Robust Eigenvalue Computation for Integral Operators with Smooth Kernel [1]

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Abstract

Eigenfunction oscillations for a compact integral operator $\mathcal{K}$ on a bounded domain are investigated, in terms of the kernel regularity. The results are used to obtain robust quasi-relative Galerkin discretization error estimates for the eigenvalue problem in the case of a nonnegative $\mathcal{K}$ with smooth kernel. Both the h (for the smooth kernel) and the p (for the analytic kernel) finite element methods (FEM) are considered. As a consequence, robust trace discretization error estimates for arbitrarily small positive powers of $\mathcal{K}$ are derived.

Keywords: Integral operator, Eigenvalue computation, Galerkin FEM, Real interpolation

Subject Classification: 45C05, 65N30, 47G10, 46B70

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1 Introduction

For a bounded domain $D \subset \mathbb{R}^d$ we consider a symmetric kernel $K \in L^2(D \times D)$ defining a nonnegative compact integral operator

$$\mathcal{K} : L^2(D) \to L^2(D) \qquad (\mathcal{K}u)(x) = \int_D K(x,x')u(x')\,dx' \quad \forall x \in D. \tag{1.1}$$

Such operators arise frequently in statistics and the theory of random fields as covariance operators (a typical example is given by the gaussian kernel $K(x,x') = \exp(-|x-x'|^2)$) and the computation of their spectral decomposition (eigenelements) is relevant e.g. for an economical random field representation via the *Karhunen-Loève* expansion (see e.g. [9]). This in turn has an impact on the complexity of many practical algorithms based on the Karhunen-Loève expansion of a random field, including e.g. solving pde's with stochastic data via polynomial chaos and stochastic Galerkin methods (see also [5] and the references therein for details and further examples).

The discretization error analysis of the eigenvalue problem for $\mathcal{K}$ in the presence of a Galerkin scheme follows in general from abstract results on compact operators in Hilbert spaces (see e.g. [3], section 6.2 or [11] for similar results in Banach spaces). Denoting by $\hbar \in \mathfrak{H} \subset \mathbb{R}$ the discretization parameter, by $P_\hbar$ the $L^2(D)$ orthogonal projection onto the finite element space $S_\hbar$ and by $(\lambda_m, \phi_m)_{m\geq 1}/(\lambda_{\hbar,m}, \phi_{\hbar,m})_{m\geq 1}$ the exact/discrete eigenelements of $\mathcal{K}$, the exact asymptotics of the eigenvalue convergence rates are ([3]):

$$0 \leq \lambda_m - \lambda_{\hbar,m} = c_{\hbar,m} \|(I - P_\hbar)E^{\mathcal{K}}_{\{\lambda_m\}}\phi_{\hbar,m}\|^2_{L^2(D)}\lambda_m \quad \forall m \geq 1, \forall \hbar \in \mathfrak{H}_m, \tag{1.2}$$

where $E^{\mathcal{K}}_A$ is the spectral projector of $\mathcal{K}$ onto the Borel set $A \subset \mathbb{R}$ and the positive constant $c_{\hbar,m} \to 1$ as the finite elment space $S_\hbar$ gets refined.
Note the rather strong *asymptotic* character of (1.2): for a fixed $m \geq 1$, (1.2) holds in general only for fine enough finite element spaces depending on m ($\hbar \in \mathfrak{H}_m$) and with a constant $c_{\hbar,m}$ depending again on m, and in fact on the spectrum gap around λ_m: the smaller the gap, the larger the estimated constant $c_{\hbar,m}$ for $\hbar \notin \mathfrak{H}_m$ and the pre-asymptotic domain $\mathfrak{H} \setminus \mathfrak{H}_m$.

The purpose of this work is to provide *robust* eigenvalue convergence rates in (1.2) for the particular case under consideration, that of an integral operator with smooth kernel, if standard finite element spaces (corresponding to the h/p version of FEM) are employed. The main result is a robust quasi-relative error estimate

$$0 \leq \lambda_m - \lambda_{\hbar,m} \leq c_{K,s}\lambda_m^{1-s}\Phi_s(\hbar) \quad \forall m \geq 1, \forall \hbar \in \mathfrak{H}, \forall s > 0, \tag{1.3}$$

where the functional $\Phi_s : \mathfrak{H} \to \mathbb{R}$ qualitatively preserves the approximation rate of the finite elements used (algebraic or exponential for the h or p method respectively), but the parameters involved are allowed to depend on s.

Questions regarding possible generalizations of (1.3) to the case of a kernel K with only finite differentiable regularity, as well as similar estimates for eigenspaces (known to be more sensitive to perturbations than the eigenvalues) will be addressed in a forthcoming paper.

The article proceeds as follows. After reviewing standard eigenvalue decay rates in dependence of the kernel regularity (finite differentiability or analyticity), we investigate eigenfunction oscillations for smooth kernels, which are shown to be milder than the eigenvalue decay rate. The result is a direct consequence of the Gagliardo-Nirenberg inequalities (see e.g. [1]). A refinement of these inequalities for analytic functions, showing in particular that any interpolation space between complex analytic functions in a given neighbourhood of $\overline{D}$ and any Sobolev space on D consists of analytic functions in a smaller neighbourhood of $\overline{D}$ (Theorem 2.23), is needed in the context of the p FEM. The third section is devoted to a proof of (1.3) in an abstract setting, for an arbitrary nonnegative compact operator in a Hilbert space with a fast eigenvalue decay ($(\lambda_m)_{m\geq 1} \in \cap_{p>0}\ell_p$), if the discretization spaces used by the Galerkin scheme approximate the eigenspaces at a convenient rate (Assumption 3.2). As a direct consequence we obtain robust eigenvalue convergence rates for arbitrarily small positive powers of $\mathcal{K}$.
Concerning notations, throughout this work generic constants are denoted by c and all the quantities on which they depend are included as subscripts.

2 Properties of the Spectral Decomposition

2.1 Eigenvalue Decay

We begin with a review of eigenvalue decay rates in terms of the kernel regularity. The results are standard (see e.g. [8], [12]), following from the abstract theory of Weyl/approximation/entropy numbers and from kernel approximation (by discrete, finite rank operators). Roughly speaking, the smoother the kernel the faster the eigenvalue decay, with analyticity implying quasi-exponential decay and finite Sobolev regularity giving rise to algebraic decay.

Remarkably, all these results hold for piecewise regular kernels on product subdomains of D, in the sense of Definition 2.1 below. Note that general piecewise regularity allowing singularities on the diagonal set of $D \times D$ ensure in general only a slower eigenvalue decay (see e.g. [8] and [5] for examples with known exact eigenelements).

Definition 2.1 *If D is a bounded domain in $\mathbb{R}^d$ and $p, q \geq 0$, a measurable function $K : D \times D \to \mathbb{R}$ is said to be* **piecewise** $H^{p,q}$ **on** $D \times D$ *if there exists a finite family $\mathcal{D} = (D_j)_{j\in\mathcal{J}}$ of subdomains of D such that*

i. $D_j \cap D_{j'} = \emptyset \quad \forall j, j' \in \mathcal{J}$ *with* $j \neq j'$

ii. $D \setminus \bigcup_{j\in\mathcal{J}} D_j$ *is a null set in* $\mathbb{R}^d$

iii. $\overline{D} \subset \bigcup_{j\in\mathcal{J}} \overline{D_j}$

iv. $K|_{D_j\times D_{j'}} \in H^{p,q}(D_j \times D_{j'}) \quad \forall j, j' \in \mathcal{J}$.

