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Seminar für Angewandte Mathematik
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**Abstract**

We describe a deterministic Finite Element (FE) solution algorithm for a stochastic elliptic boundary value problem (sbvp), whose coefficients are assumed to be random fields with finite second moments and known, piecewise smooth two-point spatial correlation function. Separation of random and deterministic variables (parametrization of the uncertainty) is achieved via a Karhunen-Loève (KL) expansion. An $O(N \log N)$ algorithm for the computation of the KL eigenvalues is presented, based on a kernel independent Fast Multipole Method (FMM). Truncation of the KL expansion gives an $(M, 1)$ Wiener Polynomial Chaos (PC) expansion of the stochastic coefficient and is shown to lead to a high dimensional, deterministic boundary value problem (dbvp). Analyticity of its solution in the stochastic variables with sharp bounds for the domain of analyticity are used to prescribe variable stochastic polynomial degree $\underline{r} = (r_1, ..., r_M)$ in an $(M, \underline{r})$ Wiener PC expansion for the approximate solution. Pointwise error bounds for the FEM approximations of KL eigenpairs, the truncation of the KL expansion and the FE solution to the dbvp are given. Numerical examples show that $M$ depends on the spatial correlation length of the random diffusion coefficient. The variable polynomial degree $\underline{r}$ in PC-stochastic Galerkin FEM allows to handle KL expansions with $M$ up to 30 and $r_1$ up to 10 in moderate time.

**Keywords:** Stochastic partial differential equations, Stochastic finite element methods

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# 1 Introduction

The rapid development of (adaptive) algorithms, hardware and software in recent years has made the accurate numerical solution of elliptic partial differential equations a routine matter in many (but not all!) engineering applications. Despite this, results of accurate FE computations often deviate significantly from the responses of the physical system under consideration. Having eliminated the *discretization error*, and assuming (boldly, perhaps) that the *modelling error* inherent in the selected partial differential equations (PDEs) is negligible (i.e. that the adopted PDEs precisely describe the physics of the system under consideration), the gap between simulation and observation must be due to uncertainty in the input data. This is closely related to the fact that, contrary to popular belief, so far a computer can only *transform* information rather than increase it.

One task for applied mathematics and scientific computing is therefore to develop tools for "uncertainty processing", i.e. the systematic and quantitative numerical representation of uncertainty in input data and its propagation to the output of a FE simulation. This requires new developments in several areas of applied mathematics and engineering: input parameters are replaced by random variables resp. by random fields with known or estimated *statistics*, and the governing PDEs must be reformulated as stochastic PDEs. Traditional deterministic FE solutions must be reformulated to allow for randomness in input data and solution.

In the present paper, we focus on this latter aspect, i.e. the formulation, design and analysis of *deterministic* FEM solutions of stochastic elliptic PDEs by the stochastic Galerkin method.

This method has been used for some time (e.g. [12, 15] and the references there) in engineering. It is based on a Ritz projection onto a finite dimensional subspace of the probability space.

Our model problem is an elliptic diffusion problem in a domain $D$ with inhomogeneous stochastic diffusion coefficient $a(x,\omega)$ which we assume for simplicity to be isotropic.

To specify assumptions on the coefficients $a(x,\omega)$, we let $(\Omega,\Sigma,P)$ be a $\sigma-$finite probability space and $D\subset\mathbb{R}^d$ a bounded open set with Lipschitz boundary $\Gamma=\partial D$. We assume that $a\in L^\infty(D\times\Omega)$ is strictly positive, with lower and upper bound $\alpha$ and $\beta$ respectively,

$$\alpha\le a(x,\omega)\le\beta,\quad \lambda\times P-\text{a.e. } (x,\omega)\in D\times\Omega. \tag{1.1}$$

We consider the following model problem, with stochastic l.h.s.,

$$\left\{\begin{array}{rcll} -\mathrm{div}(a(x,\omega)\nabla_x u(x,\omega)) &=& f(x) & \text{in } D \\ u(x,\omega)\,|_{x\in\partial D} &=& 0 & \end{array}\right. ,\quad P-\text{a.e. } \omega\in\Omega. \tag{1.2}$$

The coefficient $a(x,\omega)$ as well as the solution $u(x,\omega)$ are random fields in $D\subset\mathbb{R}^d$, i.e. jointly measurable functions from $D\times\Omega$ to $\mathbb{R}$.

We assume that the known information about the diffusion coefficient $a$ includes its mean field and its two-point correlation, given by

$$E_a(x) := \int_\Omega a(x,\omega)\, dP(\omega) \quad \text{and} \quad C_a(x,x') := \int_\Omega a(x,\omega)a(x',\omega)\, dP(\omega), \tag{1.3}$$

i.e. that $E_a(x)$ and $C_a(x,x')$ are explicitly and exactly known (this is a rather optimistic assumption since often a functional form of $C_a(x,x')$ is postulated with a finite number of free parameters which are statistically estimated from the available data). We emphasize that we do not assume ergodicity of the random field $a(x,\omega)$.

An equivalent assumption is that the mean field $E_a$ and its covariance $V_a$ are known, since by definition,

$$V_a(x,x') := C_a(x,x') - E_a(x)E_a(x'). \tag{1.4}$$

Given this information on $a(x,\omega)$ and a known deterministic source term $f(x)$ (this could be relaxed as well, see e.g. [20, 21]) our aim is to solve (1.2).

Whereas the random field $u(x,\omega)$ is, under assumption (1.1), a mathematically well-defined object, the task 'compute $u(x,\omega)$' is less obvious to realize numerically and of limited interest in practice. In applications only certain statistics and moments of $u(x,\omega)$ are of interest, and this is also our goal of computation: *given statistics $E_a$ and $C_a$ of the data, compute statistics of the random solution $u$*, like $E_u, C_u$ or probabilistic level sets,

$$D_\varepsilon^\delta := \{x \in D \mid P(|u(x,\cdot)| > \delta) < \varepsilon\}, \tag{1.5}$$

if $\varepsilon, \delta > 0$ are given.

Note that for $C_a, V_a$ to exist, $a(x,\omega)$ must have finite second moment, and this follows immediately from (1.1).

The simplest approach to a numerical solution of (1.2) is Monte Carlo (MC) simulation. This means to generate numerous samples of $a(x,\omega)$ with prescribed statistics, solve (1.2) for each sample, and to determine the statistics of $u(x,\omega)$ from the set of solutions. Due to the generally slow convergence of MC methods, this approach requires a rather large number of 'samples', i.e. a large number of solutions of deterministic, three-dimensional boundary value problems. Conceptually, MC corresponds to a 'collocation in $\omega$'.

Perturbation methods (see e.g. [3]) to solve (1.2) represent the stochastic solution as an exponentially convergent infinite series, in which each term solves a problem with the same deterministic coefficient (that is, independent of $\omega$) but different stochastic loadings. It turns out that in order to compute exactly even the simplest statistic of $u$, namely $E_u$, one has to know the distribution function of $a(x,\cdot)$ at any $x \in D$, a very strong requirement.

Here, we develop and analyze a stochastic Galerkin method for the numerical solution of (1.2) which can be understood as Galerkin discretization in probability space. Stochastic Galerkin methods have attracted considerable attention in recent years, we mention here only [1, 2, 7, 12, 13, 15] and the references there. Unlike the 'collocation' type MC approaches, in stochastic Galerkin FEM the stochastic 'variable' $\omega$ is

discretized by an orthogonal projection with respect to the probability measure $P$ onto a finite dimensional subspace of $(\Omega, \Sigma, P)$.

The idea of reducing a stochastic partial differential equation to a high dimensional deterministic one is not new – see, e.g. [12, 13, 23, 24] and the references there. The feasibility of a stochastic Galerkin discretization of (1.2) strongly depends on the availability of a basis of $L^2(\Omega, dP)$. In numerous works [12], [13], the use of a so-called *Wiener Chaos expansion* [23] has been advocated. Here, we use the *Karhunen-Loève* expansion of the random field $a(x, \omega)$ to generate coordinates in $(\Omega, \Sigma, P)$ with certain optimality conditions for the deterministic approximation of the random solution. To realize this computationally, for general covariance kernels and in general domains, we propose a kernel independent Fast Multipole Method to compute the eigenpairs of the covariance operator for $a(x, \omega)$ in log-linear complexity per eigenpair.

The stochastic Galerkin FEM is, like the deterministic FEM, based on a variational formulation of (1.2). To define it, we introduce the Hilbert space $\mathcal{H}_0^1(D)$ of $H_0^1(D)$-valued random fields with finite second moments $\mathcal{H}_0^1(D) := L^2(\Omega, dP; H_0^1(D))$. The variational form of (1.2) reads: Find $u \in \mathcal{H}_0^1(D)$ such that for every $v \in \mathcal{H}_0^1(D)$

$$\int_\Omega \left( \int_D a(x,\omega) \nabla_x u \cdot \nabla_x v dx \right) dP(\omega) = \int_\Omega \left( \int_D f(x,\omega) v(x,\omega) dx \right) dP(\omega) \qquad (1.6)$$

Under assumption (1.1), existence and uniqueness of a solution $u$ to (1.6) follow from the Lax-Milgram Lemma.

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# 2 Karhunen-Loève (KL) expansions

To reduce (1.2) to a deterministic (albeit infinite dimensional) problem, we separate, in a sense, deterministic and stochastic variables in the coeffcient $a(x, \omega)$. The theoretical tool to this end is the so-called Karhunen-Loève (KL) expansion.

## 2.1 Properties of the Karhunen-Loève expansion

If the random diffusion coefficient $a \in L^2(D \times \Omega)$, then $V_a \in L^2(D \times D)$ and its *covariance operator*

$$V_a : L^2(D) \to L^2(D), \quad (V_a u)(x) := \int_D V_a(x, x') u(x') dx' \quad \forall u \in L^2(D) \qquad (2.1)$$

is a symmetric, non-negative and compact integral operator. It therefore has a countable sequence $(\lambda_m, \phi_m)_{m \geq 1}$ of eigenpairs with

$$\mathbb{R} \ni \lambda_m \searrow 0, \quad \text{as } m \to \infty \qquad (2.2)$$

(we assume here that the KL eigenvalues are enumerated in decreasing order of magnitude: $\lambda_1 \geq \lambda_2 \geq ... \geq 0$ with multiplicity counted).

