS}}[u] &\geq 0 &&\text{in } (0,T) \times \mathbb{R} \\ u &\geq \psi &&\text{in } (0,T) \times \mathbb{R} \\ (u - \psi)\left(\frac{\partial u}{\partial \tau} + A_{\mathrm{BS}}[u] \right) &= 0 &&\text{in } (0,T) \times \mathbb{R} \\ u(0, \cdot) &= \psi(0, \cdot) &&\text{in } \mathbb{R} \,, \end{aligned} \tag{3.2.1}$$

with the Black-Scholes operator $A_{\mathrm{BS}}[\cdot]$ defined in (3.1.2). Next, we consider as proposed in [21] the excess to payoff function

$$U(\tau, x) := u(\tau, x) - \psi(\tau, x) \qquad \forall (\tau, x) \in [0, T] \times \mathbb{R}\,.$$

Like for the European option the excess to payoff function decays exponentially as illustrated in Figure 2 for an American put ($T = 1$, $K = 1$, $\sigma = 0.5$, $r = 0.05$). Note that $U = 0$ in the exercise region. Zero Dirichlet boundary conditions can

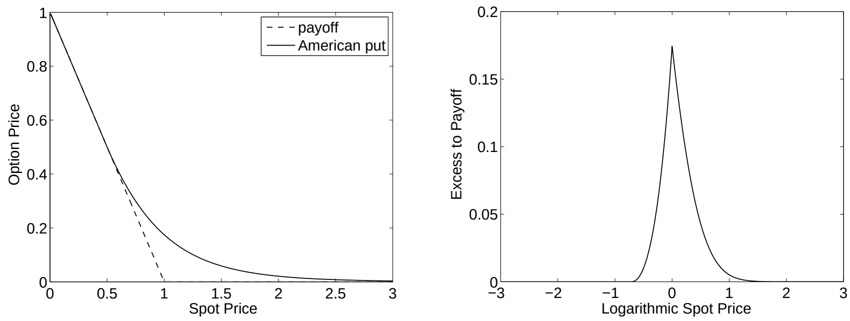

Figure 2: American put option price u (left) and the corresponding excess to payoff function U (right) in logarithmic spot price.

be imposed and we look for a solution $U_{\mathrm{R}}(\tau, \cdot) \in K_0$ where the convex cone K_0 is defined by

$$K_0 := \{v \in H_0^1(\Omega_R) \mid v \geq 0 \quad \text{a.e. } x\}\,.$$

Proposition 3.4. *For all $\tau \in (0, T)$ the excess to payoff $U_R(\tau, \cdot)$ is the solution in K_0 of*

$$\left(\frac{\partial U_R}{\partial \tau}, v - U_R\right) + a_{BS}\,(U_R, v - U_R) \geq \ \langle f, v - U_R\rangle \qquad \forall v \in K_0\,, \tag{3.2.2}$$

$$U_R(0, \cdot) = 0 \qquad\qquad \textit{in } \Omega_R$$

where $\langle \cdot, \cdot\rangle$ denotes the $\left(H^1(\Omega_R)\right)^* \times H^1(\Omega_R)$ duality pairing, the bilinear form $a_{BS}(\cdot, \cdot)$ is given by (3.1.6) and the function f is

$$f(\tau, \cdot) := -\frac{\partial \psi}{\partial \tau}(\tau, \cdot) - A_{BS}[\psi(\tau, \cdot)] \qquad \textit{in } \Omega_R\,. \tag{3.2.3}$$

Proof. See [21]. □

Example 3.5. For the standard American put option with $\psi(\tau, x) = (K - e^x)^+$ the function f in (3.2.3) is

$$f(\tau, \cdot) = \frac{\sigma^2}{2} K \delta_{\log(K)} - rK\mathbf{1}_{(-\infty, \log K]}. \tag{3.2.4}$$

The continuity and coercivity of the bilinear form $a_{\mathrm{BS}}(\cdot, \cdot)$ imply according to [14] existence and uniqueness of the solution of the variational form (3.2.2). We next discretize the problem (3.2.2) by Finite Elements.

Discretization. Space and time are discretized as presented in Section 3.1 and to ensure $v_h \in \mathcal{V}_h^q \cap K_0$ a.e. on Ω_R only linear basis functions are considered. By defining

$$\underline{K}_0 := \{\underline{v}_h \in \mathbb{R}^N \mid v_h^i \geq 0 \quad \text{for } i = 1, \ldots, N\} ,$$

and using the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{A}$ and the load vector $\underline{f}$ as defined in (3.1.9), a sequence of linear complementarity problems is obtained for (3.2.2). The solutions $\underline{U}_h(\tau_m) \in \underline{K}_0$ are for $m = 1, \ldots, M$ determined by