We denote by $H^{p,q}_{\mathcal{D}}(D^2)$ *the space of piecewise* $H^{p,q}$ *functions on* $D \times D$ *in the sense given above.*
Moreover, if there exists also a finite family $\mathcal{G} = (G_j)_{j\in\mathcal{J}}$ *of open sets in* $\mathbb{R}^d$ *such that*

v. $\overline{D_j} \subset G_j \quad \forall j \in \mathcal{J}$

vi. $K|_{D_j\times D_{j'}}$ *has an analytic/*$H^{p,q}$ *continuation to* $G_j \times G_{j'} \quad \forall j, j' \in \mathcal{J}$,

then we say that K *is* **piecewise analytic/$H^{p,q}$ on a covering of $D \times D$** *and we denote by* $\mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)/H^{p,q}_{\mathcal{D},\mathcal{G}}(D^2)$ *the corresponding spaces.*
Similarly we introduce spaces of piecewise regular functions defined on D*, which we denote by* $\mathcal{H}^p_{\mathcal{D}}(D)$, $\mathcal{H}^p_{\mathcal{D},\mathcal{G}}(D)$ *etc.*

For analytic kernels it holds

Proposition 2.2 *Let* $K \in L^2(D \times D)$ *be a symmetric kernel defining a compact nonnegative integral operator via* (1.1)*. If* $K \in \mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)$ *and* $(\lambda_m)_{m\geq 1}$ *denotes the eigenvalue sequence of* $\mathcal{K}$*, then there exist constants* $c_{1,K}, c_{2,K} > 0$ *such that*

$$0 \leq \lambda_m \leq c_{1,K} e^{-c_{2,K} m^{1/d}} \quad \forall m \geq 1. \tag{2.1}$$

One is often interested in gaussian kernels of the form

$$K(x, x') := \sigma^2 \exp(-|x - x'|^2/(\gamma^2\Lambda^2)) \quad \forall (x, x') \in D \times D, \tag{2.2}$$

where $\sigma, \gamma > 0$ are real parameters (standard deviation, correlation length) and Λ is the diameter of the domain D. Since this kernel admits an analytic continuation to the whole complex space $\mathbb{C}^d$, the eigenvalue decay is in this case even faster than in (2.1).

Proposition 2.3 *If* $K \in L^2(D\times D)$ *is given by* (2.2)*, then for the eigenvalue sequence* $(\lambda_m)_{m\geq 1}$ *of* $\mathcal{K}$ *defined by* (1.1) *it holds*

$$0 \leq \lambda_m \leq c_{\sigma,\gamma} \frac{(1/\gamma)^{m^{1/d}}}{\Gamma(m^{1/d}/2)} \quad \forall m \geq 1. \tag{2.3}$$

Proposition 2.4 *Let* $D \subset \mathbb{R}^d$ *be a bounded domain and* $K \in L^2(D\times D)$ *be a symmetric kernel defining a compact nonnegative integral operator* $\mathcal{K}$ *via* (1.1)*. If* $K \in \mathcal{H}^{p,0}_{\mathcal{D},\mathcal{G}}(D^2)$ *for some* $p \geq 0$*, then there exists a constant* $c_K > 0$ *such that*

$$0 \leq \lambda_m \leq c_K m^{-p/d} \quad \forall m \geq 1. \tag{2.4}$$

Corollary 2.5 *Let* $D \subset \mathbb{R}^d$ *be a bounded domain and* $K \in L^2(D \times D)$ *be a symmetric kernel defining a compact nonnegative integral operator* $\mathcal{K}$ *via* (1.1)*. If* K *is piecewise smooth (i.e. piecewise* $\mathcal{H}^{p,q}_{\mathcal{D},\mathcal{G}}(D^2)\ \forall p, q \geq 0$*) on a covering of* $D \times D$ *and* $(\lambda_m)_{m\geq 1}$ *denotes the eigenvalue sequence of* $\mathcal{K}$*, then for any* $s > 0$ *there exists a constant* $c_{K,s} > 0$ *such that*

$$0 \leq \lambda_m \leq c_{K,s} m^{-s} \quad \forall m \geq 1. \tag{2.5}$$

2.2 Eigenfunction Oscillations

We show next that the smoothness assumption on the kernel allows also a good control of the eigenfunctions and their derivatives in the $L^\infty(D)$ norm. Roughly speaking, the eigenfunctions are shown to be bounded from above, asymptotically as $m \to \infty$, by any negative power of the corresponding eigenvalue. In other words, the eigenfunction oscillations are much weaker than the eigenvalue decay rate.
We start by noting that the piecewise regularity of the eigenfunctions follows from that of the kernel K.

Proposition 2.6 *If $K \in \mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)/\mathcal{H}^{p,q}_{\mathcal{D},\mathcal{G}}(D^2)$, then the eigenfunctions of $\mathcal{K}$ given by* (1.1) *corresponding to nontrivial eigenvalues belong to $\mathcal{A}_{\mathcal{D},\mathcal{G}}(D)/\mathcal{H}^{p}_{\mathcal{D},\mathcal{G}}(D)$.*

Proof. The conclusion follows at once from the eigenvalue equation

$$\phi_m(x) = \frac{1}{\lambda_m} \sum_{j' \in \mathcal{J}} \int_{D_{j'}} K(x,x')\phi_m(x')\,dx' \quad \forall x \in D_j \tag{2.6}$$

which can be naturally extended to G_j by replacing K by $K_{jj'}$. ■

Remark 2.7 *Similarly, if $K \in \mathcal{H}^{p,q}_{\mathcal{D}}(D^2)$, then the eigenfunctions of $\mathcal{K}$ corresponding to nontrivial eigenvalues belong to $\mathcal{H}^{p}_{\mathcal{D}}(D)$.*

2.2.1 Smooth Kernel

The following result due to Ehrling-Nirenberg-Gagliardo (see [1], Theorem 4.14), is essential for our analysis.

Theorem 2.8 *Let $D \subset \mathbb{R}^d$ be a bounded domain having the uniform cone property and $\varepsilon_0 \in (0,\infty)$, $n \in \mathbb{N}$, $p \in [1,\infty)$. Then there exists $c_{\varepsilon_0,n,p,D} > 0$ such that for any $\varepsilon \in (0,\varepsilon_0]$, $l \in \{0,1,\dots,n-1\}$ and $u \in W^{n,p}(D)$,*

$$|u|_{l,p} \leq c_{\varepsilon_0,n,p,D} \left\{ \varepsilon |u|_{n,p} + \varepsilon^{-l/(n-l)} |u|_{0,p} \right\}, \tag{2.7}$$

where

$$|u|^p_{l,p} := \int_D \sum_{|\alpha|=l} |\partial^\alpha u(x)|^p \, dx.$$

Theorem 2.9 *For $D \subset \mathbb{R}^d$ a bounded domain and K piecewise smooth on $D \times D$ such that the domains D_j in Definition 2.1 all have the uniform cone property, we denote by $(\lambda_m, \phi_m)_{m\geq 1}$ the eigenelements of the associated integral operator $\mathcal{K}$ via* (1.1)*, such that $\|\phi_m\|_{L^2(D)} = 1, \forall m \geq 1$. Then for any $s > 0$ and any multiindex $\alpha \in \mathbb{N}^d$ there exists $c_{K,s,\alpha} > 0$ such that*

$$\|\partial^\alpha \phi_m\|_{L^\infty(D_j)} \leq c_{K,s,\alpha} |\lambda_m|^{-s} \quad \forall m \geq 1, \forall j \in \mathcal{J}. \tag{2.8}$$

Proof. We first note that the eigenvalue equation (2.6) implies (by differentiating and applying the Cauchy-Schwarz inequality to estimate the resulting integrals) for any $\alpha \in \mathbb{N}^d$ the existence of a constant $c_{K,\alpha} > 0$ such that

$$\|\partial^\alpha \phi_m\|_{L^\infty(D_j)} \leq c_{K,\alpha}|\lambda_m|^{-1} \quad \forall m \geq 1, \forall j \in \mathcal{J}. \tag{2.9}$$