Moreover, there exists a sequence of random variables $(X_m)_{m\geq 1}$ such that

$$\int_\Omega X_m(\omega)\,dP(\omega) = 0, \quad \int_\Omega X_n(\omega)X_m(\omega)\,dP(\omega) = \delta_{nm}, \quad \forall n,m \geq 1, \tag{2.3}$$

and $a(x,\omega)$ can be expanded in a Karhunen-Loève expansion:

$$a(x,\omega) = E_a(x) + \sum_{m\geq 1} \sqrt{\lambda_m}\phi_m(x)X_m(\omega). \tag{2.4}$$

The KL expansion converges in $L^2(D\times\Omega)$ (see [14]) due to

$$\sum_{m=1}^{\infty} \lambda_m = \int_D \int_\Omega a(x,\omega)^2 < \infty. \tag{2.5}$$

**Remark 2.1** *The convergence rate of the KL series in $L^2(D\times\Omega)$ is equal to the one of the eigenvalue sum in* (2.5) *(see also Figure 1).*

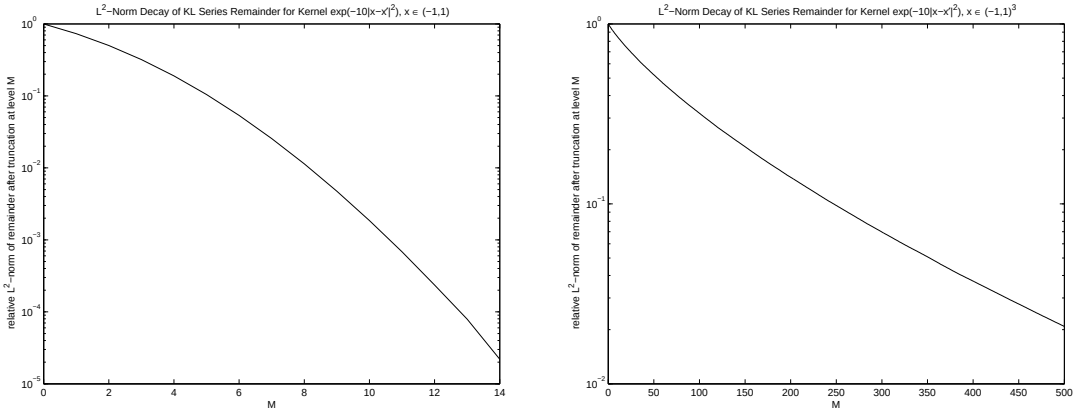


Figure 1: Convergence rates of the KL series in 1D and 3D.

If, however, the sequences $(\phi_m)_{m\geq 1}, (X_m)_{m\geq 1}$ are uniformly bounded in $L^\infty(D)$ and $L^\infty(\Omega;dP)$ respectively, and if

$$\sum_{m=1}^{\infty} \sqrt{\lambda_m} < \infty, \tag{2.6}$$

then the Karhunen-Loève expansion (2.4) converges uniformly on $D\times\Omega$.

Below, we approximate $a(x,\omega)$ by a *deterministic* function $a_M(x,\omega)$ of the first $M$ random variables $(X_m)_{1\leq m\leq M}$ by truncating the KL expansion (2.4). Since truncation of the KL expansion after $M$ terms will later be seen to lead to an $M+d$ dimensional deterministic problem, the complexity of our approach strongly depends on the size of $M$ which in turn (compare (2.6)) depends on the decay of the KL eigenvalues $\lambda_m$.

## 2.2 KL eigenvalue decay

Decay criteria for the KL eigenvalue sequence $(\lambda_m)_{m\geq 1}$ are crucial, since KL eigenvalue decay determines the stochastic regularity and the complexity of the stochastic Galerkin FEM as we shall see below.

We state decay rates for the KL eigenvalues in terms of regularity of the covariance kernel $V_a$. Roughly speaking, the smoother the covariance kernel of the coefficient, the faster the KL eigenvalue decay, with analyticity implying exponential decay and finite Sobolev regularity giving rise to algebraic decay. Remarkably, these results hold true already for *piecewise* (in the sense of the following definition) regularity of the covariance kernel. For proofs, we refer to [21] and [22].

**Definition 2.2** *If $D$ is a bounded domain of $\mathbb{R}^d$, a covariance function $V : D\times D \to \mathbb{R}$ is said to be* **piecewise analytic/smooth/$H^{p,q}$ on** *$D\times D$ if there exists a finite family $(D_j)_{1\leq j\leq J} \subset \mathbb{R}^d$ of open hypercubes such that*

$$\overline{D} \subseteq \bigcup_{j=1}^{J} \overline{D_j}, \tag{2.7}$$

*$D_j \cap D_{j'} = \emptyset, \forall j \neq j'$ and $V|_{D_j\times D_{j'}}$ has an analytic/smooth/$H^p \otimes H^q$ continuation in a neighbourhood of $\overline{D_j} \times \overline{D_{j'}}$ for any pair $(j, j')$.*

### 2.2.1 Analytic regularity

**Proposition 2.3** *Let $V \in L^2(D \times D)$ be a symmetric covariance kernel defining a self-adjoint, non-negative and compact integral operator via*

$$V : L^2(D) \to L^2(D), \quad (Vu)(x) = \int_D V(x, x')u(x')\, dx'. \tag{2.8}$$

*If $V$ is piecewise analytic on $D \times D$ in the sense of Definition 2.2, and if $(\lambda_m)_{m\geq 1}$ is the eigenvalue sequence of its associated operator (2.8), then there exist constants $c_1, c_2 > 0$ such that*

$$0 \leq \lambda_m \leq c_1 e^{-c_2 m^{1/d}}, \quad \forall m \geq 1. \tag{2.9}$$

One is often interested in Gaussian covariance kernels of the form

$$V_a(x, x') := \sigma^2 \exp(-|x - x'|^2/(\gamma^2\Lambda^2)), \quad \forall (x, x') \in D \times D, \tag{2.10}$$

where $\sigma, \gamma > 0$ are real parameters and $\Lambda$ is the diameter of the domain $D$. Note that $\sigma$ and $\gamma$ are in this case referred to as the standard deviation and the correlation length of $a$ respectively. Since this kernel admits an analytic continuation to the whole complex space $\mathbb{C}^d$, the eigenvalues decay is in this case even faster than in (2.9) (compare Figure 2).

**Proposition 2.4** *If $a \in L^2(D \times \Omega)$ and $V_a$ is given by* (2.10)*, then for the eigenvalue sequence $(\lambda_m)_{m \geq 1}$ of $V_a$ it holds*

$$0 < \lambda_m \lesssim \sigma^2 \frac{(1/\gamma)^{m^{1/d}+2}}{\Gamma(0.5m^{1/d})} \quad \forall m \geq 1. \tag{2.11}$$

Note that the decay estimate (2.11) is subexponential in dimension $d > 1$, and this is essentially due to the higher multiplicity of the eigenvalues in dimension larger than 1. To visualize this effect, we plot in Figure 2 the largest 2000 eigenvalues of the 3D factorizable kernel $V_a(x, x') = \exp(-10|x-x'|^2), x \in (-1, 1)^3$ together with an estimate obtained by dropping the (asymptotically negligible) numerator in (2.11).

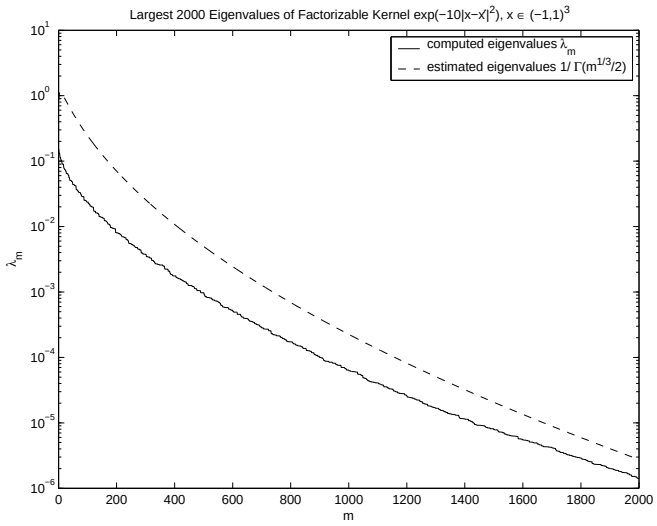


Figure 2: Eigenvalue decay for analytic covariance kernel in 3D.

### 2.2.2 Sobolev regularity

Many covariances which occur in engineering practice are piecewise analytic in the sense of Definition 2.2 (see [17]). If, however, $V(x, x')$ admits only finite Sobolev regularity, for instance if

$$V_a(x, x') := \sigma^2 \exp(-|x - x'|^{1+\delta}/(\gamma^{1+\delta}\Lambda^{1+\delta})) \quad \forall (x, x') \in D \times D, \tag{2.12}$$

for some $0 \leq \delta < 1$, algebraic decay of the KL eigenvalues can be shown (compare Figure 3).

**Proposition 2.5** *If $V \in L^2(D \times D)$ is symmetric and piecewise $H^{p,0}$ with $p \geq 0$, then for the symmetric, non-negative and compact Carleman operator defined by* (2.8) *it holds*

$$0 \leq \lambda_m \lesssim m^{-p/d} \quad \forall m \geq 1. \tag{2.13}$$

Note that in this case a small correlation length $\gamma$ can deteriorate the eigenvalue decay rate only by a multiplicative factor.
Note also that the symmetry and regularity assumptions on the covariance kernel $V$ ensure via interpolation that $V$ is piecewise $H^{p_1,p_2}, \forall p_1, p_2 \geq 0$ with $p_1 + p_2 = p$, too. For further details and typical examples of covariance kernels satisfying such regularity asumptions we refer the reader to [20], section 4.3.
For $D = (-1, 1)$ we plot in the Figure 3 the first 10 eigenvalues of the analytic kernel $V(x, x') = \exp(-|x - x'|^2)$, whose decay is well approximated by the inverse of the factorial (dashed line). The second plot shows the largest 14 eigenvalues of the kernel $V(x, x') = \exp(-|x - x'|^{1+\delta})$ for various values of $\delta$, as well as the corresponding asymptotic estimate (2.13).

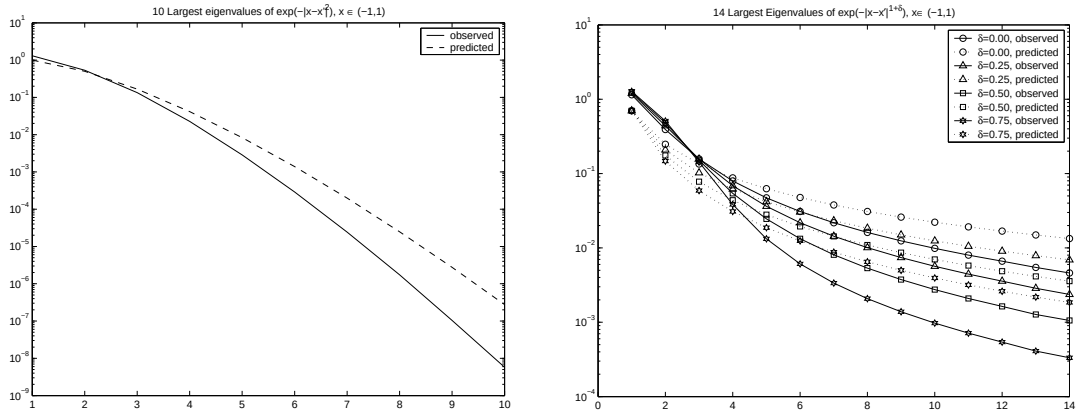


Figure 3: Eigenvalue decay in dependence on the regularity of the kernel.

