# On a Conjectured Increasing Series - Sleeping Habits of Armadillos - [1]

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Seminar für Angewandte Mathematik
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**Abstract**

The rate of total heat loss of two bodies maintained at constant temperature in a homogeneous conducting medium of low temperature is a function of the distance between the two bodies. Assuming spherical bodies with equal temperatures, this function has been explicitly computed in [4] and its monotonic increase has been verified numerically. In this note we give a rigorous completely elementary proof of this fact, and thus a positive answer to the question raised by M.L. Glasser in [3].

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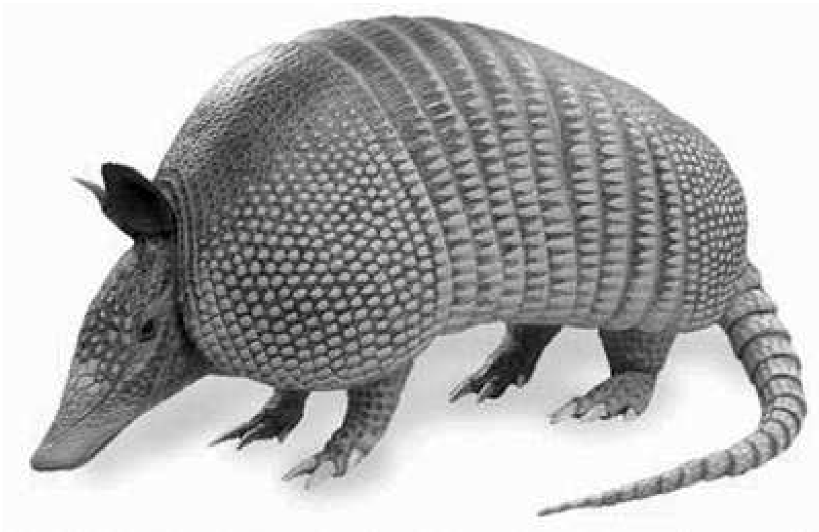

Figure 1: Nine-banded armadillo.

# 1 Problem Formulation

In [4] Glasser and Davidson proposed a mathematical model which explains the habit of many animals to huddle together at night to keep warm, a model considered especially appropriate for armadillo colonies (see Figure 1). Considering for simplicity only a pair of sleeping animals (identical spherical bodies of constant temperature) in an homogeneous environement of lower temperature, they derived an explicit expression for the total amount of heat given off by the two bodies in the steady state, in dependence of the distance between the bodies. Numerical tests confirmed the intuititively clear monotonic increase with the distance between the bodies of the heat loss, but a formal proof of this fact was not found.

Following the presentation of this problem given by Eremenko in [1], let us denote by $B_1$ and $B_2$ the two bodies of constant temperature 1 in a medium $\mathbb{R}^3$ of constant temperature 0 away from $B_1$, $B_2$. Let $2r$ be the distance between $B_1$ and $B_2$. If $T_r : \mathbb{R}^3 \setminus (B_1 \cup B_2) \to [0, \infty[$ denotes the temperature of the environement, then in steady state

$$\begin{aligned} \Delta T_r &= 0 && \text{in } \mathbb{R}^3 \setminus (B_1 \cup B_2) \\ T_r &= 1 && \text{on } \partial B_1 \cup \partial B_2 \\ T_r &\to 0 && \text{at } \infty. \end{aligned}$$

The heat loss of the two bodies is then given by

$$C(r) := \int_{\partial B_1 \cup \partial B_2} \frac{\partial T_r}{\partial n} \, dS \tag{1.1}$$

where $\partial/\partial n$ is the differentiation along the inward normal.
Explicit computation based on knowledge of the general solution to Laplace equation in $\mathbb{R}^3 \setminus (B_1 \cup B_2)$ with $B_1$, $B_2$ disjoint translates of the unit ball led Glasser and Davidson to the formula

$$C(r) = \mathrm{const} \sum_{k=1}^{\infty} (-1)^k \frac{1}{U_k(r)} \quad \forall r > 1, \tag{1.2}$$

where $(U_k)_{k\geq 1}$ are the Chebyshev polynomials of the second kind given by the recursion

$$U_0(r) = 1, \quad U_1(r) = 2r, \qquad U_{k+1}(r) = 2rU_k(r) - U_{k-1}(r) \quad \forall k \geq 1. \tag{1.3}$$