$$\begin{aligned} &(\mathbf{M} + \theta\Delta t\mathbf{A})\, \underline{U}_h(\tau_m) \geq \Delta t\underline{f}^\theta(\tau_m) + (\mathbf{M} - (1-\theta)\Delta t\mathbf{A})\, \underline{U}_h(\tau_{m-1}) \\ &\underline{U}_h(\tau_m)^T \Big((\mathbf{M} + \Delta t\mathbf{A})\, \underline{U}_h(\tau_m) - \Delta t\underline{f}^\theta(\tau_m) - (\mathbf{M} - (1-\theta)\Delta t\mathbf{A})\, \underline{U}_h(\tau_{m-1})\Big) = 0 \\ &\underline{U}_h(\tau_0) = \underline{0} , \end{aligned} \tag{3.2.5}$$

with $\underline{f}^\theta(\tau_m) = \theta\underline{f}(\tau_m) + (1-\theta)\underline{f}(\tau_{m-1})$. There are different methods available for solving linear complementarity problems. Most of these need additional requirements since $\mathbf{A}$ is in general not symmetric. For PSOR the matrix $\mathbf{M} + \theta\Delta t\mathbf{A}$ needs to be diagonal dominant [1] and in [2] conditions on the time and space discretization are imposed. We use the algorithm proposed in [21] which transforms (3.2.5) into a contracting fix-point problem where in each iteration a symmetric linear complementarity problem is solved.

3.3 Swing options

Now both described approaches are brought together and functions similar to excess to payoff functions for swing options are determined. Denote by $\psi(\tau, x)$ the payoff function and by $u^{(p)}(\tau, x)$ the swing option price with $p \in \mathbb{N}$ exercise rights in time to maturity $\tau \in [0, T]$ and logarithmic spot price $x \in \mathbb{R}$. According to Corollary 2.3 swing option prices can be determined as prices of American contingent claims. Using (3.2.1), the function $u^{(p)}$ is the solution of the variational inequality

$$\begin{aligned} \frac{\partial u^{(p)}}{\partial \tau} + A_{\mathrm{BS}}[u^{(p)}] &\geq 0 && \text{in } (0,T) \times \mathbb{R} \\ u^{(p)} &\geq \Psi^{(p)} && \text{in } (0,T) \times \mathbb{R} \\ \left(u^{(p)} - \Psi^{(p)}\right)\left(\frac{\partial u^{(p)}}{\partial \tau} + A_{\mathrm{BS}}[u^{(p)}]\right) &= 0 && \text{in } (0,T) \times \mathbb{R} \\ u^{(p)}(0,\cdot) &= \Psi^{(p)}(0,\cdot) && \text{in } \mathbb{R} , \end{aligned}$$

with p^{th} reward function

$$\Psi^{(p)}(\tau, x) := \begin{cases} \psi(\tau, x) + u_\tau^{(p)}(\delta, x) & \text{for } \tau \in [\delta, T) \\ \psi(\tau, x) & \text{for } \tau \in [0, \delta) . \end{cases} \tag{3.3.1}$$

This reward function involves a European option price $u_\tau^{(p)}$ which according to Corollary 2.3 and (3.1.1) satisfies the partial differential equation

$$\begin{aligned} \frac{\partial u_\tau^{(p)}}{\partial t} + A_{\mathrm{BS}}[u_\tau^{(p)}] &= 0 && \text{in } (0,\delta) \times \mathbb{R} , \\ u_\tau^{(p)}(0,\cdot) &= u^{(p-1)}(\tau - \delta, \cdot) && \text{in } \mathbb{R} . \end{aligned}$$

To impose zero boundary conditions an excess to payoff function $U^{(p)}$ for swing options has to be introduced. In contrast to American and European options, the p^{th} reward function $\Psi^{(p)}$ in (3.3.1) is computed iteratively and so its exact values are not known in advance. This makes the construction of an excess to payoff function using $\Psi^{(p)}$ difficult. Therefore, another function $\psi^{(p)}$ which is in general not the p^{th} reward function, but shares the same asymptotics $\lim_{x\to\pm\infty}\psi^{(p)}(\tau,x) = \lim_{x\to\pm\infty}\Psi^{(p)}(\tau,x)$ is used. A natural choice for $\psi^{(p)}$ is obtained by using iteratively the shifted payoff function of the European option instead of its price $u_\tau^{(p)}$. This results in the following iterative definition

$$\psi^{(p)}(\tau,x) := \begin{cases} \psi(\tau,x) + e^{-r\delta}\psi^{(p-1)}(\tau,x+r\delta) & \text{for } \tau \geq (p-1)\delta \\ \psi^{(p-1)}(\tau,x) & \text{for } \tau \in [0,(p-1)\delta) \end{cases}$$

with $\psi^{(0)}(\tau,x) = 0$. Using property (2.0.3), it is easily seen that the function $\psi^{(p)}$ has the same asymptotics as the p^{th} reward function $\Psi^{(p)}$. The excess to payoff function is defined as

$$U^{(p)}(\tau,x) := u^{(p)}(\tau,x) - \psi^{(p)}(\tau,x) \qquad \forall(\tau,x) \in [0,T]\times\mathbb{R}\,.$$