## 2.3 KL eigenfunction estimates

The pointwise convergence of the KL expansion is essential for the error control in (1.2) when truncating the KL expansion after $M$ terms. So, beside criteria ensuring a fast decay of eigenvalues, we need to estimate also the eigenfunctions of the covariance kernel in the $L^\infty(D)$ norm.

**Proposition 2.6** *Let $V \in L^2(D \times D)$ be symmetric and piecewise smooth in the sense of Definition 2.2 and w.r.t. the covering* (2.7). *Denote by $(\lambda_m, \phi_m)_{m\geq 1}$ the sequence of eigenpairs of the associated covariance operator via* (2.8), *such that $\|\phi_m\|_{L^2(D)} = 1, \forall m \geq 1$. Then for any $s > 0$ and any multiindex $\alpha \in \mathbb{N}^d$ it holds, with a constant depending on $s, \alpha$ and $V$,*

$$\|\partial^\alpha \phi_m\|_{L^\infty(D_j)} \lesssim |\lambda_m|^{-s}, \quad \forall 1 \leq j \leq J, \forall m \geq 1. \tag{2.14}$$

## 3 Fast Multipole Computation of KL expansions

In order to use the (truncated) KL expansion (2.4) in practice, we must be able to compute efficiently and accurately its first $M$ eigenpairs in arbitrary domains $D$. In one dimension, for particular kernels, explicit eigenfunctions are known (see, e.g., [12]). These can be used to obtain explicit eigenpairs also for multidimensional tensor product domains $D$, if $V_a(x,x')$ is separable. This is often the case in subsurface flow problems, where $D$ is a box and the covariance kernel $V_a$ of Gaussian type (2.10).

To deal with random coefficients in arbitrary geometries, however, an efficient numerical approximation of the eigenpairs of the operator associated to the covariance kernel via (2.8) is an essential step in the efficient numerical solution of problem (1.2). Note that only the eigenpairs $(\lambda_m, u_m)$ with $\lambda_m \neq 0$ are of interest. The eigenvalue problem reads in variational form: Find $0 \neq \lambda \in \mathbb{R}$ and $0 \neq u \in L^2(D)$ such that

$$\int_{D\times D} V_a(x,x')\phi_m(x')v(x)\,dx'\,dx = \lambda_m \int_D \phi_m(x)v(x)\,dx \quad \forall v \in L^2(D). \tag{3.1}$$

Since the eigenpairs of $V_a$ are used to approximate the diffusion coefficient $a$, $L^\infty$ approximations of the eigenfunctions are needed. To compute KL eigenpairs, we use finite element (FE) discretizations of (3.1) with piecewise constants on a shape regular, quasi-uniform triangulation $\mathcal{T}_h$ of $D$ with meshwidth $h$ (which will later also be used for the FE approximation of (1.2)). To avoid technicalities, we assume that $\mathcal{T}_h$ is subordinate to the covering $\{\overline{D_j}\}_{1\le j\le J}$ of $D$, i.e. that each $\overline{D_j \cap D}$ is a finite union of closed simplices in $\mathcal{T}_h$. Let $S_h^0$ denote the FE space of discontinuous, piecewise constant functions on $\mathcal{T}_h$. The Galerkin approximation of (3.1) with the FE space $S_h^0 \subset L^2(D)$ reads: find $0 \neq \lambda_m^h, \phi_m^h \in S_h^0$ such that

$$\int_{D\times D} V_a(x,x')\phi_m^h(x')v(x)\,dy\,dx = \lambda_m^h \int_D \phi_m^h(x)v(x)\,dx \quad \forall v \in S_h^0. \tag{3.2}$$

For the eigenpair approximation error we have the following pointwise bound (see also [16]).

**Proposition 3.1** *Suppose that $a \in L^\infty(D\times\Omega)$ such that $V_a \in C^1(\overline{D}, L^2(D))$. Let $(\lambda_m, u_m)$ be an eigenpair of $V_a$ with $\lambda_m \neq 0$. Then it holds, with constants depending on $V_a$ and $m$,*

$$\begin{cases} \|\phi_m - \phi_m^h\|_{L^\infty(D)} & \lesssim & h \\ |\lambda_m - \lambda_m^h| & \lesssim & h \end{cases} \qquad \text{as } h \to 0. \tag{3.3}$$

We remark that in the proof it is not essential that the eigenfunction approximation is piecewise constant – higher order approximations can be obtained provided that $\phi_m(x)$ is sufficiently smooth piecewise in $D$ and that $S_h^0$ is replaced by $S_h^p$ (see [21] for details).

The calculation of KL eigenpairs involves the solution of the dense matrix eigenproblem corresponding to (3.1), i.e. of

$$\mathbf{V}\underline{\phi} = \lambda \mathbf{M}\underline{\phi}. \tag{3.4}$$

Here both matrices $\mathbf{V}$ and $\mathbf{M}$ are symmetric and positive definite, with $\mathbf{M}$ being diagonal if we choose as basis of $S_h^0$ the characteristic functions of the elements $K \in \mathcal{T}_h$.

For physical domains $D$ in dimension $d = 3$ and realistic meshes $\mathcal{T}_h$, the size $N$ of the eigenproblem can be as large as $10^6$ and standard eigensolvers are not applicable.

We compute eigenpairs corresponding to the largest eigenvalues by an iterative Krylov subspace eigensolver [11] which requires only matrix vector multiplies. This is feasible for large $N$ since the multiplication $\underline{\phi} \to \mathbf{V}\underline{\phi}$ can be done in $O(N \log N)$ operations and memory using a variant of the fast multipole method for general, piecewise analytic correlation kernels $V_a(x, x')$, with a cluster tree subordinate to the partition of D introduced in Definition 2.2 of the piecewise smoothness (see [18], [19]).

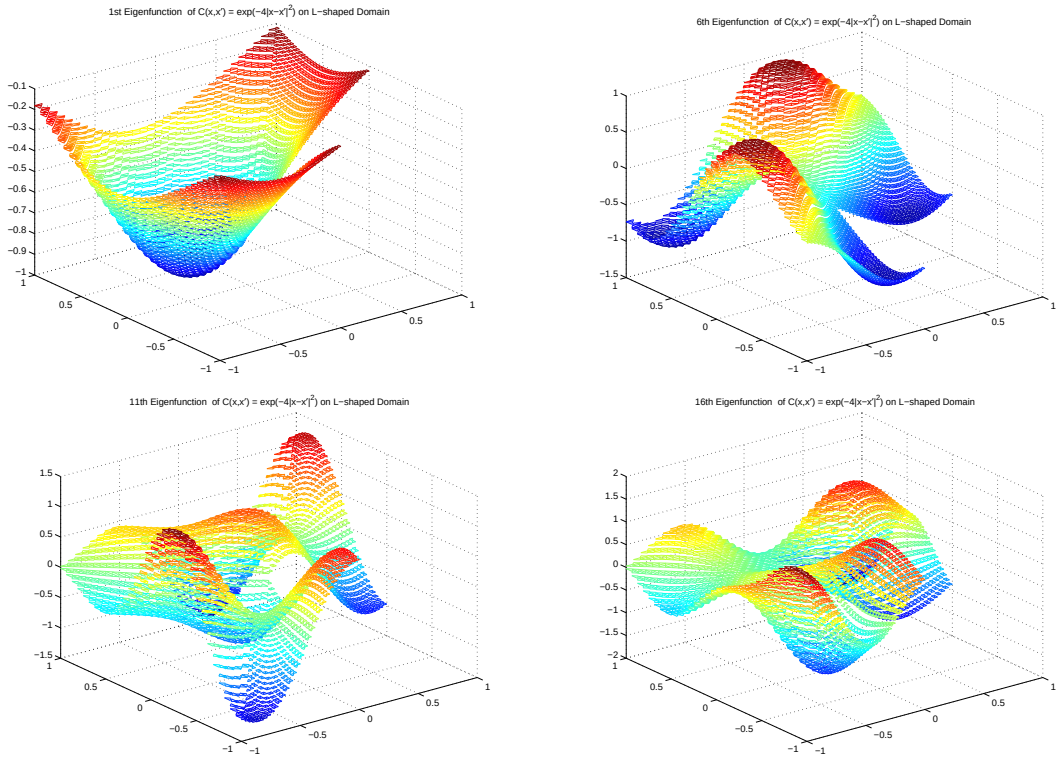


Figure 4: Eigenfunctions of the Gaussian covariance kernel on the L-shaped domain for correlation length 0.5, computed using a full matrix on a regular triangulation with 4096 elements.

# 4 Stochastic Galerkin Method

Throughout this section we assume that the diffusion coefficient $a$ satisfies (1.1) and possesses a Karhunen-Loève expansion (2.4) such that

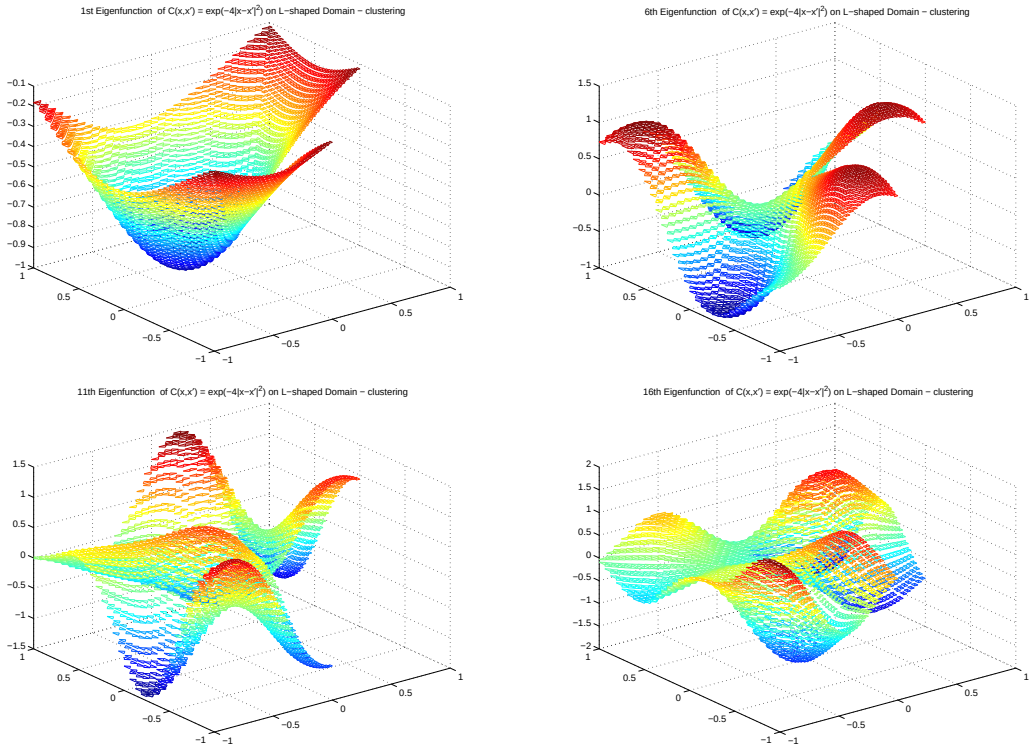


Figure 5: Eigenfunctions of the Gaussian covariance kernel on the L-shaped domain for correlation length 0.5, computed using clustering on a regular triangulation with 4096 elements.