Equivalently,

$$\begin{aligned} U_k(r) &= \left. \frac{x^{2k+2}-1}{x^{k+2}-x^k} \right|_{x=r+\sqrt{r^2-1}} \\ &= \left. \left. \frac{x}{x^2-1} \cdot \frac{1}{\sqrt{y(y+1)}} \right|_{y=\frac{1}{x^{2k+2}-1}} \right|_{x=r+\sqrt{r^2-1}} \qquad \forall r > 1 \end{aligned} \tag{1.4}$$

or (see e.g. [5]),

$$U_k(r) = \prod_{j=1}^{k} (r - \cos\frac{j\pi}{k+1}) \quad \forall r > 1. \tag{1.5}$$

The monotonicity of $C$ given by (1.2) on $[1, \infty[$ was an open question (see e.g. [2]) until very recently, when Eremenko, returning to (1.1) and avoiding in fact the explicit computation of Glasser and Davidson, found an elegant solution using basic principles of classical potential theory (see e.g. [6]). We recommend the reader to consult [1] for this solution and for an illuminating discussion of the original problem and of its possible generalizations.
In this note we propose a direct, completely elementary (but rather technical) argument showing that the monotonicity of $C$ given by (1.2) follows from the convexity properties of the function $\mathbb{N}_+ \ni k \to U_k'(r)/U_k^2(r) \in [0, \infty[$ for a fixed $r > 1$.

## 2 Solution

An equivalent formulation of the problem is to prove that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{U_k'(r)}{U_k(r)^2} > 0 \quad \forall r > 1. \tag{2.1}$$

A straight calculation based on the representation formula (1.4) shows that the general term $R_k := U_k'/U_k^2$ of the alternating series (2.1) (formula (1.5) ensures that $U_k$ is strictly increasing on $[1, \infty[$) is given by

$$R_k(r) = \left. \left. 2\sqrt{y(y+1)} \left( k - \frac{2}{x^2-1} + 2(k+1)y \right) \right|_{y=\frac{1}{x^{2k+2}-1}} \right|_{x=r+\sqrt{r^2-1}},$$

which suggests considering the natural interpolation between the integers given by $R : [0, \infty[ \times ]1, \infty[ \to [0, \infty[$,

$$R(t,x) := 2\sqrt{y(y+1)}\left(t - \frac{2}{x^2-1} + 2(t+1)y\right)\Bigg|_{y=\frac{1}{x^{2t+2}-1}}. \tag{2.2}$$

Clearly then,

$$R_k(r) = R(k, r + \sqrt{r^2-1}) \quad \forall k \in \mathbb{N}_+, \forall r > 1, \tag{2.3}$$

and further simple properties of the function $R$ are collected in the following result.

**Proposition 2.1** *For $R$ defined by* (2.2) *it holds (compare Figure 2),*
i. $R(t,x) \geq 0$, $\forall (t,x) \in [0,\infty[ \times ]1,\infty[$, *with equality only for* $t = 0$.
ii. $\lim_{t \nearrow \infty} R(t,x) = 0\,,\ \forall x \in ]1,\infty[$.
iii. $\lim_{x \searrow 1} R(t,x) = \frac{1}{3}\left(t + 1 - \frac{1}{t+1}\right)\,,\ \forall t \in [0,\infty[$.

*Proof.* i. It suffices to prove that the derivative w.r.t. $t$ of the bracketed expression in (2.2) is positive for $t \in ]0,\infty[$, which is equivalent to $x^{4t+4} > 2(t+1)x^{2t+2}\log x + 1$ or, with the substitution $z := x^{2t+2}$, to $z^2 > z \log z + 1$ for $z > 1$. This follows from $\log z < z - 1$. Property ii. is obvious while iii. is just a simple calculus exercise left to the reader. ☐

