ERRATUM: Convergence of p-FEM for Maxwell Eigenvalue Problems

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Abstract

The main result of [3] (Lemma 7.3) asserting that a p-uniform discrete Friedrichs inequality (7.1) ensures the asymptotically spectrally correct approximation of Maxwell eigenvalues in the context of a Galerkin discretization by means of the p-version of edge elements is wrong. The error can be traced back to a spurious formula in [6, Sect. 4.3].

1. The error. Let us recall the definitions of $\mathsf{T} : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}\mathbf{u}, \mathbf{curl}\,\mathbf{u}'\right)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{\epsilon}\mathbf{u}, \mathbf{u}')_{L^2(\Omega)} \quad \forall \mathbf{u}' \in \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)\ ,$$

and of $\mathsf{T}_h : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h)$ by

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}_h\mathbf{u}, \mathbf{curl}\,\mathbf{u}_h'\right)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{\epsilon}\mathbf{u}, \mathbf{u}_h')_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{u}_h' \in \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h)\ .$$

By the very definition of the spaces, we cannot expect $\boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h) \subset \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$. Therefore it is not possible to add both variational equations, which means that

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}(\mathsf{T}_h - \mathsf{T})\mathbf{u}, \mathbf{curl}\,\mathbf{u}_h'\right)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \mathbf{u}_h' \in \mathcal{W}^1_{p,0}(\Omega_h)\ ,$$

cannot be true. As a consequence

$$\mathsf{T}_h \neq \mathsf{F}_p \circ \mathsf{T}\ . \tag{1.1}$$

There is another argument that instantly refutes equality in (1.1): We have $\mathrm{Ker}(\mathsf{T}) = \boldsymbol{H}_0(\mathbf{curl}\,0; \Omega)$, but, of course, the kernel of T_h is much smaller.

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2. Counterexamples. The fact that a discrete Poincaré-Friedrichs inequality is not sufficient for convergence of Maxwell eigenvalues has been observed repeatedly. In [1, Sect. 6.2] numerical experiments confirm a discrete Poincaré-Friedrichs inequality for nodal elements on criss-cross meshes, but, nevertheless, spurious eigenvalues crop up. In [4] another couterexample is given for the p-version of nodal elements.

3. Related work.

- In [2] a rather general theory of spectrally correct approximations of Maxwell eigenvalues has been developed. The authors give necessary and sufficient criteria. In particular a discrete Poincaré-Friedrichs inequality is identified as a necessary condition, but it is not sufficient. Only *discrete compactness* of the trial spaces is shown to guarantee convergence of the eigenvalues.
- Considerations similar to those in Sects. 6 and 7 of the report can be found in [5].

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Abstract

The paper considers the solution of Maxwell eigenvalue problems by the p-version of **curl**-conforming finite elements on tetrahedral meshes. The asymptotic quasi-optimal convergence of discrete eigenvalues and eigenvectors as $p \to \infty$ is proved. The proof relies on a novel technique combining tools from the calculus of differential forms with techniques for simplicial complexes.

Keywords: Maxwell eigenvalue problem, p-version of edge elements, discrete Poincaré-Friedrichs inequality, Poincaré map

Subject Classification: 65N30, 78M10

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4. Introduction. Resonant electromagnetic modes of a bounded dielectric cavity $\Omega \subset \mathbb{R}^3$ with perfectly conducting walls can be computed as solutions of the eigenvalue problem

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{e} &= \omega^2 \boldsymbol{\epsilon}\mathbf{e} && \text{in } \Omega\,, \\ \mathbf{e} \times \mathbf{n} &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{4.1}$$

that belong to non-zero eigenvalues ω^2. Here, $\mathbf{n}$ is the exterior unit normal vector on $\partial\Omega$, $\mathbf{e}$ is the electric field, and $\boldsymbol{\mu}$, $\boldsymbol{\epsilon}$ are the magnetic permeability tensor and the dielectric tensor, respectively. The metric tensors $\boldsymbol{\mu}, \boldsymbol{\epsilon} \in (L^\infty(\Omega))^{3\times 3}$ are assumed to be uniformly symmetric positive definite and piecewise smooth with respect to a Lipschitz partitioning of Ω. Moreover, the computational domain Ω itself is supposed to be a polyhedron with Lipschitz continuous boundary. This assumption is merely made for the sake of simplicity: using parametric finite elements the results can easily be extended to curvilinear Lipschitz polyhedra.

The Maxwell eigenvalue problem (4.1) is special in the sense that it features a large eigenspace of "unphysical solutions", namely the eigenspace for $\omega = 0$. This compounds difficulties in the analysis of conforming finite element schemes for (4.1) compared with eigenvalue problems associated with positive definite second order differential operators like $-\Delta$.

Meanwhile, much insight has been gained into the convergence of the h-version of **curl**-conforming finite elements for (4.1). Starting with Kikuchi's work [22], the bulk of the studies centered on the the concepts of discrete compactness and collective compactness [1]. Among others, the works [24, 11, 17, 16] fall in this category. An approach based on a mixed formulation of (4.1) was pursued in [7], based on techniques introduced in [5]. Later, in [4, 3] a more refined analysis was given. However, all attempts to establish the crucial discrete compactness property for spectral **curl**-conforming finite elements have failed so far, see [6] for preliminary two-dimensional investigations.

In this article we study the pure p-version of **curl**-conforming finite elements for the Galerkin discretization of (4.1) based on a uniform polynomial degree $p \in \mathbb{N}_0$ ($\mathbb{N}_0$ is the set of non-negative integers). We are mainly interested in establishing the convergence of approximate eigenvalues and eigenvectors as the polynomial degree p of the local finite element trial spaces becomes large, while the **triangulation is kept fixed**. In other words, asymptotic estimates for $p \to \infty$ will be the main focus of the paper.

Local interpolation estimates are instrumental for establishing the discrete compactness property for the h-version of **curl**-conforming finite elements (*cf.* [20, Sect. 4.4]). However, the lack of suitable inverse estimates in spaces of polynomials [28, Sect. 3.6] foil this approach to the p-version. Thus, we are forced to resort to completely different techniques, that combine tools from the calculus of differential forms with techniques for simplicial complexes. The drawback of these arguments is their inherent non-locality, which renders these ideas irrelevant for an investigation of the hp-version of finite elements, see [14].

The main result will be a proof of asymptotic quasi-optimal convergence of eigenvalues and eigenvectors as $p \to \infty$. By this we mean that, provided that p is large enough, the best approximation of eigenmodes in the finite element space will determine the discretization error in eigenvalues/eigenspaces.

5. Variational formulation and function spaces. Subjecting (4.1) to integration by parts, one derives the $\mathbf{e}$-based variational formulation: find $\mathbf{e} \in$

$\boldsymbol{H}_0(\mathbf{curl};\Omega)$, $\omega \neq 0$, such that

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{e},\mathbf{curl}\,\mathbf{e}'\right)_{\boldsymbol{L}^2(\Omega)} = \omega^2\,(\boldsymbol{\epsilon}\mathbf{e},\mathbf{e}')_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{e}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\;. \tag{5.1}$$

Here, we adopted the usual notation $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ for the Hilbert space of $\boldsymbol{L}^2(\Omega)$-vectorfields with **curl** in $\boldsymbol{L}^2(\Omega)$ and vanishing tangential components on $\partial\Omega$.