**Assumption 4.1** *The family $X = (X_m)_{m\geq 1}$ of random variables is uniformly bounded in $L^\infty(\Omega, dP)$, i.e.*

$$\exists c_X > 0, \quad \|X_m\|_{L^\infty(\Omega,dP)} \leq c_X \in \mathbb{R}, \ \forall m \geq 1. \tag{4.1}$$

Assumption 4.1 coupled with the eigenvalue and eigenfunction estimates derived in Propositions 2.3, 2.5, 2.6 allow then a good control of the truncation error in the KL series.

## 4.1 Truncation of the KL expansion of $a$

Under Assumption 4.1 we give next sufficient conditions for the uniform convergence on $D \times \Omega$ of the KL expansion of $a$.

For any $M \in \mathbb{N}$ we define the truncated coefficient

$$a_M(x,\omega) = E_a(x) + \sum_{m=1}^{M} \sqrt{\lambda_m}\phi_m(x)X_m(\omega). \tag{4.2}$$

We have the following pointwise error estimates for the truncated coefficient in dependence on the smoothness of $a(x,\omega)$.

**Proposition 4.2** [piecewise analytic/smooth kernel] *If $V_a$ is piecewise analytic/smooth on $D \times D$ and* (4.1) *holds, then the KL expansion of $a$ converges uniformly on $D \times \Omega$ at the rate*

$$\|a - a_M\|_{L^\infty(D\times\Omega)} \lesssim \begin{cases} e^{-c_2(1/2-s)M^{1/d}} & \text{if } V_a \text{ pw analytic} \\ M^{1-p(1-s)/d} & \text{if } V_a \text{ pw smooth} \end{cases} \qquad \forall M \in \mathbb{N}, \tag{4.3}$$

*for any $s > 0, p \geq 0$ and with a constant depending on $d, s, c_1, c_2, c_X, J$.*

Note that since $p$ can be chosen arbitrarily large and $s$ arbitrarily close to 0 in (4.3), the tail of the KL series of $a$ is rapidly decaying, uniformly on $D \times \Omega$.
The well-posedness of the stochastic pde with truncated diffusion coefficient $a_M$ follows easily from Proposition 4.2.

**Proposition 4.3** *If $V_a$ is piecewise smooth on $D \times D$, then the random field $a_M(x, \omega)$ is bounded and positive for $M$ large enough, depending on $\alpha, \beta$ in* (1.1) *and $V$,*

$$\alpha/2 \leq a_M(x, \omega) \leq 2\beta, \quad \lambda \times P - a.e.\ (x, \omega) \in D \times \Omega. \tag{4.4}$$

A Strang-type argument allows the control of the error in the solution $u$.

**Proposition 4.4** [Strang] *Assuming* (1.1), *let $u$ and $u_M$ be the unique solutions in $\mathcal{H}^1_0(D) := L^2(\Omega, dP; H^1_0(D))$ of*

$$-\mathrm{div}(a(x, \omega)\nabla_x u(x, \omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D)) \tag{4.5}$$

*and*

$$-\mathrm{div}(a_M(x, \omega)\nabla_x u_M(x, \omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D)) \tag{4.6}$$

*respectivley. Then*

$$\|u - u_M\|_{\mathcal{H}^1_0(D)} \leq (2/\alpha) \cdot \|a - a_M\|_{L^\infty(D\times\Omega)} \cdot \|u\|_{\mathcal{H}^1_0(D)}. \tag{4.7}$$

Combining Propositions 4.2, 4.4, we obtain that the error due to replacing the diffusion coefficient $a$ by its truncated KL expansion $a_M$ in (1.2) is rapidly decaying as $M \to \infty$, at least in the case of piecewise analytic/smooth covariance kernel $V_a$. This is essential since the number $M$ of terms retained in the KL expansion will later on determine the deterministic dimension necessary for the stochastic Galerkin method.

**Corollary 4.5** *Consider a diffusion coefficient $a$ satisfying* (1.1) *and such that $V_a$ is piecewise analytic/smooth on $D \times D$. If $u$ and $u_M$ are the solutions of* (4.5) *and* (4.6) *respectively, then*

$$\|u - u_M\|_{\mathcal{H}^1_0(D)} \lesssim \|u\|_{\mathcal{H}^1_0(D)} \cdot \begin{cases} e^{-c_2(1/2-s)M^{1/d}} & \text{if } V_a \text{ pw analytic} \\ M^{1-p(1-s)/d} & \text{if } V_a \text{ pw smooth,} \end{cases} \tag{4.8}$$

*for $M$ large enough and any $s > 0$, $p \geq 0$.*

## 4.2 Associated deterministic problem

In this section we study the equation (4.6) obtained by truncation at level $M$ of the KL expansion of the diffusion coefficient $a$ in (1.2). Without loss of generality, we suppose in the following $c_X = 1/2$, so that for $(X_m)_{m\geq 1}$ in (2.4),

$$\text{Ran } X_m \subset I := [-1/2, 1/2], \quad \forall m \geq 1. \tag{4.9}$$

The reduction of the stochastic boundary value problem (1.2) with truncated diffusion coefficient $a_M$ to a deterministic elliptic problem will be achieved under

**Assumption 4.6** *The family $X = (X_m)_{m\geq 1}$ of random variables is independent.*

We denote by $\rho_m$ the probability measure associated to the random variable $X_m$,

$$\rho_m(\mathcal{B}) := P(X_m \in \mathcal{B}) \quad \text{for any Borel set } \mathcal{B} \subseteq I \tag{4.10}$$

and, for all $M \geq 1$, we define a probability measure on $I^M$ by

$$\rho := \rho_1 \times \rho_2 \times \ldots \times \rho_M. \tag{4.11}$$

To $a_M$ we associate the $(M, 1)$ PC expansion $\tilde{a}_M$ by

$$\tilde{a}_M : D \times I^M \to \mathbb{R}, \quad \tilde{a}_M(x, y_1, y_2, \ldots, y_M) = E_a(x) + \sum_{m=1}^{M} \sqrt{\lambda_m}\phi_m(x)y_m. \tag{4.12}$$

We consider the following deterministic elliptic problem, in variational form: Find $\tilde{u}_M \in H_0^1(D) \otimes L^2(I^M, d\rho)$ such that

$$-\text{div}(\tilde{a}_M(x, y)\nabla\tilde{u}_M(x, y)) = f(x) \quad \text{in } H^{-1}(D) \otimes L^2(I^M, d\rho). \tag{4.13}$$

The uniform ellipticity of all truncates $a_M$ ensures the well-posedness of (4.13). The solution of (4.6) can be then obtained from the solution of (4.13) by backward substitution, as follows

**Proposition 4.7** *If $\tilde{u}_M$ is the solution of* (4.13) *and $u_M$ solves* (4.6)*, then*

$$u_M(x, \omega) = \tilde{u}_M(x, X_1(\omega), X_2(\omega), \ldots, X_M(\omega)), \tag{4.14}$$

$\lambda \times P -$ *a.e.* $(x, \omega) \in D \times \Omega$.

## 4.3 Stochastic Regularity

In this section we study the deterministic elliptic problem (4.13), assuming, for simplicity, that $\rho \sim \lambda$, the Lebesgue measure on $I^M$. The solution $\tilde{u}_M$ solves then also: Find $\tilde{u}_M \in H_0^1(D) \otimes L^2(I^M)$ such that

$$-\text{div}(\tilde{a}_M(x, y)\nabla\tilde{u}_M(x, y)) = f(x) \quad \text{in } H^{-1}(D) \otimes L^2(I^M). \tag{4.15}$$

If the number $M$ of terms retained in the truncated KL expansion is large, the number of degrees of freedom necessary for the accurate solution of Problem (4.15) appears to be prohibitive. However, this is not so in general due to favourable regularity properties of the solution $\tilde{u}_M(x,y)$ with respect to $y$.

We formulate here a result on 'stochastic regularity' which will allow us below to show that the computational effort in solving (4.15) is moderate, even for large $M$.

We assume the covariance kernel $V_a$ to be piecewise analytic in the sense of Definition 2.2. Obviously, due to definition (4.12), $\tilde{a}_M(x,y)$ has, as a function of $y \in I^M$, an $L^\infty(D)$-valued analytic extension on $\mathbb{C}^M$. The solution $\tilde{u}_M$ posseses also an analytic continuation to a cylinder-neighbourhood of $I^M$ in the complex plane, as follows from the next result. To precisely describe this neighbourhood, we introduce the quantities

$$\nu_m := \sqrt{\lambda_m}\|\phi_m\|_{L^\infty(D)}, \quad \forall m \geq 1. \tag{4.16}$$

By Propositions 2.3 and 2.6, we have, for piecewise analytic $V_a(x,x')$ and any $s > 0$, the decay estimate

$$0 \leq \nu_m \lesssim e^{-c_2(1/2-s)m^{1/d}}, \quad \forall m \geq 1. \tag{4.17}$$

Due to the decay estimate (4.17), the size of the domain of analyticity of $\tilde{u}_M$ increases in each direction $y_m$ as $m$ increases from 0 to $M$. More precisely, it holds

**Proposition 4.8** *Let $\tilde{u}_M$ be the solution of* (4.13). *Then, for all $M \geq 1$, $\tilde{u}_M$ can be analytically extended, as an $H_0^1(D)$-valued function on $I^M$, to*

$$G^M := \{(z_1, z_2, \ldots, z_M) \in \mathbb{C}^M \mid \sum_{m=1}^{M} \nu_m |z_m| \leq c\}, \tag{4.18}$$

*where $c > 0$ depends on the diffusion coefficient $a$.*

## 4.4 Polynomial Chaos Discretization

The analyticity of $\tilde{u}_M$ as a function of $y$ ensures an exponential convergence rate of its finite element (FE) approximations obtained by a $p$-method w.r.t. $y$ which is the projection onto a "polynomial chaos" subspace of the probability space.
To this end we define, for $r \in \mathbb{N}$, the space of polynomials of degree at most $r$,

$$\mathcal{P}_r := \text{span}\,\{1, t, t^2, \ldots, t^r\} \subset L^2(I) \tag{4.19}$$

and, for $\mathbf{r} = (r_1, r_2, \ldots, r_M) \in \mathbb{N}^M$, an anisotropic polynomial space by

$$\mathcal{P}_{\mathbf{r}} := \mathcal{P}_{r_1} \otimes \mathcal{P}_{r_2} \otimes \cdots \otimes \mathcal{P}_{r_M} \subset L^2(I^M). \tag{4.20}$$