Plots of $R(\cdot, x)$ are shown in Figure 2 for several values of $x$. The simple but crucial observation is that for any $x > 1$ the function $R(\cdot, x)$ has exactly one inflection point $t_x \in ]0,\infty[$ and this is what we prove in Theorem 2.2 below.
Clearly, (2.1) becomes with the notations introduced above,

$$\sum_{k=1}^{\infty} (-1)^{k+1} R(k,x) > 0 \quad \forall x > 1. \tag{2.4}$$

Before proceeding, let us explain the proof idea. We distinguish two cases, as follows.
I. For large values of $x$ we show that the sequence $(R(k,x))_{k\geq 1}$ is strictly decreasing (this can be seen in Figure 2 already for $x = 3$), so that (2.4) holds trivially.
II. For $x$ close to 1 a more refined argument is needed. Essentially, what we use is that for a concave function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ and a sequence $x_k = x_0 + kh, k = 0, 1, 2, \ldots$ of equidistant points ($h > 0$) in $I$ we have

$$f(x_1) - f(x_2) + f(x_3) - \ldots + f(x_{2n-1}) \geq \frac{1}{2}\left(f(x_{2n}) + f(x_0)\right) \quad \forall n \geq 1, \tag{2.5}$$

which follows easily by summing over $1 \leq j \leq n$ the obvious inequalities $f(x_{2j-1}) - f(x_{2j-2}) \geq (f(x_{2j}) - f(x_{2j-2}))/2$. A reversed inequality holds of course for convex functions.
We employ (2.5) twice for the function $R(\cdot, x)$, on both the concavity and convexity domains of $R(\cdot, x)$, avoiding the region around the inflection point $t_x$. The part of the

sum (2.4) corresponding to $k \sim t_x$ is then estimated using a sharp lower bound on the derivative $dR/dt(\cdot, x)$ (see Lemma 2.4). Finally we conclude the proof by noting that this negative contribution to the series (2.4) is compensated by the positivity of the sum (2.5) corresponding to the concavity domain $[0, t_x]$ of $R(\cdot, x)$.

**Theorem 2.2** *For any $x > 1$ there exists a unique $t_x \in ]0, \infty[$ such that $R(\cdot, x)$ is strictly concave on $[0, t_x[$ and strictly convex on $]t_x, \infty[$.*

*Proof.* A straight calculation based on formula (2.2) shows that

$$\frac{dR}{dt}(t,x) = \sqrt{y(y+1)}\left((1+2y)c(x) - (1+8y+8y^2)\log(1+\frac{1}{y})\right)\bigg|_{y=\frac{1}{x^{2t+2}-1}} \tag{2.6}$$

where $c : [1, \infty[ \to [4, \infty[$ is strictly increasing, given by

$$c(x) := 2(1 + \log x + \frac{2}{x^2-1}\log x) \quad \forall x > 1.$$

Further,

$$\frac{d^2R}{dt^2}(t,x) = -(\log x)\sqrt{y(y+1)}\,\Big((1+8y+8y^2)(c(x)+2) - (1+2y)(1+24y+24y^2)\log(1+\frac{1}{y})\Big)\bigg|_{y=\frac{1}{x^{2t+2}-1}}. \tag{2.7}$$

Due to $R(t,x) \searrow 0$ as $t \nearrow \infty$, it suffices to prove (see Corollary 2.3 below) that for any $x > 1$ there exists a unique zero $t_x \in ]0, \infty[$ of $d^2R/dt^2(\cdot, x)$. In view of (2.7) and with the substitution

$$u := 2(t+1)\log x/(c(x)+2) \tag{2.8}$$

the zeros of $d^2R/dt^2(\cdot, x)$ in $]-1, \infty[$ are given via (2.8) by

$$\frac{1}{1+2y}\left(\frac{1}{3} + \frac{2}{3(1+24y+24y^2)}\right)\bigg|_{y=\frac{1}{e^{(c(x)+2)u}-1}} = u > 0. \tag{2.9}$$

Equation (2.9) can be viewed as a fixed point problem

$$(g \circ h)(u) = u \in ]0, \infty[ \tag{2.10}$$

where

$$h : ]0, \infty] \to ]0, \infty[, \quad h(u) = \frac{1}{e^{(c(x)+2)u} - 1}$$

and

$$g : ]0, \infty[ \to ]0, 1[, \quad g(y) = \frac{1}{1+2y}\left(\frac{1}{3} + \frac{2}{3(1+24y+24y^2)}\right).$$