We apply now Theorem 2.8 on D_j with $p = 2$, $\varepsilon_0 = \max_{m\geq 1}|\lambda_m|$ and choose in (2.7) $\varepsilon = \lambda_m$, $u = \phi_m$ for an arbitrary $m \geq 1$. It follows that for any $n \in \mathbb{N}$ there exists $c_{\varepsilon,n,D_j} > 0$ such that for all $l \in \{0, 1, \dots, n-1\}$

$$\begin{aligned} |\phi_m|_{D_j}|_{l,2} &\leq c_{\varepsilon_0,n,D_j}\left\{\lambda_m|\phi_m|_{n,2} + \lambda_m^{-l/(n-l)}|\phi_m|_{0,2}\right\} \\ &\leq c_{\varepsilon_0,n,D_j,K}\left\{1 + \lambda_m^{-l/(n-l)}\right\} \leq c_{\varepsilon_0,n,D_j,K}\lambda_m^{-l/(n-l)}, \end{aligned} \tag{2.10}$$

due to (2.9).
Now, for any $s > 0$ and $\alpha \in \mathbb{N}^d$ we choose $l = \lceil d/2 \rceil + |\alpha|$ and $n > l$ such that $l/(n-l) < s$. From (2.10) and the Sobolev embedding theorems we deduce then

$$\begin{aligned} \|\partial^\alpha \phi_m\|_{L^\infty(D_j)} &\leq c_{\alpha,D_j}\|\phi_m\|_{H^l(D_j)} \leq c_{\alpha,D_j}\sum_{k=0}^{l}|\phi_m|_{D_j}|_{k,2} \\ &\leq c_{\varepsilon_0,n,D_j,K,\alpha}\sum_{k=0}^{l}\lambda_m^{-k/(n-k)} \\ &\leq c_{\varepsilon_0,n,D_j,K,\alpha}\lambda_m^{-l/(n-l)} \leq c_{\varepsilon_0,n,D_j,K,\alpha}\lambda_m^{-s} \end{aligned}$$

for all $m \geq 1$, and the proof is concluded. ■

Remark 2.10 *Under the regularity assumptions of Proposition 2.9 the estimate* (2.8) *is optimal in the sense that for any α it fails to hold with $s = 0$. This can be seen e.g. on $D :=]0,1[$ by taking $K := \sum_{m\geq 1}\lambda_m \cdot \phi_m \otimes \phi_m$, with $\lambda_m := e^{-m}$ and $\phi_m(x) := m\phi(m^2x - m)\,\forall x \in]0,1[, \forall m \geq 1$, where $\phi \in C_0^\infty(]0,1[)$ satisfies $\|\phi\|_{L^2(]0,1[)} = 1$.*

Remark 2.11 *It can be shown that further assumptions, like stationarity of the kernel i.e. $K(x,x') = k(x-x')$ for some $k : D \to \mathbb{R}$, lead to the uniform L^∞ boundedness of the eigenfunctions (but not of their derivatives).*

Remark 2.12 *If K is not piecewise smooth in the sense of Definition 2.1 (for instance, if the singularities of K lie on the diagonal set of $D \times D$, as it is the case for some usual stationary kernels like e.g. $K(x,x') = k(x-x') = \exp(-|x-x'|^{1+\delta})$ with $0 \leq \delta < 1$), then estimates of type* (2.8) *hold true only for $s > c_k > 0$, where the constant $c_k \in]0,1]$ depends on the Sobolev regularity of k in D (see also [5] for further examples with known exact eigenelements).*

2.2.2 Analytic Kernel

We prove in the following a refinement of Theorem 2.9 for analytic functions, which allows us a better control (explicit dependence on α) of the constant $c_{K,s,\alpha}$ in (2.8).

Theorem 2.13 *Suppose that $K \in \mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)$ is a symmetric kernel and denote by $(\lambda_m, \phi_m)_{m\geq 1}$ the eigenelements of $\mathcal{K}$ defined by* (1.1)*, such that $\|\phi_m\|_{L^2(D)} = 1, \forall m \geq 1$. Then for any $s > 0$ there exist constants $c_{K,s}, r_{K,s} > 0$ such that*

$$\|\partial^\alpha \phi_m\|_{L^\infty(D_j)} \leq c_{K,s}\lambda_m^{-s} r_{K,s}^{-|\alpha|}\alpha! \quad \forall \alpha \in \mathbb{N}^d, \forall j \in \mathcal{J}. \tag{2.11}$$

Theorem 2.13 will be a direct consequence of an interpolation result (Theorem 2.23 below) which we prove in the following.

We start with a lemma of Gagliardo-Ehrling-Nirenberg type. Note that the statement is global, therefore weaker than that of Theorem 2.8, but at the same time stronger, allowing a better control of the constant involved (inspection of the induction proof of Theorem 2.8 as presented in [1] reveals that the estimated $c_{\varepsilon_0,m,p,D}$ increases with $m, p \to \infty$ at a pessimistic super-exponential rate).

Lemma 2.14 *For any $\alpha \in \mathbb{N}^d, k \in \mathbb{N}, k \geq 1$ and $u \in H^{k|\alpha|}(\mathbb{R}^d)$ it holds*

$$\|\partial^\alpha u\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2}\|\partial^{k\alpha}u\|_{L^2(\mathbb{R}^d)}^{1/k}\|u\|_{L^2(\mathbb{R}^d)}^{(k-1)/k}. \tag{2.12}$$

Proof. It suffices to check that for any $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}, k \geq 1$, $\varepsilon > 0$ and $u \in H^{k|\alpha|}(\mathbb{R}^d)$ it holds

$$\|\partial^\alpha u\|_{L^2(\mathbb{R}^d)}^2 \leq \varepsilon\|\partial^{k\alpha}u\|_{L^2(\mathbb{R}^d)}^2 + \varepsilon^{-1/(k-1)}\|u\|_{L^2(\mathbb{R}^d)}^2, \tag{2.13}$$

since we can take then $\varepsilon := (\|u\|_{L^2(\mathbb{R}^d)}/\|\partial^{k\alpha}u\|_{L^2(\mathbb{R}^d)})^{2-2/k}$ and obtain (2.12).

To this end, we note that $\sup_{x\geq 0} x/(x^k+1) \in]0,1[$ and set here $x = \varepsilon^{1/(k-1)}\xi^{2\alpha}$ for $\xi \in \mathbb{R}^d$. Upon multiplying the resulting inequality $\xi^{2\alpha} \leq \varepsilon\xi^{2k\alpha} + \varepsilon^{-1/(k-1)}$ by $|\hat{u}(\xi)|^2$, integrating $d\xi$ and taking inverse Fourier transforms we obtain (2.13). ■

We localize Lemma 2.14 using cut-off functions which are analytic up to a given order (see [6], Theorem 1.4.2 for a proof of Lemma 2.16 below, or [13], Proposition 1.4.10).

Definition 2.15 *If $G \subset \mathbb{R}^d$, we define for $\delta > 0$ the δ-neighbourhood of G by*

$$G_\delta := \{x \in \mathbb{R}^d \mid \inf_{y\in G}\max_{1\leq n\leq d}|x_n - y_n| < \delta\}.$$

Further, for a bounded domain $D \subset \mathbb{R}^d$,

$$\delta_D := \inf\{\delta > 0 \mid D \subset (\partial D)_\delta\}.$$

Lemma 2.16 *If $D \subset \mathbb{R}^d$ is a bounded domain, then for any $\alpha \in \mathbb{N}^d$ there exists $\psi_{\delta,\alpha} \in C_0^\infty(D)$ such that $0 \leq \psi_{\delta,\alpha} \leq 1$, $\psi_{\delta,\alpha} = 1$ on $D \setminus (\partial D)_\delta$ and*

$$\|\partial^\beta \psi_{\delta,\alpha}\|_{L^\infty(D)} \leq \left(\frac{3|\alpha|}{\delta}\right)^{|\beta|} \quad \forall \beta \in \mathbb{N}^d, \beta \leq \alpha \tag{2.14}$$

where $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

We turn now to the proof of a local version of Lemma 2.14 for analytic functions.

Definition 2.17 *For D bounded domain in $\mathbb{R}^d$ we define for $r > 0$ the set U_r as the open neighbourhood of $\overline{D}$ in $\mathbb{C}^d$ given by*

$$U_r := \{z = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d \mid \inf_{x \in D} \max_{1 \le n \le d} |z_n - x_n| < r\}.$$

By $\mathcal{A}(\overline{U_r})$ we further denote the space of all complex analytic functions on $\overline{U_r}$.

Lemma 2.18 *If $D \subset \mathbb{R}^d$ is a bounded domain, then for any $r > 0$, $\phi \in \mathcal{A}(\overline{U_r})$, $\delta > 0$, $\alpha \in \mathbb{N}^d$ and $k \in \mathbb{N}^\star$ it holds*

$$\|\partial^\alpha \phi\|_{L^2(D \setminus (\partial D)_\delta)} \le \sqrt{2}(2^{d+1}\lambda(D)^{1/2})^{1/k} \|\phi\|_{L^\infty(U_r)}^{1/k} \|\phi\|_{L^2(D)}^{(k-1)/k} (\mu/32kd)^{-|\alpha|}\alpha! \quad (2.15)$$

where $\mu := \min\{\delta, 2rd\}$.