Further, for $\mathbf{r} \in \mathbb{N}^M$, we denote by $\tilde{u}_{M,\mathbf{r}}$ the solution of the $y$-semidiscretization of (4.15) in the subspace $H_0^1(D) \otimes \mathcal{P}_{\mathbf{r}}$: Find $\tilde{u}_{M,\mathbf{r}} \in H_0^1(D) \otimes \mathcal{P}_{\mathbf{r}}$ such that

$$\int_{I^M} \int_D \tilde{a}_M(x,y) \nabla_x \tilde{u}_{M,\mathbf{r}}(x,y) \cdot \nabla_x v(x,y)\, dx\, dy = \int_{I^M} \int_D f(x) v(x,y)\, dx\, dy, \tag{4.21}$$

$\forall v \in H_0^1(D) \otimes \mathcal{P}_{\mathbf{r}}$.
Based on the quasi-optimality of any Galerkin projection of (4.15) and on Proposition 4.8, we estimate the convergence rate of the $y$-semidiscretization of (4.15) in terms of the overall number $N_{\mathbf{r}}$ of deterministic problems to be solved, *independently of the number $M$ of terms retained in the truncated KL expansion $\tilde{a}_M(x,y)$.*

**Theorem 4.9** *Let $a \in L^\infty(D \times \Omega)$ satisfy (1.1). Suppose that $V_a$ is piecewise analytic so that (2.9) holds with two strictly positive constants $c_1, c_2$. Define*

$$r_m := \lceil M^{1/d}/m^{1/d} \rceil, \quad \forall 1 \le m \le M. \tag{4.22}$$

*Then, with constants depending only on $c_1, c_2$ and with an absolute constant $c_3$,*

$$N_{\mathbf{r}} := \dim \mathcal{P}_{\mathbf{r}} \lesssim e^{c_3 M} \tag{4.23}$$

*and*

$$\|\tilde{u}_M - \tilde{u}_{M,\mathbf{r}}\|_{H_0^1(D)\otimes L^2(I^M)} \lesssim e^{-c_2 M^{1/d}} \lesssim e^{-c_2 c_3^{-1/d} (\log N_{\mathbf{r}})^{1/d}}. \tag{4.24}$$

**Remark 4.10** *Due to the $y$-analyticity of $\tilde{u}_M$, one can show that the asymptotic error estimate (4.24) holds also in $H_0^1(D) \otimes L^\infty(I^M)$.*

**Remark 4.11** *The convergence rate (4.24) is, for $d = 1$, algebraic and, for $d > 1$, sub-algebraic, which makes the computation in the latter case rather expensive. Using different polynomial FE spaces for the $y$-discretization, the convergence rate can be improved.*

**Remark 4.12** *Theorem 4.9 assumed exact eigenpairs $(\lambda_m, \phi_m)_{1\le m\le M}$ of the KL expansion. However, a similar result also holds for computed eigenpairs $(\lambda_m^h, \phi_m^h)_{1\le m\le M}$ after choosing $M \simeq |\log h|^d$, in order to balance the KL truncation and eigenvalue discretization errors (see [22] for details).*

**Remark 4.13** *Figures 6 to 9 show strong dependence of the constant $c_2$ in (2.9), (4.3),(4.8) on spatial correlation length $\gamma$ in (2.10); (4.22) gives a sound basis for selection of variables and orders in the polynomial chaos approximation (4.31) of the random solution $u$.*

## 4.5 Adaptive Selection of Stochastic Degree

Proposition 4.9 gave an error estimate of the spectral discretization in the stochastic variable based on the assumption of piecewise analyticity of the covariance function $V_a(x,x')$ in $D \times D$ and based on the a-priori selection (4.22) of the stochastic polynomial degrees $r_m$ which is based on the stochastic regularity result Proposition 4.8.

Alternatively, one can determine numerically the polynomial degree $\mathbf{r}$, using the following Algorithm which successively identifies the coordinates $y_m$ in which the largest change in the FE solution occurs when the polynomial degree $r_m$ is increased.

**Algorithm 4.14**
*1.* **Choose** *a steering parameter* $0 < \theta \leq 1$
*2.* **Choose** *an overkill level* $K \in \mathbb{N}$ *(in practice* $K \leq 15$*)*
*3.* **Compute**

$$\tilde{u}_{M,k\cdot\mathbf{e}_m} \qquad \forall 1 \leq k \leq K, 1 \leq m \leq M,$$

*with* $(\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_M)$ *the standard basis of* $\mathbb{R}^M$
*4.* **Compute** *the decay rate of the relative error (size of domain of analyticity of* $\tilde{u}_M$*) w.r.t.* $y_k$

$$\eta_{m,k} := \frac{\|\tilde{u}_{M,k\cdot\mathbf{e}_m} - \tilde{u}_{M,(k-1)\cdot\mathbf{e}_m}\|}{\|\tilde{u}_{M,(k-1)\cdot\mathbf{e}_m}\|}$$

*for all* $1 \leq k \leq K, 1 \leq m \leq M$
*5.* **Initialize** *the polynomial degree* $\mathbf{r} := (0, 0, \ldots, 0) \in \mathbb{N}^M$
*6.* **Determine** *the 'active' stochastic dimensions*

$$\mathcal{M}_\theta := \{m \mid \eta_{m,r_m+1} \geq \theta \cdot \max_{1\leq n\leq M} \eta_{n,r_n+1}\}$$

*7.* **Compute** *the new polynomial degree (raise* $r_m$ *for all* $m \in \mathcal{M}_\theta$*)*

$$\mathbf{r}_{\text{new}} := \mathbf{r} + \sum_{m\in\mathcal{M}_\theta} \mathbf{e}_m$$

*8.* **If** $\max_m r_m < K$ *goto 6* **else** *stop*

**Remark 4.15** *One can use a simplified (algebraic) version of the Algorithm* (4.14) *to generate the adaptive polynomial degree* $\mathbf{r}$*, by replacing the stochastic pde to be solved in step 3 by a stochastic algebraic equation,*

$$(\beta_0 + \sum_{m=1}^{M} \beta_m y_m)u = 1$$

*where*

$$\beta_0 := \inf_D E_a, \quad \beta_m := \sqrt{\lambda_m}\|\phi_m\|_{L^\infty(D)} \simeq \sqrt{\lambda_m},\ m \geq 1.$$

**Remark 4.16** *Moreover, one can compute* $(\eta_{m,k})_{1\leq k\leq K}$ *only for a small value of* $K$ *and use a-priori knowledge (exponential decay in* $k$*) to predict* $\eta_{m,k}$ *for all* $k > K$ *by linear regression on* $(\log(\eta_{m,k}))_{1\leq k\leq K}$*.*

The next figures show the results obtained using the algebraic version of Algorithm 4.14 (see Remark 4.15) for different correlation lengths $\gamma$ in (2.10), on the unit square and the L-shaped domain, respectively.

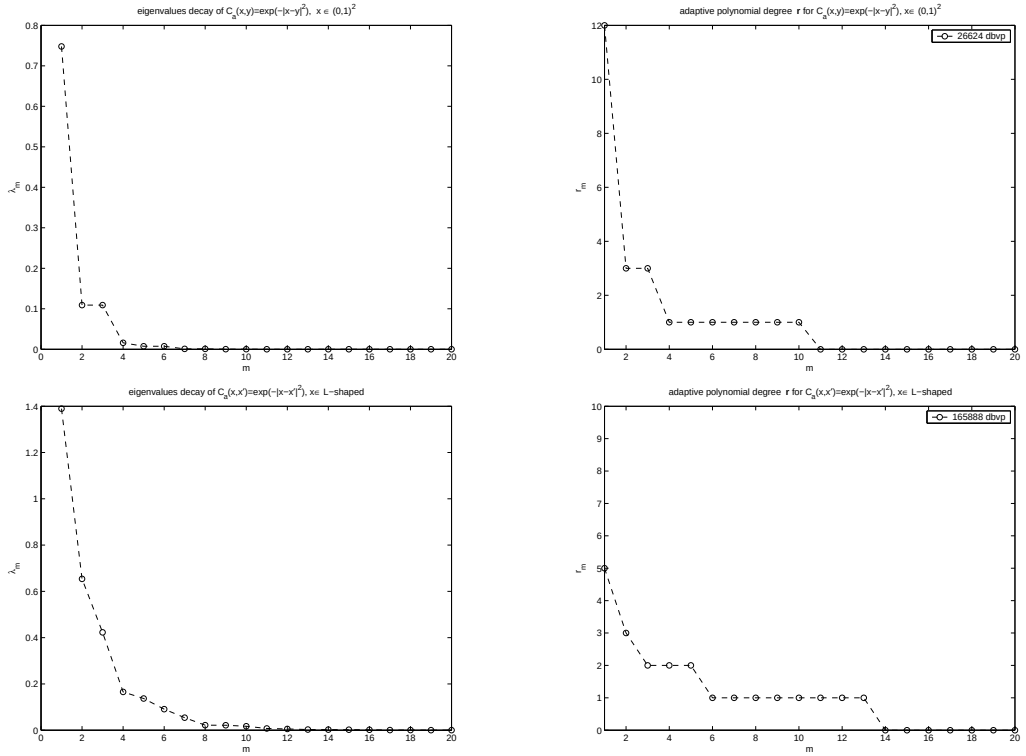


Figure 6: Exponential eigenvalue decay and adapted polynomial degree (obtained using the algebraic version of the algorithm) on the unit square and L-shaped domain for correlation length 1.