Since $\operatorname{Ran} g \subset ]0,1[$, all solutions $u$ to (2.10) lie in $]0,1[$. Moreover, the existence of such a solution follows by observing that $(g \circ h)(1) < 1$ (trivially) and $(g \circ h)(u) > u$ for $u > 0$ small enough. Indeed, the latter is equivalent to

$$\frac{1}{1+2y}\left(\frac{1}{3} + \frac{2}{3(1+24y+24y^2)}\right) > \frac{1}{c(x)+2}\log\left(1+\frac{1}{y}\right)$$

for $y > 0$ large enough and this holds asymptotically as $y \to \infty$ due to $c(x) > 4$.
It remains therefore to show that the solution is unique. To this end we compute the derivative of $g \circ h$,

$$(g \circ h)'(u) = (c(x)+2)i(z)\big|_{z=y(y+1)}\big|_{y=\frac{1}{e^{u(c(x)+2)}-1}},$$

where $i : [0,\infty[ \to [0,\infty[$ is given by

$$i(z) := z\left(\frac{2}{3(4z+1)} + \frac{1}{3(4z+1)(6z+1/4)} + \frac{1}{(6z+1/4)^2}\right).$$

It is easy to see that $i$ is strictly increasing on $[0, z_{\max}]$ and strictly decreasing to $1/6$ on $[z_{\max}, \infty[$ with $z_{\max}$ the unique positive solution of $84z^2 - 19z/4 - 27/32 = 0$. This implies the existence of a unique $z_x^* \in ]0, z_{\max}[$ such that $i < 1/(c(x)+2)$ on $[0, z_x^*[$, $i(z_x^*) = 1/(c(x)+2)$, and $i > 1/(c(x)+2)$ on $]z_x^*, \infty[$.
Equivalently, $(g \circ h)' > 1$ on $]0, u_x^*[$ and $0 < (g \circ h)' < 1$ on $]u_x^*, \infty[$, where

$$u_x^* = \frac{1}{c(x)+2}\log\left(1 + \frac{1+\sqrt{1+4z_x^*}}{2z_x^*}\right).$$

In particular, $g \circ h$ is a contraction on $[u_x^* + \epsilon, 1]$ for any $\epsilon > 0$ small enough, such that also $(g \circ h)(u_x^* + \epsilon) > u_x^* + \epsilon$ holds. The uniqueness of a fixed point in $[u_x^* + \epsilon, 1]$ for $g \circ h$ is therefore proven. Besides, $g \circ h$ can not have further fixed points in $]0, u_x^* + \epsilon[$ if $\epsilon$ is small enough to ensure $(g \circ h)' > 1$ on $]0, u_x^* + \epsilon[$, too.
Due to (2.8), the corresponding inflection point $t_x$ of $R(\cdot, x)$ belongs to $[-1, \infty[$. But the properties i. and ii. in Proposition 2.1 ensure the existence of at least one inflection point of $R(\cdot, x)$ in $]0, \infty[$, so that we conclude $t_x \in ]0, \infty[$. □

From Proposition 2.1 and Theorem 2.2 we obtain

**Corollary 2.3** *For any $x > 1$ there exists a unique $T_x \in ]0, t_x[$ such that $R(\cdot, x)$ is strictly increasing on $[0, T_x]$ and strictly decreasing on $[T_x, \infty[$.*

*Proof.* For any $x > 1$ properties i. and ii. in Proposition 2.1 ensure the existence of a global maximum of $R(\cdot, x)$ while Theorem 2.2 implies the existence of at most one extremal point for $R(\cdot, x)$. □

Theorem 2.2 and Corollary 2.3 mathematically describe the observed shape of $R(\cdot, x)$ (see Figure 2), which consists of a bump followed by a long convex (and quite flat, as we shall see next) tail. It can be inferred from Figure 2 that the steepest descent of $R(\cdot, x)$ has an approximate slope of $-1/10$, independent of $x$. Indeed, we have

**Lemma 2.4** *It holds*

$$\frac{dR}{dt}(t,x) \geq \frac{dR}{dt}(t_x,x) \geq -0.107 \quad \forall t \in [0,\infty], \forall x \geq 1. \tag{2.11}$$

*Proof.* Noting that $c(x) \geq 4$ for any $x \geq 1$, we deduce from formula (2.6) that for any $t, x$ as in (2.11) we have

$$\frac{dR}{dt}(t,x) \geq \inf_{y\geq 0} \sqrt{y(y+1)} \left(4(1+2y) - (1+8y+8y^2)\log(1+\frac{1}{y})\right). \tag{2.12}$$

The infimum in (2.12) (which is also the unique local minimum of the corresponding function; this follows from the proof of Theorem 2.2) can be found numerically and it equals $-0.1061...$ (see Figure 3). □

## 2.1 The case $x$ away from 1

As Figure 2 for $x = 3$ suggests, in this case the sequence $(R(k,x))_{k\in\mathbb{N}_+}$ is decreasing (the bump migrates towards $t = 0$ as $x \nearrow \infty$), so that the positivity of (2.4) follows trivially.