As can be seen by testing with irrotational functions, due to $\omega \neq 0$, (5.1) can be equivalently stated on the Hilbert space

$$\boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega) := \{\mathbf{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) :\ (\boldsymbol{\epsilon}\mathbf{u},\mathbf{z})_{\boldsymbol{L}^2(\Omega)} = 0\ \forall \mathbf{z} \in \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega)\}\;, \tag{5.2}$$

where $\boldsymbol{H}_0(\mathbf{curl}\,0;\Omega) = \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \mathrm{Ker}(\mathbf{curl})$. This results in: seek $\mathbf{e} \in \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$, $\omega \neq 0$, such that

$$\left(\mu^{-1}\,\mathbf{curl}\,\mathbf{e},\mathbf{curl}\,\mathbf{e}'\right)_{\boldsymbol{L}^2(\Omega)} = \omega^2\,(\boldsymbol{\epsilon}\mathbf{e},\mathbf{e}')_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{e}' \in \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)\;. \tag{5.3}$$

For the analysis of the variation problem (5.3) we have to rely on the Hilbert spaces

$$\begin{aligned} \boldsymbol{X}_N(\boldsymbol{\epsilon},\Omega) &:= \{\mathbf{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) :\ \mathrm{div}(\boldsymbol{\epsilon}\mathbf{u}) \in L^2(\Omega)\}\;, \\ \boldsymbol{X}_T(\boldsymbol{\mu}^{-1},\Omega) &:= \{\mathbf{u} \in \boldsymbol{H}_0(\mathrm{div};\Omega) :\ \mathbf{curl}(\boldsymbol{\mu}^{-1}\mathbf{u}) \in L^2(\Omega)\}\;. \end{aligned}$$

Importance is bestowed on these spaces by the following embedding result, see [29, 27, 21, 13] and [20, Sect. 4.1].

LEMMA 5.1. *The embedding* $\boldsymbol{X}_N(\boldsymbol{\epsilon},\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{X}_T(\boldsymbol{\mu}^{-1},\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ *is compact.*

Then standard arguments yield the following estimate, *cf.* [20, Thm. 4.7].

COROLLARY 5.2 (Poincaré-Friedrichs-type inequality). *With a constant* $C > 0$ *depending on* Ω *only, holds true*

$$\|\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} \leq C\,\|\mathbf{curl}\,\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{u} \in \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)\;.$$

Following [24], for the analysis of the eigenvalue problem (5.3) it is convenient to introduce the operator $\mathsf{T} : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$ by

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}\mathbf{u},\mathbf{curl}\,\mathbf{u}'\right)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{\epsilon}\mathbf{u},\mathbf{u}')_{L^2(\Omega)} \quad \forall \mathbf{u}' \in \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)\;. \tag{5.4}$$

By Corollary 5.2 it is well defined, continuous and $\boldsymbol{L}^2(\Omega)$-selfadjoint.

LEMMA 5.3. *The operator* $\mathsf{T} : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$ *is compact.*

Proof. By its variational definition (5.4), the vectorfield $\mathsf{T}\mathbf{u}$, $\mathbf{u} \in \boldsymbol{L}^2(\Omega)$, satisfies

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}\mathbf{u} &= \boldsymbol{\epsilon}\mathbf{u} && \text{in } \Omega\;, \\ \mathrm{div}(\boldsymbol{\epsilon}\mathsf{T}\mathbf{u}) &= 0 && \text{in } \Omega\;, \\ \mathsf{T}\mathbf{u} \times \mathbf{n} &= 0 && \text{on } \partial\Omega\;. \end{aligned} \tag{5.5}$$

This yields, thanks to $\mathrm{div} \circ \mathbf{curl} = 0$ and $\mathbf{curl}(\mathsf{T}\mathbf{u}) \cdot \mathbf{n} = \mathrm{div}_\Gamma(\mathsf{T}\mathbf{u} \times \mathbf{n})$, that

$$\mathsf{T}\mathbf{u} \in \boldsymbol{X}_N(\boldsymbol{\epsilon},\Omega) \quad , \quad \mathbf{curl}\,\mathsf{T}\mathbf{u} \in \boldsymbol{X}_T(\boldsymbol{\mu}^{-1},\Omega)\;. \tag{5.6}$$

Since $\boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$ is a closed subspace of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, Lemma 5.1 implies the assertion. ∎

By means of T the eigenvalue problem (5.3) can be recast as

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{e}, \mathbf{curl}\,\mathbf{e}'\right)_{\boldsymbol{L}^2(\Omega)} = \omega^2 \left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}\mathbf{e}, \mathbf{curl}\,\mathbf{e}'\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{e}' \in \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)\;,$$

and converted into the operator eigenvalue problem

$$\mathsf{T}\mathbf{e} = \omega^{-2}\mathbf{e}\;. \tag{5.7}$$

Thus, the Riesz-Schauder theory for the spectrum of compact, self-adjoint operators in Hilbert space implies that (5.1) has an increasing sequence $0 < \omega_1^2 < \omega_2^2 < \ldots$ of nonzero real Maxwell eigenvalues tending to ∞. They all have finite multiplicity and the corresponding eigenspaces are mutually $\boldsymbol{L}^2(\Omega)$-orthogonal. Hence, by switching to the complement $\boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$ of Ker($\mathbf{curl}$) we have recovered a situation typical of second order elliptic eigenproblems.

6. The discrete eigenvalue problem. Let T be a non-degenerate tetrahedron in $\mathbb{R}^3$ and write $\mathcal{P}_p(T)$ for the space of 3-variate polynomials with total degree $\leq p$, $p \in \mathbb{N}_0$, on T. $\mathbb{N}_0$ is the set of non-negative integers. Taking the cue from [18], for any $p \in \mathbb{N}_0$, we chose the local trial space

$$\mathcal{W}_p^1(T) := \left\{ \begin{array}{l} \mathbf{x} \in T \mapsto \mathbf{u}_1(\mathbf{x}) + \mathbf{u}_2(\mathbf{x}) :\; \mathbf{u}_1 \in (\mathcal{P}_p(T))^3, \\ \mathbf{u}_2 \in (\mathcal{P}_{p+1}(T))^3,\; \mathbf{u}_2(\mathbf{x}) \cdot \mathbf{x} = 0 \quad \forall \mathbf{x} \in T. \end{array} \right\}\;.$$

Next, let Ω be equipped with a fixed conforming tetrahedral triangulation $\mathcal{M}_h$ and introduce

$$\begin{aligned} \mathcal{W}_p^1(\mathcal{M}_h) &:= \{\mathbf{u}_h \in \boldsymbol{H}(\mathbf{curl};\Omega) :\; \mathbf{u}_{h|T} \in \mathcal{W}_p^1(T) \quad \forall T \in \mathcal{M}_h\}\;, \\ \mathcal{W}_{p,0}^1(\mathcal{M}_h) &:= \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \mathcal{W}_p^1(\mathcal{M}_h)\;. \end{aligned} \tag{6.1}$$

