Dual-Primal FETI algorithms for edge element approximations: Three-dimensional h finite elements on shape-regular meshes[1]

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Abstract

A family of dual-primal FETI methods for edge element approximations in three dimensions is proposed and analyzed. The key part of this work relies on the observation that for these finite element spaces there is a strong coupling between degrees of freedom associated to subdomain edges and faces and a local change of basis is therefore necessary. The primal constraints are associated with subdomain edges. We propose three methods. They ensure a condition number that is independent of the number of substructures and possibly large jumps of one of the coefficients of the original problem, and only depends on the number of unknowns associated with a single substructure, as for the corresponding methods for continuous nodal elements. A polylogarithmic dependence is shown for two algorithms. Numerical results validating our theoretical bounds are given.

Keywords: Edge elements, Maxwell's equations, finite elements, domain decomposition, FETI, preconditioners, heterogeneous coefficients

Subject Classification: 65F10, 65N22, 65N30, 65N55

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1. Introduction. In this paper, we consider the boundary value problem

$$\begin{aligned} L\mathbf{u} := \mathbf{curl}\,(A\,\mathbf{curl}\,\mathbf{u}) + B\,\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with Ω a bounded polyhedral domain in $\mathbb{R}^3$ and **curl** the three-dimensional curl operator; see, e.g., [23]. The domain Ω has unit diameter and $\mathbf{n}$ is its unit normal. The coefficient matrices A and B are symmetric, uniformly positive definite with entries $A_{ij}, B_{ij} \in L^\infty(\Omega)$, $1 \leq i, j \leq 3$.

The weak formulation of problem (1.1) requires the introduction of the Hilbert space $H(\mathbf{curl}\,;\Omega)$, defined by

$$H(\mathbf{curl}\,;\Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^3 | \quad \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3 \right\}.$$

The space $H(\mathbf{curl}\,;\Omega)$ is equipped with the following inner product and graph norm,

$$(\mathbf{u}, \mathbf{v})_{\mathbf{curl}} := \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx + \int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, dx, \qquad \|\mathbf{u}\|^2_{\mathbf{curl}} := (\mathbf{u}, \mathbf{u})_{\mathbf{curl}}.$$

The tangential component $\mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$, of a vector $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)^3$; see [4, 23]. We note that the vectors $\mathbf{n} \times (\mathbf{u} \times \mathbf{u})$ and $\mathbf{u} \times \mathbf{n}$ are perpendicular, have the same length, and are both perpendicular to $\mathbf{n}$. Boundary conditions can therefore be equivalently expressed in terms of either one. The trace space can be further characterized and this will be done in section 7. The subspace of vectors in $H(\mathbf{curl}\,;\Omega)$ with vanishing tangential component on $\partial\Omega$ is denoted by $H_0(\mathbf{curl}\,;\Omega)$.

For any $\mathcal{D} \subset \Omega$, we define the bilinear form

$$a_{\mathcal{D}}(\mathbf{u}, \mathbf{v}) := \int_{\mathcal{D}} (A\,\mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} + B\,\mathbf{u} \cdot \mathbf{v})\, dx, \quad \mathbf{u}, \mathbf{v} \in H(\mathbf{curl}\,;\Omega). \tag{1.2}$$

The variational formulation of Equation (1.1) is:
Find $\mathbf{u} \in H_0(\mathbf{curl}\,;\Omega)$ such that

$$a_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \quad \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega). \tag{1.3}$$

Neumann and/or inhomogeneous conditions can also be considered and the generalization of our algorithms to these cases is straightforward.

The purpose of this work is to construct and analyze a dual-primal FETI (FETI-DP) preconditioner for three-dimensional h finite element approximations of Problem (1.3). Neumann-Neumann (NN) and FETI algorithms are particular domain decomposition (DD) methods of iterative substructuring type: they rely on a nonoverlapping partition into subdomains. They are among the most popular and heavily tested DD methods and are now employed for the solution of huge problems on parallel architectures; see, e.g, [21, 13, 11, 33, 3]. The rate of convergence is often independent of possibly large jumps of the coefficients.

The motivation of this work lies in the fact that no iterative substructuring methods (and in particular no NN or FETI preconditioners) that are robust with respect to the number of unknowns, the number of subdomains, and large jumps of the coefficients have been developed or proposed so far for edge element approximations of three dimensional problems. Some methods are available for two dimensional approximations. In [32], a domain decomposition preconditioner was proposed, which is based on a standard coarse space and local spaces associated to the subdomain edges. NN preconditioners with standard coarse spaces were studied in [27]. One level FETI methods were developed in [29, 25], thanks to the introduction of suitable local functions which are the analog of constants and rigid body modes for the Laplace equation and linear elasticity, respectively. These functions were then employed to construct a Balancing NN method in [28]. FETI-DP algorithms were proposed in [30]. Standard coarse spaces however are not in general suitable for quasioptimal preconditioners in three dimensions and the search for suitable local functions in three dimensions for Balancing NN and one level FETI methods has produced no results so far.

A first important attempt to solve this problem was made in [16] where a wire basket algorithm was proposed and studied in three dimensions. See also [17] for a generalization to a saddle-point problem. Local components are associated with the faces of the partition and two wire basket coarse spaces are then considered. The underlying idea is to employ two coarse spaces that should reduce two error components associated with a discrete Helmholtz decomposition. The corresponding preconditioned operator is shown to be scalable and its condition number to grow polylogarithmically. Independence of coefficient jumps is not guaranteed due to the overlap between the coarse and local spaces. No numerical result is provided. We believe that the size of the coarse space in [16] associated to the subspace of discrete divergence free vectors can be reduced. We also refer to [8], which gives a fine survey of algebraic solvers currently employed in large scale computational electromagnetics. We note in particular that some FETI algorithms are sometimes employed in practice (see equation (4.4)) without preconditioning.

In this work, we first show that the difficulty of iterative substructuring for edge element approximations mainly lies in the strong coupling between degrees of freedom associated with subdomain edges and faces and that no efficient and robust iterative substructuring strategy is therefore possible unless a change of basis is performed. The situation is here the same as for general p and hp continuous finite element approximations for diffusive problems, for which strong coupling exists between edge and face components, unless very special bases are employed; see, e.g., [20]. The surprising feature is that strong coupling appears here even for the simple h version, something that has no analog with nodal finite elements.

We next propose a change of basis, which is local and is only associated to the tangential degrees of freedom on the subdomain edges. This change of basis is much simpler than those necessary for the p versions, produces very sparse matrices, and does not rely on the fact that the subdomains have a simple shape.

Once a change of basis is performed we are able to devise robust FETI-DP algorithms, by selecting suitable primal constraints quite straightforwardly. We propose three algorithms: one with a minimal coarse space that does not exhibit good convergence properties for large problems, one with an improved, low dimensional coarse space for which good convergence is assessed, and a third one with a very large coarse space, which has the advantage of not requiring a change of basis and can therefore be potentially suitable for some future extensions.

An analysis of the resulting condition numbers is performed: it relies on some generalizations of certain discrete Sobolev type inequalities for the more complicated traces spaces associated to the subdomains Ω_i, conforming in $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_S;\Omega_i)$. The bounds are not optimal, but we present them here in view of the novelty of these methods and of these theoretical tools.

This paper is organized as follows:
in section 2, we introduce our discrete problems, the subdomain partition, and local and global finite element spaces. Tangential vector finite element spaces are described in section 3. We introduce FETI-DP algorithms and an abstract framework for their analysis in section 4. The strong coupling between edge and face degrees of freedom is shown in section 5. Our FETI-DP algorithms, which rely on a change of basis and suitable sets of primal constraints, are introduced in section 6. Section 7 is devoted to the technical tools necessary for the analysis: they consist of trace and inverse inequalities for discrete tangential vectors, decompositions associated to the face boundaries and the face interiors, and relations between these decomposition for neighboring substructures. The proof of two of the main results are given in the appendices. Condition number bounds are given in section 8 and section 9 is devoted to some numerical tests.

2. Partitions and discrete spaces. We discretize this problem using edge elements, also known as Nédélec elements; see [24]. These are vector finite elements that only ensure the continuity of the tangential component across the elements, as is physically required for the electric and magnetic fields, solutions of Maxwell's equations. We refer to [23] for a fine presentation of approximations of electromagnetic problems, the Sobolev space $H(\mathbf{curl}\,;\Omega)$, and edge elements.

We introduce a shape-regular triangulation $\mathcal{T} = \mathcal{T}_h$ of the domain Ω, made of affinely mapped cubes. In particular, if $\widehat{Q} = (-1,1)^3$ is a reference cube, for each element $K \in \mathcal{T}$, there exists an affine mapping $F_K : \widehat{Q} \to K$, such that K is the image of $\widehat{Q}$. Here we only consider meshes built on affinely mapped cubes for simplicity but our results are equally valid for approximations on tetrahedral meshes.

Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}$. For every edge $e \in \mathcal{E}_h$, we fix a direction, given by a unit vector $\mathbf{t}_e$, tangent to e. The length of the edge e is denoted by $|e|$ and, in the following, we will always denote the measure of a region $\mathcal{D}$ by $|\mathcal{D}|$.

We next consider a nonoverlapping partition of the domain Ω into subdomains (substructures),

$$\{\Omega_i \mid \quad 1 \leq i \leq N, \quad \cup_{i=1}^N \overline{\Omega_i} = \overline{\Omega}\},$$

such that each Ω_i is connected. The substructures Ω_i are unions of elements in $\mathcal{T}$. We denote the diameter and the local meshsize of Ω_i by H_i and and h_i, respectively. We define H as the maximum of the diameters of the subdomains:

$$H := \max_{1\leq i\leq N} \{H_i\}.$$

We assume that the coefficients A and B are constant in each substructure Ω_i and denote them by A_i and B_i, respectively. Since jumps of both coefficients will play a role in the rate of convergence of our algorithms, we only consider jumps in one of them:

$$\begin{array}{ll} \text{Case 1:} & \left\{ \begin{array}{l} A = \mathrm{diag}\{a,a,a\}, \\ 0 < \beta_i |\mathbf{x}|^2 \leq \mathbf{x}^t B_i \mathbf{x} \leq \gamma_i |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^3, \end{array} \right. \\ \text{Case 2:} & \left\{ \begin{array}{l} 0 < \beta_i |\mathbf{x}|^2 \leq \mathbf{x}^t A_i \mathbf{x} \leq \gamma_i |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^3, \\ B = \mathrm{diag}\{b,b,b\}, \end{array} \right. \end{array} \tag{2.1}$$

for $i = 1, \cdots, N$, where $|\cdot|$ denotes the standard Euclidean norm. Scaling matrices for our FETI algorithms will be constructed with the values $\{\gamma_i\}$.

We always assume that the substructures are images of a reference square under sufficiently regular maps, which effectively means that their aspect ratios remain uniformly bounded. In addition, we assume that the ratio of the diameters of two adjacent subregions is bounded away from zero and infinity. Further assumptions, necessary for the analysis but not for the definition of the algorithms, are made at the beginning of section 7.

We next define the local spaces

$$\mathrm{H}_\star(\mathrm{curl}\,;\Omega_i) := \{\mathbf{u}_i \in \mathrm{H}(\mathrm{curl}\,;\Omega_i)|\ \mathbf{n}\times(\mathbf{u}_i\times\mathbf{n}) = 0 \text{ on } \partial\Omega\cap\partial\Omega_i\}$$

and the following polynomial spaces on the reference square,

$$ND(\widehat{Q}) = \mathbb{Q}_{0,1,1}(\widehat{Q})\otimes\mathbb{Q}_{1,0,1}(\widehat{Q})\otimes\mathbb{Q}_{1,1,0}(\widehat{Q}),$$

with $\mathbb{Q}_{k_1,k_2,k_3}(\widehat{Q})$ the space of polynomials of degree k_i in the i-th variable. We note that the tangential component of a vector in $ND(K)$ over a face of $\widehat{Q}$ perpendicular to, e.g., the x axis is vector function of $\mathbb{Q}_{0,1}\times\mathbb{Q}_{1,0}$.

On an affinely mapped element $K\in\mathcal{T}$, we take

$$ND(K) = \{\mathbf{u} = J_{F_K}^{-T}\widehat{\mathbf{u}}\ |\quad \widehat{\mathbf{u}}\in ND(\widehat{Q})\}, \tag{2.2}$$

with J_{F_K} the Jacobian of the transformation F_K. The tangential component of a vector in $ND(K)$ can also be characterized in this case.

On each subdomain Ω_i, the lowest-order Nédélec finite element spaces are defined as

$$X_i = ND^h(\Omega_i) := \{\mathbf{u}\in\mathrm{H}_\star(\mathrm{curl}\,;\Omega_i)|\ \mathbf{u}_{|K}\in ND(K),\ K\in\mathcal{T}_h,\ K\subset\Omega_i\}.$$

Higher polynomial degrees can also be considered and our results and bounds will remain valid with constants that depend on the polynomial degree. See, e.g., [23] for more details. Functions in X_i have a constant tangential component along the fine edges in $\mathcal{E}_h$. The degrees of freedom are normally chosen as the constant values of the tangential component on the fine edges in $\mathcal{E}_h$: for $\mathbf{u}\in X_i$,

$$\lambda_e(\mathbf{u}) := \mathbf{u}\cdot\mathbf{t}_{e|_e} = |e|^{-1}\int_e \mathbf{u}\cdot\mathbf{t}_e\,ds,\quad e\in\mathcal{E}_h,\ \ e\subset\overline{\Omega}_i. \tag{2.3}$$

These edge averages also define local interpolation operators. The degrees of freedom (2.3) can be naturally partitioned into three classes according to where the corresponding edge lies: face, edge, and interior.

We next consider the product space

$$X = X(\Omega) := \prod_{i=1}^N X_i \subset \prod_{i=1}^N \mathrm{H}_\star(\mathrm{curl}\,;\Omega_i),$$

which consists of vectors that have in general a discontinuous tangential component across the subdomain boundaries. The discrete solution is sought in the conforming space

$$\widehat{X} := X\cap\mathrm{H}_0(\mathrm{curl}\,;\Omega),$$

of vectors with a continuous tangential components across the subdomain faces and edges.

Finally, we will also employ the standard finite element spaces of scalar, continuous, piecewise trilinear functions on the subdomains $V^h(\Omega_i) \subset H^1(\Omega_i)$. We note that

$$\nabla V^h(\Omega_i) \subset ND^h(\Omega_i). \tag{2.4}$$

3. Interface functions. We define the boundaries $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ and the interface Γ as their union. We remark that Γ is the union of the interior subdomain *faces*, regarded as open sets, which are shared by two subregions, and interior subdomain *edges* and *vertices*, which are shared by more than two subregions. In the following, we tacitly assume that points on $\partial\Omega$ are excluded from the geometrical objects that we consider, or, in other words, we will only deal with geometrical objects (faces, edges, vertices, ...) that belong to Γ. We denote the faces of Ω_i by F_{ij} and its edges by E_{ij} and also use faces and edges with one or no subscript. We will always assume that a face F_{ij} does not coincide with a connected component of $\partial\Omega_i$; this implies that the boundary ∂F_{ij} is not empty.