**Proposition 2.5** *If* $x \geq 3$, *then*

$$R(k,x) > R(k+1,x) \quad \forall k \in \mathbb{N}_+. \tag{2.13}$$

*In particular* $R(2n-1,x) > R(2n,x)$ *for any* $n \in \mathbb{N}_+$, *so that* (2.4) *holds for* $x \geq 3$.

*Proof.* Fix $x \geq 3$ and note first that the inequality (2.13) for $k = 1$ can be proved easily using (2.3) and the explicit formulae $R_1(r) = 1/2r^2$, $R_2(r) = 8r/(4r^2-1)^2$ for $r \geq 1.5$ (ensured by $x \geq 3$). From Corollary 2.3 we deduce then $T_x \leq 1$, so that $R(\cdot,x)$ is decreasing on $[1,\infty[$, which gives (2.13).
Alternatively, due to (2.3) it suffices to show for any $k \geq 2$ that

$$\frac{U_k'(r)}{U_k^2(r)} > \frac{U_{k+1}'(r)}{U_{k+1}^2(r)} \quad \forall r \geq 1.5. \tag{2.14}$$

Differentiating the recursive formula (1.3) we see that

$$U_{k+1}'(r) = 2U_k(r) + 2rU_k'(r) - U_{k-1}'(r). \tag{2.15}$$

Dividing (2.15) by $U_k'(r)$, and using the resulting formula as well as the recursion (1.3), we obtain that (2.14) is equivalent to

$$2r + 2\frac{U_k(r)}{U_k'(r)} - \frac{U_{k-1}'(r)}{U_k'(r)} < \left(2r - \frac{U_{k-1}(r)}{U_k(r)}\right)^2. \tag{2.16}$$

But (1.5) and the Jensen inequality for the convex function $]-\infty, r[\ni s \to 1/(r-s) \in \mathbb{R}_+$ ensure

$$\frac{U_k(r)}{U_k'(r)} = \left(\sum_{j=1}^{k} \frac{1}{r-\cos(j\pi/(k+1))}\right)^{-1} \leq \left(\frac{k}{r}\right)^{-1} = \frac{r}{k},$$

so that a sufficient condition for (2.16) to hold is

$$2r + 2\frac{r}{k} \leq \left(2r - \frac{U_{k-1}(r)}{U_k(r)}\right)^2. \tag{2.17}$$

From (1.4) we deduce that

$$\frac{U_{k-1}(r)}{U_k(r)} = \left.\frac{x + x^2 + \cdots + x^{2k}}{1 + x + \cdots + x^{2k+1}}\right|_{x=r+\sqrt{r^2-1}} \leq \frac{1}{r + \sqrt{r^2 - 1}} \leq \frac{1}{r},$$

so that (2.17) holds if ($k \geq 2$)

$$3r \leq \left(2r - \frac{1}{r}\right)^2.$$

This inequality is equivalent to $4r^4 - 3r^3 - 4r^2 + 1 \geq 0$, which holds for $r \geq 1.5$ due to $2r^4 \geq 3r^3$ and $2r^4 \geq 4r^2$. □

## 2.2 The case $x$ close to 1

**Lemma 2.6** *For any* $1 \leq x \leq 3.7$ *it holds*

$$2R(1, x) - R(2, x) \geq 1/9 > 0.111. \tag{2.18}$$

*Proof.* Due to (2.3) it suffices to prove that $2R_1(r) - R_2(r) \geq 1/9$ for any $1 \leq r \leq 2$. But $R_1(r) = 1/2r^2$ and $R_2(r) = 8r/(4r^2 - 1)^2$, so that the claimed inequality becomes, after some simple algebra, $(r - 1)(16r^5 + 16r^4 - 136r^3 - 64r^2 + 9) \leq 0$. This holds for $1 \leq r \leq 2$ due to $16r^5 \leq 64r^3$, $16r^4 \leq 64r^2$ and $9 \leq 9r^3$. □

**Proposition 2.7** *For any* $1 \leq x \leq 3.7$ *it holds*

$$\sum_{k \geq 1} (-1)^{k+1} R(k, x) \geq \frac{1}{2}\left(1/9 + \frac{dR}{dt}(t_x, x)\right) > 0.002 > 0. \tag{2.19}$$