This yields the first family of Nedéléc's $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming finite elements, see [26] for suitable local and global degrees of freedom. These spaces are closely linked with the space of continuous Lagrangian finite elements of degree $p+1$

$$\begin{aligned} \mathcal{W}_p^0(\mathcal{M}_h) &:= \{u_h \in H^1(\Omega) :\; u_{h|T} \in \mathcal{P}_{p+1}(T) \quad \forall T \in \mathcal{M}_h\}\;, \\ \mathcal{W}_{p,0}^0(\mathcal{M}_h) &:= \mathcal{W}_p^0(\mathcal{M}_h) \cap H_0^1(\Omega)\;, \end{aligned} \tag{6.2}$$

by the relationships

$$\begin{aligned} \mathbf{grad}\,\mathcal{W}_p^0(\mathcal{M}_h) &\subset \mathcal{W}_p^1(\mathcal{M}_h) \cap \boldsymbol{H}(\mathbf{curl}\,0;\Omega)\;, \\ \mathbf{grad}\,\mathcal{W}_{p,0}^0(\mathcal{M}_h) &\subset \mathcal{W}_{p,0}^1(\mathcal{M}_h) \cap \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega)\;. \end{aligned} \tag{6.3}$$

Please note that (6.3) becomes an equality, if the second Betti number of Ω vanishes, see [19, Sect. 6] and [20, Thm 3.7].

Based on $\mathcal{W}_{p,0}^1(\mathcal{M}_h)$ the Galerkin discretization of (5.1) is straightforward and leads to: seek $\mathbf{e}_h \in \mathcal{W}_{p,0}^1(\Omega_h)$, $\omega_p \neq 0$, such that

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{e}_h, \mathbf{curl}\,\mathbf{e}_h'\right)_{\boldsymbol{L}^2(\Omega)} = \omega_p^2 \left(\boldsymbol{\epsilon}\mathbf{e}_h, \mathbf{e}_h'\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{e}_h' \in \mathcal{W}_{p,0}^1(\mathcal{M}_h)\;. \tag{6.4}$$

Parallel to the continuous case, se (5.3), a variational problem equivalent to (6.4) can be posed on the space

$$\boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h) = \{\mathbf{u}_h \in \mathcal{W}_{p,0}^1(\mathcal{M}_h) :\; (\boldsymbol{\epsilon}\mathbf{u}_h, \mathbf{z}_h)_{\boldsymbol{L}^2(\Omega)} = 0\; \forall \mathbf{z}_h \in \mathcal{W}_{p,0}^1(\mathcal{M}_h) \cap \mathrm{Ker}(\mathbf{curl})\}\;.$$

Does this pave the way for studying the approximation of Maxwell eigenvalues along the same lines as for the discrete Laplacian, namely by appealing to the theory of eigenvalue approximation for self-adjoint positive definite operators with compact resolvent? Unfortunately, this hope is dashed by the observation that, in general,

$$\boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h) \not\subset \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)\ .$$

Bluntly speaking, in terms of (5.3) the variational problem (6.4) when restricted to $\boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h)$ is a *non-conforming* discretization.

7. Convergence of discrete eigenvalues. Theorem 4.7 in [20] asserts that the following *discrete Poincaré-Friedrichs inequality* holds true

$$\exists C = C(\mathcal{M}_h,p) > 0: \quad \|\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C\,\|\mathbf{curl}\,\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{u}_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)\ . \tag{7.1}$$

Therefore, we can define meaningful operators $\mathsf{T}_p : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h)$ by

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{T}_p\mathbf{u},\mathbf{curl}\,\mathbf{u}'_h\right)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{\epsilon}\mathbf{u},\mathbf{u}'_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{u}'_h \in \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h)\ . \tag{7.2}$$

It can be used to rewrite the discrete eigenvalue problem (6.4) in analogy to (5.7) in the form

$$\mathsf{T}_p\mathbf{e}_h = \omega_p^{-2}\mathbf{e}_h\ . \tag{7.3}$$

We aim to appeal to the powerful abstract theory of [2, Sect. 7] that enables us to assess the convergence of the eigenvalues and eigenvectors of T_p to those of T as soon as *uniform convergence*

$$\|\mathsf{T}_p - \mathsf{T}\|_{\boldsymbol{L}^2(\Omega)} \to 0 \quad \text{for} \quad p \to \infty \tag{7.4}$$

is established: Assuming (7.4) and writing $\mathbf{v}_1,\dots,\mathbf{v}_m$ for the orthonormalized eigenfunctions of T belonging to an eigenvalue $\omega^{-2} > 0$ of multiplicity $m \in \mathbb{N}$, we conclude from [2, Theorem 7.3] that **for all sufficiently large** $p > p_0$, $p_0 \in \mathbb{N}_0$, we will find m discrete eigenvalues $\omega_{p,1}^{-2},\dots,\omega_{p,m}^{-2}$ of T_p such that, with $C > 0$ independent of p,

$$\begin{aligned} |\omega^{-2} - \omega_{p,k}^{-2}| \leq C\Big(&\sum_{n,l=1}^{m} ((\mathsf{T}-\mathsf{T}_p)\mathbf{v}_n,\mathbf{v}_l)_{\boldsymbol{L}^2(\Omega)} + \\ &+ \left\|(\mathsf{T}-\mathsf{T}_p)_{|\,\mathrm{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_m\}}\right\|^2_{\boldsymbol{L}^2(\Omega)\to\boldsymbol{L}^2(\Omega)}\Big)\ . \end{aligned} \tag{7.5}$$

Moreover, for each $\mathbf{v}_k$ we find a discrete eigenfunction $\mathbf{v}_p \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)$ such that, with $C > 0$ independent of p,

$$\|\mathbf{v}_k - \mathbf{v}_p\|_{\boldsymbol{L}^2(\Omega)} \leq C\left\|(\mathsf{T}-\mathsf{T}_p)_{|\,\mathrm{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_m\}}\right\|_{\boldsymbol{L}^2(\Omega)\to\boldsymbol{L}^2(\Omega)} \quad . \tag{7.6}$$

Note that, in both cases the constants are independent of the choice of the polynomial degree p.

To gauge the difference between T and T_p we introduce a *Fortin projector* $\mathsf{F}_p : \boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega) \mapsto \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h)$ according to [3]. It can be defined via the saddle point problem: seek $\mathsf{F}_p\mathbf{u} \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)$, $\mathbf{w}_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h) \cap \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega)$ such that

$$\begin{aligned} \left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathsf{F}_p\mathbf{u},\mathbf{curl}\,\mathbf{u}'_h\right)_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{\epsilon}\mathbf{u}'_h,\mathbf{w}_h)_{\boldsymbol{L}^2(\Omega)} &= \left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{u}'_h\right)_{\boldsymbol{L}^2(\Omega)}\ , \\ (\boldsymbol{\epsilon}\mathsf{F}_p\mathbf{u},\mathbf{w}'_h)_{\boldsymbol{L}^2(\Omega)} &= 0\ , \end{aligned} \tag{7.7}$$

for all $\mathbf{u}'_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)$, $\mathbf{w}'_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h) \cap \boldsymbol{H}_0(\mathbf{curl}\, 0;\Omega)$. Owing to the discrete Poincaré–Friedrichs inequality (7.1), F_p is a well-defined continuous operator.