Using weighted Sobolev spaces and similar arguments like in [22], it can be easily shown that $U^{(p)}$ decays exponentially. Therefore, zero Dirichlet boundary conditions on $U_{\mathrm{R}}^{(p)}$ are imposed on the truncated domain Ω_R. Before stating the variational formulation we define according to (3.2.3)

$$\begin{aligned} f^{(p)}(\tau,x) &:= -\frac{\partial\left(\psi^{(p)}(\tau,x)\right)}{\partial\tau} - A_{\mathrm{BS}}\left[\psi^{(p)}(\tau,x)\right] \\ &= \begin{cases} f^{(1)}(\tau,x) + e^{-r\delta}f^{(p-1)}(\tau-\delta,x+r\delta) & \text{for } \tau \geq (p-1)\delta \\ f^{(p-1)}(\tau,x) & \text{for } \tau \in [0,(p-1)\delta), \end{cases} \end{aligned}$$

with $f^{(0)}(\tau,x) := 0$ for $(\tau,x) \in (-\delta,T)\times\mathbb{R}$ and according to (3.1.4)

$$\begin{aligned} g_\tau^{(p)}(t,x) &:= -\frac{\partial\left(e^{-rt}\psi^{(p)}(\tau-\delta,x+rt)\right)}{\partial t} - A_{\mathrm{BS}}\left[e^{-rt}\psi^{(p)}(\tau-\delta,x+rt)\right] \\ &= \begin{cases} g_\tau^{(1)}(t,x) + e^{-r\delta}g_\tau^{(p-1)}(t,x+r\delta) & \text{for } \tau \geq (p-1)\delta \\ g_\tau^{(p-1)}(t,x) & \text{for } \tau \in [0,(p-1)\delta), \end{cases} \end{aligned}$$

with $g_\tau^{(0)}(t,x) := 0$ for $(t,x) \in (0,\delta)\times\mathbb{R},\ \tau \in [\delta,T)$.

Remark 3.6. For swing put options with $\psi(\tau,x) = (K-e^x)^+$ the functions $g_\tau^{(1)}$ and $f^{(1)}$ are given by (3.1.7) and (3.2.4), respectively.

The following Proposition results from the variational forms (3.1.5), (3.2.2) and the iterative construction above.

Proposition 3.7. *For all $\tau \in (0,T)$ the excess to payoff function $U_R^{(p)}(\tau,\cdot)$ solves*

$$\left(\frac{\partial U_R^{(p)}}{\partial\tau}, v - U_R^{(p)}\right) + a_{BS}\left(U_R^{(p)}, v - U_R^{(p)}\right) \geq \langle f^{(p)}, v - U_R^{(p)}\rangle\,, \tag{3.3.2}$$

$$U_R^{(p)}(0,\cdot) = 0\,,$$

```
for l = 1 : p
    for τ = (l − 1)δ : Δt : T
        if l > 1
            calculate U_τ^(l)(δ, ·) using (3.3.3)
        else
            U_τ^(l)(δ, ·) = 0
        endif
        calculate U^(l)(τ, ·) using (3.3.2)
    end
end
Set u^(p)(0, ·) = U^(p)(0, ·) + ψ^(p)(0, ·)
```

Table 1: Pseudo code for swing option prices

for all $v \in K_0^{(p)}(\tau)$ *where the cone* $K_0^{(1)}(\tau) := K_0$ *and for* $p > 1$

$$K_0^{(p)}(\tau) := \left\{ v \in H_0^1(\Omega_R) \mid v \geq \left\{ \begin{array}{ll} U_\tau^{(p)}(\delta,\cdot) & \textit{for } \tau \geq \delta \\ 0 & \textit{for } \tau \in [0,\delta) \end{array} \right\} \qquad a.e.\ x \right\},$$

with the function $U_\tau^{(p)}(t,\cdot)$ *satisfying for all* $t \in (0,\delta)$

$$\begin{aligned} \left(\frac{\partial U_\tau^{(p)}}{\partial t}, w \right) + a_{BS}\left(U_\tau^{(p)}, w \right) &= \langle g_\tau^{(p-1)}, w \rangle , \qquad \forall w \in H_0^1(\Omega_R) \qquad (3.3.3) \\ U_\tau^{(p)}(0,\cdot) &= U_R^{(p-1)}(\tau-\delta,\cdot) \qquad \textit{in } \Omega_R . \end{aligned}$$

Combining the variational formulations of European and American options a variational form for swing option prices in excess to payoff has been derived. With the performed transformations artificial time-dependent non-zero boundary conditions are avoided. The derived algorithm is illustrated as pseudo code in Table 1.

Remark 3.8. Assuming a general spot price model (2.0.1), the only difference in the variational formulations (3.3.2), (3.3.3) is the bilinear form a_{BS} and the resulting $f^{(1)}$, $g^{(1)}$.

Discretization. The space is discretized as presented in Sections 3.2 with linear basis functions. To obtain the initial condition for (3.3.3) the time discretization has to be chosen such that $\delta/\Delta t \in \mathbb{N}$. Then, the variational inequality (3.3.2) transforms to a system of linear complementarity problems similar to (3.2.5) and for the variational form (3.3.3) a system of linear equations similar to (3.1.11) is obtained.