Proof. We first note that, due to the Cauchy integral formula,

$$\|\partial^\alpha \phi\|_{L^\infty(D)} \le \|\phi\|_{L^\infty(U_r)} r^{-|\alpha|}\alpha! \quad \forall \alpha \in \mathbb{N}^d. \quad (2.16)$$

For $\delta > 0, \alpha \in \mathbb{N}^d$ and every $k \in \mathbb{N}_+$ we construct $\psi_{\delta,k\alpha}$ on D as in Lemma 2.16 and then apply Lemma 2.14 with $u := \psi_{\delta,k\alpha} \cdot \phi\big|_D$. It follows

$$\|\partial^\alpha \phi\|_{L^2(D \setminus (\partial D)_\delta)} \le \|\partial^\alpha(\psi_{\delta,k\alpha} \cdot \phi\big|_D)\|_{L^2(\mathbb{R}^d)} \le \sqrt{2}\|\partial^{k\alpha}(\psi_{\delta,k\alpha} \cdot \phi\big|_D)\|_{L^2(\mathbb{R}^d)}^{1/k} \|\phi\|_{L^2(D)}^{(k-1)/k}. \quad (2.17)$$

But, using (2.14) and (2.16),

$$\begin{aligned}\|\partial^{k\alpha}(\psi_{\delta,k\alpha}\phi\big|_D)\|_{L^\infty(\mathbb{R}^d)} &= \|\sum_{\beta \le k\alpha} \binom{k\alpha}{\beta} \partial^\beta \psi_{\delta,k\alpha} \partial^{k\alpha-\beta}\phi\|_{L^\infty(D)} \\ &\le \|\phi\|_{L^\infty(U_r)} \sum_{\beta \le k\alpha} \binom{k\alpha}{\beta} (3k|\alpha|)^{|\beta|} \delta^{-|\beta|} r^{-|k\alpha-\beta|}(k\alpha-\beta)!.\end{aligned}$$

Further the obvious inequalities $n^m \le (n+m)!/n! \le m!\,2^{n+m}$ (with $n = 3k|\alpha|$, $m = |\beta|$), $|\beta|! \le d^{|\beta|}\beta!\ \forall \beta \in \mathbb{N}^d$ and $(k\alpha)! \le (\alpha!)^k k^{k|\alpha|}$ ensure

$$\begin{aligned}\|\partial^{k\alpha}(\psi_{\delta,k\alpha}\phi\big|_D)\|_{L^\infty(\mathbb{R}^d)} &\le \|\phi\|_{L^\infty(U_r)} \sum_{\beta \le k\alpha} \binom{k\alpha}{\beta} \beta!\, 2^{3k|\alpha|+|\beta|} (d/\delta)^{|\beta|} r^{-|k\alpha-\beta|}(k\alpha-\beta)! \\ &\le \|\phi\|_{L^\infty(U_r)} (\alpha!)^k k^{k|\alpha|} (r/8)^{-k|\alpha|} \sum_{\beta \le k\alpha} (2rd/\delta)^{|\beta|}. \quad (2.18)\end{aligned}$$

The sum on the r.h.s. of (2.18) can be estimated by

$$\sum_{\beta \le k\alpha} (2rd/\delta)^{|\beta|} \le \binom{k|\alpha|+d+1}{d+1} \max\{1, (2rd/\delta)^{k|\alpha|}\} \le 2^{k|\alpha|+d+1}(\mu/2rd)^{-k|\alpha|}. \quad (2.19)$$

Inserting (2.19) in (2.18) and using the resulting inequality in (2.17), we obtain the conclusion. ■

Lemma 2.19 *There exists a constant $c_{1,d} > 0$ such that if $D \subset \mathbb{R}^d$ is a bounded domain, then for any $r > 0$, $\phi \in \mathcal{A}(\overline{U_r})$, $\delta > 0$, $\alpha \in \mathbb{N}^d$ and $k \in \mathbb{N}^\star$ it holds*

$$\|\partial^\alpha \phi\|_{L^\infty(D\setminus(\partial D)_{2\delta})} \le c_{1,d}\lambda(D)^{1/2k}(\mu/k)^{-d/2}\|\phi\|^{1/k}_{L^\infty(U_r)}\|\phi\|^{(k-1)/k}_{L^2(D)}(\mu/64kd)^{-|\alpha|}\alpha! \tag{2.20}$$

where $\mu := \min\{\delta, 2rd\}$.

Proof. Clearly,

$$\partial^\alpha\phi(x) = \sum_{\beta\in\mathbb{N}^d}\frac{\partial^{\alpha+\beta}\phi(x')}{\beta!}(x-x')^\beta \quad \forall x,x' \in \overline{D}, \sup_{1\le n\le d}|x_n - x'_n| < r, \tag{2.21}$$

where the series converges absolutely. We consider $0 < \rho < r$ to be chosen later and an arbitrary $x \in D\setminus(\partial D)_{\delta+\rho}$. Integrating (2.21) w.r.t. x' on the d dimensional hypercube centered at x and of size 2ρ we obtain via the Cauchy-Schwarz inequality

$$2^d\rho^d|\partial^\alpha\phi(x)| \le \sum_{\beta\in\mathbb{N}^d}\frac{\|\partial^{\alpha+\beta}\phi\|_{L^2(D\setminus(\partial D)_\delta)}}{\beta!}2^{d/2}\rho^{|\beta|+d/2}. \tag{2.22}$$

From (2.15), (2.22) and with $\mu := \min\{\delta, 2rd\}$ we deduce that

$$|\partial^\alpha\phi(x)| \le c_{1,d}\lambda(D)^{1/2k}\rho^{-d/2}\|\phi\|^{1/k}_{L^\infty(U_r)}\|\phi\|^{(k-1)/k}_{L^2(D)}\cdot \\ \cdot(\mu/32kd)^{-|\alpha|}\sum_{\beta\in\mathbb{N}^d}(\mu/32\rho kd)^{-|\beta|}\frac{(\alpha+\beta)!}{\beta!} \tag{2.23}$$

where $c_{1,d} = 2^{(d+3)/2}$. Since $(\alpha+\beta)!/\beta! \le \alpha!\,2^{|\alpha|+|\beta|}$, the conclusion follows from (2.23) by choosing $\rho = \mu/128kd \le r/64$. ■

Theorem 2.20 *Let $D \subset \mathbb{R}^d$ be a bounded domain. If $r > 0$ and $s \in]0,1]$, then there exist $V_{D,s,r}$ open neighbourhood of $\overline{D}$ in $\mathbb{C}^d$ and a constant $c_{D,s,r} > 0$ such that for any $\phi \in \mathcal{A}(\overline{U_r})$*

$$\|\phi\|_{L^\infty(V_{D,s,r})} \le c_{D,s,r}\|\phi\|^s_{L^\infty(U_r)}\|\phi\|^{1-s}_{L^2(D)}. \tag{2.24}$$

Proof. Consider $\phi \in \mathcal{A}(\overline{U_r})$ arbitrary. From the Cauchy integral formula we have

$$\|\partial^\alpha\phi\|_{L^\infty(D)} \le \|\phi\|_{L^\infty(U_r)}r^{-|\alpha|}\alpha! \quad \forall\alpha \in \mathbb{N}^d. \tag{2.25}$$