LEMMA 7.1. *If* (7.1) *holds with a constant* independent of p, *then* $\mathsf{F}_p \to \mathrm{Id}$ *pointwise on* $\boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$ *as* $p\to\infty$.

Proof. We verify the assumption of Babuška-Brezzi theory for Galerkin solutions of variational saddle point problems, see [8, Ch. 3]. Since

$$(\boldsymbol{\epsilon}\mathsf{F}_p\mathbf{u},\mathbf{w}'_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \mathbf{w}'_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)\cap \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega) \quad \Leftrightarrow \quad \mathsf{F}_p\mathbf{u} \in \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon},\mathcal{M}_h)\ ,$$

the p-uniform "ellipticity on the kernel" is an immediate consequence of the assumption of the lemma and the uniform positivity of $\boldsymbol{\mu}$. Moreover, with $C = C(\boldsymbol{\epsilon}) > 0$, we get for all $\mathbf{w}_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)\cap \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega)$,

$$\sup_{\mathbf{v}_h\in\mathcal{W}^1_{p,0}(\mathcal{M}_h)} \frac{(\boldsymbol{\epsilon}\mathbf{v}_h,\mathbf{w}_h)_{\boldsymbol{L}^2(\Omega)}}{\|\mathbf{w}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq \frac{(\boldsymbol{\epsilon}\mathbf{w}_h,\mathbf{w}_h)_{\boldsymbol{L}^2(\Omega)}}{\|\mathbf{w}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq C\,\|\mathbf{w}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = C\,\|\mathbf{w}_h\|_{\boldsymbol{L}^2(\Omega)}\ .$$

This confirms quasi-optimality of solutions of (7.7). Next, note that testing the first equation of (7.7) with $\mathbf{u}'_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)\cap \boldsymbol{H}_0(\mathbf{curl}\,0;\Omega)$ reveals $\mathbf{w}_h = 0$. This implies, for $\mathbf{u}\in\boldsymbol{Z}_0(\boldsymbol{\epsilon},\Omega)$,

$$\|\mathbf{u}-\mathsf{F}_p\mathbf{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq C \inf_{\mathbf{v}_h\in\mathcal{W}^1_{p,0}(\mathcal{M}_h)} \|\mathbf{u}-\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\ ,$$

with a constant $C>0$ independent of p. We know that compactly supported smooth vectorfields in $(C_0^\infty(\Omega))^3$ are dense in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, see [18, Ch. 2]. Therefore, by simple interpolation estimates,

$$\inf_{\mathbf{v}_h\in\mathcal{W}^1_{p,0}(\mathcal{M}_h)} \|\mathbf{u}-\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \to 0 \quad \text{as} \quad p\to\infty\ .$$

This gives the assertion of the lemma. $\square$

¿From Sect. 10.2 of [23], in particular, Corollary 10.4, we immediately deduce the following result, whose elementary proof is provided for the sake of completeness.

LEMMA 7.2. *Let X,Y,Z be Banach spaces and $T : Z \mapsto X$ be a* compact *linear operator. If the bounded linear operators $F_n : X \mapsto Y$, $n\in\mathbb{N}$, are pointwise convergent for $n\to\infty$ with limit operator $F : X\mapsto Y$, then*

$$\lim_{n\to\infty} \sup\{\|(F_n - F)Tz\|_Y :\ z\in Z,\ \|z\|_Z \leq 1\} = 0\ ,$$

that is, $(F_n - F)\circ T \to 0$ uniformly.

Proof. By the uniform boundedness principle there is $C_F > 0$ such that

$$\max\{\|F\|_{X\mapsto Y}, \sup_{n\in\mathbb{N}} \|F_n\|_{X\mapsto Y}\} \leq C_F\ .$$

Fix $\epsilon>0$ and write $B = \{z\in Z : \|z\|_Z\leq 1\}$. As $T(B)$ is pre-compact, there is $M\in\mathbb{N}$ and $z_1,\dots,z_M\in B$ such that

$$T(B) \subset \bigcup_{k=1}^{M} \{x\in X : \|x - T(z_k)\|_X \leq \frac{\epsilon}{3C_F}\}\ .$$

Thanks to pointwise convergence of $F_n\to F$, there is $N\in\mathbb{N}$ such that

$$\|F_n(T(z_k)) - F(T(z_k))\|_Y \leq \tfrac{1}{3}\epsilon \quad \forall k = 1,\dots,M\ ,\ \forall n > N\ .$$

Then, for any $z \in B$,

$$\begin{aligned}
&\|F_n(Tz) - F(Tz)\|_Y \\
&\quad \leq \|F_n(Tz) - F_n(Tz_k)\|_Y + \|F_n(Tz_k) - F(Tz_k)\|_Y + \|F(Tz_k) - F(Tz)\|_Y \\
&\qquad\qquad \leq C_F\|Tz - Tz_k\|_X + \|F_n(Tz_k) - F(Tz_k)\|_Y + C_F\|Tz - Tz_k\|_X \leq \epsilon \;,
\end{aligned}$$

provided that $n > N$. □

LEMMA 7.3. *If* $\mathsf{F}_p \to \mathrm{Id}$ *pointwise on* $\boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$*, then* $\mathsf{T}_p \to \mathsf{T}$ *uniformly on* $\boldsymbol{L}^2(\Omega)$.

Proof. The definitions of T and T_p in (5.4) and (7.2), respectively, imply

$$\left(\boldsymbol{\mu}^{-1}\,\mathbf{curl}(\mathsf{T}_p - \mathsf{T})\mathbf{u}, \mathbf{curl}\,\mathbf{u}'_h\right)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \mathbf{u}'_h \in \mathcal{W}^1_{p,0}(\Omega_h) \;,$$

which can be expressed through the Fortin projector as

$$\mathsf{T}_p = \mathsf{F}_p \circ \mathsf{T} \;. \tag{7.8}$$

From Lemma 5.3 we recall that $\mathsf{T} : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$ is compact. Then, Lemma 7.2 with $Z := \boldsymbol{L}^2(\Omega)$, $X = \boldsymbol{Z}_0(\boldsymbol{\epsilon}, \Omega)$, $Y = \boldsymbol{L}^2(\Omega)$, $F = \mathrm{Id}$, $n := p$, and $F_n := \mathsf{F}_p$ finishes the proof. □

Summing up, in order to show the asymptotically quasi-optimal convergence of discrete eigenvalues and eigenvectors according to (7.5) and (7.6), we have to establish that the constant in the discrete Poincaré-Friedrichs inequality (7.1) can be chosen independent of p. This will be tackled in the remainder of the article.