4 Numerical results

In this section the derived algorithm is applied in the Black-Scholes model as well as for a mean reverting, seasonal spot price. The obtained numerical results show a smooth and stable behavior allowing to interpret the optimal exercise boundary and to analyze prices of additional rights in swing contracts. Additionally, the proposed Finite Element procedure is compared to lattice and Monte Carlo methods. Finally, we extend the algorithm to Lévy models and give a numerical example.

4.1 Valuation in a Black-Scholes market

Let $(W_t)_{t\geq 0}$ denote a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ with associated, complete filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. The spot price process $(S_t)_{t\geq 0}$ is assumed to have the following risk neutral dynamics

$$dS_t = S_t r dt + S_t \sigma dW_t, \qquad S_0 = s, \tag{4.1.1}$$

where r and σ are positive constants, representing the interest rate and the volatility.

It is well known that in the case of no dividends, the American call option price is equal to the price of a European call option as there is never any value in exercising early. Thus the optimal exercise strategy of a swing call option starting at time t with p exercise rights and maturity T is given by

$$\tau^* = (T-(p-1)\delta, \ldots, T-\delta, T) \in \mathcal{T}_t^{(p)} \qquad \text{for } t \leq T-(p-1)\delta.$$

Consequently, swing derivatives with multiple American put rights and payoff functions

$$\phi(t,s) = (K-s)^+ \qquad \text{for } (t,s) \in [0,T]\times\mathbb{R}^+,\ K>0 \tag{4.1.2}$$

are examined in the sequel. The exercise regions of such derivatives are according to Corollary 2.3 characterized by

$$V^{(p)}(t,s) = \Phi^{(p)}(t,s) \Longleftrightarrow 0 < s \leq s_p^*(t)\,.$$

It is optimal to exercise the p^{th} right at time t, if the current spot price s is equal or below the boundary value $s_p^*(t)$. The influence of the refraction period on option prices (see (2.0.3)) also affects the exercise boundary values for which the following equation holds

$$s_p^*(t) = s_{p-1}^*(t) \quad \text{for } t \in (T-(p-1)\delta, T]\,, \tag{4.1.3}$$

for all $p \geq 2$. Exercise regions coincide as soon as the number of exercisable rights is limited.

For the numerical results the following parameter values and discretizations

$$\begin{aligned} &T=1, \quad K=100, \quad \sigma=0.3, \quad r=0.05, \quad \delta=0.1 \\ &\Delta t = 10^{-3}, \quad N=4000, \quad \Omega_R = [-3,3]\cdot\ln K. \end{aligned} \tag{4.1.4}$$

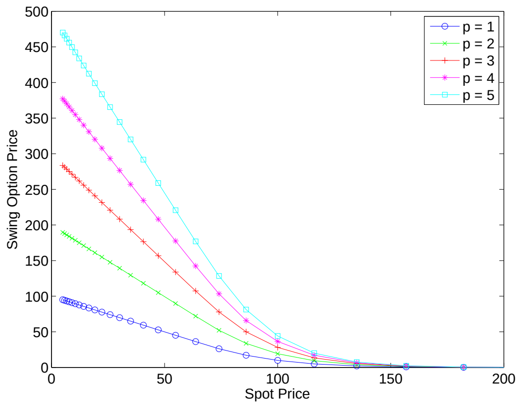

Figure 3: Finite horizon swing put option prices at time $t = 0$ for up to 5 exercise rights using the Black-Scholes model.

are used. The finite element approximation of fair swing put option prices with up to 5 exercise rights are presented in Figure 3. It is not surprising that swing and American put option values are similar in appearance. The influence of the refraction period is not visually observable on swing prices for up to 5 rights. In contrast, the impact of the refraction period on exercise regions characterized in (4.1.3) can be seen in Figure 4 where the calculated exercise boundary values are presented. Moreover, it is observed that for $p, p' \in \mathbb{N}$ with $p \geq p'$

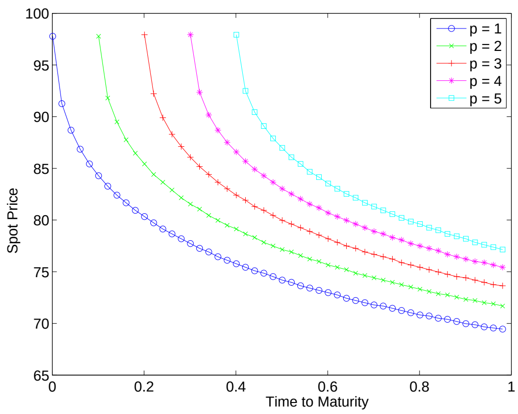

Figure 4: Exercise boundary values of a finite horizon swing put option for up to 5 exercise rights using the Black-Scholes model.

$$s_p^*(t) \geq s_{p'}^*(t) \quad \forall\ t \in [0, T]$$

This monotonicity, which has been proved by [5] for the perpetual case, yet remains open for finite horizon swing put options. Another property of the exercise boundary functions $s_p^*(\cdot)$ is that they are strictly increasing on the time interval $[0, T - (p-1)\delta]$.