## 4.6 Algorithms

The analyticity of the covariance kernel $V_a$ can be exploited to semidiscretize (4.21) w.r.t. $y$. We show next that the solution of the semidiscrete problem (4.21) can be obtained numerically by solving a large number (depending on $\mathbf{r}$) of *independent* deterministic elliptic bvps with different data and we derive the corresponding algorithm. Consequently, to compute the solution of (4.21), one can use any of the already available deterministic solvers combined with the algorithm we derive in the following.
The semidiscretization of (4.21) w.r.t. $y$ can be done using any basis of $\mathcal{P}_{\mathbf{r}}$, but this results in general in a coupled system of deterministic elliptic bvps. There exists, however, a choice of basis which leads to a decoupled system. To describe it, let us denote, for $1 \leq m \leq M$ and $r_m \in \mathbb{N}$, by $(\mu_{j,r_m}, P_{j,r_m})_{0\leq j\leq r_m}$ the eigenpairs of the symmetric bilinear form

$$(u,v) \to \int_{-1/2}^{1/2} u(t)v(t)\, t\rho_m(t)dt \tag{4.25}$$

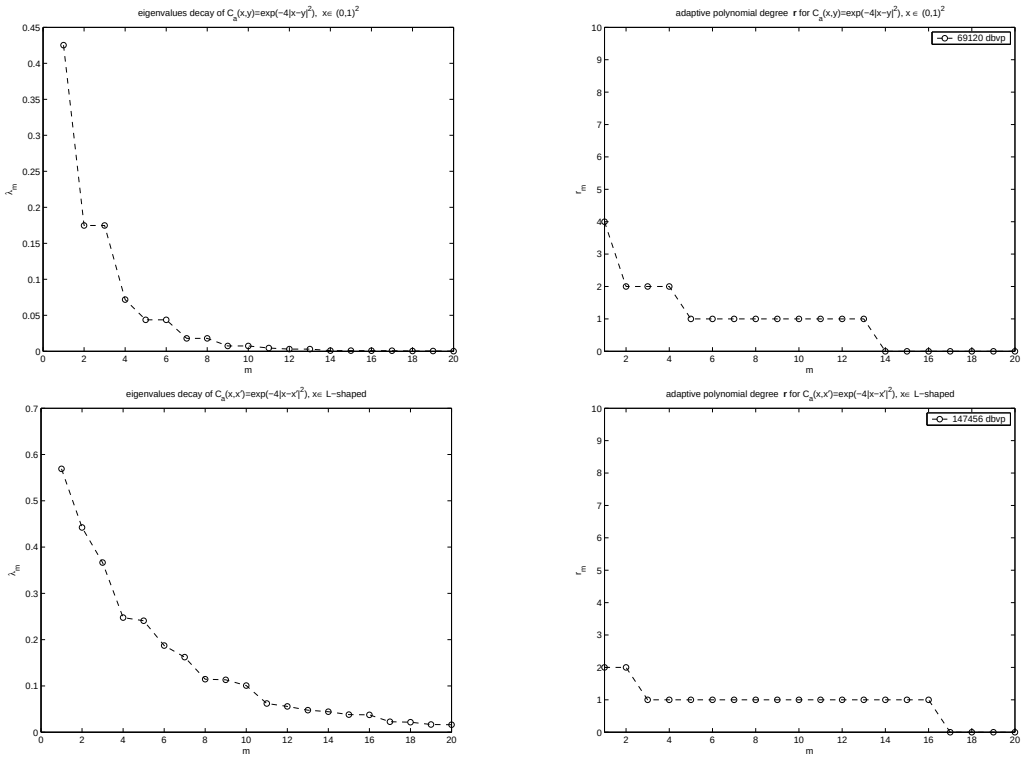


Figure 7: Exponential eigenvalue decay and adapted polynomial degree (obtained using the algebraic version of the algorithm) on the unit square and L-shaped domain for correlation length 0.5.

over $\mathcal{P}_{r_m} := \text{span}\{1, t, t^2, \ldots, t^{r_m}\}$.
For notational convenience we define on the index set $\mathbb{N}^M$ the ordering

$$\mathbf{j} \leq \mathbf{r} \iff 0 \leq j_m \leq r_m, \forall 1 \leq m \leq M, \tag{4.26}$$

$\forall \mathbf{j} = (j_1, j_2, \ldots, j_M), \mathbf{r} = (r_1, r_2, \ldots, r_M) \in \mathbb{N}^M$. We further set

$$P_{\mathbf{j},\mathbf{r}} := P_{j_1,r_1} \otimes P_{j_2,r_2} \otimes \cdots \otimes P_{j_M,r_M} \tag{4.27}$$

for $\mathbf{j} \leq \mathbf{r} \in \mathbb{N}^M$. Clearly, $P_{\mathbf{j},\mathbf{r}}$ is a polynomial in $y = (y_1, y_2, \ldots, y_M)$ and

$$\mathcal{P}_{\mathbf{r}} = \text{span}\{P_{\mathbf{j},\mathbf{r}} \mid 0 \leq j_m \leq r_m, \forall 1 \leq m \leq M\}. \tag{4.28}$$

$(P_{\mathbf{j},\mathbf{r}})_{\mathbf{j} \leq \mathbf{r}}$ is then the basis of $\mathcal{P}_{\mathbf{r}}$ we use to decouple the semidiscrete problem.
Inserting (4.12) and (4.31) in (4.21), we find

**Proposition 4.17** *For a given $\mathbf{r} \in \mathbb{N}^M$, let $\tilde{u}_{M,\mathbf{r}}$ be the solution of* (4.21). *For every multiindex $\mathbf{j} \leq \mathbf{r}$ we denote by $\tilde{u}_{M,\mathbf{j}} \in H_0^1(D)$ the solution of the deterministic diffusion problem in $D$*

$$-\text{div}(\tilde{a}_{M,\mathbf{j}} \nabla \tilde{u}_{M,\mathbf{j}}) = f_{\mathbf{j}} \quad \text{in } H^{-1}(D), \tag{4.29}$$

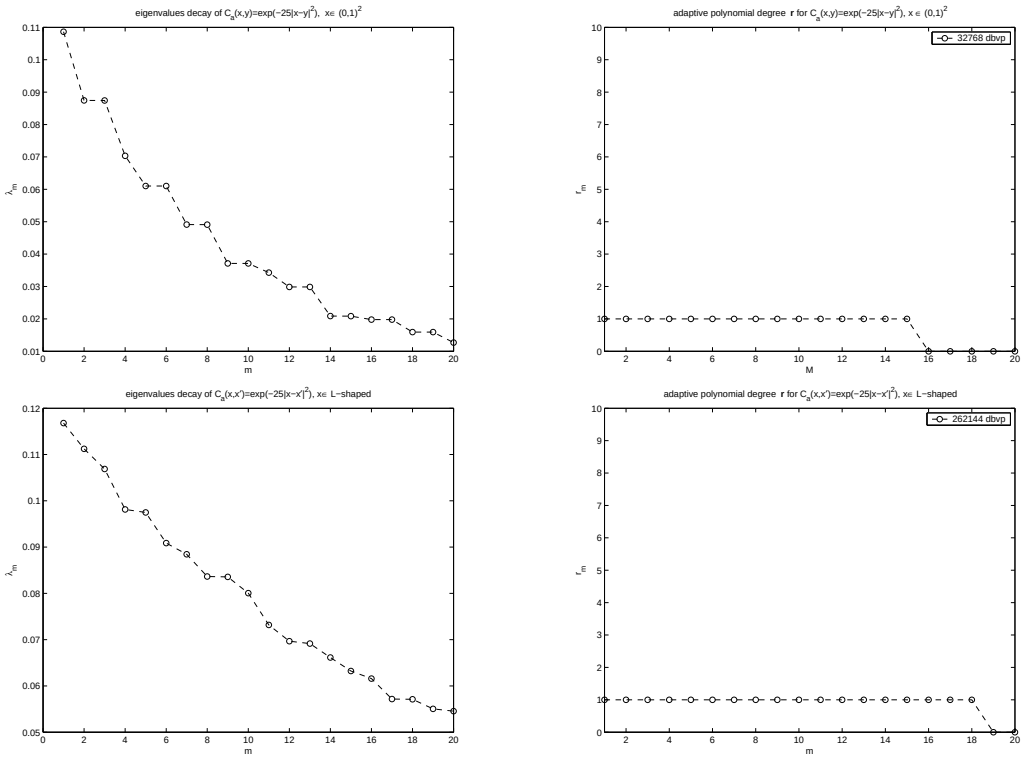


Figure 8: Exponential eigenvalue decay and adapted polynomial degree (obtained using the algebraic version of the algorithm) on the unit square and L-shaped domain for correlation length 0.2.

*with*

$$\begin{cases} \tilde{a}_{M,\mathbf{j}}(x) & := & E_a(x) + \displaystyle\sum_{m=1}^{M} \sqrt{\lambda_m} \cdot \phi_m(x) \mu_{j_m, r_m} \\ f_{\mathbf{j}}(x) & := & f(x) \cdot \displaystyle\prod_{m=1}^{M} \int_{-1/2}^{1/2} P_{j_m, r_m}(t)\, dt \end{cases} \tag{4.30}$$

*Then*

$$\tilde{u}_{M,\mathbf{r}}(x,y) = \sum_{\mathbf{j} \le \mathbf{r}} \tilde{u}_{M,\mathbf{j}}(x) P_{\mathbf{j},\mathbf{r}}(y). \tag{4.31}$$

*The semidiscretized problem* (4.21) *requires therefore the numerical solution of* $N_{\mathbf{r}}$ *(compare* (4.23)*) deterministic problems for the accuracy* (4.24)*.*

The statistics of $u_M$ solution to (4.6) can be then obtained by backward substitution, via Proposition 4.7. For the simplest statistics, the mean and the correlation, it is easy to verify

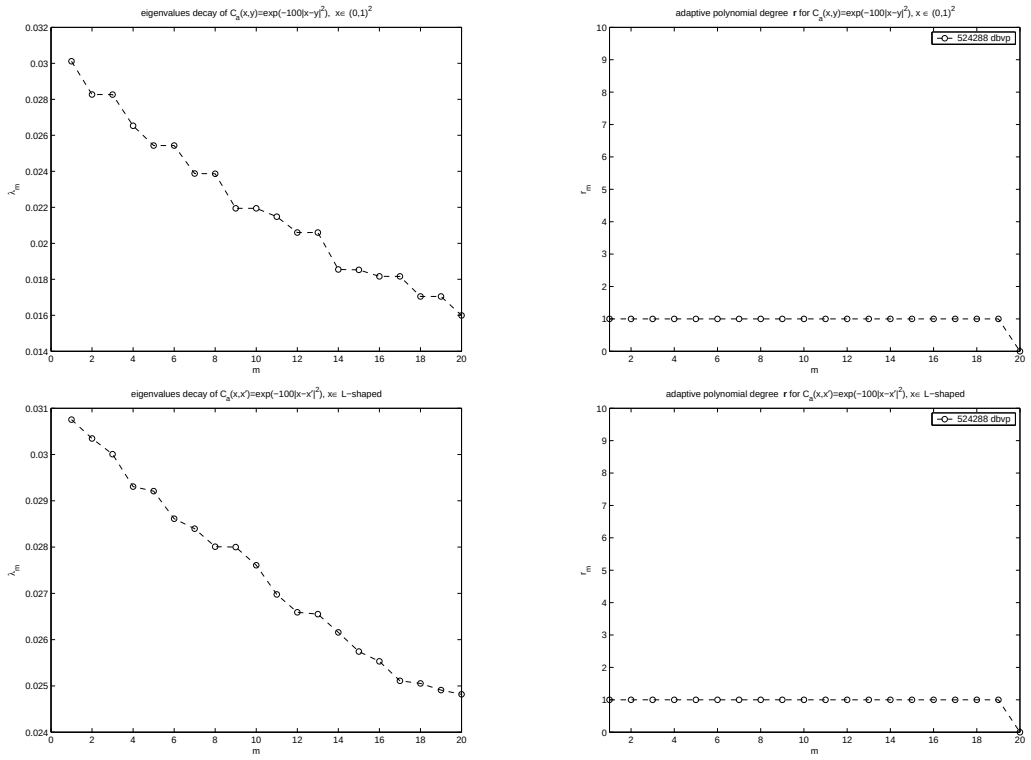


Figure 9: Exponential eigenvalue decay and adapted polynomial degree (obtained using the algebraic version of the algorithm) on the unit square and L-shaped domain for correlation length 0.1.