8. The Poincaré map. The Poincaré map is an important tool in the calculus of differential forms. Here we will review some of its properties only briefly and refer to [12, Ch. 2] for an introduction to differential forms and more details about the Poincaré map.

Let D be a domain in $\mathbb{R}^3$ and write $\mathcal{DF}^l(D)$ for the vector space of continuous differential forms of degree l, $0 \leq l \leq 3$, on D, that is, the space of continuous mappings from D into the space of alternating l-multilinear forms on $\mathbb{R}^3$.

DEFINITION 8.1. *If D is star-shaped with respect to the origin, we define the Poincaré map* $\mathfrak{K}^l : \mathcal{DF}^l(D) \mapsto \mathcal{DF}^{l-1}(D)$ *by*

$$(\mathfrak{K}^l\omega)(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_{l-1}) := \int_0^1 t^{l-1}\omega(t\mathbf{x})(\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_{l-1})\, dt \tag{8.1}$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_{l-1} \in \mathbb{R}^3$*,* $\mathbf{x} \in D$*,* $\omega \in \mathcal{DF}^l(D)$*.* According to [12], Formula (2.13.2), it satisfies

$$\boldsymbol{d}(\mathfrak{K}^l\omega) + \mathfrak{K}^{l+1}(\boldsymbol{d}\omega) = \omega \;, \tag{8.2}$$

where $\boldsymbol{d}$ is the exterior derivative.

Below, we will need the Poincaré mapping for $l = 2$ only. Moreover, we will switch from the calculus of differential forms to the more traditional vector analytic point of view. This is possible, because vectorfields and functions provide models for differential forms. The isomorphisms that we are going to use relate differential forms and functions/vectorfields on D are summarized in Table 8.

Using the translation rules, for $l = 2$ we can rewrite the Poincaré map as

$$(\mathfrak{K}^2\mathbf{w})(\mathbf{x})\mathbf{v} = \int_0^1 t\,\mathbf{w}(t\mathbf{x}) \cdot (\mathbf{x} \times \mathbf{v})\, \mathrm{d}t \quad \forall \mathbf{v} \in \mathbb{R}^3 \;,$$

l	Differential form	Related function u/vectorfield $\mathbf{u}$
0	$\mathbf{x} \mapsto \omega(\mathbf{x})$	$u(\mathbf{x}) := \omega(\mathbf{x})$
1	$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \omega(\mathbf{x})(\mathbf{v})\}$	$\mathbf{u}(\mathbf{x}) \cdot \mathbf{v} := \omega(\mathbf{x})(\mathbf{v})$
2	$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(\mathbf{x}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)$
3	$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

TABLE 8.1

Relationship between differential forms and vectorfields in three-dimensional Euclidean space ($\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$). *The operation* $\cdot$ *is the canonical inner product in Euclidean space.*

where $\mathbf{w} : D \mapsto \mathbb{R}^3$ is continuous. This means, in terms of vectorfields,

$$(\mathfrak{K}^2\mathbf{w})(\mathbf{x}) = \int_0^1 t\,(\mathbf{w}(t\mathbf{x})) \times \mathbf{x}\,\mathrm{d}t \,. \tag{8.3}$$

Of course, we can also express (8.2) as an identity for vectorfields.

LEMMA 8.2. *Let D be star-shaped with respect to the origin. If $\mathbf{w} \in (C^1(\overline{D}))^3$, has zero divergence, then*

$$(\mathbf{curl}\,\mathfrak{K}^2\mathbf{w})(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \quad \forall \mathbf{x} \in D \,.$$

Proof. Writing $\mathbf{x} = (x_1, x_2, x_3)^T$, we obtain by straightforward calculation and the condition $\operatorname{div}\mathbf{w} = 0$ that

$$\mathbf{curl}(\mathbf{w}(t\mathbf{x}) \times \mathbf{x}) = t\sum_{i=1}^{3} x_i \frac{\partial \mathbf{w}}{\partial x_i}(t\mathbf{x}) + 2\mathbf{w}(t\mathbf{x}) = t\frac{d}{dt}\mathbf{w}(t\mathbf{x}) + 2\mathbf{w}(t\mathbf{x}) \,.$$

Thus, by integration by parts,

$$\begin{aligned}(\mathbf{curl}\,\mathfrak{K}^2\mathbf{w})(\mathbf{x}) &= \int_0^1 t\ \mathbf{curl}(\mathbf{w}(t\mathbf{x}) \times \mathbf{x})\,\mathrm{d}t \\ &= \int_0^1 t^2 \frac{d}{dt}\mathbf{w}(t\mathbf{x})\,\mathrm{d}t + \int_0^1 2t\mathbf{w}(t\mathbf{x})\,\mathrm{d}t = \left[t^2 w(t\mathbf{x})\right]_0^1 = \mathbf{w}(x).\end{aligned}$$

This completes the proof. $\square$

We will rely on the Poincaré map because it provides a *continuous* mapping from $\boldsymbol{H}(\operatorname{div}0; D)$ into $\boldsymbol{L}^2(D)$. This was first observed by L. Demkowicz in [15, Sect. 3] in two dimensions and, for the sake of completeness, we repeat the arguments.

LEMMA 8.3. *Assume that the domain $D \subset \mathbb{R}^3$ is bounded and star-shaped with respect to the origin. Then, with* $\operatorname{diam}(D) := \sup\{|\mathbf{x} - \mathbf{y}|,\ \mathbf{x}, \mathbf{y} \in D\}$, *we have*

$$\left\|\mathfrak{K}^2\mathbf{w}\right\|_{\boldsymbol{L}^2(D)} \le \operatorname{diam}(D)\ \|\mathbf{w}\|_{\boldsymbol{L}^2(D)} \quad \forall \mathbf{w} \in (C^1(\overline{D}))^3 \,.$$

Proof. The proof boils down to repeatedly applying the Cauchy-Schwarz inequality. With $\mathbf{u} := \mathfrak{K}^2\mathbf{w}$,

$$\begin{aligned}\int_D |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} &= \int_D \left|\int_0^1 t\mathbf{w}(t\mathbf{x}) \times \mathbf{x}\,\mathrm{d}t\right|^2 \,\mathrm{d}\mathbf{x} \\ &\leq \int_D \int_0^1 t^2 |\mathbf{w}(t\mathbf{x})|^2 |\mathbf{x}|^2 \,\mathrm{d}t\,\mathrm{d}\mathbf{x} \\ &\leq \operatorname{diam}(D)^2 \int_0^1 t^2 \int_D |\mathbf{w}(t\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= \operatorname{diam}(D)^2 \int_0^1 t^2 \int_{tD} |\mathbf{w}(\mathbf{y})|^2 t^{-1} \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\ &\leq \operatorname{diam}(D)^2 \int_0^1 t \int_D |\mathbf{w}(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \leq \tfrac{1}{2} \operatorname{diam}(D)^2 \left\|\mathbf{w}\right\|^2_{\boldsymbol{L}^2(D)} \ .\end{aligned}$$

This completes the proof. ☐

Another key property is that the Poincaré map $\mathfrak{K}^2$ can be used to construct the local trial spaces $\mathcal{W}^1_p(T)$ on a tetrahedron T, see [19].