For a face F and an edge E of a substructure Ω_i, we introduce unit vectors tangent to ∂F and E, denoted $\mathbf{t}_{\partial F}$ and $\mathbf{t}_E$, respectively. The sets of fine edges (and the corresponding degrees of freedom) on Γ_i and Γ are denoted by $\Gamma_{i,h}$ and Γ_h, respectively. We note that there are no degrees of freedom associated with the subdomain vertices and that subdomain faces and edges that lie on $\partial\Omega$ do not belong to the interface.

REMARK 3.1. *In case Neumann boundary conditions are imposed on $\partial\Omega$, edges (but not faces) lying on $\partial\Omega$ are part of the interface Γ and also need to be employed in the definition of our algorithms.*

We now introduce some trace spaces consisting of tangential components on the boundaries of the substructures. A tangential vector $\mathbf{w}$, defined on $\partial\Omega_i \setminus \partial\Omega$, belongs to W_i if and only if there exists $\mathbf{u} \in X_i$ such that, on the closure of each face $F \subset \Gamma_i$,

$$\mathbf{w} = \mathbf{n} \times (\mathbf{u} \times \mathbf{n}), \quad \text{on } \overline{F}.$$

We note that a function $\mathbf{w} \in W_i$ belongs to the lowest order, two-dimensional edge element space on each face F and that the tangential component of $\mathbf{w}$ along an edge shared by two faces must be the same when calculated on either one of the faces.

Similarly, given a face F of a substructure Ω_i, we consider the tangential component along ∂F

$$u = \mathbf{u} \cdot \mathbf{t}_{\partial F}, \quad \text{on } \partial F.$$

The function u is piecewise constant. A similar definition holds for the tangential component along an edge $E \subset \partial F$.

We will use the following convention: given a vector $\mathbf{u}$ defined in Ω_i, we denote its tangential component on $\partial\Omega_i$ by the same bold letter $\mathbf{u}$. Its tangential component along an edge or the boundary of a face is denoted by u.

We will employ the product space of functions defined on Γ, $W := \prod_i W_i$, and its continuous subspace $\widehat{W}$ consisting of tangential traces of vectors in $\widehat{X}$.

Vectors in the spaces W_i and W are uniquely defined by the degrees of freedom in $\Gamma_{i,h}$ and Γ_h, respectively. For each fine edge $e \in \Gamma_h$, let $\mathcal{N}_e$ be the set of indices of the subdomains that have e on their boundary. Throughout this paper, we will use the same notation for a vector in W_i or X_i and the corresponding column vector of degrees of freedom. Similarly for the corresponding spaces and for global vectors in X and W.

We recall that, if $\mathbf{u} = \nabla\phi \in H(\mathbf{curl}\,;\Omega_i)$, for $\phi \in H^1(\Omega_i)$, and $\mathbf{v} \in H(\mathbf{curl}\,;\Omega_i)$, then, on a face F, we have, see [7, Sect. 1],

$$\mathbf{u} = \mathbf{n} \times (\nabla\phi \times \mathbf{n}) = \nabla_S\phi = \nabla_S(\phi|_F),$$
$$\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n} = \mathrm{curl}_S\mathbf{v} = \mathrm{curl}_S(\mathbf{n}\times(\mathbf{v}\times\mathbf{n})),$$

with ∇_S and curl_S the surface gradient and curl on F. For the whole of $\partial\Omega_i$, tangential gradients and curls are taken face by face.

Finally, for $i = 1, \cdots, N$, we define the extensions into the interior of the Ω_i

$$\mathcal{H}_i :\ W_i \longrightarrow X_i,$$

that are discrete harmonic with respect to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$. We recall that $\mathbf{u}^{(i)} = \mathcal{H}_i\mathbf{w}^{(i)}$ minimizes the energy $a_{\Omega_i}(\mathbf{u}^{(i)},\mathbf{u}^{(i)})$ among all the vectors of X_i with tangential component equal to $\mathbf{w}^{(i)}$ on Γ_i. We will refer to $\mathcal{H}_i$ as the *Maxwell* discrete harmonic extension.

4. FETI-DP methods. In this section, we introduce a first dual-primal FETI method for the solution of the linear system arising from the edge element discretization of problem (1.3). Throughout the paper, given two column vectors $\mathbf{u}$ and $\mathbf{w}$ of degrees of freedom, we denote their scalar product in l^2 by $< \mathbf{u}, \mathbf{w} >:= \mathbf{u}^T\mathbf{w}$. We recall that dual-primal FETI methods were originally introduced in [11]. The first theoretical result was given in [22] for two dimensional problems and then later in [19] for three dimensions. Extensive work and analysis has been performed for linear elasticity problems in [12, 18]. Algorithms for two-dimensional edge element approximations have been proposed in [30]. See also the forthcoming [31, Ch. 6 and 8].

We first assemble the local stiffness matrices, relative to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$, and the local load vectors. The degrees of freedom that belong only to one substructure can be eliminated in parallel by block Gaussian elimination. We note that these are degrees of freedom associated to edges e in the interior of the substructures. We are then left with the degrees of freedom involving the *tangential* component along the substructure boundaries. Let $\mathbf{f}^{(i)}$ be the resulting right hand sides and $S^{(i)}$ the Schur complement matrices

$$S^{(i)} :\ W_i \longrightarrow W_i,$$

relative to the tangential degrees of freedom on Γ_i.

We recall that the local Schur complements satisfy the following property

$$|\mathbf{u}^{(i)}|^2_{S^{(i)}} := \langle \mathbf{u}^{(i)}, S^{(i)}\mathbf{u}^{(i)}\rangle = a_{\Omega_i}(\mathcal{H}_i\mathbf{u}^{(i)}, \mathcal{H}_i\mathbf{u}^{(i)}); \tag{4.1}$$

see, e.g., [26, 27]. Since the local bilinear forms are positive definite, so are the local Schur complements $S^{(i)}$. We write

$$\mathbf{u} := \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N)} \end{bmatrix} \in W, \quad S := \mathrm{diag}\{S^{(1)}, \cdots, S^{(N)}\}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N)} \end{bmatrix}.$$

The solution $\mathbf{u} \in W$ to the discrete problem can then be found by minimizing the energy

$$\frac{1}{2}\langle \mathbf{u}, S\mathbf{u}\rangle - \langle \mathbf{f}, \mathbf{u}\rangle$$

subject to the constraint that $\mathbf{u}$ is continuous, i.e., it belongs to $\widehat{W}$.

For dual-primal FETI methods we work in a subspace $\widetilde{W} \subset W$ of functions satisfying a certain number of continuity constraints. We have

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta.$$

Here the primal space $\widehat{W}_\Pi \subset \widehat{W}$ consists of continuous tangential vectors determined by degrees of freedom (*primal variables*) associated to the substructures. Choices for primal constraints are given in section 6.2.

The dual space $\widetilde{W}_\Delta$ is the product space of spaces associated to the substructures

$$\widetilde{W}_\Delta := \prod_{i=1}^{N} \widetilde{W}_{\Delta,i}$$

of functions for which the functionals given by the primal variables vanish.

The primal degrees of freedom can then be eliminated together with the internal ones, at the expenses of solving one coarse problem. We are then left with a problem involving interface functions with vanishing primal degrees of freedom, and, consequently, in the dual space $\widetilde{W}_\Delta$. Let $\widetilde{S} : \widetilde{W}_\Delta \to \widetilde{W}_\Delta$ be the corresponding Schur complement and $\widetilde{\mathbf{f}}_\Delta$ the corresponding load vector. We then look for $\mathbf{u}_\Delta \in \widetilde{W}_\Delta$, such that

$$\frac{1}{2}\langle \mathbf{u}_\Delta, \widetilde{S}\mathbf{u}_\Delta\rangle - \langle \widetilde{\mathbf{f}}_\Delta, \mathbf{u}_\Delta\rangle \longrightarrow \min$$

subject to the constraint that $\mathbf{u}_\Delta$ is continuous. The continuity constraint is expressed by the equation

$$B_\Delta \mathbf{u}_\Delta = 0,$$

where B_Δ is constructed from $\{0, 1, -1\}$ and evaluates the difference between all the corresponding tangential degrees of freedom on Γ. We employ the same matrix as in our previous paper [29] and then enforce redundant conditions associated with the substructure edges. The matrix B_Δ has the following block structure:

$$B_\Delta = [B_\Delta^{(1)} \quad B_\Delta^{(2)} \quad \dots \quad B_\Delta^{(N)}],$$

where each block corresponds to a substructure.

We obtain the saddle point problem

$$\begin{array}{rcrcl} \widetilde{S}\mathbf{u}_\Delta & + & B_\Delta^T \lambda & = & \widetilde{\mathbf{f}}_\Delta \\ B_\Delta \mathbf{u}_\Delta & & & = & 0 \end{array} \tag{4.2}$$

with $\mathbf{u}_\Delta \in \widetilde{W}_\Delta$ and $\lambda \in V := Range(B_\Delta)$.

We note that $\widetilde{S}$ can be obtained from the restriction of S to the space $\widetilde{W}$, by eliminating the primal degrees of freedom. We have therefore the minimization property

$$\langle \mathbf{u}_\Delta, \widetilde{S}\mathbf{u}_\Delta\rangle = \min\langle \mathbf{u}, S\mathbf{u}\rangle, \tag{4.3}$$

where the minimum is taken over all the functions $\mathbf{u} = \mathbf{u}_\Delta + \mathbf{w}_\Pi$, $\mathbf{w}_\Pi \in \widehat{W}_\Pi$. This property ensures that $\widetilde{S}$ is also positive definite.

Since the Schur complement $\widetilde{S}$ is invertible, an equation for λ can easily be found:

$$F\lambda = d, \tag{4.4}$$

with

$$F := B_\Delta \widetilde{S}^{-1} B_\Delta^T, \quad d := B_\Delta \widetilde{S}^{-1} \widetilde{f}_\Delta. \tag{4.5}$$

Once λ is found, the primal variables are given by

$$\mathbf{u}_\Delta = \widetilde{S}^{-1}(\widetilde{\mathbf{f}}_\Delta - B_\Delta^T \lambda) \in \widetilde{W}.$$

In order to define a preconditioner for (4.4), we need to define scaling matrices and functions defined on the subdomain boundaries. As opposed to our previous work [29, 30], they are constructed with the coefficient that has jumps across the subdomains. For either case considered in (2.1), we define, on each substructure, $\delta_i^\dagger \in W_i$, such that on each fine edge $e \in \Gamma_{i,h}$,

$$\delta_{i\,|_e}^\dagger = \gamma_i^\chi \,/ \sum_{j \in \mathcal{N}_e} \gamma_j^\chi, \tag{4.6}$$

for an arbitrary but fixed $\chi \in [1/2, +\infty)$; see (2.1). By direct calculation, we find

$$\gamma_i {\delta_j^\dagger}^2 \leq \min(\gamma_i, \gamma_j). \tag{4.7}$$

For each substructure Ω_i, we next introduce a diagonal matrix $D_\Delta^{(i)} : V \to V$. The diagonal entry corresponding to the Lagrange multiplier that enforces the equality of the degree of freedom associated with a fine edge e between Ω_i and a second substructure Ω_j is set equal to the value of $\delta_j^\dagger$ along e. The remaining values are zero. We next define the scaled matrix

$$B_{D,\Delta} = [D_\Delta^{(1)} B_\Delta^{(1)} \quad D_\Delta^{(2)} B_\Delta^{(2)} \ \ldots \ D_\Delta^{(N)} B_\Delta^{(N)}] : \widetilde{W}_\Delta \to V.$$

We solve the dual system (4.4) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := B_{D,\Delta} S B_{D,\Delta}^T = \sum_{i=1}^N D_\Delta^{(i)} B_\Delta^{(i)} S^{(i)} B_\Delta^{(i)T} D_\Delta^{(i)}; \tag{4.8}$$

see [11, 22, 19].

We now recall an abstract framework for the analysis of FETI-DP algorithms, which was originally given in [19] and recalled in [30]. It turns out that condition number bounds rely on one stability estimate for the following jump operator

$$P_\Delta := B_{D,\Delta}^T B_\Delta : \widetilde{W} \longrightarrow \widetilde{W}.$$

We summarize the properties of P_Δ proven in [19, Sect. 6] in the following lemma.

LEMMA 4.1. *The operator P_Δ is a projection and preserves the jump of any function $w \in \widetilde{W}$, i.e.,*

$$B_\Delta P_\Delta \mathbf{w} = B_\Delta \mathbf{w}.$$

If $\mathbf{v} := P_\Delta \mathbf{w}$*, for* $\mathbf{w} \in \widetilde{W}$*, then on every fine edge* $e \in \Gamma_{i,h}$*, we have*

$$\mathbf{v}^{(i)} = \sum_{j \in \mathcal{N}_e} \delta_j^\dagger (\mathbf{w}^{(i)} - \mathbf{w}^{(j)}). \tag{4.9}$$

Finally, $P_\Delta \mathbf{w} = 0$*, if* $\mathbf{w} \in \widehat{W}$*.*

The following fundamental result can be found in [19, Th. 1]; see also [31, Sect. 6.4.3]. It employs the norms

$$|\mathbf{v}|_S^2 := \langle \mathbf{v}, S\mathbf{v} \rangle = \sum_{i=1}^N \langle \mathbf{v}^{(i)}, S^{(i)} \mathbf{v}^{(i)} \rangle, \quad |\mathbf{v}|_{\widetilde{S}}^2 := \langle \mathbf{v}, \widetilde{S}\mathbf{v} \rangle. \tag{4.10}$$

THEOREM 4.2. *Let* C_{P_Δ} *be such that*

$$|P_\Delta \mathbf{w}_\Delta|_S^2 \le C_{P_\Delta} |\mathbf{w}_\Delta|_{\widetilde{S}}^2, \qquad \mathbf{w}_\Delta \in \widetilde{W}_\Delta. \tag{4.11}$$

Then, if $\widetilde{S}$ *and* M^{-1} *are invertible,*

$$\langle M\lambda, \lambda \rangle \le \langle F\lambda, \lambda \rangle \le C_{P_\Delta} \langle M\lambda, \lambda \rangle, \quad \lambda \in V. \tag{4.12}$$

5. Remarks on iterative substructuring for edge element approximations. Preconditioners for Schur complement systems rely on decoupling degrees of freedom associated to geometrical objects associated to subdomains, typically vertices, edges, and faces for three-dimensional continuous nodal elements; see, e.g., [10, Sect. 5]. In our case, we only need to consider subdomain edges and faces. The performance of the corresponding preconditioned iterative method depends on how weak the coupling between the different blocks of degrees of freedom is and depends on the particular basis chosen. This decoupling may appear explicitly in the construction of finite element subspaces as in wire basket methods, [10], but it may also be hidden in the algorithm and may not appear explicitly in the subspaces considered, as in Neumann-Neumann or FETI methods. Indeed, the scaled matrix B_Δ acts on vectors of degrees of freedom and is constructed with the scaling functions $\delta_i^\dagger$. These functions are constant on the edges and faces of a subdomain and formula (4.9) naturally decouples degrees of freedom on edges and faces.