**Proposition 4.18** *If $u_M$ solves* (4.6) *and $\tilde{u}_{M,\mathbf{j}}$ solves* (4.29) *for all $\mathbf{j} \leq \mathbf{r}$, then*

$$E_{u_M}(x) = \sum_{\mathbf{j} \leq \mathbf{r}} \tilde{u}_{M,\mathbf{j}}(x) \prod_{m=1}^{M} \int_{-1/2}^{1/2} P_{j_m,r_m}(y_m)\, \rho_m(y_m) dy_m \tag{4.32}$$

$$C_{u_M}(x,x') = \sum_{\mathbf{j} \leq \mathbf{r}} \tilde{u}_{M,\mathbf{j}}(x) \tilde{u}_{M,\mathbf{j}}(x') \prod_{m=1}^{M} \int_{-1/2}^{1/2} \left(P_{j_m,r_m}(y_m)\right)^2 \rho_m(y_m) dy_m \tag{4.33}$$

We summarize the steps of the algorithm we developed for solving (1.2).

**Algorithm 4.19**

1. `[Computation of the deterministic part of the KL expansion of` $a$`]`
   **Given** $V_a$
   **Choose** *truncation order* $M$
   **Compute** *the first* $M$ *KL approximate eigenpairs* $(\lambda_m^h, \phi_m^h)_{1 \leq m \leq M}$ *of* $V_a$ *by generalized FMM*

2. `[Computation of the basis for PC semidiscretization]`
**Compute** *anisotropic polynomial degree* $\mathbf{r} = (r_m)_{1\le m\le M} \in \mathbb{N}^M$ *by* (4.22) *or Algorithm 4.14*
**Compute** *for* $m = 1 : M$ *eigenpairs* $(\mu_{j,r_m}, P_{j,r_m})_{0\le j\le r_m}$ *of*

$$(u, v) \to \int_{-1/2}^{1/2} u(t)v(t)\,\rho_m(t)dt \tag{4.34}$$

*in* $\mathcal{P}_{r_m} :=$ *span*$\{1, t, t^2, \ldots, t^{r_m}\}$

3. `[Semidiscretization]`
**Given** $E_a$
**Given** $f$
**Compute** $\tilde{u}^h_{M,\mathbf{j}}$ *solution of the deterministic diffusion problem in* $D$ *approximately, using FE in* $D$

$$-\mathrm{div}(\tilde{a}^h_{M,\mathbf{j}}\nabla\tilde{u}^h_{M,\mathbf{j}}) = f_{\mathbf{j}} \quad \text{in } H^{-1}(D), \tag{4.35}$$

*where*

$$\begin{cases} \tilde{a}^h_{M,\mathbf{j}}(x) &= E_a(x) + \sum_{m=1}^{M} \sqrt{\lambda^h_m}\cdot\phi^h_m(x)\mu_{j_m,r_m} \\ f_{\mathbf{j}}(x) &= f(x)\cdot\prod_{m=1}^{M}\int_{-1/2}^{1/2} P_{j_m,r_m}(t)\,dt \end{cases} \tag{4.36}$$

*and for all* $\mathbf{j} = (j_1, j_2, \ldots, j_M) \in \mathbb{N}^M$ *with* $0 \le j_m \le r_m$, $\forall 1 \le m \le M$

4. `[Post-processing]`
**Given** $(X_m)_{1\le m\le M}$
**Compute** *statistics of* $u_M(x,\omega)$ *via backward substitution*

$$u^h_{M,\mathbf{r}}(x,\omega) = \sum_{\mathbf{j}\le\mathbf{r}} \tilde{u}^h_{M,\mathbf{j}}(x)\cdot\prod_{m=1}^{M} P_{j_m,r_m}(X_m(\omega)) \tag{4.37}$$

# 5 Implementation

The stochastic Galerkin FEM described above can be implemented efficiently on parallel processors of BEOWULF (see [5]) type. There are three key computational tasks: i) computation of the KL eigenpairs (3.1), ii) determination of the orthogonal polynomial basis (4.27) and iii) the solution of the deterministic FE problems (4.29).

In our implementation, we use for i) a generalized Fast Multipole Method ([18, 19]). This step is performed serially since the complexity of the FMM is log-linear in $N$, the number of degrees of freedom in domain $D$. We are able to treat reasonably large FE meshes with several hundred thousand degrees of freedom with this serial implementation. As Ansatz functions, piecewise constants or piecewise linears are

used. The eigenproblem is solved using JDBSYM by Roman Geus and Oscar Chinellato [10, 11].

The computation of the orthogonal polynomials (4.27) amounts to solving matrix eigenvalue problems of the type $\mathbf{A}x = \lambda \mathbf{B}x$ with symmetric and positive semidefinite matrices $\mathbf{A}, \mathbf{B}$ of size $r_m$. Due to the product form (4.27) of the shape functions in the stochastic variable, only the univariate form (4.25) needs to be discretized. Since in the adaptive algorithms for the selection of the stochastic polynomial degrees the optimal degrees are apriori unknown, the univariate generalized eigenproblems need to be solved for polynomial degrees $r_m$ which could possibly occur. In our implementation this was done in MATLAB for degrees $r_m$ between 1 and 20 and univariate probability densities $\rho_m(t) = 1$, for all $1 \leq m \leq M$. The eigenpairs were then stored on disk for the main calculation.

For the deterministic FE solutions (4.29) in step iii), we assume that they are performed on the same mesh for all coefficients $\tilde{a}_{M,\mathbf{j}}$ which was also used for the eigenvalue computations (or a uniformly refined version of this mesh).[1] This is done using the C++ class library Concepts [9, 6, 8]. We use linear FEM on the same triangular mesh as for step i) . The resulting linear system is solved using a a diagonally preconditioned CG. As the deterministic FEM problems are all completely independent, they can be solved in parallel. This parallelization is achieved by a simple shell script (referenced as the *master script*) which runs on the administration node of a BEOWULF cluster.

The *master script* is given the total number of deterministic problems to be solved, the input data and the size of the blocks in which the deterministic problems should be grouped. Using this data, the master script sets up a job for every block of problems in the queueing system of the Beowulf cluster using a *job script*. [2] The queueing system calls the job script when enough CPUs on the cluster are available. As we only request one CPU per job, this is not a problem.

The *job script* sets up an archive including the input data (which is received from the master script), the FEM solver itself and the indices of the problems to be run. This archive is then sent to the allocated CPU, unpacked and run. One block of problems is solved by executing the FEM solver once. It reads in the mesh and additional data and computes othogonal polynomial basis (step ii)) at start-up. Then, all the deterministic problems are solved serially one after the other. For every problem, the system matrices and the load vector have to be computed separately. Every deterministic FEM solve generates one vector of coefficients. The resulting data of all deterministic problems in the block is again archived and sent back to the administration node of the cluster.

When all jobs generated by the master script have run, a result archive from every block should have arrived on the administration node. It is up to the queueing system to parallelize the whole process. If enough CPUs are available, all blocks are solved in

[1] Note that this assumption was made only for convenience of implementation—if an adaptive FE solver for the equation (4.29) is available, then for each $\mathbf{j}$, a different mesh adapted to the coefficient $\tilde{a}_{M,\mathbf{j}}$ could be created. This, however, would require more sophisticated postprocessing when computing mean and variances of the stochastic Galerkin solution.

[2] A typical size of a block gives a run time of an hour or less, depending on the total number of problems.

parallel. If the cluster is nearly full (only very few CPUs available at the same time), it might even happen that the whole problem is solved serially - this is the worst case.

# 6 Numerical Experiments

We present here some numerical results we have obtained using Algorithm 4.19 in a two-dimensional physical domain $D$ (compare Figure 10 and Figure 11 for $D$ the $L$-shaped domain and the unit square respectively). In each case the input data $(E_a, V_a, f)$, as well as the number of dofs in $D$ and the adaptive polynomial degree $r$ are mentioned in the caption. Note that the same physical mesh in $D$ has been used for the computation of the Karhunen-Loève expansion (step 1, Algorithm 4.19) and for the discretization of the deterministic diffusion problems (step 3, Algorithm 4.19). We have used the postprocessing formulas in Proposition 4.18 to compute the mean field and the trace of the standard deviation on $D$ of the stochastic solution. The corresponding plots are in the top left and bottom right corners of each figure respectivley. The second plot of the top row represents the solution of only one deterministic problem in $D$ with $E_a$ as diffusion coefficient. The relative difference between this solution and the mean field of the computed stochastic solution (which is due to the stochastic fluctuation of $a$) is plotted (pointwise, and set to zero on $\partial D$) in both cases in the top right corner. Note that this difference is usually significant if the correlation length is small, reaching for instance 20% in the case presented in Figure 10. The same relative difference does not exceed 4% in the second example (Figure 11) and this is a consequence of the fact that $E_a$ is large, compared to the fluctuation (since the spatial correlation length is large, only a few terms in the KL expansion are relevant). Further, plots of the min/max values of the solutions computed in step 3, Algorithm 4.19, as well as of the autocorrelation of the stochastic solution are to be found on the bottom line (left, middle) of each figure.
An overkill solution in the case of a Gaussian covariance kernel and correlation length $\gamma = 0.2$ has been computed on the unit square with $M = 17$ and $r = (1, 1, \ldots, 1) \in \mathbb{N}^{17}$ by solving $2^{17} = 131072$ deterministic diffusion problems on a triangulation of the unit square with 8192 elements. The error in the mean field between the overkill solution and solutions corresponding to $M = 3, M = 8$ are plotted in Figure 12.

# 7 Conclusion

The present work allows the following conclusions. Stochastic elliptic problems with random diffusion coefficients $a(x, \omega) \in L^2(\Omega; L^\infty(D))$ whose spatial correlations are known explicitly can be numerically solved by a deterministic Finite Element Method in the physical variable $x$ as well as in the stochastic variables. To this end, we represent the random field $a(x, \omega)$ as a Karhunen-Loève series.

A generalized Fast Multipole type algorithm is proposed which allows the Finite Element approximation of the first $M$ eigenpairs of the spatial covariance operator for $a(x, \omega)$ in log-linear complexity per eigenpair, in polyhedral domains $D \subset \mathbb{R}^d$.