LEMMA 8.4. *Let $\mathcal{S}_h$ be a subset of tetrahedra of the mesh $\mathcal{M}_h$ that share a vertex (which will be assumed to coincide with the origin of a Cartesian coordinate system) and write S for the domain triangulated by $\mathcal{S}_h$. If*

$$\mathbf{w}_h \in \{\mathbf{v}_h \in \boldsymbol{H}(\mathrm{div}; S) : \ \mathbf{v}_{h|T} \in (\mathcal{P}_p(T))^3 \ \forall T \in \mathcal{S}_h\} \ ,$$

then $\mathfrak{K}^2\mathbf{w}_h \in \mathcal{W}^1_p(\mathcal{S}_h)$.

Proof. It is obvious from the definition of $\mathfrak{K}^2\mathbf{w}_h$ in (8.3) that $\mathfrak{K}^2\mathbf{w}_h$ will be a piecewise polynomial vectorfield on $\mathcal{S}_h$ of local degree $\leq p+1$. Moreover, it is straightforward that $(\mathfrak{K}^2\mathbf{w}_h(\mathbf{x})) \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in S$. In short,

$$\mathfrak{K}^2\mathbf{w}_{h|T} \in \mathcal{W}^1_p(T) \quad \forall T \in \mathcal{S}_h \ .$$

In light of definition (6.1) it remains to confirm the tangential continuity of $\mathfrak{K}^2\mathbf{w}_h$. To that end consider a face F shared by two tetrahedra in $\mathcal{S}_h$. Without loss of generality assume that

$$F \subset \{\mathbf{x} \in \mathbb{R}^3 : \ x_3 = 0\} \ .$$

Next, remember that $\mathbf{w}_h \in \boldsymbol{H}(\mathrm{div}; S)$ implies the normal continuity of $\mathbf{w}_h = (w_1, w_2, w_3)^T$ across F, in this case the continuity of w_3. The components u_1, u_2 of $\mathbf{u}_h = (u_1, u_2, u_3)^T := \mathfrak{K}^2\mathbf{w}_h$ are tangential to F. By (6.1) they are given by

$$\begin{aligned} u_1(\mathbf{x}) &= -\int_0^1 t\, w_3(t\mathbf{x}) x_2 \,\mathrm{d}t \ , \\ u_2(\mathbf{x}) &= \int_0^1 t\, w_3(t\mathbf{x}) x_1 \,\mathrm{d}t \ , \end{aligned}$$

and, obviously, they only depend on the continuous function w_3. This implies the continuity of u_1, u_2 across F. ☐

The previous lemma gives a stable discrete vector potential on a group of tetrahedra sharing a vertex (a "patch"). In the next section we are going to generalize this to the entire mesh $\mathcal{M}_h$.

9. p-uniform discrete Poincaré-Friedrichs inequality. In order to obtain global stable discrete vector potentials, we will make heavy use of stable polynomial extensions. These have been constructed in [25], see, in particular, Theorem 1 in this article and its proof.

LEMMA 9.1. *Let T be a tetrahedron in $\mathbb{R}^3$. Then, for any union F of triangular faces of T there is a continuous extension operator $\mathfrak{E}_F^T : H^{\frac{1}{2}}(F) \mapsto H^1(T)$ such that*

$$\mathfrak{E}_F^T(\widetilde{\mathcal{P}}_p(F)) \subset \mathcal{P}_p(T) \ ,$$

where

$$\widetilde{\mathcal{P}}_p(F) := \{p \in C^0(F) : \ p_{|\triangle} \in \mathcal{P}_p(\triangle) \text{ for any triangle } \triangle \subset F\} \ .$$

LEMMA 9.2. *Let $\mathcal{D}_h$ be a subset of tetrahedra of $\mathcal{M}_h$ and $\mathcal{D}'_h \subset \mathcal{D}_h$. Write D and D' for the domains $\subset \Omega$ triangulated by $\mathcal{D}_h$ and $\mathcal{D}'_h$, respectively, of which D is assumed to be connected. Then there is an continuous extension operator $\mathfrak{E}_{D'\mapsto D}$: $\mathcal{W}_p^0(\mathcal{D}'_h) \mapsto \mathcal{W}_p^0(\mathcal{D}_h)$ that fulfills*

$$\|\mathfrak{E}_{D'\mapsto D}(v_h)\|_{H^1(D)} \leq C \|v_h\|_{H^1(D')} \quad \forall v_h \in \mathcal{W}_p^0(\mathcal{D}'_h) \ ,$$

with $C > 0$ independent of p (, but, of course, dependent on $\mathcal{D}_h$).

Proof. The proof will rely on induction with respect to the number $\sharp(\mathcal{D}_h \setminus \mathcal{D}'_h)$ of tetrahedra in $\mathcal{D}_h \setminus \mathcal{D}'_h$. To begin with, we pick some $v_h \in \mathcal{W}_p^0(\mathcal{D}'_h)$.

If $\mathcal{D}_h \setminus \mathcal{D}'_h = \{T\}$, the tetrahedron T will be adjacent to D', $\partial T \cap \partial D'$ will be a union F of faces of T, and, by the previous lemma, $\mathfrak{E}_F^T v_{h|F}$ will be the desired extension, because

$$\left\|v_{h|F}\right\|_{H^{\frac{1}{2}}(F)} \leq C \|v_h\|_{H^1(D')} \ ,$$

where the constant only depends on D' and T.

If $\sharp(\mathcal{D}_h \setminus \mathcal{D}'_h) > 1$, then remove a tetrahedron $\widetilde{T}$ from $\mathcal{D}_h \setminus \mathcal{D}'_h$ such that the domain covered by $\mathcal{D}_h \setminus \{\widetilde{T}\}$ remains connected. By the induction hypothesis there is a H^1-stable piecewise polynomial extension of v_h to $D \setminus \{\widetilde{T}\}$. Then apply the same reasoning as in the case $\sharp(\mathcal{D}_h \setminus \mathcal{D}'_h) = 1$. ☐

Beside the polynomial extension results we have to invoke discrete topology in order to glue together the discrete vector potentials on individual patches of $\mathcal{M}_h$. To that end we have to fix some notions: as usual a tetrahedral triangulation $\mathcal{T}_h$ will be regarded as the set of its individual tetrahedra. The *simplicial complex* $\mathrm{SC}(\mathcal{T}_h)$ associated with $\mathcal{T}_h$ is the set comprising all tetrahedra, triangular faces, straight edges and vertices of tetrahedra in $\mathcal{T}_h$.

A simplicial complex supports a co-homology of co-chains, see [20, Sect. 3] and the references therein. If a simplicial complex arising from a tetrahedral triangulation $\mathcal{T}_h$ of a domain $D \subset \mathbb{R}^3$ has vanishing first Betti number, then any edge cycle is the boundary of an orientable surface composed of faces. In this case $\mathcal{W}_p^1(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}\,0; D) = \mathbf{grad}\,\mathcal{W}_p^0(\mathcal{T}_h)$. This topological fact forces us to assume a special structure of our underlying mesh $\mathcal{M}_h$.