Applying Lemma 2.19 we find that for any $\delta > 0$, $\alpha \in \mathbb{N}^d$ and $k \in \mathbb{N}^\star$ it holds

$$\|\partial^\alpha \phi\|_{L^\infty(D\setminus(\partial D)_{2\delta})} \le c_{1,d}\lambda(D)^{1/2k}(\mu/k)^{-d/2}\|\phi\|^{1/k}_{L^\infty(U_r)}\|\phi\|^{(k-1)/k}_{L^2(D)}(\mu/64kd)^{-|\alpha|}\alpha! \tag{2.26}$$

where $\mu := \min\{\delta, 2rd\}$.
For a given $s \in]0,1]$ and $k > 1/s$ set $t := (ks-1)/(k-1) \in]0,s]$. Interpolation between (2.25) and (2.26) with logarithmic weight t yields

$$\|\partial^\alpha\phi\|_{L^\infty(D\setminus(\partial D)_{2\delta})} \le c^{1-t}_{1,d}\lambda(D)^{(1-t)/2k}(\mu/k)^{-d(1-t)/2}\cdot \\ \cdot\|\phi\|^s_{L^\infty(U_r)}\|\phi\|^{1-s}_{L^2(D)}(\mu/64kd)^{-(1-t)|\alpha|}r^{-t|\alpha|}\alpha! \tag{2.27}$$

for any $\alpha \in \mathbb{N}^d$ and $\delta > 0$.
Consider now $z \in U_{2\delta}$ arbitrary. For any $0 < \delta < \min\{r/4, \delta_D/2\}$ the analyticity of ϕ implies the existence of an $x \in D \setminus (\partial D)_{2\delta}$ such that

$$\phi(z) = \sum_{\alpha \in \mathbb{N}^d} \frac{\partial^\alpha \phi(x)}{\alpha!}(z-x)^\alpha. \tag{2.28}$$

From (2.28) and (2.27) we obtain ($\mu = \delta$ in this case)

$$|\phi(z)| \leq c_{1,d}^{1-t} \lambda(D)^{(1-t)/2k} (\delta/k)^{-d(1-t)/2} \|\phi\|^s_{L^\infty(U_r)} \|\phi\|^{1-s}_{L^2(D)} \sum_{\alpha \in \mathbb{N}^d} \left((64kdr/\delta)^t/256kd\right)^{-|\alpha|} \tag{2.29}$$

for any $0 < \delta < \min\{r/4, \delta_D/2\}$ and $\alpha \in \mathbb{N}^d$.
Choosing $\delta = \min\{64kdr/(512kd)^{1/t}, \delta_D/3\} < \min\{r/4, \delta_D/2\}$ the sum on the r.h.s. of (2.29) is finite and we obtain the desired estimate (2.24) with $V_{D,s,r} = U_{2\delta}$ and

$$c_{D,s,r} = 2^d c_{1,d}^{1-t} \lambda(D)^{(1-t)/2k} (\delta/k)^{-d(1-t)/2}.$$

■

From Theorem 2.20 and the Cauchy integral formula it immediately follows

Corollary 2.21 *Let $D \subset \mathbb{R}^d$ be a bounded domain. If $r > 0$ and $s \in]0,1]$, then there exist constants $c_{1,D,s,r}, c_{2,D,s,r} > 0$ such that for any $\phi \in \mathcal{A}(\overline{U_r})$ and $\alpha \in \mathbb{N}^d$ it holds*

$$\|\partial^\alpha \phi\|_{L^\infty(D)} \leq c_{1,D,s,r} \|\phi\|^s_{L^\infty(U_r)} \|\phi\|^{1-s}_{L^2(D)} c_{2,D,s,r}^{-|\alpha|} \alpha!. \tag{2.30}$$

Theorem 2.13 follows now as a consequence of Corollary 2.21.
Proof of Theorem 2.13. Let us fix $j \in \mathcal{J}$ and note that the eigenvalue equation on D_j

$$\phi_m(x) = \frac{1}{\lambda_m} \int_{D_j} \sum_{j' \in \mathcal{J}} K(x,x') \chi_{j'}(x') \phi_m(x')\, dx' \quad \forall x \in D_j, \tag{2.31}$$

where χ_j denotes for $j \in \mathcal{J}$ the indicator function of the set D_j, can be analytically continued to G_j by replacing K in (2.31) by $K_{jj'}$. Denoting by $\tilde{\phi}_m$ the extension of $\phi_m\big|_{D_j}$ to G_j we clearly have, for $r_K > 0$ small enough,

$$\tilde{\phi}_m \in \mathcal{A}(U_{r_K}), \quad \|\tilde{\phi}_m\|_{L^\infty(U_{r_K})} \leq c_K \lambda_m^{-1} \quad \forall m \geq 1. \tag{2.32}$$

The desired estimate (2.11) follows using (2.32) in (2.30), since $\|\phi_m\|_{L^2(D_j)} \leq 1$. ■

Remark 2.22 *Using an appropriate version of Lemma 2.14, the L^2 norm on the r.h.s. of* (2.24) *or* (2.30) *can be replaced by any norm of ϕ on the Sobolev scale.*

In view of Remark 2.22, Theorem 2.13 can be reformulated as an interpolation result, as follows (for definitions, standard notations and general results we refer the reader to [14]). The proof is straightforward, in view of Lemma 2.5.1. in [2].

Theorem 2.23 *Let $D \subset \mathbb{R}^d$ be a bounded domain and let $r > 0$, $\theta \in [0,1[$ and $t \in \mathbb{R}$ be arbitrary parameters. Then there exists $r_{D,r,\theta,t} > 0$ such that*

$$(\mathcal{A}(U_r) \cap C(\overline{U_r}), H^t(D))_{\theta,p} \hookrightarrow \mathcal{A}(U_{r_{D,r,\theta,t}}) \cap C(\overline{U_{r_{D,r,\theta,t}}}) \tag{2.33}$$

for any $p \in [1, +\infty]$, where all spaces are equipped with usual norms.

3 Eigenvalue Computation

3.1 Discretization of the Eigenvalue Problem

Using the minimax principle we investigate in the following the convergence of the discretized eigenvalues of the integral operator $\mathcal{K}$ given by (1.1).
Let $\hbar \in \mathfrak{H}$ be a discretization parameter and let $\mathcal{S} := (S_\hbar)_{\hbar\in\mathfrak{H}} \subset L^2(D)$ denote an arbitrary finite element space family. The discrete eigenvalue problem reads, in variational form: Find $(\lambda_{\hbar,m}, \phi_{\hbar,m})_{m\geq 1} \subset \mathbb{R} \times S_\hbar$ such that

$$\int_{D\times D} K(x,x')\phi_{\hbar,m}(x')\psi(x)\,dx'dx = \lambda_{\hbar,m}\int_D \phi_{\hbar,m}(x)\psi(x)\,dx \quad \forall\psi \in S_\hbar. \tag{3.1}$$

(3.1) shows that the sequence $(\lambda_{\hbar,m}, \phi_{\hbar,m})_{m\geq 1}$ is nothing but the eigenvalue sequence of the compact nonnegative operator $P_\hbar \mathcal{K} P_\hbar$ in $L^2(D)$, where $P_\hbar$ denotes the $L^2(D)$ orthogonal projection onto $S_\hbar$.

We derive in the following discretization error estimates for the eigenvalue problem (3.1) for arbitrary compact nonnegative operators with a fast eigenvalue decay, acting in a separable Hilbert space $(H, \langle,\rangle_H)$. The results will be then seen to apply to the case of an integral operator with smooth kernel. Concerning notations, $\mathcal{B}(H)$ will be the space of bounded linear operators acting in the Hilbert space $(H, \langle,\rangle_H)$, and we will occasionally use traditional subscripts (e.g. $\mathcal{B}_\infty$, $\mathcal{B}_{\infty,p}$, $\mathcal{B}_{\text{sym}}$) for spaces of operators with additional properties (compactness, compactness and ℓ_p summability of the eigenvalue sequence for $p > 0$, symmetry).
As a straightforward consequence of the minimax principle it holds,

Proposition 3.1 *If $(H, \langle,\rangle_H)$ is a separable Hilbert space, $\mathcal{K} \in \mathcal{B}_\infty(H)$ is nonnegative and $S_\hbar$ is an arbitrary closed subspace of H, then*

$$\lambda_{\hbar,m} \leq \lambda_m \quad \forall m \geq 1, \tag{3.2}$$

where $(\lambda_{\hbar,m})_{m\geq 1}$ denotes the eigenvalue sequence of $P_\hbar \mathcal{K} P_\hbar$ in H.