4.2 Valuation in electricity markets

As swing options are widely used in commodity and especially electricity markets, we now turn to the mean reverting seasonal spot price model proposed in [20]. Here, the risk neutral spot price $(S_t)_{t\geq 0}$ is assumed to be the solution to the stochastic differential equation

$$dS_t = \kappa(b(t) - \log(S_t))S_t dt + \sigma S_t dW_t, \qquad S_0 = s\,, \tag{4.2.1}$$

where $b(t)$ is defined as

$$b(t) := \frac{1}{\kappa}\,\left(\frac{\sigma^2}{2} + \frac{df}{dt}(t)\right) + f(t) - \frac{\lambda\kappa}{\sigma}\,,$$

for a deterministic, continuously differentiable function $f(t)$. Remark, that the mean reversion level is a time-dependent function which captures the seasonal patterns observed in electrical spot prices. In [20] the authors suggest

$$f(t) := \delta + \gamma \cos\left((t+\omega)\frac{2\pi}{365}\right)\,, \tag{4.2.2}$$

for weekdays and report the following parameter estimates

$$\kappa = 0.016,\ \sigma = 0.086,\ \lambda = 0.036,\ \delta = 4.867,\ \gamma = 0.306,\ \omega = 0.836. \tag{4.2.3}$$

These are obtained from daily electrical spot and futures price observations from 01/01/1993 to 31/12/1999 at the Nordic Power Exchange, Nord Pool ASA. For illustration one path of this model using the above parameter estimates is displayed in Figure 5. The large influence of the seasonal mean reversion level

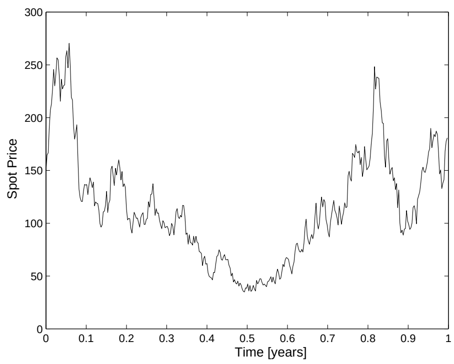

Figure 5: Spot price scenario of the model (4.2.1), (4.2.2), (4.2.3).

on the risk neutral price process $(S_t)_{t\geq 0}$ is clearly visible.

Due to different demands and the non-storable nature of electrical power, swing call options are of special interest in electricity markets. One type of swing call option is the so-called virtual hydro storage. This derivative gives the contract holder a fixed number of exercise rights for virtually or physically producing electrical power at almost no production costs, like it is done by running a hydro power plant. Assuming production costs $K \geq 0$ and a finite time horizon $[0, T]$ its payoff function is given by

$$\phi(t, s) = (s - K)^{+} \qquad \text{for } (t, s) \in [0, T] \times \mathbb{R}^{+} . \tag{4.2.4}$$

For the numerical valuation the electrical spot price model (4.2.1) with estimates (4.2.3) as well as the following parameter values and discretizations

$$T = 1, \quad \delta = 0.1, \quad r = 0.05, \quad K = 10,$$
$$\Delta t = 10^{-3}, \quad N = 4000, \quad \Omega_R = [-9, 9].$$

are considered. The finite element approximation of virtual hydro storage prices at time $t = 0$ for up to 7 exercise rights are presented in Figure 6. Remark that

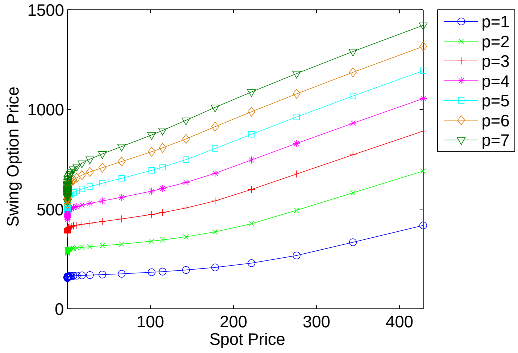

Figure 6: Virtual hydro storage prices at time $t = 0$ for up to 7 exercise rights using the mean reverting, seasonal spot price model (4.2.1).

option prices seem to increase linearly with initial spot prices. Moreover, the value of an additional right seems to depend on the already available number of exercise rights in a contract but not on the initial spot price. This might results from the high volatility and the mean reverting property of the chosen spot price model. However, the impact of the seasonality on the exercise boundary values presented in Figure 7 is non negligible. By comparing the form of the function $s_1^*(\cdot)$ plotted in time to maturity with the simulated spot price scenario in Figure 5, we observe that both inherit the structure of the time-dependent mean reversion level.