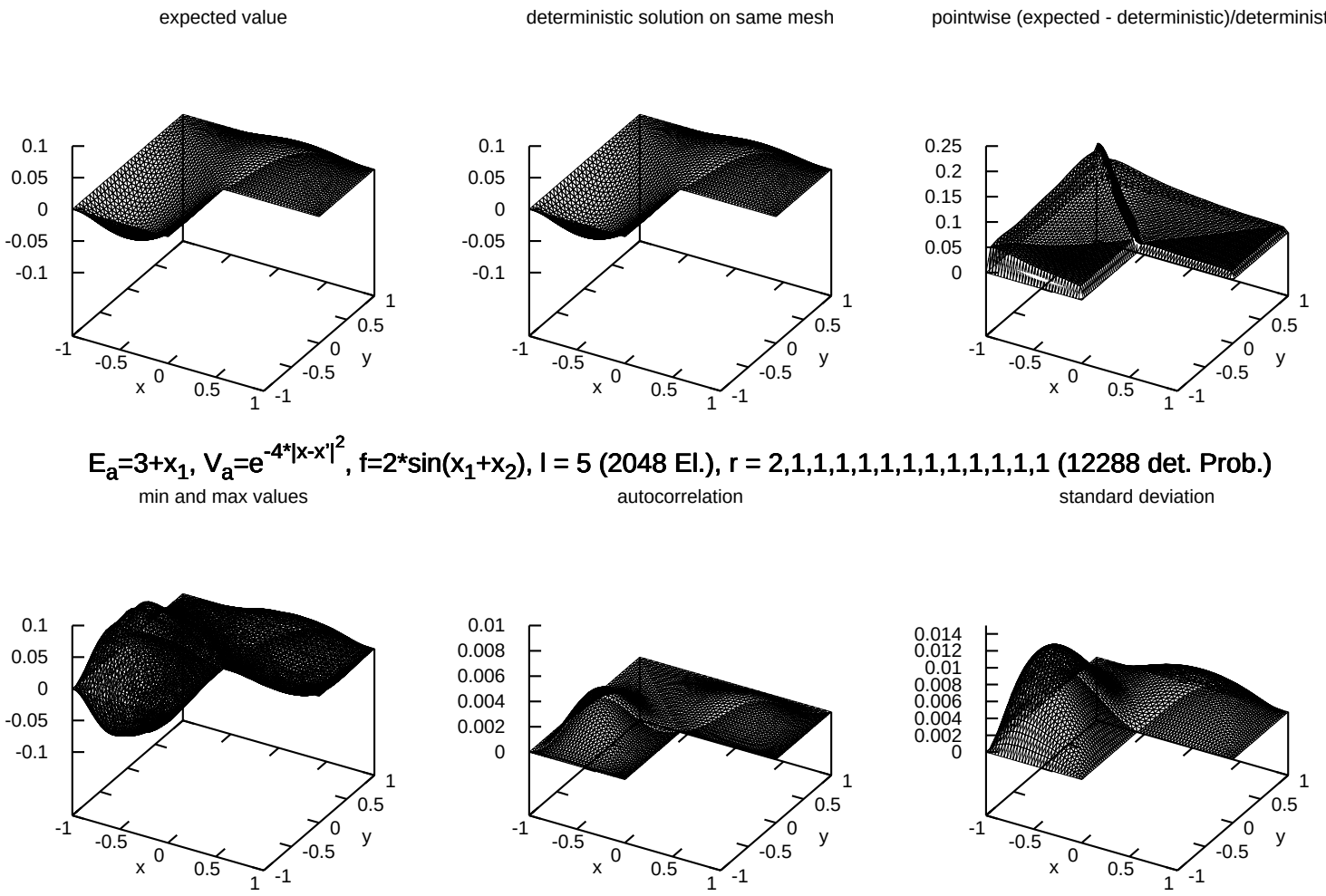


Figure 10: Statistics of the solution to sbvp on the L-shaped domain

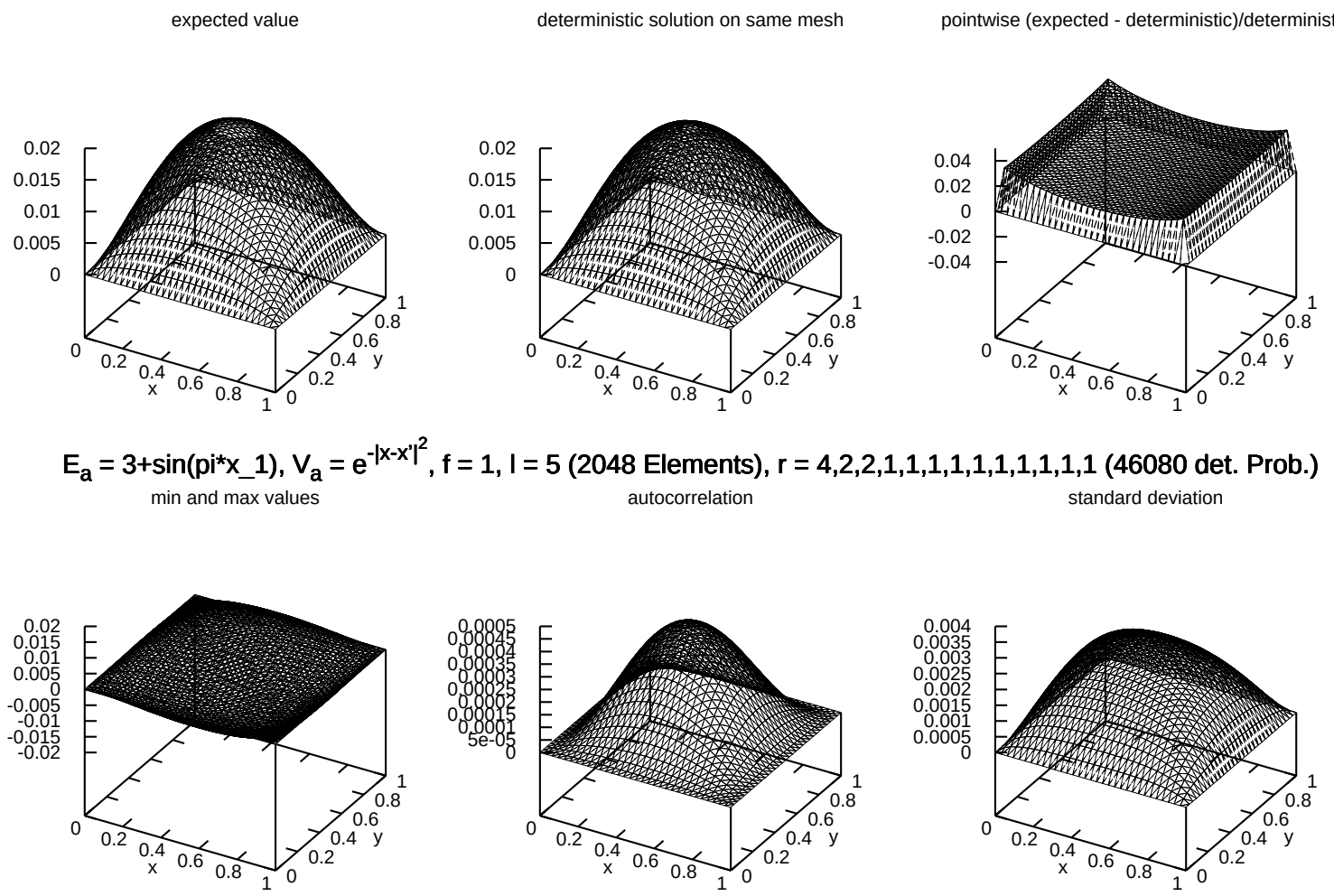


Figure 11: Statistics of the solution to sbvp on the unit square

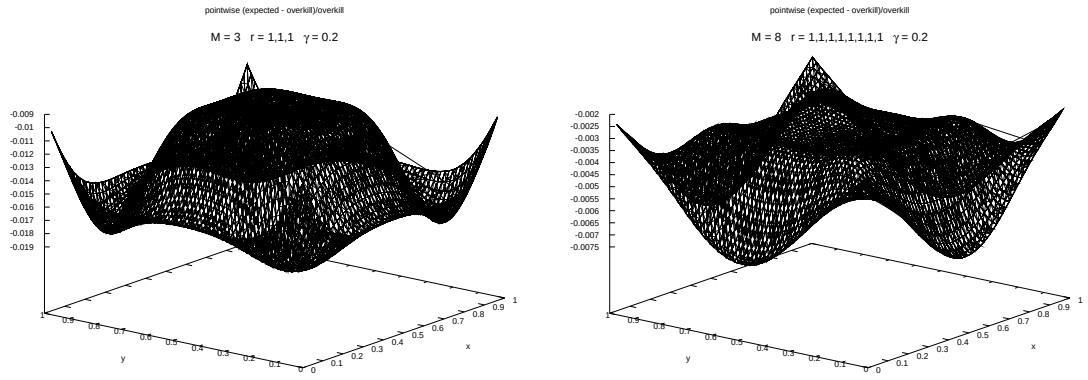


Figure 12: Error in the mean field

Adopting the independent random variables $X_m(\omega)$ in the KL expansion as new coordinates and truncating it after $M$ terms, we approximate the random diffusion coefficient by a Wiener PC expansion of degree 1 in $M$ deterministic variables $y_1, ..., y_M$.

Piecewise analyticity of the covariance kernel of $a(x,\omega)$ in $D \times D$ implies exponential decay of KL eigenvalues and exponential convergence in $M$ of the $(M,1)$ Wiener PC approximation of $a(x,\omega)$; the rate of exponential convergence is shown to depend on the spatial correlation length of $a(x,\omega)$.

Equilibration of this PC truncation error with an $O(h^p)$ $h$-version FEM error estimate allows to approximate the stochastic elliptic problem by a deterministic one in $d+M$ dimensions with error $O(h^p)$ where the number of extra, stochastic variables equals $M = O(|\log h|^d)$.

The exact solution $u_M(x,y)$ of this approximate, deterministic problem is analytic in the stochastic variables $y_m$ which suggests approximating $u_M$ by a $(M,\mathbf{r})$ PC expansion in $y$ and by a standard $h$-version FEM in $x$.

The size of the domains of analyticity of $u_M$ w.r. to $y_m$ is shown to increase rapidly with $m$ which allows to derive sharp apriori bounds on the polynomial degrees $\mathbf{r}$ necessary in the PC approximation of $u_M(x,y)$.

Orthogonalization of the PC shape functions w.r. to the probability measure is shown to decouple the linear system for the PC solution, reducing the solution complexity to that of a Monte Carlo expansion. An implementation on a BEOWOLF type architecture is described and numerical examples are presented in physical dimension $d=2$ with PC expansions of degree $r$ up to 4 in up to $M=13$ stochastic variables, for spatial correlation lenghts down to 0.1.

Mean fields and second moments of the randon solution and its derivatives are computable throughout the physical domain $D$ within minutes.

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