Lower bounds for $\lambda_{\hbar,m}$ in terms of λ_m follow from the approximability of the eigenfunctions $(\phi_m)_{m\geq 1}$ through the finite element family $\mathcal{S}$. We make therefore the following

Assumption 3.2 *The eigenelements $(\lambda_m, \phi_m)_{m\geq 1}$ of $\mathcal{K}$ and the finite element space family $\mathcal{S}$ have the property that for any $s > 0$ there exists $c_{\mathcal{K},\mathcal{S},s} > 0$ and $\Phi_s : \mathfrak{H} \to \mathbb{R}$ such that*

$$\|(I - P_\hbar)\phi_m\|_H \leq c_{\mathcal{K},\mathcal{S},s}\lambda_m^{-s}\Phi_s(\hbar) \quad \forall m \geq 1, \forall \hbar \in \mathfrak{H}. \tag{3.3}$$

Remark 3.3 *Later we will see that Assumption 3.2 is satisfied in the case of an integral operator* (1.1) *with a piecewise smooth/analytic kernel K and in the context of the standard h/p methods. Moreover, the functional $\Phi_s : \mathfrak{H} \to \mathbb{R}$ qualitatively preserves the convergence rate of the finite element spaces employed (algebraic or exponential, with parameters allowed to depend on s).*

Based on Assumption 3.2 we prove next that the trace discretization error is optimal for the class of compact operators with fast eigenvalue decay.

Theorem 3.4 *If* $\mathcal{K} \in \cap_{p>0}\mathcal{B}_{\infty,p}(H)$ *is nonnegative and Assumption 3.2 is satisfied, then for any* $0 < s < 1/2$ *there exists a constant* $c_{\mathcal{K},\mathcal{S},s} > 0$ *such that*

$$0 \leq \text{Tr } \mathcal{K} - \text{Tr } P_\hbar \mathcal{K} P_\hbar \leq c_{\mathcal{K},\mathcal{S},s} \Phi_s(\hbar)^2 \quad \forall \hbar \in \mathfrak{H}. \tag{3.4}$$

Proof. Fix $\hbar \in \mathfrak{H}$. The lower bound follows immediately from Proposition 3.1. Further, the obvious identity

$$\mathcal{K} - P_\hbar \mathcal{K} P_\hbar = (I - P_\hbar)\mathcal{K} + \mathcal{K}(I - P_\hbar) - (I - P_\hbar)\mathcal{K}(I - P_\hbar) \tag{3.5}$$

and the fact that $\mathcal{K}$ is nonnegative ensure that ($H := L^2(D)$)

$$\langle (\mathcal{K} - P_\hbar \mathcal{K} P_\hbar)\phi, \phi\rangle_H \leq 2|\langle \mathcal{K}\phi, (I - P_\hbar)\phi\rangle_H| \quad \forall \phi \in H. \tag{3.6}$$

Using (3.6) and Assumption 3.2 it follows

$$\begin{aligned} \text{Tr } \mathcal{K} - \text{Tr } P_\hbar \mathcal{K} P_\hbar &= \sum_{m\geq 1} \langle (\mathcal{K} - P_\hbar \mathcal{K} P_\hbar)\phi_m, \phi_m\rangle_H \leq 2 \sum_{m\geq 1} |\langle \mathcal{K}\phi_m, (I - P_\hbar)\phi_m\rangle_H| \\ &\leq 2 \sum_{m\geq 1} \lambda_m \|(I - P_\hbar)\phi_m\|_H^2 \leq c_{\mathcal{K},\mathcal{S},s} \Phi_s(\hbar)^2 \sum_{m\geq 1} \lambda_m^{1-2s}. \end{aligned} \tag{3.7}$$

The assumption $\mathcal{K} \in \cap_{p>0}\mathcal{B}_{\infty,p}(H)$ ensures the summability of the series on the r.h.s. of (3.7) for $s < 1/2$, which concludes the proof. ∎

We derive in the following individual discretization error estimates for the eigenvalues.

Theorem 3.5 *If* $\mathcal{K} \in \cap_{p>0}\mathcal{B}_{\infty,p}(H)$ *is nonnegative and Assumption 3.2 is satisfied, then for any* $s > r > 0$ *there exists a constant* $c_{\mathcal{K},\mathcal{S},s,r} > 0$ *such that*

$$0 \leq \lambda_m - \lambda_{\hbar,m} \leq c_{\mathcal{K},\mathcal{S},s,r}(\Phi_r(\hbar)^2 \lambda_m^{1-2s} + \Phi_r(\hbar)^4 \lambda_m^{-4s}) \quad \forall m \geq 1, \forall \hbar \in \mathfrak{H}, \tag{3.8}$$

where $(\lambda_m)_{m\geq 1}$ *and* $(\lambda_{\hbar,m})_{m\geq 1}$ *are the eigenvalue sequences of* $\mathcal{K}$ *and* $\mathcal{K}_\hbar := P_\hbar \mathcal{K} P_\hbar$ *respectively.*

Proof. Fix $m \geq 1$ and $\hbar \in \mathfrak{H}$. From the minimax principle,

$$\lambda_{\hbar,m} = \max_{\substack{U\subset H \\ \dim U \geq m}} \min_{\substack{\phi\in U \\ \|\phi\|_H = 1}} \{\langle \mathcal{K}\phi, \phi\rangle_H + \langle (\mathcal{K}_\hbar - \mathcal{K})\phi, \phi\rangle_H\}. \tag{3.9}$$

The identity (3.5) and the fact $\mathcal{K}$ is nonnegative ensure then

$$\langle (\mathcal{K}_\hbar - \mathcal{K})\phi, \phi\rangle_H \geq -2|\langle \mathcal{K}\phi, (I - P_\hbar)\phi\rangle_H| \quad \forall \phi \in H. \tag{3.10}$$

Using (3.10) in (3.9) we obtain

$$\begin{aligned} \lambda_{\hbar,m} &\geq \max_{\substack{U\subset H \\ \dim U \geq m}} \min_{\substack{\phi\in U \\ \|\phi\|_H = 1}} \{\langle \mathcal{K}\phi, \phi\rangle_H - 2|\langle \mathcal{K}\phi, (I - P_\hbar)\phi\rangle_H|\} \\ &\geq \max_{\substack{U\subset H \\ \dim U \geq m}} \min_{\substack{\phi\in U \\ \|\phi\|_H = 1}} \{\langle \mathcal{K}\phi, \phi\rangle_H - 2\|(I - P_\hbar)\mathcal{K}\phi\|_H \|(I - P_\hbar)\phi\|_H\}. \end{aligned} \tag{3.11}$$

At this stage we choose U to be the subspace of H spanned by the first m eigenfunctions $\phi_1, \phi_2, \ldots, \phi_m$ of $\mathcal{K}$. Expanding $\phi = \sum_{j=1}^m \alpha_j \phi_j$ and using (3.3) we obtain

$$\|(I - P_\hbar)\mathcal{K}\phi\|_H \leq c_{\mathcal{K},\mathcal{S},r}\Phi_r(\hbar)\sum_{j=1}^m |\alpha_j|\lambda_j^{1-r} \leq c_{\mathcal{K},\mathcal{S},r}\Phi_r(\hbar)\lambda_m^{-r}\sum_{j=1}^m |\alpha_j|\lambda_j \quad (3.12)$$

for any $r > 0$. Similarly, $(\lambda_m)_{m\geq 1} \in \cap_{p>0}\ell_p$ ensures

$$\begin{aligned}\|(I - P_\hbar)\phi\|_H &\leq c_{\mathcal{K},\mathcal{S},r}\Phi_r(\hbar)\sum_{j=1}^m |\alpha_j|\lambda_j^{-r} \\ &\leq c_{\mathcal{K},\mathcal{S},r}\Phi_r(\hbar)\left(\sum_{j=1}^m \lambda_j^{-2r}\right)^{1/2} \leq c_{\mathcal{K},\mathcal{S},r,\delta}\Phi_r(\hbar)\lambda_m^{-r-\delta} \quad (3.13)\end{aligned}$$

for any $r, \delta > 0$. From (3.11), (3.12), (3.13) we obtain

$$\lambda_{\hbar,m} \geq \min_{\sum_{j=1}^m |\alpha_j|^2=1} \left\{\sum_{j=1}^m \lambda_j|\alpha_j|^2 - 2c_{\mathcal{K},\mathcal{S},r,\delta}\Phi_r(\hbar)^2\lambda_m^{-2r-\delta}\sum_{j=1}^m \lambda_j|\alpha_j|\right\} \quad \forall r, \delta > 0. \quad (3.14)$$