4.3 Comparison to other methods

Next, the derived algorithm is compared to lattice and Monte Carlo methods with respect to accuracy and computational time. Note that the algorithms considered for the comparison need to be able to handle swing options of the form (2.0.2). This is the case for the Monte Carlo approach presented in [5]. For lattice methods, to the best of our knowledge there is no specific algorithm in the literature that considers this type of swing options. Thus, we apply for

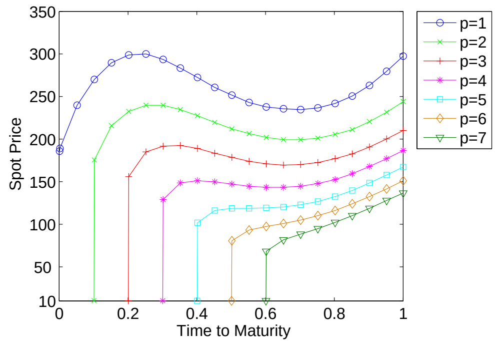

Figure 7: Exercise boundary values of virtual hydro storages in time to maturity for up to 7 exercise rights using the mean reverting, seasonal spot price model (4.2.1).

the lattice approach Corollary 2.3 and standard binomial tree methods. All computations are performed on a Dual-Core AMD Opteron(tm) Processor 2218 with 2.61GHz using MATLAB 7.4.

For the comparison a swing put option with up to five exercise rights and initial spot price $s = 100$ is considered. The other parameters used for the computation are

$$T = 1, \quad K = 100, \quad \sigma = 0.3, \quad r = 0.05, \quad \delta = 0.1\,.$$

As a benchmark solution we take the results of the Finite Element algorithm with 4000 mesh points, 10^3 time steps and $\theta = 0.5$. These swing option prices are $V^{(1)}(0,100) = 9.8700$, $V^{(2)}(0,100) = 19.2550$, $V^{(3)}(0,100) = 28.1265$, $V^{(4)}(0,100) = 36.4505$, $V^{(5)}(0,100) = 44.1843$.

For the computations with the Monte Carlo method we use 20 time points and M number of simulations. The swing option prices are calculated with 20 different seeds and compared to the above benchmark solutions. The absolute errors, [standard deviation] and the elapsed time (of one run) in seconds are presented in Table 2.

	$M_1 = 2000$		$M_2 = 4000$		$M_3 = 8000$		$M_4 = 16000$	
	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	3.3e-01 [0.15]	6.53	1.8e-01 [0.11]	25.51	6.1e-02 [0.07]	100.61	1.1e-02 [0.04]	468.28
$p = 2$	8.9e-01 [0.41]	18.79	4.5e-01 [0.35]	73.40	8.6e-02 [0.21]	289.58	8.1e-02 [0.12]	1349.56
$p = 3$	1.8e+00 [0.78]	31.06	9.0e-01 [0.67]	121.29	2.1e-01 [0.38]	478.55	1.3e-01 [0.21]	2230.34
$p = 4$	2.8e+00 [1.20]	43.33	1.5e+00 [1.04]	169.18	3.8e-01 [0.59]	667.58	1.5e-01 [0.31]	3111.24
$p = 5$	3.7e+00 [1.68]	55.59	2.0e+00 [1.44]	217.07	5.6e-01 [0.80]	856.62	1.8e-01 [0.44]	3991.97

Table 2: Absolute errors and elapsed time using the Monte Carlo method for a swing option with up to five exercise rights.

For the binomial tree method we use $\mathcal{M}$ time points and compare the results with the benchmark solutions. The absolute errors and the elapsed time in seconds are presented in Table 3.

	$\mathcal{M}_1 = 200$		$\mathcal{M}_2 = 400$		$\mathcal{M}_3 = 800$		$\mathcal{M}_4 = 1600$	
	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	6.9e-03	0.02	3.4e-03	0.05	1.6e-03	0.19	8.0e-04	0.69
$p = 2$	1.5e-02	0.06	7.3e-03	0.23	3.7e-03	1.10	2.0e-03	5.85
$p = 3$	2.3e-02	0.10	1.2e-02	0.39	6.0e-03	1.86	3.3e-03	10.11
$p = 4$	3.2e-02	0.13	1.6e-02	0.53	8.1e-03	2.49	4.4e-03	13.54
$p = 5$	4.1e-02	0.16	2.0e-02	0.64	1.0e-02	2.99	5.1e-03	16.21

Table 3: Absolute errors and elapsed time using the binomial tree method for a swing option with five exercise rights.

Finally, the option prices are computed with the derived Finite Element algorithm. We consider 10^3 time points, $\theta = 0.5$ and N number of mesh points in the interval $\Omega_R = [-3, 3] \cdot \ln K$. The absolute errors and the elapsed time in seconds are presented in Table 4.