*Proof.* Set $k_x := \lfloor t_x \rfloor \in \mathbb{N}$ and note that the concavity of $R(\cdot, x)$ on $[0, k_x]$ implies that the function $[1, k_x] \ni t \to R(t, x) - R(t - 1, x) \in \mathbb{R}$ is decreasing (having negative derivative). Coupling this property with (2.18) we obtain, with $\delta_{0k}$ the Kronecker symbol,

$$R(2k + 1, x) - R(2k, x) \geq R(2k + 2, x) - R(2k + 1, x) + \delta_{0k}\frac{1}{9},$$

for $k = 0$ and any $1 \leq k \leq (k_x - 2)/2$. Adding $R(2k + 1, x) - R(2k, x)$ on both sides and summing over $k$ we obtain, with $l_x := \max\{2, k_x\}$,

$$\sum_{k=1}^{l_x - 1} (-1)^{k+1} R(k, x) \geq \frac{1}{2}\left(R(l_x, x) + \frac{1}{9}\right), \tag{2.20}$$

where we have assumed $l_x$ to be an even integer (the case $l_x$ odd follows analogously). A similar argument based on the convexity of $R(\cdot, x)$ on $[l_x + 1, \infty[$ leads to

$$\sum_{k=l_x+2}^{\infty} (-1)^{k+1} R(k, x) \geq -\frac{1}{2} R(l_x + 1, x). \tag{2.21}$$

Summing inequalities (2.20) and (2.21) we obtain a lower estimate for the part of the series (2.4) corresponding to $k \notin \{l_x, l_{x+1}\}$. The conclusion follows then by adding $-R(l_x, x) + R(l_x + 1, x)$ on both sides of the resulting estimate, taking into account that, due to Lemma 2.4 and to the mean value theorem,

$$R(l_x + 1, x) - R(l_x, x) \geq \frac{dR}{dt}(t_x, x) > -0.107.$$

Note that in order to estimate the terms of (2.4) with $k$ around $k_x$ in the case $l_x$ odd ($l_x = k_x$ then) one should first derive estimates of type (2.20) and (2.21) for $k \leq k_x - 2$ and $k \geq k_x + 3$ respectively, and use then the inequalities

$$\begin{aligned} -R(k_x - 1, x) + R(k_x, x) &\geq \frac{-R(k_x, x) + R(t_x, x)}{t_x - k_x} \\ -R(k_x + 1, x) + R(k_x + 2, x) &\geq \frac{-R(t_x, x) + R(k_x + 1, x)}{k_x + 1 - t_x} \end{aligned}$$

(which follow from the concavity/convexity of $R(\cdot, x)$ on $[0, t_x]/[t_x, \infty[$ respectively) to control the terms of (2.4) for $k \in \{k_x - 1, k_x, k_x + 1, k_x + 2\}$. □

Propositions 2.5 and 2.7 cover the case $x > 1$, so that the proof of (2.4) is complete.