Decompositions into edge and face components are fairly harmless (i.e., logarithmically stable) operations for continuous nodal h finite elements but turn out to be disastrous for edge element approximations, as it can be seen in Figure 5.1, left. More precisely, we refer to Figure 5.1, right, and consider the gradient of a continuous, scalar, piecewise trilinear function ϕ_E with vanishing nodal values on the closure of a subdomain Ω_i except at one node on a coarse edge E where it is one. Since ϕ_E decreases linearly from one to zero along an edge of length $O(h)$, its tangential component is $O(h^{-1})$. This vector is curl free and has a low energy:

$$\|\nabla \phi_E\|_{H(\mathbf{curl}\,;\Omega_i)}^2 = \|\nabla \phi_E\|_{L^2(\Omega_i)^3}^2 = O(h^{-2} \cdot h^3) = O(h).$$

We recall that the square of the L^2 norm of a basis function is $O(h^3)$ while that of its curl is $O(h)$. When we put to zero the degrees of freedom on the two faces adjacent to E, we obtain a vector $\mathbf{w}$ with a nonvanishing curl and therefore with a much larger energy

$$\|\mathbf{w}\|_{H(\mathbf{curl}\,;\Omega_i)}^2 \sim \|\mathbf{curl}\,\mathbf{w}\|_{L^2(\Omega_i)^3}^2 = O(h^{-2} \cdot h) = O(1/h).$$

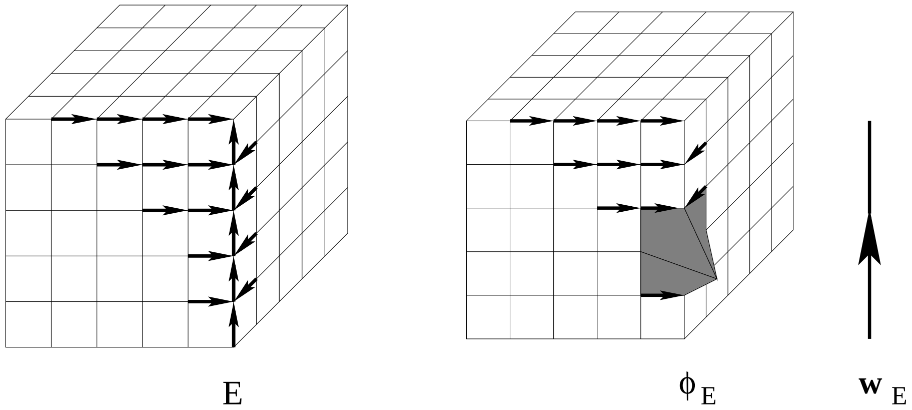

FIG. 5.1. *Two types of basis functions associated to a subdomain edge: the standard basis (left) and one consisting of gradients of continuous, scalar, nodal functions associated to nodes internal to the coarse edge and one vector function with unitary tangential component along the coarse edge* .

The constant C_{P_Δ} in Theorem 4.2 is therefore expected to grow at least as h^{-2}, thus resulting in a condition number that grows at least as h^{-2}! The situation is the same as for general p and hp approximations, for which strong coupling exists between edge and face components, unless very special bases are employed; see, e.g., [20].

We stress that this will happen independently of the chosen primal constraints, unless they impose the continuity of *all* the fine degrees of freedom across the coarse edges; in the latter case, the difference $\mathbf{w}^{(i)} - \mathbf{w}^{(j)}$ in formula (4.9) already vanishes on the coarse edges and no decoupling takes place. Any Neumann-Neumann or FETI method which employs the standard three-dimensional edge element basis is bound to show a factor h^{-2} in the condition number.

6. FETI-DP algorithms for edge element approximations. The discussion in the previous section suggests that before devising effective FETI methods, a change of basis is necessary. We will therefore employ a local change of basis on the local spaces X_i. New basis functions are only needed for the degrees of freedom associated to the edges of the subdomains. In this new basis, it will be relatively straightforward to find a good set of primal constraints that ensure scalability and a coarse problem of relatively small size.

6.1. A change of basis. We consider a local space X_i associated to a substructure Ω_i and introduce the following new basis,

DEFINITION 6.1 (New basis).

1. *The basis functions associated with the interior edges e in $\Omega_{i,h}$ and with those in the interior of the faces are the same as for the standard basis;*
2. *the basis functions associated with a subdomain edge E are:*
   (a) *one vector function $\mathbf{\Phi}_E$ with unitary tangential component along E and vanishing tangential component along all the fine edges $e \in \mathcal{E}_h$ that lie on the remaining coarse edges, on the faces, and in the interior of Ω_i;*
   (b) *the gradients of continuous, scalar, nodal functions associated to the interior nodes of E; these scalar functions take the value zero at all the nodes in the closure of Ω_i except at one node in the interior of E, where they are equal to one.*

This new basis is described in Figure 5.1 and it has the same dimension as the old one. The degree of freedom associated to a basis function $\mathbf{\Phi}_E$ is the average over the coarse edge E, while those associated to the basis functions at Point 2.b are values at the nodes internal to the subdomain edges. The new basis functions are introduced only for the coarse subdomain edges and it can be easily proven that they are linearly independent. The new degrees of freedom can still be partitioned into interior, face, and edge. For an interface function $\mathbf{w} \in W_i$ they give rise to the following decomposition:

$$\mathbf{w} = \sum_E (\mathbf{w}_E + \nabla_S \, \phi_E) + \sum_F \mathbf{w}_F, \quad \text{on } \partial\Omega_i. \tag{6.1}$$

A face component $\mathbf{w}_F$ vanishes on all faces except F and has a vanishing tangential component along all coarse edges. The component $\mathbf{w}_E$ has a constant tangential component along the edge E and vanishing degrees of freedom on all faces and remaining edges; we will refer to it in the following as *constant component along the edge E*. Finally, the scalar function ϕ_E vanishes at all the nodes except at those in the interior of E; we will refer to $\nabla_S \, \phi_E$ as *the gradient component along E*.

Given a local vector $\mathbf{u} \in X_i$, the corresponding vectors of degrees of freedom in the new and old bases are related by

$$\mathbf{u}^{new} = Q_i \mathbf{u}^{old}, \quad \mathbf{u}^{old} = Q_i^{-1} \mathbf{u}^{new} = R_i \mathbf{u}^{new}. \tag{6.2}$$

The matrix R_i can easily be found by noting that its columns are associated to the new basis functions and the entries of each column are the degrees of freedom in the old basis associated with one new basis function.

DEFINITION 6.2 (Change of basis). *The matrix R_i is defined as follows:*

1. *Columns for the new functions associated with the degrees of freedom in the interior of Ω_i or in the interior of a face (cf. Point 1 in Definition 6.1) consist of zeros except for one entry corresponding to a fine edge where they take the value one.*
2. *The column for a basis function that has unitary tangential component along a coarse edge E (cf Point 2.a in Definition 6.1) is zero except for the entries relative to the fine edges $e \subset E$, where it takes the value 1 or -1, depending on the convention chosen for the direction of e and E.*
3. *The column for the gradient of a scalar nodal function associated with a node x_k on an edge E (cf Point 2.b in Definition 6.1) is zero except for the entries relative to the fine edges $e \subset \overline{\Omega}_i$ that have x_k as an end point; these entries are equal to $\pm|e|^{-1}$ depending on the convention chosen for the direction of e.*

We note that R_i has very few nonzero entries and its inverse is also sparse. The new basis and the definition of the matrix R_i do not rely on the fact that the substructures are elements of a coarse mesh or have a special shape, but can straightforwardly be defined for less regular subdomains produced by, e.g., practical mesh partitioners.

6.2. Primal constraints. We consider the FETI-DP algorithm introduced in section 4. We assume from now on that vectors of primal degrees of freedom are relative to the new basis introduced in the previous section and recall that we still have a partition into edge and face degrees of freedom. The matrix B_Δ still consists of zeros and ones, the scaling functions are defined by associating to each new degree of freedom the ratio in (4.6), which only depends on the subdomain partition and the coefficients A or B, and the Schur complements are those obtained from stiffness matrices in the new basis.

We are only left with the definition of suitable primal constraints. Our choices are related to the decomposition (6.1) of the degrees of freedom for the new basis. We first impose that, for each coarse edge of the interface Γ, the degree of freedom associated with the term $\mathbf{w}_E$ in the decomposition (6.1) is the same for each subdomain that contains E. The degree of freedom associated to this term is the average of the tangential component along the edge E:

$$a_E(\mathbf{w}) = a_E(w) := |E|^{-1} \int_E \mathbf{w} \cdot \mathbf{t}_E \, ds = |E|^{-1} \int_E w \, ds. \tag{6.3}$$

DEFINITION 6.3 (Algorithm A). *The space $\widetilde{W} = \widetilde{W}_A$ is the subspace of W of interface vectors for which the averages $a_E(\mathbf{w})$ of the tangential component along the coarse edges E are equal, independently of which component of $\mathbf{w} \in \widetilde{W}_A$ is used in the evaluation of these averages. The local spaces $\widetilde{W}_{\Delta,i}$ consist of vectors with a vanishing average for the tangential component along the edges, while the primal subspace $\widehat{W}_\Pi$ consists of continuous vectors with a constant tangential component along each coarse edge and zero components $\mathbf{w}_F$ and $\nabla_S \, \phi_E$ in the decomposition (6.1).*

We note that, in case the subdomain partition coincides with a coarse mesh, the primal constraints are the degrees of freedom for the edge element coarse space. Algorithm A is not however expected to perform well. Indeed, we see that there are no constraints acting on the scalar functions ϕ_E in the decomposition (6.1), except from the requirement that they vanish at the end points of the edges E. We therefore expect that Algorithm A performs at least as bad as the corresponding FETI-DP for continuous, scalar finite elements where only primal constraints at the subdomain vertices are imposed; cf Algorithm A in [19].

Together with averages along the edges, we need to add additional constraints involving the gradient components along the edges. Looking at which algorithms are effective for scalar approximations, averages of the scalar functions ϕ_E

$$< \phi_E >_E := |E|^{-1} \int_E \phi_E \, ds, \tag{6.4}$$

along the subdomain edges should be imposed to be continuous as well; see Algorithm C in [19] and [31, Sect. 6.4.2].

Before giving a precise definition, we want to express these second averages in terms of the vector $\mathbf{w} \in W_i$ itself. We assume that the vector $\mathbf{w}$ in (6.1) has a vanishing average (6.3) for the tangential component along an edge E. We denote by s the arc length along E. Since ϕ_E vanishes at the end points of E, integration by parts yields

$$\begin{aligned} < \phi_E >_E &= |E|^{-1} \int_E \phi_E \, ds = -|E|^{-1} \int_E \phi_E' \, s \, ds = -|E|^{-1} \int_E (\nabla \phi_E \cdot \mathbf{t}_E) \, s \, ds \\ &= -|E|^{-1} \int_E (\mathbf{w} - \mathbf{w}_E) \cdot \mathbf{t}_E \, s \, ds = -|E|^{-1} \int_E (\mathbf{w} \cdot \mathbf{t}_E) \, s \, ds. \end{aligned}$$

When $\mathbf{w} \cdot \mathbf{t}_E$ has a vanishing mean value, the averages of the functions ϕ_E are therefore equal to *first order moments* of the original vector $\mathbf{w}$ along the edge. For each coarse edge E, we define the continuous, scalar function θ_E to be zero at all the nodes of the interface Γ except at those internal to E where it is equal to one.

DEFINITION 6.4 (Algorithm B). *The space $\widetilde{W} = \widetilde{W}_B$ is the subspace of W of interface vectors for which the averages $a_E(\mathbf{w})$ and $< \phi_E >_E$ along the coarse edges, or, equivalently, $a_E(\mathbf{w})$ and the first order moments*

$$|E|^{-1} \int_E (\mathbf{w} \cdot \mathbf{t}_E)\, s\, ds,$$

are equal, independently of which component of $\mathbf{w} \in \widetilde{W}_B$ is used in the evaluation of these averages. The local spaces $\widetilde{W}_{\Delta,i}$ consist of vectors with vanishing average and first order moment for the tangential component along the edges, while the primal subspace $\widehat{W}_\Pi$ is spanned the same functions as for Algorithm A and in addition the gradients $\nabla_S\, \theta_E$ associated to the coarse edges.

Algorithm B will provide a very favorable bound. We note that, despite the difficulty of this problem, the size of the global problem that needs to be solved (equal to the number of primal constraints) is very reasonable: we have two constraints per coarse edge. In case, for instance, the substructures are elements of a coarse, cubical, uniform mesh, each subdomain has twelve edges, which is shared by four substructures. We have therefore *six* coarse degrees of freedom for each substructure.

We finally define an algorithm with a very large coarse space.

DEFINITION 6.5 (Algorithm C). *The space $\widetilde{W} = \widetilde{W}_C$ is the subspace of W of interface vectors for which the tangential component along the fine edges on the coarse edges*

$$\lambda_e(\mathbf{w}) = |e|^{-1} \int_e \mathbf{w} \cdot \mathbf{t}_e\, ds, \quad e \subset E,$$

are equal. The local spaces $\widetilde{W}_{\Delta,i}$ consist of vectors with a vanishing tangential component along the edges, while the primal subspace $\widehat{W}_\Pi$ is spanned by the continuous basis functions associated to the fine edges that lie on a coarse edge.

A practical implementation of FETI-DP algorithms for Problem (1.1) is given in [30, Sect. 6]. While one should in principle employ matrices and vectors associated to the new basis for Algorithms A and B, it is possible to employ the original operators in the standard basis and work with the local matrices Q_i in (6.2). We note in particular that primal constraints can be expressed in terms of the old basis and the matrix B_Δ, which evaluates the difference between corresponding degrees of freedom, is the same for the two bases in case, for instance, the same orientation for the corresponding fine edges is employed on different subdomains.

7. Technical tools. In this section, we will prove some decomposition and comparison results for edge element vectors. Some of the ideas of this analysis have been suggested in [5]. Many of our estimates depend on a logarithmic factor

$$\omega := 1 + \log(H/h), \tag{7.1}$$

where the ratio H/h is a shorthand notation for the maximum over the substructures of H_i/h_i. As is often customary in the analysis of iterative substructuring methods, we require that the substructures are elements of a shape-regular coarse mesh $\mathcal{T}_H$ or that they are union of a uniformly bounded number of coarse shape-regular elements.