	$N_1 = 100$		$N_2 = 200$		$N_3 = 400$		$N_4 = 800$	
	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	2.5e-03	0.08	3.0e-04	0.12	2.4e-04	0.19	1.1e-04	0.47
$p = 2$	3.7e-03	2.03	4.2e-03	2.85	1.6e-03	4.49	2.3e-04	7.84
$p = 3$	1.5e-02	3.75	1.1e-02	5.33	3.4e-03	8.26	7.1e-04	14.41
$p = 4$	4.0e-02	5.27	1.9e-02	7.43	3.8e-03	11.57	9.9e-04	20.16
$p = 5$	7.4e-02	6.57	2.6e-02	9.23	6.5e-03	14.40	3.4e-03	25.10

Table 4: Absolute errors and elapsed time using the Finite Element method for a swing option with five exercise rights.

In addition to the absolute error, we analyze the experimental convergence rate of our algorithm $r_{N_i} = \left(\ln \text{error}_{N_{i+1}} - \ln \text{error}_{N_i}\right) / \left(\ln N_{i+1} - \ln N_i\right)$ with $i = 1, 2, 3$. For the swing option with five exercise rights (last row in Table 4) the rates are $r_{N_1} = 1.5,\ r_{N_2} = 2.0,\ r_{N_3} = 1.8$ and match the expected rate $\mathcal{O}(\frac{1}{N^2})$ well. Similarly for the lattice method we obtain the rates $r_{\mathcal{M}_1} = 1.0,\ r_{\mathcal{M}_2} = 1.0,\ r_{\mathcal{M}_3} = 1.0$.

Looking at the computational results, it is observed that the accuracy of the Finite Element and binomial tree methods are very high even though the computational times are low. The computation time is higher for the Monte Carlo method, but the results' accuracy is lower in comparison to the other procedures.

Apart from the computational results, some remarks about the differences of the methods should be made. It is important to note that by using Monte Carlo simulations or lattice methods the swing option price for only one initial spot price is found. In contrast, the whole swing option price curve is obtained by applying a Finite Element method. The knowledge of the option curve allows to interpret values of additional rights in swing contracts and to observe the dependence on initial spot prices. Another aspect is the computation of the exercise boundary values. In the Finite Element approach, exercise boundary values are obtained simultaneous and so no additional computational time is needed. Using Monte Carlo simulations or lattice methods, swing option prices for several initial spot prices would have to be computed to obtain reasonable information about the exercise regions.

4.4 Extension to valuation in a multiperiod Lévy market

For the theoretical results in Section 2 we only considered spot prices of the form (2.0.1). Because the proof of Corollary 2.3 in [5] is only given for these continuous processes. Nevertheless, the Algorithm 1 can easily be adapted to Lévy processes.

Let X be a Lévy process with Lévy measure ν. We consider the spot price process

$$dS_t = S_t r dt + \int_{\mathbb{R}} S_{t-}(e^z - 1)\widetilde{J}(\mathrm{d}t, \mathrm{d}z) \qquad S_0 = s, \tag{4.4.1}$$

where $\widetilde{J}$ is the compensated jump measure of X (see [7]). The corresponding bilinear form, derived in [22] is given by

$$a_{\mathrm{jump}}(\varphi, \eta) = -\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\varphi(x+z) - \varphi(x) - (e^z - 1)\frac{\partial \varphi}{\partial x}(x) \right) \eta(x)\nu(\mathrm{d}z)\,\mathrm{d}x\,,$$

for $\varphi,\ \eta \in H_0^1(\Omega_R)$. With this bilinear form, European and American options can be priced similarly as described in Section 3.1 and 3.2. We here are not going further into details, but refer the reader to the papers [22, 21]. Rather we present an example to illustrate the differences to continuous processes.

For the numerical example a so-called CGMY [6] process is considered where the Lévy density $k(z)$ is given by

$$k(z) = C\left(\frac{e^{Gz}}{|z|^{1+Y}} 1_{\{z<0\}} + \frac{e^{-Mz}}{|z|^{1+Y}} 1_{\{z>0\}} \right). \tag{4.4.2}$$

with $C = 1$, $G = 10$, $M = 10$ and $Y = 0.5$. All the other parameter are like in the Black-Scholes model.

$$T = 1, \quad K = 100, \quad r = 0.05, \quad \delta = 0.1$$
$$\Delta t = 10^{-3}, \quad N = 4000, \quad \Omega_R = [-3, 3] \cdot \ln K.$$

The computed swing put option prices with up to five exercise rights look similarly to the prices calculated in the Black-Scholes model (Figure 3) and are thus not presented here. Only the exercise boundary of the swing put option is illustrated in Figure 8. In contrast to the result in the Black-Scholes model (see Figure 4), the exercise boundary values in a Lévy model never reach the option's strike price which is well known for American options [18, 21].

We would like to emphasize again that Theorem 2.2 and Corollary 2.3 are not proven in the literature for Lévy models (4.4.1). Therefore, it is not known if the multiple stopping time problem can be reduced to a cascade of single stopping time problems also for Lévy processes. This is a topic for further research.