Fix $r, \delta > 0$, choose also an arbitrary $\rho > 0$ and define $\varepsilon := c_{\mathcal{K},\mathcal{S},r,\delta}\Phi_r(\hbar)^2\lambda_m^{-2r-\delta}$. If $\varepsilon \geq 1/\sqrt{m}$ then $\Phi_r(\hbar)^2 > c_{\mathcal{K},k,r,\delta,\rho}\lambda_m^{2r+\delta+\rho}$, so that it holds, with $s := r + (\delta + \rho)/2$,

$$\lambda_{\hbar,m} \geq 0 \geq \lambda_m - c_{\mathcal{K},\mathcal{S},s,r}\Phi_r(\hbar)^2\lambda_m^{1-2s}. \quad (3.15)$$

Otherwise $\varepsilon < 1/\sqrt{m}$ and we apply Lemma 3.7 below to obtain from (3.14)

$$\begin{aligned}\lambda_{\hbar,m} \geq \lambda_m &- c_{\mathcal{K},\mathcal{S},r,\delta}\sqrt{m}\lambda_m^{1-2r-\delta}\Phi_r(\hbar)^2 - \\ &- c_{\mathcal{K},\mathcal{S},r,\delta,\rho}\max\{c_{\mathcal{K},\mathcal{S},r,\delta}\lambda_1\lambda_m^{-4r-2\delta-\rho}\Phi_r(\hbar)^4, \lambda_m^{1-2r-\delta-\rho}\Phi_r(\hbar)^2\}\end{aligned}$$

which is equivalent to the desired estimate (3.8), with $s = r + (\delta + \rho)/2$, since δ, ρ are arbitrarily small. ∎

Corollary 3.6 *If $\mathcal{K} \in \cap_{p>0}\mathcal{B}_{\infty,p}(H)$ is nonnegative and Assumption 3.2 is satisfied, then for any $s > r > 0$ there exists a constant $c_{\mathcal{K},\mathcal{S},s,r} > 0$ such that*

$$0 \leq \lambda_m - \lambda_{\hbar,m} \leq c_{\mathcal{K},\mathcal{S},s,r}(\Phi_r(\hbar)^2\lambda_m^{1-2s} + \Phi_r(\hbar)^{4\alpha}\lambda_m^{1-\alpha-4s\alpha}) \quad (3.16)$$

$\forall m \geq 1, \forall \hbar \in \mathfrak{H}, \forall \alpha \in [0,1]$, where $(\lambda_m)_{m\geq 1}$ and $(\lambda_{\hbar,m})_{m\geq 1}$ are the eigenvalue sequences of $\mathcal{K}$ and $\mathcal{K}_\hbar := P_\hbar\mathcal{K}P_\hbar$ respectively.

Proof. Consider an arbitrary $t \geq 0$. If $\Phi_r(\hbar)^{4\alpha} \geq \lambda_m^t$, then clearly

$$\lambda_{\hbar,m} \geq 0 \geq \lambda_m - \Phi_r(\hbar)^{4\alpha}\lambda_m^{1-t}. \quad (3.17)$$

Otherwise $\Phi_r(\hbar)^{4\alpha} < \lambda_m^t$ and (3.8) becomes

$$0 \leq \lambda_m - \lambda_{\hbar,m} \leq c_{\mathcal{K},\mathcal{S},s,r}(\Phi_r(\hbar)^2\lambda_m^{1-2s} + \Phi_r(\hbar)^{4\alpha}\lambda_m^{t(1-\alpha)/\alpha-4s}). \tag{3.18}$$

Choose now t to balance the estimates (3.17) and (3.18), that is, $t = (1+4s)\alpha$. It follows then from (3.17) and (3.18) that

$$0 \leq \lambda_m - \lambda_{h,m} \leq c_{\mathcal{K},\mathcal{S},s,r}(\Phi_r(\hbar)^2\lambda_m^{1-2s} + \Phi_r(\hbar)^{4\alpha}\lambda_m^{1-(1+4s)\alpha})$$

for any $m, \hbar, \alpha$ as in (3.16), which concludes the proof. ∎

Lemma 3.7 *If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ is a non-increasing sequence of nonnegative real numbers and $\varepsilon \in [0, 1/\sqrt{m}[$, then*

$$\min_{\sum_{j=1}^m t_j^2=1} \sum_{j=1}^m \lambda_j(t_j^2 - 2\varepsilon t_j) \geq \lambda_m - \sqrt{m}\varepsilon\lambda_m - m\varepsilon \max\{\lambda_1\varepsilon, \lambda_m\}. \tag{3.19}$$

Proof. Obviously, the minimum on the l.h.s. of (3.19) is attained at a location $t := (t_1, t_2, \ldots, t_m)$ on the unit m dimensional sphere, with nonnegative coordinates. Using Lagrange multipliers for $f, g : \mathbb{R}^m \to \mathbb{R}$ given by

$$f(t) := \sum_{j=1}^m \lambda_j(t_j^2 - 2\varepsilon t_j) \quad \text{and} \quad g(t) := \sum_{j=1}^m t_j^2 - 1$$

we obtain the existence of a real $\lambda \neq 0$ such that for the location t of the minimum of f restricted to $g^{-1}(\{0\})$ it holds $\nabla f - \lambda\nabla g = 0$, that is,

$$t_j = \frac{\varepsilon\lambda_j}{\lambda_j - \lambda} \quad \forall 1 \leq j \leq m. \tag{3.20}$$

Imposing $g(t) = 0$ we obtain that λ solves the equation

$$\sum_{j=1}^m \frac{\lambda_j^2}{(\lambda_j - \lambda)^2} = \frac{1}{\varepsilon^2}. \tag{3.21}$$

In order to estimate λ, we first remark that $t_j \geq 0, \forall 1 \leq j \leq m$ implies that λ is the unique solution of (3.21) situated in the interval $]-\infty, \lambda_m[$. Further, the assumption on ε ensures the positivity of λ, therefore we have $\lambda \in]0, \lambda_m[$. As a consequence, the m-th term of the sum in (3.21) is the largest one, which implies then

$$\lambda_m - \sqrt{m}\varepsilon\lambda_m \leq \lambda \leq \lambda_m - \varepsilon\lambda_m. \tag{3.22}$$

In view of (3.22), computing f at t given by (3.20) yields,

$$\begin{aligned}\min_{\substack{t\in\mathbb{R}^m\\ g(t)=0}} f(t) = \lambda - \sum_{j=1}^m \frac{\varepsilon^2\lambda_j^2}{\lambda_j - \lambda} \quad &\geq \quad \lambda_m - \sqrt{m}\varepsilon\lambda_m - \varepsilon^2 m \max_{1\leq j\leq m} \frac{\lambda_j^2}{\lambda_j - \lambda} \\ &\geq \quad \lambda_m - \sqrt{m}\varepsilon\lambda_m - \varepsilon^2 m \max\{\lambda_1, \frac{\lambda_m}{\varepsilon}\},\end{aligned}$$

due to the convexity of $x \to x^2/(x - \lambda)$ on $]\lambda, \infty[$. ∎

Remark 3.8 *In the case of an integral operator* (1.1) *with smooth kernel K, the regularity condition $\mathcal{K} \in \cap_{p>0}\mathcal{B}_{\infty,p}(L^2(D))$ follows from Corollary 2.5.*

3.1.1 Smooth Kernel - h FEM

For an integral operator (1.1) with smooth kernel K, we consider a fixed polynomial degree $k \in \mathbb{N}$, $\hbar := h \in]0, \infty]$ and $S_\hbar = S_h := S_h^k$, the space of discontinuous piecewise polynomials of total degree at most k on a regular mesh in $\overline{D}$ of width h and subordinate to $\mathcal{D}$, i.e. to the covering $(\overline{D_j})_{j \in \mathcal{J}}$. With this choice it holds,