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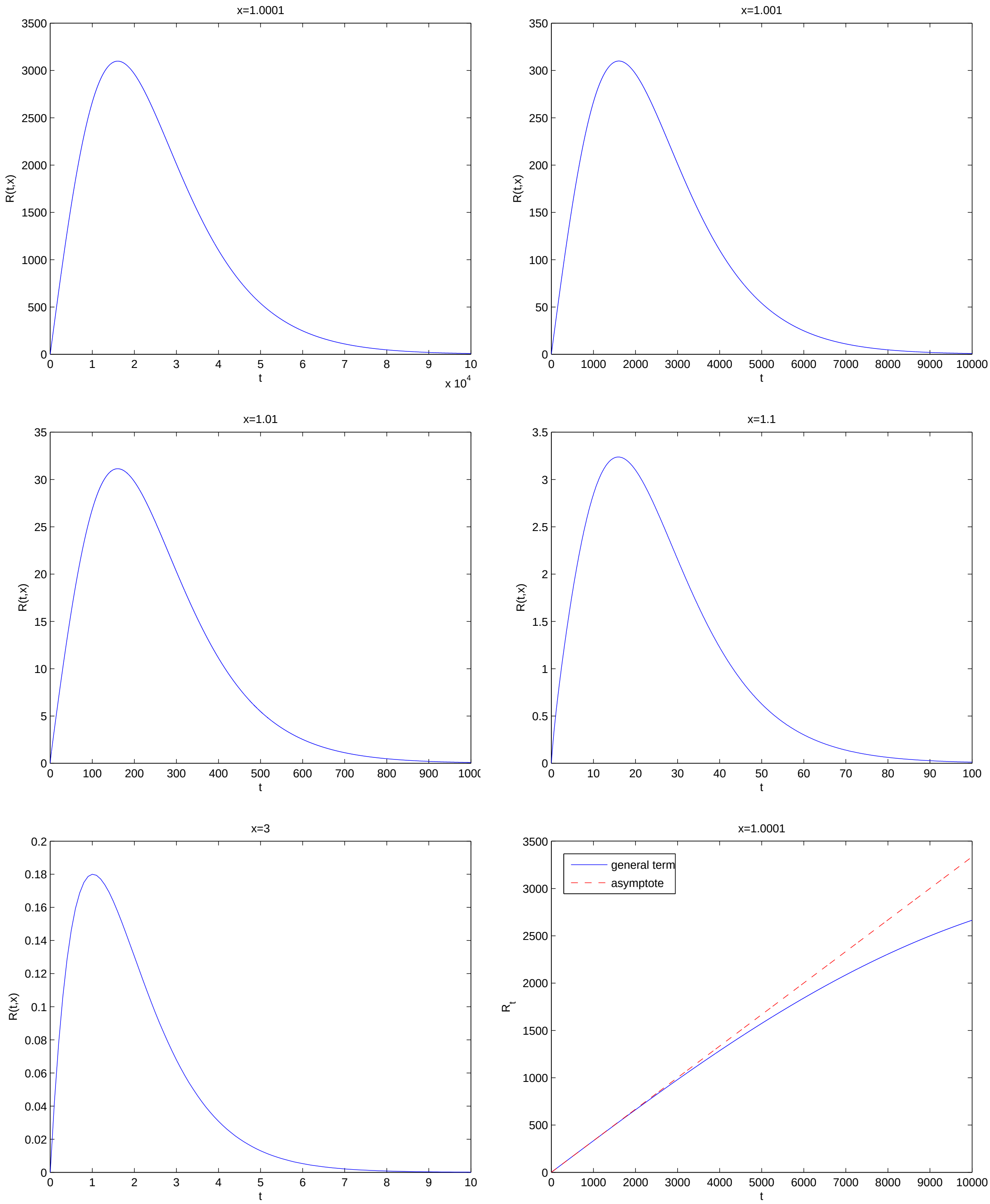


Figure 2: Plots of $R(\cdot, x)$ for $x = 1.0001, 1.001, 1.01, 1.1, 3$ and asymptotic behaviour of $R(t, x)$ as $x \searrow 1$.

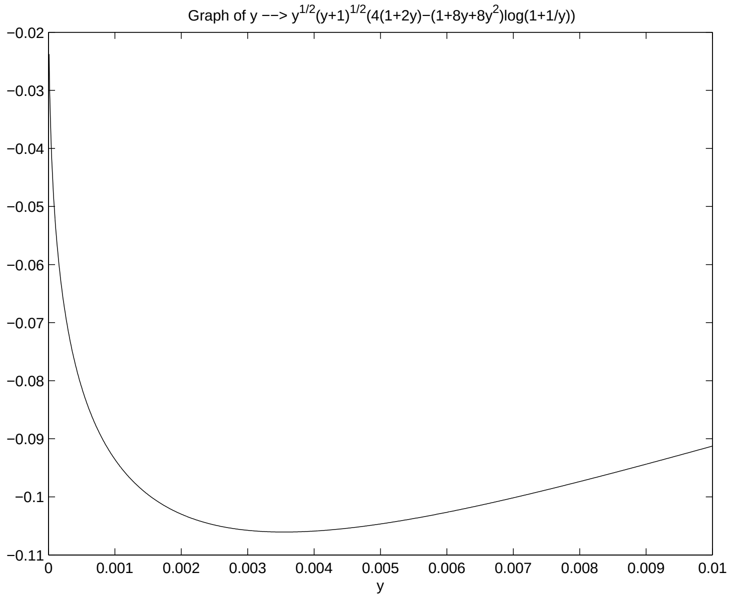


Figure 3: The r.h.s. of (2.12) in a neigbourhood of its extremal point.