DEFINITION 9.3. *A tetrahedral triangulation $\mathcal{T}_h$ of a polyhedron called* fragmentable, *if one of the following is true:*

(i) $\bigcap\{\overline{T},\ T \in \mathcal{T}_h\} \neq \emptyset$, *that is all tetrahedra of $\mathcal{T}_h$ share at least one vertex.*

(ii) *$\mathcal{T}_h$ is the union of two fragmentable subsets $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}_h$, $\mathcal{T}_1 \neq \mathcal{T}_h$, $\mathcal{T}_2 \neq \mathcal{T}_h$ such that*
- *there is a partition $S_1 \cup \cdots \cup S_N$, $N \in \mathbb{N}$, of the simplicial complex $\mathrm{SC}(\mathcal{T}_1 \cap \mathcal{T}_2)$, for which all simplicial complexes S_j are mutually disjoint and have vanishing first Betti number.*
- *$\mathrm{SC}(\mathcal{T}_1) \cap \mathrm{SC}(\mathcal{T}_2) \subset \mathrm{SC}(\mathcal{T}_1 \cap \mathcal{T}_2)$.*

For a two-dimensional example illustrating the kind of decomposition of a mesh required in the above definition see Figure 9.1. The assumption that the triangulation be fragmentable does not seem to be very restrictive for practical geometries, see Figure 9.2 for examples.

An example of a tetrahedral mesh that need not be fragmentable is a "minimal" triangulation of a thick spherical shell: for instance, defining $\mathcal{T}_1$ and $\mathcal{T}_2$ by sets of tetrahedra clustered around two opposite poles of the sphere will invariable leave us with a torus shaped overlap region, whose first Betti number does not vanish. Unfortunately, we failed to show that all triangulations of domains with vanishing second Betti number, that is, without cavities, are fragmentable.

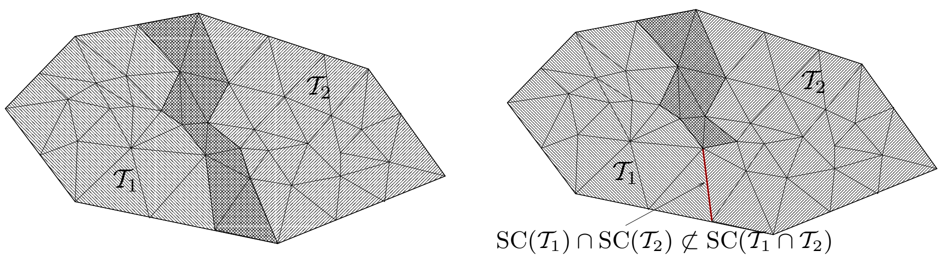

FIG. 9.1. *Two dimensional example of a decomposition of a triangular mesh. Left: valid splitting according to Definition 9.3. Right: insufficient overlap of sub-meshes.*

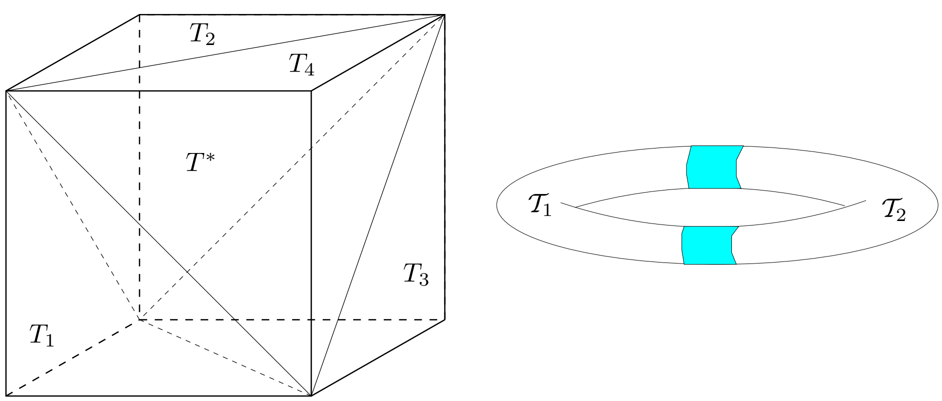

FIG. 9.2. *Left: Splitting of a triangulated cube $\mathcal{T}_1 = \{T_1, T_2, T^*\}$, $\mathcal{T}_2 = \{T_3, T_4, T^*\}$ with overlap T^*. Right: Splitting of a triangulated torus, for which the overlap (shaded) can be decomposed into two disjoint parts with vanishing first Betti number each.*

THEOREM 9.4. *If $\mathcal{M}_h$ is fragmentable, then for any*

$$\mathbf{w}_h \in \{\mathbf{v}_h \in \boldsymbol{H}(\mathrm{div}\,0;\Omega) : \ \mathbf{v}_{h|T} \in \mathcal{P}_p(T) \quad \forall T \in \mathcal{M}_h\}$$

we can find $\widehat{\mathbf{u}}_h \in \mathcal{W}_p^1(\mathcal{M}_h)$ *such that*

$$\mathbf{curl}\,\widehat{\mathbf{u}}_h = \mathbf{w}_h \quad \textit{and} \quad \|\widehat{\mathbf{u}}_h\|_{L^2(\Omega)} \le C\,\|\mathbf{w}_h\|_{\boldsymbol{L}^2(\Omega)} \ ,$$

where the constant $C > 0$ *only depends on* $\mathcal{M}_h$. *In particular,* C *is independent of the polynomial degree* $p \in \mathbb{N}_0$.

Proof. The proof will be conducted by induction with respect to the number of tetrahedra $\sharp\mathcal{M}_h$ in $\mathcal{M}_h$. First, we fix a $\mathcal{M}_h$-piecewise polynomial (of degree p) vectorfield $\mathbf{w}_h$ with $\operatorname{div}\mathbf{w}_h = 0$.

If $\mathcal{M}_h$ fits case (i) in Definition 9.3, in particular, if $\sharp\mathcal{M}_h = 1$, then we can simply invoke Lemma 8.3.to finish the proof.

If $\cap\{\overline{T},\ T \in \mathcal{T}_h\} = \emptyset$, then by the assumption that $\mathcal{M}_h$ is fragmentable, it can be split into $\mathcal{M}_1$ and $\mathcal{M}_2$ as in case (ii) of Definition 9.3. Denote by $\Omega_i \subset \Omega$ the domain triangulated by $\mathcal{M}_i$. We point out that Definition 9.3 ensures $\Omega = \Omega_1 \cup \Omega_2$. The induction hypothesis guarantees the existence of $\widehat{\mathbf{u}}_h^i \in \mathcal{W}_p^1(\mathcal{M}_i)$ such that

$$\left\|\widehat{\mathbf{u}}_h^i\right\|_{\boldsymbol{L}^2(\Omega_i)} \le C_i\,\|\mathbf{w}_h\|_{\boldsymbol{L}^2(\Omega_i)} \ , \tag{9.1}$$

with $C_i = C_i(\mathcal{M}_i)$, $i = 1, 2$.