7.1. Trace spaces. For a substructure Ω_i, we need to consider tangential traces of functions in $H(\mathbf{curl}\,; \Omega_i)$ and the corresponding edge element spaces on the boundary, on a face F, or along ∂F. We refer to, e.g., [1, 6, 7] for more details and definitions.

In the following, Sobolev spaces of vectors on $\partial\Omega_i$ and F are always understood as spaces of tangential vectors; we will use a notation with bold letters for them. We define

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}\,_S;\partial\Omega_i) = \{\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\partial\Omega_i)\,,\ \mathrm{curl}\,_S\mathbf{u} \in H^{-1/2}(\partial\Omega_i).\}$$

The space $H^{-1/2}(\partial\Omega_i)$ is the dual of $H^{1/2}(\partial\Omega_i)$, while $\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)$ is the dual of $\mathbf{H}_\perp^{1/2}(\partial\Omega_i)$. The precise definition of $\mathbf{H}_\perp^{1/2}(\partial\Omega_i)$ and its norm are given in Appendix A since they are not important at this stage: it is, roughly speaking, the space of tangential vectors on $\partial\Omega_i$, the restriction of which to a face F belongs to $\mathbf{H}^{1/2}(F)$ and with a *normal* component along the face boundaries in $H^{1/2}(\partial\Omega_i)$; see [1, 6, 7].

For a face F, we define

$$\mathbf{H}_{\perp,00}^{-1/2}(\mathrm{curl}\,_S;F) = \{\mathbf{u} \in \mathbf{H}_{\perp,00}^{-1/2}(F)\,,\ \mathrm{curl}\,_S\mathbf{u} \in H_{00}^{-1/2}(F)\}$$

where $\mathbf{H}_{\perp,00}^{-1/2}(F)$ and $H_{00}^{-1/2}(F)$ are the duals of $\mathbf{H}_{\perp,00}^{1/2}(F)$ and $H_{00}^{1/2}(F)$, respectively, consisting of functions in $\mathbf{H}^{1/2}(F)$ and $H^{1/2}(F)$ for which the extensions by zero to $\partial\Omega_i$ belong to $\mathbf{H}_\perp^{1/2}(\partial\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$, respectively. If $\mathcal{E}\mathbf{u}$ and $\mathcal{E}\phi$ denote these extensions, we employ the norms

$$\|\mathbf{u}\|_{\mathbf{H}_{\perp,00}^{1/2}(F)} = \|\mathcal{E}\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\partial\Omega_i)}, \qquad \|\phi\|_{H_{00}^{1/2}(F)} = \|\mathcal{E}\phi\|_{H^{1/2}(\partial\Omega_i)}.$$

The space $\mathbf{H}^{-1/2}(\mathrm{curl}\,_S;F)$ is defined in a similar way:

$$\mathbf{H}^{-1/2}(\mathrm{curl}\,_S;F) = \{\mathbf{u} \in \mathbf{H}^{-1/2}(F)\,,\ \mathrm{curl}\,_S\mathbf{u} \in H^{-1/2}(F)\}.$$

The spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}\,_S;\partial\Omega_i)$, $\mathbf{H}_{\perp,00}^{-1/2}(\mathrm{curl}\,_S;F)$, and $\mathbf{H}^{-1/2}(\mathrm{curl}\,_S;F)$ are equipped with the graph norms. We note that $H^{-1/2}(F)$, $\mathbf{H}^{-1/2}(F)$, and $\mathbf{H}^{-1/2}(\mathrm{curl}\,_S;F)$ are proper subspaces of $H_{00}^{-1/2}(F)$, $\mathbf{H}_{\perp,00}^{-1/2}(F)$, and $\mathbf{H}_{\perp,00}^{-1/2}(\mathrm{curl}\,_S;F)$, respectively. In addition, by simple computation,

$$\begin{aligned} \|\phi\|_{H_{00}^{-1/2}(F)} &\le C\,\|\mathcal{E}\phi\|_{H^{-1/2}(\partial\Omega_i)} \le C\,\|\phi\|_{H^{-1/2}(F)}, \\ \|\mathbf{w}\|_{\mathbf{H}_{\perp,00}^{-1/2}(F)} &\le C\,\|\mathcal{E}\mathbf{w}\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} \le C\,\|\mathbf{w}\|_{\mathbf{H}^{-1/2}(F)}. \end{aligned} \tag{7.2}$$

In the following, we will also employ $H^{-1}(\partial F)$, the dual of $H^1(\partial F)$.

Throughout, we will work with scaled norms for the spaces $H^s(\mathcal{D})$, $s > 0$, obtained from the definition of the Sobolev norm on a region with diameter one and a dilation. Thus, if $H_\mathcal{D}$ is the diameter of a region $\mathcal{D} \subset \mathbb{R}^n$, we define

$$\|u\|_{H^1(\mathcal{D})}^2 = |u|_{H^1(\mathcal{D})}^2 + H_\mathcal{D}^{-2}\|u\|_{L^2(\mathcal{D})}^2,$$

for a substructure $\mathcal{D} = \Omega_i$ or the boundary of one of its faces $\mathcal{D} = \partial F$, and

$$\|u\|_{H^{1/2}(\mathcal{D})}^2 = |u|_{H^{1/2}(\mathcal{D})}^2 + H_\mathcal{D}^{-1}\|u\|_{L^2(\mathcal{D})}^2,$$

for the boundary of a substructure $\mathcal{D} = \partial\Omega_i$ or one of its faces $\mathcal{D} = F$. Analogous definitions hold for the spaces of vectors. The definition of dual norms employs these scaled norms. As is standard in the analysis of domain decomposition methods, inequalities are obtained for regions of unit diameter and then by a scaling argument that provides an explicit dependence on the diameter of the regions.

We have the following trace estimates. The first two are a straightforward consequence of the fact that $\mathbf{curl}\,\mathbf{u} \in L^2(\Omega_i)^3$ has a vanishing divergence and therefore its normal component is well-defined; see, e.g., [4, Lem. 1.2, Chap. III]. The others can be found in, e.g., [6], Theorems 3.6 and 5.6, and Proposition 5.3.

LEMMA 7.1. *Let Ω_i be a substructure and F one of its faces. Then, there exists a constant C, independent of H_i, such that, for $\mathbf{u} \in H(\mathbf{curl}\,;\Omega_i)$,*

$$\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(\partial\Omega_i)} \leq C\,\|\mathbf{curl}\,\mathbf{u}\|^2_{L^2(\Omega_i)},$$

$$\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}_{00}(F)} \leq C\,\|\mathbf{curl}\,\mathbf{u}\|^2_{L^2(\Omega_i)},$$

$$\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} + H_i^2\,\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(\partial\Omega_i)} \leq C\,(\|\mathbf{u}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{u}\|^2_{L^2(\Omega_i)}),$$

$$\|\mathbf{u}\|^2_{\mathbf{H}_{\perp,00}^{-1/2}(F)} + H_i^2\,\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}_{00}(F)} \leq C\,(\|\mathbf{u}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{u}\|^2_{L^2(\Omega_i)}).$$

In addition, for $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{curl}\,_S;F)$, its tangential component $u = \mathbf{u}\cdot\mathbf{t}_{\partial F}$ satisfies

$$\|u\|^2_{H^{-1}(\partial F)} \leq C\,(\|\mathbf{u}\|^2_{\mathbf{H}^{-1/2}(F)} + H_i^2\,\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(F)}).$$

We note that tangential components along face boundaries are not defined for every vector in $H(\mathbf{curl}\,;\Omega_i)$ or its appropriate trace space. They can be defined however for edge element vectors and the link between the stronger and weaker norms is given by the inverse inequalities of the following lemma.

LEMMA 7.2. *Let Ω_i be a substructure and F one of its faces and assume that the mesh on F is quasi-uniform. Let $\mathbf{u} \in W_i$. Then there exists a constant, independent of h and H_i, such that*

$$\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(F)} \leq C\,\omega^2\,\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(\partial\Omega_i)},$$

$$\|\mathbf{u}\|^2_{\mathbf{H}^{-1/2}(F)} \leq C\,\omega^2\,\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)},$$

$$\|\mathbf{u}\|^2_{\mathbf{H}^{-1/2}(F)} + H_i^2\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(F)} \leq C\omega^2(\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} + H_i^2\|\mathrm{curl}\,_S\mathbf{u}\|^2_{H^{-1/2}(\partial\Omega_i)}),$$

with ω the logarithmic factor defined in (7.1).

We note that a stronger version of the first inequality, involving a piecewise constant function on $\partial\Omega_i$, was already proven in [35, Lem. 4.4] by using an equivalent dual norm for $H^{-1/2}(\partial\Omega_i)$ with a supremum taken over a finite element space of piecewise bilinear, continuous functions augmented with one bubble for each boundary element. That proof however cannot be easily generalized to the edge element tangential space in $\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)$. The lemma is proven in Appendix A, using a localization result and an inverse inequality for norms in Sobolev spaces with negative exponents.

7.2. Decompositions associated with the edges. For a substructure Ω_i, we now consider the decomposition (6.1) in more detail. The edge contributions can be found in the following way:
We consider $\mathbf{w} \in W_i$ and an edge $E \subset \partial F$, $F \subset \partial\Omega_i$. We assume that the edge tangent vectors $\mathbf{t}_E$ have the same direction as $\mathbf{t}_{\partial F}$. Let w be its tangential component along ∂F. We have

$$\mathbf{w}_E = a_E(\mathbf{w})\,\mathbf{\Phi}_E,$$

or, equivalently

$$w_E = \mathbf{w}_E \cdot \mathbf{t}_E = a_E(\mathbf{w}) = a_E(w).$$

Here, $\mathbf{\Phi}_E$ is the basis function of Point 2.a, Definition 6.1. Let $\Phi_E(s)$ be the function that is one on E and zero on the remaining of ∂F. By definition, we have

$$\int_E (w - w_E)\, ds = \int_E (w - a_E(w))\, ds = 0.$$

If $s \in [0, |E|]$ is the arc length along E, there then exists a continuous, piecewise linear scalar function $\phi_E(s)$, vanishing on $\partial F \setminus E$, such that

$$\phi_E'(s) = w(s) - a_E(w), \quad \phi_E(s) = \int_0^s (w(s') - a_E(w))\, ds', \qquad \text{on } E.$$

We have therefore the decomposition

$$w = \sum_{E \subset \partial F} (a_E(w)\, \Phi_E + \phi_E'), \quad \text{on } \partial F, \tag{7.3}$$

which gives the tangential component along ∂F of the decomposition (6.1). A function ϕ_E defined on the whole of Ω_i can then be obtained as a zero extension, by setting to zero all the nodal values that do not lie inside of E. We have therefore found the terms $\nabla \phi_E$ in the decomposition (6.1).

We note that the decomposition (7.3) is not logarithmically stable. Roughly speaking, $a_E(w)$ is not bounded linear functional in $H^{-1}(\partial F)$, the appropriate space of the tangential component w along ∂F (apart from a logarithmic factor). For this reason, we are forced to consider a second decomposition. We define the average

$$a_{\partial F}(\mathbf{w}) = a_{\partial F}(w) := |\partial F|^{-1} \int_{\partial F} w\, ds.$$

There then exists a continuous, piecewise linear scalar function $\phi_{\partial F}(s)$, such that

$$\phi_{\partial F}'(s) = w(s) - a_{\partial F}(w), \quad \phi_{\partial F}(s) = \int_0^s (w(s') - a_{\partial F}(w))\, ds' + C, \qquad \text{on } \partial F. \tag{7.4}$$

This gives the second decomposition

$$w = a_{\partial F}(w) + \phi_{\partial F}', \quad \text{on } \partial F. \tag{7.5}$$

We note that $\phi_{\partial F}$ does not necessarily vanish at all the vertices on ∂F and is defined up to an additive constant. Once this constant is fixed and a suitable extension to F is defined, the corresponding edge element function $\nabla \phi_{\partial F}$ provides an extension of the tangential component $\phi_{\partial F}'$ inside Ω_i. The terms associated to the two decompositions are illustrated in Figure 7.1 for the case of a triangular face. We make the following choice.

DEFINITION 7.3. *The constant in (7.4) is chosen such that the scalar function* $\phi_{\partial F}$ *has a vanishing mean value* $< \phi_{\partial F} >_E$ *over one of the edges* $E \subset \partial F$*;* $\phi_{\partial F}$ *is then extended by zero to all the remaining nodes in* $\overline{\Omega}_i \setminus \partial F$*.*

A more intuitive and equivalent definition of $\phi_{\partial F}$ is the following:
we consider the function

$$\phi_{\partial F}(s) = \int_0^s (w(s') - a_{\partial F}(w))\, ds', \quad s \in [0, |\partial F|],$$

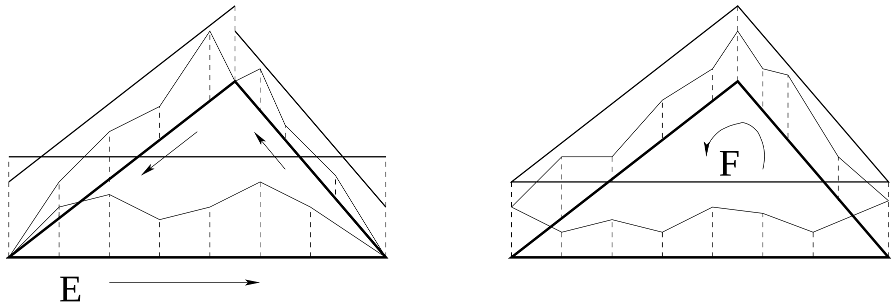

FIG. 7.1. *Terms associated to two decompositions of the tangential component w along the boundary of a face: $\{a_E(w)\Phi_E,\, \phi_E\}$ for (7.3) (left) and $a_{\partial F}(w)$ and $\phi_{\partial F}$ for (7.5) (right).*

defined on ∂F and vanishing at one of the vertices. We then take its mean value along an edge E and assign this value at all the internal nodes on F and the remaining ones in $\overline{\Omega}_i$. This is related to the similar coarse interpolant in [10, Sect. 6.2], where averages along face boundaries are employed. Since only the gradient of this function has a meaning here, we can subtract the edge average on $\overline{\Omega}_i$ and obtain the function in Definition 7.3. Here, we have chosen to work with an edge average instead of one on ∂F since for the proof of Lemma 8.2 we will need to compare gradient contributions for two different substructures that may only have an edge in common but not an entire face.

We now consider the decomposition (7.5) in more detail and show that it is logarithmically stable. The following is a trivial property of the zero extension from a subdomain edge; see [10, Lem. 4.7].