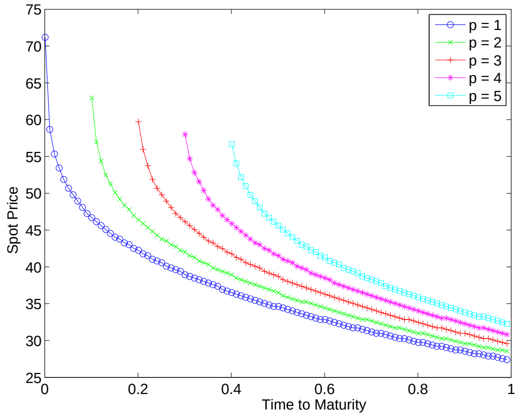

Figure 8: Exercise boundary values of a finite horizon swing put option for up to 5 exercise rights using Lévy model (4.4.1), (4.4.2).

5 Conclusions

The transformation of multiple stopping time problems to series of single stopping time problems allows to apply Finite Element methods for pricing swing options. The proposed procedure is highly flexible with respect to different spot price models and payoff functions. In particular, the presented numerical results show a smooth and stable behavior, allowing to interpret the exercise boundary values and to analyze prices of addition exercise rights in swing contracts. Moreover, computational speed and accuracy are superior to Monte Carlo simulations and similar to binomial tree methods. The algorithm is based on a very general setting for European and American option prices and can thus be adapted to Lévy models under the assumption that swing contracts in these markets can be reduced to optimal stopping time problems with multiple periods.

References

[1] Y. Achdou and O. Pironneau. *Computational methods for option pricing*, volume 30 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[2] A. Borici and H.-J. Lüthi. Fast solutions of complementarity formulations in american put pricing. *Journal of Computational Finance*, 9(1):63–81, 2005.

[3] B. Bouchard and N. Touzi. Discrete-time approximation and monte carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111:175–206, 2004.

[4] R. Carmona and S. Dayanik. Optimal multiple stopping of linear diffusions and swing options. Technical report, Department of Operations Research and Financial Engineering, Princeton, 2003.

[5] R. Carmona and N. Touzi. Optimal multiple stopping and valuation of swing options. *Mathematical Finance*, 2006. to appear.

[6] P. Carr, H. Geman, D.B. Madan, and M. Yor. The fine structure of assets returns: An empirical investigation. *Journal of Business*, 75(2):305–332, 2002.

[7] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[8] M. Dahlgren. A continuous time model to price commodity-based swing options. *Review of Derivatives Research*, 8:27–47, 2005.

[9] M. Dahlgren and R. Korn. The swing option on the stock market. *International Journal of Theoretical and Applied Finance*, 8(1):123–139, 2005.

[10] J. Hinz. Valuing virtual production capacities on flow commodities. *Mathematical Methods of Operations Research*, 2006. to appear.

[11] J. C. Hull and A. White. Efficient procedures for valuing european and american path dependent options. *Journal of Derivatives*, 1:21–31, 1993.

[12] A. Ibanez. Valuation by simulation of contingent claims with multiple early exercise opportunities. *Mathematical Finance*, 14(2):223–248, 2004.

[13] A. Ibanez and F. Zapatero. Valuation of american options through computation of the optimal exercise boundary. *Journal of financial and quantitative analysis*, 39(21), 2004.

[14] P. Jaillet, D. Lamberton, and B. Lapeyre. Variational inequalities and the pricing of american options. *Acta Applicandae Mathematicae*, 21:263–289, 1990.

[15] P. Jaillet, E.I. Ronn, and S. Tompaidis. Valuation of commodity-based swing options. *Management Science*, 50(7):909–921, 2004.

[16] I. Karatzas and S. E. Shreve. *Methods of mathematical finance*. Springer, 1998.

[17] J. Keppo. Pricing of electricity swing options. *Journal of Derivatives*, 11:26–43, 2004.

[18] S. Z. Levendorskiĭ. Early exercise boundary and option prices in Lévy driven models. *Quant. Finance*, 4(5):525–547, 2004.

[19] F. Longstaff and E. Schwartz. Valuing american options by simulation: A simple least squares approach. *Review of Financial Studies*, 14:113–148, 2001.

[20] J. J. Lucia and E. S. Schwartz. Electricity prices and power derivatives: Evidence from the nordic power exchange. *Review of Derivatives Research*, 5(1):5–50, 2002.

[21] A.-M. Matache, P.-A. Nitsche, and C. Schwab. Wavelet galerkin pricing of american options on Lévy driven assets. *Quantitative Finance*, 5(4):403–424, 2005.

[22] A.-M. Matache, T. von Petersdorff, and C. Schwab. Fast deterministic pricing of options on Lévy driven assets. *M2AN Math. Model. Numer. Anal.*, 38(1):37–71, 2004.

[23] N. Meinshausen and B. M. Hambly. Monte carlo methods for the valuation of multiple-exercise options. *Mathematical Finace*, 14(4):557–583, 2004.

[24] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 1997.

[25] R. T. Thanawalla. Valuation of swing options using an extended least squares monte carlo algorithm. Technical report, Heriot-Watt University, England, 2005.

[26] A. C. Thompson. Valuation of path-dependent contigent claims with multiple exercise decisions over time: The case of take-or-pay. *Journal of Financial and Quantitative Analysis*, 30(2):271–293, 1995.