Proposition 3.9 *If $K \in L^2(D \times D)$ is piecewise smooth on $D \times D$ defining a compact nonnegative integral operator $\mathcal{K}$ via (1.1) and $k \in \mathbb{N}$, $S_\hbar = S_\hbar^k \ \forall \hbar = h \in]0, \infty]$, then Assumption 3.2 holds with $\Phi_s(\hbar) = \Phi(\hbar) := h^{k+1}$ (independent of $s > 0$) and the assumptions of Theorem 3.5 are met.*

Proof. In view of Remark 3.8, we only have to check (3.3). To this end, we note that the standard h FEM approximation property holds for S_h,

$$\|\phi - P_\hbar \phi\|_{L^2(D)} \leq c_{k,\mathcal{D}} h^{k+1} |\phi|_{k+1} \quad \forall h > 0, \forall \phi \in H_{\mathcal{D}}^{k+1}(D), \tag{3.23}$$

where

$$|\phi|_{k+1}^2 := \sum_{j \in \mathcal{J}} \sum_{|\alpha| = k+1} \|\partial^\alpha \phi\|_{L^2(D_j)}^2 \quad \forall \phi \in H_{\mathcal{D}}^{k+1}(D).$$

The conclusion follows then by applying (3.23) to $\phi := \phi_m \in H_{\mathcal{D}}^{k+1}(D)$ (in view of Remark 2.7), and using Proposition 2.9 to estimate $|\phi_m|_{k+1}$ in terms of a given $s > 0$ and λ_m. ■

3.1.2 Analytic Kernel - p FEM

For the case of a kernel K piecewise analytic on a covering of $D \times D$ we consider $\hbar := p \in \mathbb{N}$ and $S_\hbar := S^p$ the space of discontinuous piecewise polynomials of total degree at most p on a fixed mesh $\mathcal{T}$ in $\overline{D}$ subordinate to $\mathcal{D}$, i.e. to the covering $(\overline{D_j})_{j \in \mathcal{J}}$. We further assume that each element of $\mathcal{T}$ can be included in a hypercube which is in turn contained in one of the sets $(G_j)_{j \in \mathcal{J}}$ in Definition 2.1 (this can be ensured by a mesh refinement).
The p FEM error estimate we shall use in the following reads (see [10], Lemma 3.2.7 and [4], Theorem 12.4.7)

Lemma 3.10 *If $T \in \mathbb{R}^d$ is a hypercube and ϕ is analytic on T satisfying therefore with some $c_\phi, r = r_\phi > 0$*

$$\|\partial^\alpha \phi\|_{L^\infty(T)} \leq c_\phi r^{-|\alpha|} \alpha! \quad \forall \alpha \in \mathbb{N}^d \setminus (0, 0, \ldots, 0), \tag{3.24}$$

then there exist $c_{r,T}, b = b_{r,T} > 0$ such that

$$\|\phi - P_p \phi\|_{L^\infty(T)} \leq c_{r,T} c_\phi e^{-bp}, \tag{3.25}$$

where P_p denotes the $L^2(T)$ orthogonal projection on the space of polynomials in d variables and of total degree at most p (restricted to T).

Using Lemma 3.10 and Theorem 2.13 we immediately check Assumption 3.2 if the kernel K is piecewise analytic on a covering of $D \times D$.

Proposition 3.11 *If K is piecewise analytic on a covering of $D\times D$ defining a compact nonnegative integral operator $\mathcal{K}$ via* (1.1)*, then Assumption 3.2 holds with $\hbar = p \in \mathbb{N}$, $\Phi_s(\hbar) := e^{-b_{K,\mathcal{S},s}p}$ and the assumptions of Theorem 3.5 are met.*

3.2 $\mathcal{K}^\epsilon$ Spectrum Approximation

For a given $\epsilon > 0$, a simple argument based on the Lipschitz continuity of $]0,\infty[\ni x \to x^\epsilon \in]0,\infty[$ and Corollary 3.6 shows that the computable eigenvalues of $\mathcal{K}^\epsilon$, namely $(\lambda^\epsilon_{\hbar,m})_{m\geq 1}$, are good approximations of the exact eigenvalues $(\lambda^\epsilon_m)_{m\geq 1}$ of $\mathcal{K}^\epsilon$.

Theorem 3.12 *If $K \in L^2(D\times D)$ is piecewise smooth on $D \times D$ defining a compact nonnegative integral operator $\mathcal{K}$ via* (1.1) *such that Assumption 3.2 is satisfied and $0 < \epsilon \leq 1$, then for any $r < \epsilon/2$ and $\alpha \in]0, \epsilon/(1+4r)[$*

$$0 \leq \sum_{m=1}^{\infty} (\lambda^\epsilon_m - \lambda^\epsilon_{\hbar,m}) \leq c_{K,\mathcal{S},r,\alpha} \max\{\Phi_r(\hbar)^2, \Phi_r(\hbar)^{4\alpha}\} \quad \forall \hbar \in \mathfrak{H}. \tag{3.26}$$

Proof. From the Lipschitz condition

$$\lambda^\epsilon_m - \lambda^\epsilon_{\hbar,m} \leq \lambda^{\epsilon-1}_m (\lambda_m - \lambda_{\hbar,m})$$

and (3.16) we obtain that for any $s > r > 0$ there exists a constant $c_{K,\mathcal{S},s,r} > 0$ such that

$$0 \leq \lambda^\epsilon_m - \lambda^\epsilon_{\hbar,m} \leq c_{K,\mathcal{S},s,r} (\Phi_r(\hbar)^2 \lambda^{\epsilon-2s}_m + \Phi_r(\hbar)^{4\alpha} \lambda^{\epsilon-\alpha-4s\alpha}_m) \tag{3.27}$$

$\forall m \geq 1, \forall \hbar \in \mathfrak{H}, \forall \alpha \in [0,1]$. We fix $r < \epsilon/2$ and $\alpha \in]0, \epsilon/(1+4r)[$ and take $s \in]r, \epsilon/2[\cap]0, (\epsilon-\alpha)/4\alpha[$ arbitrary. Summing (3.27) over $m \geq 1$ we obtain (3.26). ■

Remark 3.13 *For the finite element spaces considered in sections 3.1.1, 3.1.2,* (3.26) *becomes*

$$0 \leq \sum_{m=1}^{\infty} (\lambda^\epsilon_m - \lambda^\epsilon_{h,m}) \leq c_{K,\mathcal{S},\alpha} h^{(k+1)\min\{2,4\alpha\}} \quad \forall h > 0, \forall \alpha < \epsilon, \tag{3.28}$$

for the h-version, and

$$0 \leq \sum_{m=1}^{\infty} (\lambda^\epsilon_m - \lambda^\epsilon_{p,m}) \leq c_{K,\mathcal{S},\epsilon} e^{-b_{K,\mathcal{S},\epsilon}p} \quad \forall p \in \mathbb{N}, \tag{3.29}$$

for the p-version respectively.

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References

[1] W. Adams. *Sobolev Spaces*, Academic Press, 1978.

[2] W.O. Amrein, A. B. de Monvel and V. Georgescu. *C_0-groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser, 1996.

[3] F. Chatelin. *Spectral Approximations of Linear Operators*, Academic Press, 1983.

[4] P.J. Davis. *Interpolation & Approximation*, Dover Publications, 1975.

[5] R. Ghanem and P.D. Spanos, *Stochastic Finite Elements: a Spectral Approach*, Springer, 1991.

[6] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*, Grundlehren **256**, Springer, 1983.

[7] T. Kato. *Perturbation Theory for Linear Operators*, Grundlehren **132**, Springer, 1984.

[8] H. König. *Eigenvalue Distribution of Compact Operators*, Operator Theory: Advances and Applications **16**, Birkhäuser, 1986.

[9] M. Loève. *Probability Theory* Vol. I+II, Springer, New York 1978.

[10] M. Melenk. *hp-Finite Element Methods for Singular Perturbations*, Lecture Notes in Mathematics **1796**, Springer, 2002.

[11] J.E. Osborn. *Spectral approximation for compact operators*, Math. Comput. **29**(1975), 712-725.

[12] A. Pietsch. *Eigenvalues and s-Numbers*, Mathematik und ihre Anwendungen **43**, Geest & Portig, 1987.

[13] L. Rodino. *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific, 1993.

[14] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.