LEMMA 7.4. *Let Ω_i be a substructure and F one of its faces. If ϕ is a continuous, piecewise trilinear function that vanishes at all nodes in $\overline{\Omega}_i$ except those on ∂F, then*

$$\|\nabla\phi\|^2_{L^2(\Omega_i)} \leq C\, \|\phi\|^2_{L^2(\partial F)}.$$

We next find a bound for the face average.

LEMMA 7.5. *Let Ω_i be a substructure and F one of its faces. There exists a constant, independent of H_i and the meshsize, such that, for $\mathbf{w} \in X_i$,*

$$|a_{\partial F}(w)|^2 \leq C\, {H_i}^{-1}\, \omega^2\, \|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}.$$

Proof. We have

$$\begin{aligned} |a_{\partial F}(w)| &= |\partial F|^{-1}\, |\int_{\partial F} \mathbf{w}\cdot\mathbf{t}_{\partial F}\, ds| = |\partial F|^{-1}\, |\int_F \mathrm{curl}_S\, \mathbf{w}\, dS| \\ &\leq C\, H_i^{-1}\, \|\mathrm{curl}_S\mathbf{w}\|_{H^{-1/2}(F)}\, \|1\|_{H^{1/2}(F)}. \end{aligned}$$

The proof is concluded by using the first inequality of Lemma 7.2, the trace estimate in Lemma 7.1, and the fact that $\|1\|_{H^{1/2}(F)} \leq C\sqrt{H_i}$. □

We note that a better bound can be proven; see the proof of [35, Lem. 4.1]. Here, we chose a different proof that relies on Lemma 7.2, in order to give a self-contained presentation. Indeed, there is no difference in our final result.

We now need the following technical result.

LEMMA 7.6. *Let F be a face of a substructure and $E \subset \partial F$ one of its edges. If ϕ is a continuous, piecewise linear function on ∂F and $<\phi>_E$ its mean value on E, then*

$$\|\phi- <\phi>_E\|_{L^2(\partial F)} \le C\,\|\phi'\|_{H^{-1}(\partial F)}.$$

Proof. We have, for an arbitrary constant c,

$$\|\phi- <\phi>_E\|_{L^2(\partial F)} = \|(\phi-c)- <\phi-c>_E\|_{L^2(\partial F)} \le C\,\|\phi-c\|_{L^2(\partial F)}.$$

Choosing $c = <\phi>_{\partial F}$, the mean value over ∂F, therefore gives

$$\begin{aligned}
&\|\phi- <\phi>_E\|_{L^2(\partial F)} \le C\,\|\phi- <\phi>_{\partial F}\|_{L^2(\partial F)} \\
&= C \sup_{\psi\in L^2(\partial F)} \frac{\int_{\partial F}(\phi- <\phi>_{\partial F})\,\psi\,ds}{\|\psi\|_{L^2(\partial F)}} = C \sup_{\psi\in L^2(\partial F)} \frac{\int_{\partial F}\phi\,(\psi- <\psi>_{\partial F})\,ds}{\|\psi\|_{L^2(\partial F)}} \\
&\le C \sup_{\psi\in L^2(\partial F)} \frac{\int_{\partial F}\phi\,(\psi- <\psi>_{\partial F})\,ds}{\|\psi- <\psi>_{\partial F}\|_{L^2(\partial F)}} = C \sup_{\substack{\eta\in H^1(\partial F)\\ \eta(0)=0}} \frac{\int_{\partial F}\phi\,\eta'\,ds}{|\eta|_{H^1(\partial F)}} \\
&= C \sup_{\substack{\eta\in H^1(\partial F)\\ \eta(0)=0}} \frac{-\int_{\partial F}\phi'\eta\,ds}{|\eta|_{H^1(\partial F)}} \le C\|\phi'\|_{H^{-1}(\partial F)}.
\end{aligned}$$

□

REMARK 7.7. *We note that it is necessary to subtract an average from ϕ for the previous lemma, in order to obtain a good bound; see also Definition 7.3 and Lemma 7.8 below. If P is a node on ∂F, indeed we only have*

$$\|\phi-\phi(P)\|^2_{L^2(\partial F)} \le C\,(H/h)\,\|\phi'\|^2_{H^{-1}(\partial F)}.$$

This bound can be obtained by using a similar reasoning and the trivial bound for a linear function on a small edge $e \subset \partial F$: $\|\phi\|^2_{L^\infty(e)} \le (C/h)\,\|\phi\|^2_{L^2(e)}$.

We now give a discrete trace estimate for a face boundary.

LEMMA 7.8. *Let F be a face of a substructure Ω_i and $\mathbf{w} \in W_i$. If $\phi_{\partial F}$ is chosen according to Definition 7.3, then there exists a constant, independent of H_i and the meshsize, such that*

$$\|\phi_{\partial F}\|^2_{L^2(\partial F)} \le C\,\omega^2\,(\|\mathbf{w}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}).$$

Proof. We recall that Definition 7.3 ensures that the mean value $<\phi_{\partial F}>_E$ vanishes for an edge $E \subset \partial F$. Using Lemma 7.6 and (7.5), we can then write

$$\begin{aligned}
\|\phi_{\partial F}\|_{L^2(\partial F)} &\le \|\phi'_{\partial F}\|_{H^{-1}(\partial F)} = \|w - a_{\partial F}(w)\|_{H^{-1}(\partial F)} \\
&\le \|w\|_{H^{-1}(\partial F)} + \|a_{\partial F}(w)\|_{H^{-1}(\partial F)}.
\end{aligned}$$

The first term can be bounded using the trace estimates in Lemma 7.1 and the inverse inequalities in Lemma 7.2, while the second using Lemma 7.5 and the fact that $\|1\|_{H^{-1}(\partial F)} \le CH_i^{3/2}$ since we employ scaled norms. □

Bounds for the gradient component associated to ∂F can then be found.

LEMMA 7.9. *Let F be a face of a substructure Ω_i and $\mathbf{w} \in X_i$. If $\phi_{\partial F}$ is chosen according to Definition 7.3, then there is a constant, independent of H_i and the meshsize, such that*

$$\|\nabla \phi_{\partial F}\|^2_{L^2(\Omega_i)} \leq C\,\omega^2\,(\|\mathbf{w}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}),$$
$$\|\nabla_S\,\phi_{\partial F}\|^2_{\mathbf{H}_{\perp,00}^{-1/2}(F)} \leq C\,\omega^2\,(\|\mathbf{w}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}),$$
$$\|\nabla_S\,\phi_{\partial F}\|^2_{\mathbf{H}^{-1/2}(F)} \leq C\,\omega^4\,(\|\mathbf{w}\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}).$$

Proof. The first bound is a straightforward application of Lemmas 7.4 and 7.8. The second can be then found thanks to the third trace estimate in Lemma 7.1 and the third by then using the second inverse inequality in Lemma 7.2. □

We are only left with the task of associating a tangential vector to the average $a_{\partial F}(w)$. If Ω_i is an element of a coarse triangulation, this can be done by considering the coarse, edge element basis functions in $ND(\Omega_i)$ associated to the edges $E \subset \partial F$ and associate with them the degree of freedom $a_{\partial F}(w)$:

$$\mathbf{w}_{\partial F} := a_{\partial F}(w) \sum_{E \subset \partial F} \mathbf{\Phi}_E^H =: a_{\partial F}(w)\,\mathbf{\Phi}_{\partial F}^H;$$

see Figure 7.1 for a triangular face. The generalization to the case where Ω_i is the union of coarse elements is straightforward. The following lemma relies on Lemma 7.5 and a scaling argument.

LEMMA 7.10. *Let Ω_i be a substructure and F one of its faces. There exists a constant, independent of H_i and the meshsize, such that, for $\mathbf{w} \in X_i$,*

$$\|\mathbf{w}_{\partial F}\|^2_{\mathbf{H}^{-1/2}(F)} + H_i^2\,\|\mathrm{curl}_S \mathbf{w}_{\partial F}\|^2_{H^{-1/2}(F)} \leq C\,{H_i}^2\,\omega^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)}.$$

7.3. Decompositions associated with the faces. After we have found the components associated with the edges, we are left with a tangential vector that vanishes along the the face boundary:

$$\mathbf{w}_F := \mathbf{w} - \sum_{E \subset \partial F} (\mathbf{w}_E + \nabla_S \phi_E), \quad \text{on } F. \tag{7.6}$$

Good bounds cannot be found for $\mathbf{w}_F$ since edge components are not logarithmically stable. We therefore consider a second face component associated to the better behaved decomposition related to ∂F:

$$\tilde{\mathbf{w}}_F := \mathbf{w} - (\mathbf{w}_{\partial F} + \nabla_S \phi_{\partial F}), \quad \text{on } F. \tag{7.7}$$

The following stability result is a direct consequence of Lemmas 7.9 and 7.10, and the triangle inequality.

LEMMA 7.11. *Let F be a face of $\partial\Omega_i$. Then, for $\mathbf{w} \in X_i$,*

$$\|\mathrm{curl}_S \tilde{\mathbf{w}}_F\|^2_{H^{-1/2}(F)} \leq C\,\omega^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L_2(\Omega_i)},$$
$$\|\tilde{\mathbf{w}}_F\|^2_{\mathbf{H}_{\perp}^{-1/2}(F)} \leq C\,\omega^4\,(\|\mathbf{w}\|^2_{L_2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{w}\|^2_{L_2(\Omega_i)}.)$$

Because of the H_i^2 term in front of the norm of the curl, good bounds for the Maxwell discrete harmonic extension of $\tilde{\mathbf{w}}_F$ cannot be found directly from the previous lemma. We need to further decompose $\tilde{\mathbf{w}}_F$ into a curl free and a discrete divergence free component. The following lemma is proven in Appendix B.

LEMMA 7.12. *Let F be a face of $\partial\Omega_i$ and $\tilde{\mathbf{w}}_F \in W_i$ be a tangential vector that vanishes on all faces of Ω_i except on F. There exist $\phi_F \in V^h(\Omega_i)$ and $\mathbf{u}_F \in ND^h(\Omega_i)$, that vanish on $\partial\Omega_i \setminus F$, such that*

$$\tilde{\mathbf{w}}_F = \nabla_S\, \phi_F + \mathbf{u}_F, \quad \text{on } F, \tag{7.8}$$

and

$$\begin{aligned} &\|\mathbf{curl}\,\mathbf{u}_F\|^2_{L^2(\Omega_i)} \le C\,\|\mathrm{curl}_S\,\mathbf{u}_F\|^2_{H^{-1/2}(F)} = C\,\|\mathrm{curl}_S\,\tilde{\mathbf{w}}_F\|^2_{H^{-1/2}(F)},\\ &\|\mathbf{u}_F\|^2_{L^2(\Omega_i)} \le C\,H_i^2\,\|\mathbf{curl}\,\mathbf{u}_F\|^2_{L^2(\Omega_i)}, \\ &\|\nabla\phi_F\|^2_{L^2(\Omega_i)} \le C\,(\|\tilde{\mathbf{w}}_F\|^2_{\mathbf{H}^{-1/2}(F)} + H_i^2\,\|\mathrm{curl}_S\tilde{\mathbf{w}}_F\|^2_{H^{-1/2}(F)}). \end{aligned} \tag{7.9}$$

We note that no logarithmic factors appear in the first and third bounds since we have employed the stronger norms in $\mathbf{H}^{-1/2}(F)$ and $H^{-1/2}(F)$.

7.4. Comparisons for different substructures. In this section, we relate decompositions of functions associated to neighboring substructures. We recall that for Algorithm A only the edge averages $a_E(\mathbf{w})$ are continuous and that for Algorithms B and C also $< \phi_E >_E$ are.

We first need to relate the two decompositions (7.3) and (7.5) for the tangential component along a face boundary. The relation is given by the following lemma.

LEMMA 7.13. *Let Ω_i be a substructure and $\mathbf{w} \in W_i$. Then,*

$$|\partial F|\, a_{\partial F}(w) = \sum_{E\subset\partial F} |E|\, a_E(w).$$

On every edge E, we have

$$\phi'_{\partial F}(s) = \phi'_E(s) + (a_E(w) - a_{\partial F}(w)), \quad s \in [0, |E|],$$

and, if in addition E is the edge of Definition 7.3,

$$\phi_{\partial F}(s) = (\phi_E(s) - < \phi_E >_E) + (a_E(w) - a_{\partial F}(w))\,(s - |E|/2), \quad s \in [0, |E|].$$

Proof. The first and second equalities are an immediate consequence of the definition of average and of (7.3) and (7.5), respectively. The third is obtained by integrating over $[0, s]$,

$$\phi_{\partial F}(s) = \phi_{\partial F}(0) + \phi_E(s) + (a_E(w) - a_{\partial F}(w))\, s.$$

The constant $\phi_{\partial F}(0)$ is then found by imposing that $\phi_{\partial F}(s)$ has a zero mean value along E. □

We have the following corollary.

COROLLARY 7.14. *Let F be a face shared by two substructures Ω_i and Ω_j and E an edge of F. Then, for Algorithms A, B, and C, if $\mathbf{w} \in \widetilde{W}$,*

$$a_E(\mathbf{w}^{(i)}) = a_E(\mathbf{w}^{(j)}), \qquad a_{\partial F}(\mathbf{w}^{(i)}) = a_{\partial F}(\mathbf{w}^{(j)}).$$

We are now ready to relate the gradient contributions from different substructures.

LEMMA 7.15. *Let Ω_i and Ω_k be two substructures that share a common edge E. Assume that $E \subset \partial F_i$ and $E \subset \partial F_k$, with F_i and F_k faces of Ω_i and Ω_k, respectively, that do not necessarily coincide. Let $\mathbf{w} \in \widetilde{W}$, with local components $\mathbf{w}^{(i)} \in W_i$ and $\mathbf{w}^{(k)} \in W_k$. If the edge E is that of Definition 7.3, then for Algorithms A, B, and C and $s \in [0, |E|]$,*

$$\begin{aligned} \phi_E^{(i)} - \phi_E^{(k)} &= (\phi_{\partial F_i}^{(i)} - \phi_{\partial F_k}^{(k)}) + (<\phi_E^{(i)}>_E - <\phi_E^{(k)}>_E) \\ &+ (a_{\partial F_i}(w^{(i)}) - a_{\partial F_k}(w^{(k)}))\,(s - |E|/2). \end{aligned}$$

In particular, if the two substructures share the face $F = F_i = F_k$, then

$$\phi_E^{(i)} - \phi_E^{(k)} = (\phi_{\partial F}^{(i)} - \phi_{\partial F}^{(k)}) + (<\phi_E^{(i)}>_E - <\phi_E^{(k)}>_E).$$

Proof. The result is a direct consequence of Lemma 7.13 and Corollary 7.14. □

We also have a comparison result for face components. It employs the continuous, piecewise trilinear function $\theta_{\partial F}$ that is identically one on ∂F and vanishes at all the nodes inside F.

LEMMA 7.16. *Let Ω_i and Ω_j be two substructures that share a common face F. Let in addition $\mathbf{w} \in \widetilde{W}$, with local components $\mathbf{w}^{(i)} \in W_i$ and $\mathbf{w}^{(k)} \in W_k$. For Algorithms A, B, and C, we have, on F,*

$$\mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)} = \tilde{\mathbf{w}}_F^{(i)} - \tilde{\mathbf{w}}_F^{(j)} - (<\phi_E^{(i)}>_E - <\phi_E^{(j)}>_E)\,\nabla_S \theta_{\partial F},$$

with E the edge of Definition 7.3.

Proof. The proof relies on (7.6) and (7.7). We have,

$$\begin{aligned} \mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)} &= (\tilde{\mathbf{w}}_F^{(i)} - \tilde{\mathbf{w}}_F^{(j)}) + (\mathbf{w}_{\partial F}^{(i)} - \mathbf{w}_{\partial F}^{(j)}) - \sum_{E \subset \partial F} (\mathbf{w}_E^{(i)} - \mathbf{w}_E^{(j)}) \\ &- \sum_{E \subset \partial F} (\nabla_S \phi_E^{(i)} - \nabla_S \phi_E^{(j)}) + (\nabla_S \phi_{\partial F}^{(i)} - \nabla_S \phi_{\partial F}^{(j)}). \end{aligned}$$

Corollary 7.14 implies that the second and third term on the right hand side vanish and thus

$$\begin{aligned} \mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)} &= (\tilde{\mathbf{w}}_F^{(i)} - \tilde{\mathbf{w}}_F^{(j)}) + \nabla_S r, \\ r &= (\phi_{\partial F}^{(i)} - \sum_{E \subset \partial F} \phi_E^{(i)}) - (\phi_{\partial F}^{(j)} - \sum_{E \subset \partial F} \phi_E^{(j)}). \end{aligned} \tag{7.10}$$

We now give a closer look to the remainder $\nabla_S r$. We first note that r vanishes at all the nodes inside F. We next consider the tangential component along ∂F. Using Lemma 7.13, we find, along ∂F,

$$\nabla_S r \cdot \mathbf{t}_{\partial F} = \sum_{E \subset \partial F} (a_E(w^{(i)}) - a_E(w^{(j)}))\,\Phi_E - (a_{\partial F}(w^{(i)}) - a_{\partial F}(w^{(j)})),$$

with Φ_E the characteristic function of E. Corollary 7.14 implies that the right hand side is zero and r is therefore constant on ∂F. In order to find this constant value, we find the mean value $<r>_E$ with E the edge of Definition 7.3. We have

$$r = <r>_E = -(<\phi_E^{(i)}>_E - <\phi_E^{(j)}>_E), \quad \text{on } F,$$

which concludes the proof. □

We stress that the differences between components over different substructures depend on the averages $< \phi_E >_E$, which are continuous for Algorithms B and C, or the averages $a_{\partial F}(w)$ for which good bounds have been found in Lemma 7.5.

8. Main result. The FETI operator and the preconditioner are invertible for the three algorithms considered. The proof is the same as that of [30, Lem. 4.5] and is a direct consequence of the invertibility of the local bilinear forms on the subdomains.

LEMMA 8.1. *The Schur complement $\widetilde{S}$ and the preconditioner M^{-1} for Algorithms A, B, and C are invertible.*

We are now ready to prove our main result, which is a logarithmic bound for the condition number of Algorithms B and C, independent of the coefficient jumps; see Theorem 4.2.

LEMMA 8.2 (Algorithms B and C). *For Algorithms B and C, there is a constant, independent of h, H, and the coefficients A and B, such that, for $\mathbf{w}_\Delta \in \widetilde{W}_\Delta$,*

$$|P_\Delta \mathbf{w}_\Delta|_S^2 \leq C\,\eta\,(1+\log(H/h))^4\,|\mathbf{w}_\Delta|_{\widetilde{S}}^2,$$

where

$$\eta := \begin{cases} \max\limits_{1\leq i\leq N} \max\left\{\dfrac{\gamma_i}{\beta_i}, 1+\dfrac{H_i^2\gamma_i}{a}\right\} & \text{for Case 1,} \\ \max\limits_{1\leq i\leq N} \max\left\{\dfrac{\gamma_i}{\beta_i}, 1+\dfrac{H_i^2 b}{\beta_i}\right\} & \text{for Case 2.} \end{cases}$$

Therefore, the condition number of the corresponding preconditioned FETI operators satisfies

$$\kappa(M^{-1}F) \leq C\,\eta\,(1+\log(H/h))^4.$$

Proof. Here, we only consider the coefficient distribution of Case 1 in full detail; see (2.1). Case 2 can be dealt with in the same way. Using the minimization property in (4.3), we consider the element $\mathbf{w} = \mathbf{w}_\Delta + \mathbf{w}_\Pi$, $\mathbf{w}_\Pi \in \widehat{W}_\Pi$ such that

$$|\mathbf{w}_\Delta|_{\widetilde{S}}^2 = |\mathbf{w}|_S^2 = \sum_{i=1}^N |\mathbf{w}^{(i)}|_{S^{(i)}}^2 = \sum_{i=1}^N a_{\Omega_i}(\mathcal{H}_i\mathbf{w}^{(i)}, \mathcal{H}_i\mathbf{w}^{(i)}), \tag{8.1}$$

where for the last inequality we have used the minimization property of the discrete harmonic extension. We note that, since $\mathbf{w}_\Pi$ is continuous, Lemma 4.1 ensures

$$\mathbf{v} := P_\Delta \mathbf{w}_\Delta = P_\Delta \mathbf{w}.$$

We then need to calculate

$$|P_\Delta \mathbf{w}|_S^2 = \sum_{i=1}^N |\mathbf{v}^{(i)}|_{S^{(i)}}^2 = \sum_{i=1}^N a_{\Omega_i}(\mathcal{H}_i\mathbf{v}^{(i)}, \mathcal{H}_i\mathbf{v}^{(i)}).$$

On each subdomain Ω_i, the tangential vector $\mathbf{v}^{(i)}$ is given by formula (4.9). We recall that this formula is to be understood for vectors of degrees of freedom, which are those in the new basis of Definition 6.1. These are the averages along the subdomain edges, the nodal values that determine the gradient components along the edges, and the tangential components along fine edges that lie in the interior of the faces, and they determine the decomposition (6.1) into edge and face components. In addition, the vector of degrees of freedom $\delta_j^\dagger$ takes the same value $\delta_j^\dagger(F)$ on a subdomain face F and the same value $\delta_j^\dagger(E)$ on a subdomain edge E. It is therefore natural to decompose $\mathbf{v}^{(i)}$ into terms associated to single edges and faces; see (6.1). We obtain

$$\begin{aligned}
\mathbf{v}^{(i)} &= \sum_E \mathbf{v}_E + \sum_E \nabla_S\, \phi_E + \sum_{F_{ij}} \mathbf{v}_{F_{ij}} \\
&= \sum_E \sum_k \delta_k^\dagger(E)\, (\mathbf{w}_E^{(i)} - \mathbf{w}_E^{(k)}) + \sum_E \sum_k \delta_k^\dagger(E)\, \nabla_S\, (\phi_E^{(i)} - \phi_E^{(k)}) \\
&+ \sum_{F_{ij}} \delta_j^\dagger(F_{ij})\, (\mathbf{w}_{F_{ij}}^{(i)} - \mathbf{w}_{F_{ij}}^{(j)}),
\end{aligned} \tag{8.2}$$

where a face F_{ij} is assumed to be shared by Ω_i and Ω_j and the sums over k are taken over the subdomains Ω_k that share the edge E. Since for functions in $\widetilde{W}$ averages of the tangential component along an edge E are all equal for all the algorithms considered, the constant edge contributions $\mathbf{w}_E^{(i)}$ and $\mathbf{w}_E^{(j)}$ are equal and the corresponding sums on the right hand side of (8.2) vanish; cf. Corollary 7.14. We are therefore left with two terms, which we consider separately.

Edge terms. We first note that for Algorithm C, these terms vanish since all degrees of freedom are continuous across a coarse edge in this case. For Algorithm B, we consider an edge E and two subdomains Ω_i and Ω_k that share the edge E.

We first assume that the two substructures also share a whole face F. We consider the term

$$\delta_k^\dagger(E)\, \nabla_S\, (\phi_E^{(i)} - \phi_E^{(k)}).$$

It involves a tangential vector on the boundary of Ω_i and in order to evaluate its $|\cdot|_{S^{(i)}}$ norm, we need to evaluate the energy of its discrete Maxwell extension into Ω_i. Using the minimizing property of the Maxwell extension and Lemma 7.4, we find

$$|\delta_k^\dagger(E)\, \nabla_S\, (\phi_E^{(i)} - \phi_E^{(k)})|_{S^{(i)}}^2 \leq C\, \gamma_i\, \delta_k^\dagger(E)^2\, \|\phi_E^{(i)} - \phi_E^{(k)}\|_{L^2(E)}^2.$$

We assume that we have chosen E as the edge of Definition 7.3. Using Lemma 7.15 with $F = F_i = F_k$, and the fact that $< \phi_E^{(i)} >_E = < \phi_E^{(k)} >_E$, we can write along E,

$$\phi_E^{(i)} - \phi_E^{(k)} = \phi_{\partial F}^{(i)} - \phi_{\partial F}^{(k)}, \tag{8.3}$$

and, thus

$$\begin{aligned}
|\delta_k^\dagger(E)\, \nabla_S\, (\phi_E^{(i)} - \phi_E^{(k)})|_{S^{(i)}}^2 &\leq C\, \gamma_i\, \delta_k^\dagger(E)^2\, \|\phi_{\partial F}^{(i)} - \phi_{\partial F}^{(k)}\|_{L^2(E)}^2 \\
&\leq C\, \gamma_i\, \delta_k^\dagger(E)^2\, (\|\phi_{\partial F}^{(i)}\|_{L^2(\partial F)}^2 + \|\phi_{\partial F}^{(k)}\|_{L^2(\partial F)}^2).
\end{aligned}$$

The last two terms can then be bounded using Lemma 7.8 and inequality (4.7). For the first, we find

$$\gamma_i\, \delta_k^\dagger(E)^2\, \|\phi_{\partial F}^{(i)}\|_{L^2(\partial F)}^2 \leq C\, \gamma_i\, \omega^2\, (\|\mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 + H_i^2 \|\mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2) \leq C\, \eta\, \omega^2\, |\mathbf{w}^{(i)}|_{S^{(i)}}^2,$$

where we have used the same notation for $\mathbf{w}^{(i)} \in W_i$ and the corresponding Maxwell extension in Ω_i. A similar reasoning on Ω_k gives a bound for the second term. We have therefore found

$$|\delta_k^\dagger(E)\, \nabla_S\, (\phi_E^{(i)} - \phi_E^{(k)})|_{S^{(i)}}^2 \leq C\, \eta\, (1 + \log(H/h))^2\, (|\mathbf{w}^{(i)}|_{S^{(i)}}^2 + |\mathbf{w}^{(i)}|_{S^{(k)}}^2). \tag{8.4}$$

We now consider the case of two substructures Ω_i and Ω_k that share an edge E but not a full face. We note that in this case (8.3) does not hold. Lemma 7.15 gives

$$\phi_E^{(i)} - \phi_E^{(k)} \;=\; \phi_{\partial F_i}^{(i)} - \phi_{\partial F_k}^{(k)} + (a_{\partial F_i}(w^{(i)}) - a_{\partial F_k}(w^{(k)}))\, (s - |E|/2).$$

The first two terms on the right hand side can be bounded in the same way as before. We are left with two terms involving the face averages. Using Lemma 7.5 yields

$$\begin{aligned}
&\gamma_i\, \delta_k^\dagger(E)^2\, \|a_{\partial F_i}(w^{(i)})\, (s - |E|/2)\|_{L^2(E)}^2 \\
\leq\;& C\, \gamma_i\, |a_{\partial F_i}(w^{(i)})|^2\, \|s - |E|/2\|_{L^2(E)}^2 \\
\leq\;& C\, \gamma_i\, \omega^2 H_i^2\, \|\mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 \leq C\, \eta\, \omega^2\, |\mathbf{w}^{(i)}|_{S^{(i)}}^2 .
\end{aligned}$$

An analogous bound holds for the term corresponding to Ω_k and therefore (8.4) also holds in this case.

Face terms. We now consider the sum involving face terms on the right hand side of (8.2). We fix a face $F = F_{ij}$. Lemma 7.16 ensures

$$\mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)} = \tilde{\mathbf{w}}_F^{(i)} - \tilde{\mathbf{w}}_F^{(j)}.$$

We employ the decomposition of Lemma 7.12 and find

$$\delta_j^\dagger(F)\, (\mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)}) = \delta_j^\dagger(F)\, (\nabla_S \phi_F^{(i)} - \nabla_S \phi_F^{(j)}) + \delta_j^\dagger(F)\, (\mathbf{u}_F^{(i)} - \mathbf{u}_F^{(j)}). \tag{8.5}$$

For the gradient term $\nabla_S \phi_F^{(i)}$, Lemmas 7.12 and 7.11 ensure

$$\begin{aligned}
|\delta_j^\dagger(F)\, \nabla_S \phi_F^{(i)}|_{S^{(i)}}^2 \;&\leq\; \delta_j^\dagger(F)^2\, a_{\Omega_i}(\nabla \phi_F^{(i)}, \nabla \phi_F^{(i)}) \leq \gamma_i\, \delta_j^\dagger(F)^2\, \|\nabla \phi_F^{(i)}\|_{L^2(\Omega_i)}^2 \\
&\leq\; C\, \gamma_i\, (\|\tilde{\mathbf{w}}_F^{(i)}\|_{\mathbf{H}_\perp^{-1/2}(F)}^2 + H_i^2\, \|\mathrm{curl}_S \tilde{\mathbf{w}}_F^{(i)}\|_{H^{-1/2}(F)}^2) \\
&\leq\; C\, \gamma_i\, \omega^4\, (\|\mathbf{w}^{(i)}\|_{L_2(\Omega_i)}^2 + H_i^2\, \|\mathbf{curl}\, \mathbf{w}^{(i)}\|_{L_2(\Omega_i)}^2) \\
&\leq\; C\, \eta\, (1 + \log(H/h))^4\, |\mathbf{w}^{(i)}|_{S^{(i)}}^2 .
\end{aligned}$$

The term involving $\nabla_S \phi_F^{(j)}$ can be bounded in a similar way. We therefore find

$$|\delta_j^\dagger(F)\, (\nabla_S \phi_F^{(i)} - \nabla_S \phi_F^{(j)})|_{S^{(i)}}^2 \leq C\, \eta\, (1 + \log(H/h))^4\, (|\mathbf{w}^{(i)}|_{S^{(i)}}^2 + |\mathbf{w}^{(j)}|_{S^{(j)}}^2). \tag{8.6}$$

The second term in (8.5) is associated to the curl operator and is the sum of two contributions, which we consider separately. For the first, Lemmas 7.12 and 7.11 give

$$\begin{aligned}
|\delta_j^\dagger(F)\, \mathbf{u}_F^{(i)}|_{S^{(i)}}^2 \;&\leq\; \delta_j^\dagger(F)^2\, a_{\Omega_i}(\mathbf{u}_F^{(i)}, \mathbf{u}_F^{(i)}) \leq \delta_j^\dagger(F)^2\, (a + \gamma_i H_i^2)\, \|\mathbf{curl}\, \mathbf{u}_F^{(i)}\|_{L^2(\Omega_i)}^2 \\
&\leq\; \delta_j^\dagger(F)^2\, (a + \gamma_i H_i^2)\, \|\mathrm{curl}_S \tilde{\mathbf{w}}_F^{(i)}\|_{H^{-1/2}(F)}^2 \\
&\leq\; C\, \delta_j^\dagger(F)^2\, \omega^2\, (a + \gamma_i H_i^2)\, \|\mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 \\
&\leq\; C\, \omega^2\, (1 + (\gamma_i H_i^2)/a)\, \|a^{1/2} \mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 \\
&\leq\; C\, \omega^2\, \eta\, \|a^{1/2} \mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 \\
&\leq\; C\, \omega^2\, \eta\, (\|B^{1/2} \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2 + \|a^{1/2} \mathbf{curl}\, \mathbf{w}^{(i)}\|_{L^2(\Omega_i)}^2) \\
&=\; C\, (1 + \log(H/h))^2\, \eta\, \|\mathbf{w}^{(i)}\|_{S^{(i)}}^2 .
\end{aligned}$$

For the term $\mathbf{u}_F^{(j)}$ in (8.5), we can reason in a similar way. Using (8.6), we obtain the bound

$$|\delta_j^\dagger(F)\,(\mathbf{w}_F^{(i)} - \mathbf{w}_F^{(j)})|_{S^{(i)}}^2 \le C\,\eta\,(1+\log(H/h))^4\,(|\mathbf{w}^{(i)}|_{S^{(i)}}^2 + |\mathbf{w}^{(j)}|_{S^{(j)}}^2). \qquad (8.7)$$

The proof is concluded by combining (8.2), (8.4) and (8.7) and summing over the substructures Ω_i. □

We note that the bound in the previous lemma is not likely to be sharp. Indeed, the results in Tables 9.2 and 9.3 are consistent with a *quadratic* growth with $\log(H/h)$, which is typical of many iterative substructuring algorithms.

REMARK 8.3. *For Algorithm A, we expect the bound*

$$\kappa(M^{-1}F) \le C\,\eta\,(H/h)\,(1+\log(H/h))^q.$$

A similar analysis as that above would provide a bound with $q = 4$, while the results in Table 9.1 are consistent with $q = 2$. More precisely, when bounding the edge terms associated with the decomposition (8.2), additional terms involving the averages $<\phi_E^{(i)}>_E$ and $<\phi_E^{(k)}>_E$ need to be considered; see Lemma 7.15. Logarithmic bounds for these averages cannot be found. Analogous considerations apply to the face contributions; see Lemma 7.16. Alternatively, we could simply ask that the functions $\phi_{\partial F}$ in Definition 7.3 only vanish at one vertex for Algorithm A: this would simplify Lemmas 7.15 and 7.16 but good bounds cannot be found in Lemma 7.8 for a function that only vanishes at a point of ∂F; see Remark 7.7.

9. Numerical Results. In this section, we present some numerical results on the performance of the three algorithms proposed in this paper.

We consider the domain $\Omega = (0,1)^3$ and uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$. The coarse triangulation $\mathcal{T}_H$ consists of N^3 cubical elements, with $H = 1/N$. The fine one $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ and consists of n^3 cubical elements, with $h = 1/n$. The substructures Ω_i are chosen as the elements of $\mathcal{T}_H$. The matrices A and B are always multiples of the identity; cf. (2.1). We employ the value $\chi = 1/2$ for the definition of the scaling matrices $D_\Delta^{(i)}$; see (4.6). We consider a conjugate gradient (CG) algorithm and estimate the condition number of the preconditioned operator using the quantities provided by CG. We stop the iteration when $\|z_k\|/\|\mathbf{f}\|$ is less than 10^{-12}, where z_k is the $k-$th preconditioned residual $M^{-1}(d - F\lambda_k)$.

The purpose of these simple tests is to assess the scalability, quasioptimality, and robustness with respect to coefficient jumps of our algorithms. In addition, they show that low condition numbers and iteration counts typical of FETI methods are also found for edge element approximations and that the bounds that we derived are not sharp, ether in terms of H/h and the parameter η.

We first consider the case where both coefficients are identically one. Tables 9.1, 9.2, and 9.3 show the estimated condition number and the number of iterations, as a function of the dimensions of the fine mesh and $H/h = n/N$, for Algorithms A, B, and C, respectively. For a fixed H/h, the condition number appears to remain bounded independently of the number of fine mesh points n. As expected, for a fixed ratio H/h, the condition number and the number of iterations are quite insensitive to the dimension of the fine mesh. However, considerably higher iteration counts and condition numbers are found for Algorithm A; Algorithms B and C show the typically low condition numbers of FETI methods; see, e.g., [29, 30] for two dimensional results on edge elements. See also [31, Ch. 6]. We note that, surprisingly, the much larger coarse space of Algorithm C does not translate into a much smaller number of iterations or condition number. On the other hand, results for the two algorithms are comparable (for the case $H/h = 2$, they provide the same results). Some cases with very small values of H/h give rise to very large coarse problems for Algorithm C and could not be run. The interest for Algorithm C is that it does not require a change of basis and can be potentially attractive for p and hp edge element approximations where the required change of basis is less local.

The results for Algorithms B and C are consistent with a quadratic growth with $\log(H/h)$, hinting that the result in Lemma 8.2 is not sharp. For Algorithm A, we refer to Remark 8.3.

In order to show the necessity of a change of basis, we report in Table 9.4 the condition numbers of Algorithm B employed without performing a change of basis, for the same cases of Table 9.2. The condition numbers become very high and are consistent with a quadratic growth in $n = 1/h$, thus confirming our analysis in section 5. For $H/h = 2$, there are only two degrees of freedom along a subdomain edge and the two primal constraints enforce therefore the continuity of the edge element vectors along an edge. We remark that adding more primal constraints, such as face averages or higher order moments along the edges, does not remove this dependence in h (the results are not presented here).

TABLE 9.1
Algorithm A. *Estimated condition number and number of CG iterations (in parentheses), versus H/h and n. Case of constant, unitary coefficients A and B.*

H/h	8	6	4	3	2
n=8	-	-	7.085 (15)	-	3.134 (18)
n=16	19.38 (20)	-	8.962 (33)	-	3.179 (19)
n=24	25.72 (37)	17.39 (43)	9.306 (35)	5.879 (27)	3.184 (18)
n=32	27.45 (46)	-	9.382 (35)	-	3.186 (17)
n=40	28.01 (57)	-	9.449 (35)	-	-
n=48	28.24 (59)	18.12 (48)	9.487 (34)	5.94 (26)	-

TABLE 9.2
Algorithm B. *Estimated condition number and number of CG iterations (in parentheses), versus H/h and n. Case of constant, unitary coefficients A and B.*

H/h	8	6	4	3	2
n=8	-	-	2.213 (12)	-	1.869 (13)
n=16	3.076 (15)	-	2.742 (18)	-	1.936 (13)
n=24	3.566 (20)	3.322 (20)	2.838 (18)	2.48 (16)	1.960 (13)
n=32	3.774 (22)	-	2.899 (18)	-	1.969 (13)
n=40	3.866 (22)	-	2.926 (18)	-	-
n=48	3.951 (22)	3.537 (20)	2.944 (18)	-	-

Despite the fact that our condition number bound blows up when the coefficient A goes to zero, the algorithms remain robust in practice (results are not shown here), in exactly the same way as for many iterative substructuring algorithms for Raviart-Thomas and 2D edge element approximations; see [32, 35, 27, 29].

We now focus our attention to Algorithm B and problems with coefficient jumps. We first consider jumps in B; see Case 1 in (2.1). We choose a $4 \times 4 \times 4$ checkerboard distribution, where B assumes two values, b_1 and b_2. For a fixed value of $n = 32$,

TABLE 9.3

Algorithm C. *Estimated condition number and number of CG iterations (in parentheses), versus H/h and n. Case of constant, unitary coefficients A and B.*

H/h	8	6	4	3	2
n=8	-	-	2.211 (12)	-	1.869 (13)
n=16	3.068 (16)	-	2.741 (17)	-	1.936 (13)
n=24	3.558 (20)	3.318 (20)	2.837 (18)	2.479 (16)	1.960 (13)
n=32	3.766 (21)	-	2.899 (18)	-	1.969 (13)
n=40	3.859 (22)	-	2.929 (18)	-	-
n=48	3.945 (22)	3.536 (20)	-	-	-

TABLE 9.4

Algorithm B without change of basis. *Condition number versus H/h and n. Case of constant, unitary coefficients A and B.*

H/h	8	6	4	3	2
n=8	-	-	151.6	-	1.869
n=16	643.7	-	607.6	-	1.936
n=24	1449	1429	1365	1429	1.960
n=32	2576	-	2427	-	1.969
n=40	4024	-	3793	-	-
n=48	5795	5712	5462	-	-

$b_1 = 100$, and $a = 1$, Table 9.5 shows the estimated condition number and the number of iterations, as a function of H/h and b_2. The algorithm appears to be robust with respect to large coefficient jumps. The results for a case of jumps in the coefficient A, see Case 2 in (2.1), are shown in Table 9.6. We choose a $4 \times 4 \times 4$ checkerboard distribution, where A assumes two values, $a_1 = 0.01$ and a_2. The same conclusions as for Table 9.5 can be drawn in this case. Algorithms A and C show an analogous behaviour (the results are not presented here). We note that for Case 1 and 2, iteration counts and condition numbers remain bounded when the coefficient A goes to zero (or B is large) on some subdomains; see η in Lemma 8.2. The case when jumps are present in both coefficients remains open.

Appendix A. Proof of Lemma 7.2. We first consider the first inequality in Lemma 7.2. We need to show that, given a piecewise constant function ζ on $\partial\Omega_i$, we have

$$\|\zeta\|^2_{H^{-1/2}(F)} \leq C\,(1+\log(H/h))^2\,\|\zeta\|^2_{H^{-1/2}(\partial\Omega_i)}. \tag{A.1}$$

The proof relies on two results. The first is an inverse inequality for Sobolev norms of negative exponent:

LEMMA A.1. *Let ζ be a piecewise constant function on $\partial\Omega_i$. Given $\epsilon \in [0,1/2]$, there exists a constant, independent of ζ, ϵ, H_i and the meshsize, such that,*

$$\|\zeta\|_{H^{-1/2+\epsilon}(\partial\Omega_i)} \leq C\,h^{-\epsilon}\,\|\zeta\|_{H^{-1/2}(\partial\Omega_i)}.$$

Proof. The proof relies on the inverse inequality

$$\|\zeta\|_{L^2(\partial\Omega_i)} \leq C\,h^{-1/2}\,\|\zeta\|_{H^{-1/2}(\partial\Omega_i)},$$

TABLE 9.5

Algorithm B. *$4 \times 4 \times 4$ checkerboard distribution for b: (b_1, b_2). Estimated condition number and number of CG iterations (in parentheses), versus H/h and b_2. Case of $n = 32$, $a = 1$, and $b_1 = 100$.*

H/h	8	4	2
b2= 1e-4	10.47 (58)	7.283 (46)	4.588 (34)
b2= 1e-3	10.46 (56)	7.282 (46)	4.587 (32)
b2= 1e-2	10.45 (56)	7.278 (44)	4.586 (32)
b2= 1e-1	10.37 (52)	7.252 (42)	4.572 (30)
b2= 1	9.728 (40)	7.075 (33)	4.477 (25)
b2= 1e+1	7.219 (33)	5.696 (29)	3.739 (22)
b2= 1e+2	2.549 (16)	2.254 (14)	1.739 (11)
b2= 1e+3	6.023 (30)	3.968 (23)	3.267 (20)
b2=1e+4	8.747 (36)	3.902 (24)	2.935 (19)
b2=1e+5	11.25 (52)	3.819 (32)	1.977 (20)
b2=1e+6	11.97 (52)	3.836 (34)	1.746 (18)

TABLE 9.6

Algorithm B. *$4 \times 4 \times 4$ checkerboard distribution for a: (a_1, a_2). Estimated condition number and number of CG iterations (in parentheses), versus H/h and a_2. Case of $n = 32$, $b = 1$, and $a_1 = 0.01$.*

H/h	8	4	2
a2=1.e-7	8.152 (30)	3.722 (20)	1.896 (12)
a2=1.e-6	8.102 (30)	3.704 (20)	1.901 (12)
a2=1.e-5	7.687 (29)	3.554 (19)	1.937 (12)
a2=1.e-4	6.282 (27)	3.061 (18)	2.229 (14)
a2=1.e-3	4.707 (23)	3.015 (17)	2.255 (14)
a2=1.e-2	2.549 (16)	2.254 (14)	1.739 (11)
a2=1.e-1	4.674 (24)	3.238 (19)	2.289 (15)
a2=1	5.450 (27)	3.782 (21)	2.680 (17)
a2=1.e+1	5.545 (27)	3.851 (22)	2.702 (22)
a2=1.e+2	5.484 (32)	3.801 (28)	2.707 (22)
a2=1.e+3	5.486 (32)	3.802 (28)	2.707 (22)

see [9, Th. 4.6], which gives a bound for the identity operator $I : H^{-1/2}(\partial\Omega_i) \to L^2(\partial\Omega_i)$ restricted to the finite element space on $\partial\Omega_i$, together with the trivial estimate for $I : H^{-1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$, and an interpolation argument. □

The second result is a well-known localization result. We give details about the proof since we need an explicit bound in ϵ.

LEMMA A.2. *Let $\psi \in H^{1/2-\epsilon}(F)$, with $\epsilon \in (0, 1/2)$, and $\mathcal{E}\psi$ its extension by zero to the whole of $\partial\Omega_i$. There exists a constant, independent of ψ and ϵ, such that*

$$\|\mathcal{E}\psi\|_{H^{1/2-\epsilon}(\partial\Omega_i)} \leq C\,\epsilon^{-1}\,\|\psi\|_{H^{1/2-\epsilon}(F)}.$$

Proof. We first note that, thanks to [14, Lem. 1.3.2.6],

$$\|\mathcal{E}\psi\|^2_{H^{1/2-\epsilon}(\partial\Omega_i)} \quad \leq \quad C\,(\|\psi\|^2_{H^{1/2-\epsilon}(F)} + \|\rho^{\epsilon-1/2}\,\psi\|^2_{L^2(F)}),$$

with $\rho(x)$ the distance from a point x to ∂F. Since, for the given ϵ, the space $H^{1/2-\epsilon}(F)$ coincides with that of its extensions by zero to $\partial\Omega_i$, it is enough to show

$$\|\rho^{\epsilon-1/2}\,\psi\|_{L^2(F)} \leq C\,\epsilon^{-1}\,\|\psi\|_{H^{1/2-\epsilon}(F)}.$$

This is indeed the statement of [14, Th. 1.4.4.4], with $\Omega = F$, $p = 2$, and $s = 1/2 - \epsilon$. The proof can be found there and it relies on the Hardy inequality

$$\int_0^{+\infty}\left(\frac{1}{t}\int_t^{+\infty}\psi(s)\,ds\right)^2 t^{2\alpha}\,dt \leq (\alpha - 1/2)^{-2}\int_0^{+\infty}\psi(t)^2\,t^{2\alpha}\,dt, \quad \alpha > \frac{1}{2},$$

for which the explicit dependence of the constant has been derived from [14, Pg. 28]. $\Box$

We are now ready to prove (A.1). Lemma A.2 and a duality argument yield

$$\|\zeta\|_{H^{-1/2+\epsilon}(F)} \leq C\,\epsilon^{-1}\,\|\zeta\|_{H^{-1/2+\epsilon}(\partial\Omega_i)}.$$

Using the inverse inequality of Lemma A.1 therefore gives

$$\|\zeta\|_{H^{-1/2}(F)} \leq C\,\|\zeta\|_{H^{-1/2+\epsilon}(F)} \leq C\,\epsilon^{-1}\,\|\zeta\|_{H^{-1/2+\epsilon}(\partial\Omega_i)} \leq C\,\epsilon^{-1}\,h^{-\epsilon}\,\|\zeta\|_{H^{-1/2}(\partial\Omega_i)}.$$

The proof is concluded by choosing $1/\epsilon = 1 + \log(H/h)$.

As previously mentioned, a stronger version of (A.1) was already proven in [35, Lem. 4.4]. The proof given here however has the advantage that only relies on the existence of the inverse inequality in Lemma A.1, which holds for more general finite element spaces. The proof can be directly applied for the second inequality of Lemma 7.2, involving the subspaces of $\mathbf{H}^{-1/2}(F)$ and $\mathbf{H}_{\perp}^{-1/2}(\partial\Omega_i)$ consisting of tangential, edge element vectors on $\partial\Omega_i$. More precisely, given a tangential vector $\mathbf{w}$ on $\partial\Omega_i$, we define

$$\|\mathbf{w}\|^2_{\mathbf{H}_{\perp}^{1/2}(\partial\Omega_i)} = \sum_{F'}\|\mathbf{w}\|^2_{\mathbf{H}^{1/2}(F')} + \sum_{F'}\sum_{F''\neq F'}\int_{F'}\int_{F''}\frac{|\mathbf{w}(x')\cdot\boldsymbol{\nu}\,' - \mathbf{w}(x'')\cdot\boldsymbol{\nu}\,''|^2}{|x' - x''|^3}\,dS'\,dS''; \tag{A.2}$$

see [6], where the last sum is taken over the faces F'' that share an edge with F'. Here, given two faces that share an edge, we have employed the vectors

$$\boldsymbol{\nu}\,' = \mathbf{t}_{\partial F'} \times \mathbf{n}', \quad \boldsymbol{\nu}\,'' = \mathbf{t}_{\partial F''} \times \mathbf{n}'',$$

with the outward unit normal vectors $\mathbf{n}'$ and $\mathbf{n}''$ to the two faces, and the assumption that $\mathbf{t}_{\partial F'}$ and $\mathbf{t}_{\partial F''}$ have the same direction along the common edge. We note that the last term in (A.2) scales like the $H^{1/2}$ seminorm and therefore no scaling factor involving the diameter H_i needs to be employed; see the definition of scaled norms in section 7.1. Given this norm, we can define a norm in $\mathbf{H}_{\perp}^{-1/2}(\partial\Omega_i)$ by duality. Lemmas A.1 and A.2 remain valid in this case. Finally, the third inequality of Lemma 7.2 is proven by a scaling argument.

Appendix B. Proof of Lemma 7.12. For this section we employ the lowest-order Raviart-Thomas space $RT^h(\Omega_i)$, conforming in $H(\mathrm{div}\,;\Omega_i)$, and need to recall a few results. We refer to [4, Sect. III.3.2] or [24, 2, 15] for an introduction. Here, $H(\mathrm{div}\,;\Omega_i)$ is the space of square summable vectors in Ω_i with square summable divergence and $H_\star(\mathrm{div}\,;\Omega_i)$ its subspace of vectors with vanishing normal component on $\partial\Omega_i \cap \partial\Omega$. For the reference cube $K = \hat{Q}$, we define

$$RT(K) = \mathbb{Q}_{1,0,0}(K) \otimes \mathbb{Q}_{0,1,0}(K) \otimes \mathbb{Q}_{0,0,1}(K);$$

for an affinely mapped element, the definition can be found in [4, Sect. III.3.2]. We then set

$$RT^h(\Omega_i) := \{\mathbf{u} \in \mathrm{H}_\star(\mathrm{div}\,;\Omega_i) |\ \mathbf{u}_{|_K} \in RT(K),\ K \in \mathcal{T}_h,\ K \subset \Omega_i\}.$$

We recall that for conforming finite elements in $H(\mathrm{div}\,;\Omega_i)$ the normal component across element boundaries is continuous. In addition, the normal component $\mathbf{u}\cdot\mathbf{n}$ of a vector $\mathbf{u} \in RT^h(\Omega_i)$ on $\partial\Omega_i$ is a piecewise constant function.

We next define $ND_F^h(\Omega_i)$ and $V_F^h(\Omega_i)$ as the subspaces of $ND^h(\Omega_i)$ and $V^h(\Omega_i)$, respectively, of tangential vectors and functions that vanish on $\partial\Omega_i \setminus F$. Analogously, we define $RT_F^h(\Omega_i)$ as the subspace of $RT^h(\Omega_i)$ of vectors with vanishing normal component outside F. We summarize some of the properties that we need in the following lemma.

LEMMA B.1.

1. *We have the inclusion* $\nabla V_F^h(\Omega_i) \subset ND_F^h(\Omega_i)$. *The following orthogonal decomposition is therefore well defined*
$$ND_F^h(\Omega_i) = \nabla V_F^h(\Omega_i) \oplus ND_F^h(\Omega_i)^\perp. \tag{B.1}$$
2. *We have* $\mathbf{curl}\, ND_F^h(\Omega_i) \subset RT_F^h(\Omega_i)$ *and, for every* $\mathbf{v} \in \mathbf{curl}\, ND_F^h(\Omega_i)$ *there is a unique* $\mathbf{u}^\perp \in ND_F^h(\Omega_i)^\perp$*, such that*
$$\mathbf{curl}\,\mathbf{u}^\perp = \mathbf{v}, \quad \|\mathbf{u}^\perp\|^2_{L^2(\Omega_i)} \leq C\, H_i^2\, \|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{L^2(\Omega_i)}. \tag{B.2}$$
Conversely, if $\mathbf{v} \in RT_F^h(\Omega_i)$ *has a vanishing divergence, there exists* $\mathbf{u} \in ND_F^h(\Omega_i)$*, such that* $\mathbf{v} = \mathbf{curl}\,\mathbf{u}$.
3. *Let* $\mathbf{u}$ *be the restriction of a tangential vector in* W_i *to the face* F*, such that* $\mathbf{u}\cdot\mathbf{t}_{\partial F} = 0$ *and* $\mathrm{curl}_S\,\mathbf{u} = 0$. *Then, there exists a continuous piecewise bilinear function* ϕ *on* F *that vanishes on* ∂F*, such that* $\mathbf{u} = \nabla_S\,\phi$.

For the proof, the results for the spaces $ND^h(\Omega_i)$ and $ND_0^h(\Omega_i)$ in [2, Sect. 4.1] can be straightforwardly generalized to $ND_F^h(\Omega_i)$. We note that we have excluded that a face coincides with a connected component of $\partial\Omega_i$; see section 3. For (B.2) we refer, in particular, to [2, Prop. 4.6], where, as usual, the dependence on H_i is obtained by a scaling argument. Point 3 is a two-dimensional result for a face F and can be proven as in [2, Lem. 4.3].

We next need an extension theorem, which was originally given in [35, Lem. 4.3].

LEMMA B.2. *Let* ζ *be piecewise constant function on* $\partial\Omega_i$ *with vanishing mean value. Then there exists an extension* $\mathbf{v}(\zeta) \in RT^h(\Omega_i)$*, such that* $\mathrm{div}\,\mathbf{v}(\zeta) = 0$ *and*

$$\mathbf{v}(\zeta)\cdot\mathbf{n} = \zeta, \qquad \text{on } \partial\Omega_i,$$
$$\|\mathbf{v}(\zeta)\|_{L^2(\Omega_i)} \leq C\, \|\zeta\|_{H^{-1/2}(\partial\Omega_i)},$$

with a constant that is independent of the meshsize and H_i.

We finally need a result for continuous functions.

LEMMA B.3. *Let* ϕ *be a continuous, piecewise bilinear functions on* F *that vanishes on* ∂F *and* $\mathcal{E}$ *the operator that defines the extension by zero to* $\partial\Omega_i$. *Then,*

$$\|\phi\|_{H_{00}^{1/2}(F)} = \|\mathcal{E}\phi\|_{H^{1/2}(\partial\Omega_i)} \leq C\, \|\mathcal{E}(\nabla_S\,\phi)\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)}.$$

Proof. We first recall that the surface divergence operator on $\partial\Omega_i$

$$\mathrm{div}_S : \mathbf{H}_\perp^{1/2}(\partial\Omega_i) \longrightarrow H^{-1/2}(\partial\Omega_i)$$

is defined as the adjoint of ∇_S and is continuous and surjective; see [7], Prop. 3.7 and Rem. 3.8. We can therefore write

$$\|\mathcal{E}\phi\|_{H^{1/2}(\partial\Omega_i)} = \sup_{\psi\in H^{-1/2}(\partial\Omega_i)} \frac{<\psi, \mathcal{E}\phi>}{\|\psi\|_{H^{-1/2}(\partial\Omega_i)}} \leq C \sup_{\mathbf{v}\in \mathbf{H}_\perp^{1/2}(\partial\Omega_i)} \frac{<\mathrm{div}_S\,\mathbf{v}, \mathcal{E}\phi>}{\|\mathbf{v}\|_{\mathbf{H}_\perp^{1/2}(\partial\Omega_i)}},$$

with $<\cdot,\cdot>$ the obvious duality pairing. Using the definition of div_S as adjoint of ∇_S and the fact that

$$\nabla_S\,(\mathcal{E}\phi) = \mathcal{E}(\nabla_S\,\phi),$$

completes the proof. □

We are now ready to prove Lemma 7.12. Some of the ideas of the proof rely on [34]. We first consider the curl of the tangential vector $\tilde{\mathbf{w}}_F$:

$$\zeta_F = \mathrm{curl}_S\,\tilde{\mathbf{w}}_F,$$

and note that, since $\tilde{\mathbf{w}}_F$ has a vanishing tangential component along ∂F, ζ_F has mean value zero on F. We then consider the extension by zero $\zeta = \mathcal{E}(\zeta_F)$ to $\partial\Omega_i$. This extension satisfies the assumptions of Lemma B.2. We can therefore define the divergence-free extension

$$\mathbf{v}_F = \mathbf{v}(\zeta),$$

in Ω_i. Since $\mathbf{v}_F \in RT_F^h(\Omega_i)$ is divergence free, Lemma B.1, Point 2, ensures that there exists a vector $\mathbf{u}_F \in ND_F^h(\Omega_i)^\perp$, such that

$$\mathbf{curl}\,\mathbf{u}_F = \mathbf{v}_F.$$

Lemma B.2 and (7.2) thus provide the first of (7.9):

$$\|\mathbf{curl}\,\mathbf{u}_F\|_{L^2(\Omega_i)} \leq C\,\|\zeta\|_{H^{-1/2}(\partial\Omega_i)} \leq C\,\|\zeta_F\|_{H^{-1/2}(F)} = C\,\|\mathrm{curl}_S\mathbf{u}_F\|_{H^{-1/2}(F)}.$$

The second inequality of (7.9) is a consequence of (B.2).

We next restrict $\mathbf{u}_F$ to F. Since

$$\mathrm{curl}_S\,(\tilde{\mathbf{w}}_F - \mathbf{u}_F) = 0,$$

Lemma B.1, Point 3, ensures that there exists a continuous, piecewise bilinear function that vanishes on ∂F, such that (7.8) holds. We then extend ϕ_F by zero to the rest of $\partial\Omega_i$ and consider its extension to Ω_i which is Laplace discrete harmonic. A standard stability result for Laplace discrete harmonic functions and Lemma B.3 yield

$$\begin{aligned}
\|\nabla\phi_F\|_{L^2(\Omega_i)} &\leq C\,|\phi_F|_{H_{00}^{1/2}(F)} \leq C\,\|\mathcal{E}(\nabla_S\,\phi_F)\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} \\
&\leq C\,\|\mathcal{E}\tilde{\mathbf{w}}_F\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} + \|\mathcal{E}\mathbf{u}_F\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} \\
&\leq C\,\|\tilde{\mathbf{w}}_F\|_{\mathbf{H}^{-1/2}(F)} + \|\mathcal{E}\mathbf{u}_F\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)},
\end{aligned}$$

where we have used (7.2) for the last inequality. We are only left with the task of finding a bound for the term in $\mathbf{u}_F$. For this, we can use the trace estimate in Lemma 7.1, the definition of $\mathbf{u}_F$ and the two already proven bounds:

$$\begin{aligned}\|\mathcal{E}\mathbf{u}_F\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} &= \|\mathbf{u}_F\|_{\mathbf{H}_\perp^{-1/2}(\partial\Omega_i)} \leq C\left(\|\mathbf{u}_F\|^2_{L^2(\Omega_i)} + H_i^2\,\|\mathbf{curl}\,\mathbf{u}_F\|^2_{L^2(\Omega_i)}\right)\\ &\leq C\,H_i^2\,\|\mathbf{curl}\,\mathbf{u}_F\|^2_{L^2(\Omega_i)} \leq C\,H_i^2\,\|\mathrm{curl}\,_S\mathbf{u}_F\|^2_{H^{-1/2}(F)}.\end{aligned}$$

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