The domain triangulated by $\mathcal{M}_1 \cap \mathcal{M}_2$ agrees with $\Omega_1 \cap \Omega_2$. By construction

$$\mathbf{curl}\,\widehat{\mathbf{u}}_h^i = \mathbf{w}_{h|\Omega_i} \quad i = 1, 2 \ ,$$

which implies

$$\mathbf{curl}(\widehat{\mathbf{u}}_h^2 - \widehat{\mathbf{u}}_h^1) = 0 \quad \text{in } \Omega_1 \cap \Omega_2 \ .$$

Since the connected components of $\mathrm{SC}(\mathcal{M}_1 \cap \mathcal{M}_2)$ have vanishing first Betti number, we can find $\psi_h \in \mathcal{W}_p^0(\mathcal{M}_1 \cap \mathcal{M}_2)$ such that

$$\mathbf{grad}\,\psi_h = \widehat{\mathbf{u}}_h^2 - \widehat{\mathbf{u}}_h^1 \quad \text{on } \Omega_1 \cap \Omega_2 \ .$$

By demanding vanishing mean of ψ_h (locally on each connected component), we can ensure that

$$\|\psi_h\|_{H^1(\Omega_1\cap\Omega_2)} \le C\left\|\widehat{\mathbf{u}}_h^2 - \widehat{\mathbf{u}}_h^1\right\|_{\boldsymbol{L}^2(\Omega_1\cap\Omega_2)} \ , \tag{9.2}$$

with $C = C(\Omega_1 \cap \Omega_2) > 0$. Now, apply Lemma 9.2 to extend ψ_h to $\widetilde{\psi}_h \in \mathcal{W}_p^0(\mathcal{M}_1)$ such that

$$\left\|\widetilde{\psi}_h\right\|_{H^1(\Omega_1)} \le C\,\|\psi_h\|_{H^1(\Omega_1\cap\Omega_2)} \ , \tag{9.3}$$

where C only depends on $\mathcal{M}_1$ and $\mathcal{M}_1 \cap \mathcal{M}_2$. Then, set

$$\widehat{\mathbf{u}}_h := \begin{cases} \widehat{\mathbf{u}}_h^2 & \text{in } \Omega_2 \ , \\ \widehat{\mathbf{u}}_h^1 + \mathbf{grad}\,\widetilde{\psi}_h & \text{in } \Omega_1 \setminus \Omega_2 \ . \end{cases} \tag{9.4}$$

This defines a unique vectorfield almost everywhere in Ω.

It remains to show that $\widehat{\mathbf{u}}_h$ given by (9.4) is **curl**-conforming. Tangential discontinuities can only occur at parts of $\partial\Omega_1 \cap \partial\Omega_2$ outside $\overline{\Omega_1 \cap \Omega_2}$. However, Definition 9.3 rules out such an arrangement for an admissible splitting of a fragmentable mesh. Hence, $\widehat{\mathbf{u}}_h \in \boldsymbol{H}(\mathbf{curl};\Omega)$, in particular $\widehat{\mathbf{u}}_h \in \mathcal{W}_p^1(\mathcal{M}_h)$, and $\mathbf{curl}\,\widehat{\mathbf{u}}_h = \mathbf{w}_h$.

Finally, combining (9.1), (9.2), and (9.3), we get

$$\|\widehat{\mathbf{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \left\|\widehat{\mathbf{u}}_h^1\right\|_{\boldsymbol{L}^2(\Omega_1)} + \left\|\widehat{\mathbf{u}}_h^2\right\|_{\boldsymbol{L}^2(\Omega_2)} + \left\|\mathbf{grad}\,\widetilde{\psi}_h\right\|_{\boldsymbol{L}^2(\Omega_1)} \leq C\,\|\mathbf{w}_h\|_{\boldsymbol{L}^2(\Omega)}\ ,$$

with C only depending on $\mathcal{M}_h, \mathcal{M}_1, \mathcal{M}_2$. ◻

The next and final theorem will give the desired discrete Poincaré-Friedrichs inequality that is uniform in the polynomial degree p.

THEOREM 9.5. *If $\mathcal{M}_h$ is fragmentable, then with $C = C(\mathcal{M}_h) > 0$ independent of p*

$$\|\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C\,\|\mathbf{curl}\,\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \mathbf{u}_h \in \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h)\ .$$

Proof. Pick an arbitrary $\mathbf{u}_h \in \boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h)$. By the previous theorem we can find $\widehat{\mathbf{u}}_h \in \mathcal{W}_p^1(\mathcal{M}_h)$ such that

$$\mathbf{curl}\,\widehat{\mathbf{u}}_h = \mathbf{curl}\,\mathbf{u}_h \quad \text{and} \quad \|\widehat{\mathbf{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C\,\|\mathbf{curl}\,\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)}\ ,$$

where $C = C(\mathcal{M}_h) > 0$ is independent of p.

We still have to enforce compliance with the homogeneous boundary conditions onto the vector potential $\widehat{\mathbf{u}}_h$. To that end note that

$$\mathrm{div}_\Gamma(\widehat{\mathbf{u}}_h \times \mathbf{n}) = \mathbf{curl}\,\widehat{\mathbf{u}}_h \cdot \mathbf{n} = \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega\ .$$

In other words, the tangential trace $\widehat{\mathbf{u}}_h \times \mathbf{n}$ is a divergence-free surface vectorfield that is piecewise polynomial with respect to the restriction $\partial\mathcal{M}_h$ of $\mathcal{M}_h$ onto $\partial\Omega$. Note that $\partial\mathcal{M}_h$ is a triangular mesh covering $\partial\Omega$.

Discrete co-homology theory [20, Thm. 3.7] tells us that there will always be a $H^{-\frac{1}{2}}(\partial\Omega)$-orthogonal decomposition

$$\widehat{\mathbf{u}}_h \times \mathbf{n} = \mathbf{grad}\,\varphi_h + \boldsymbol{\eta}_h\ ,$$

where φ_h is continuous with vanishing mean and $\partial\mathcal{M}_h$-piecewise polynomial of degree p. On the other hand, the co-homology surface vectorfield $\boldsymbol{\eta}_h$ can be chosen from a space of $\partial\mathcal{M}_h$-piecewise constant div_Γ-conforming surface vectorfields. This space has finite dimension equal to the first Betti number of Ω. Further, it can be obtained as the tangential trace of discrete co-homology vectorfields $\in \mathcal{W}_0^1(\mathcal{M}_h)$ [9]. We do not miss anything, because traces of co-homology vectorfields in the exterior of Ω do not contribute to $\boldsymbol{\eta}_h$: for any curface cycle $\gamma \subset \partial\Omega$ that bounds an oriented surfaced $\Sigma \subset \Omega$ we know that its path integral along γ will vanish:

$$\int_\gamma \widehat{\mathbf{u}}_h \cdot \mathrm{d}\vec{s} = \int_\Sigma \mathbf{curl}\,\widehat{\mathbf{u}}_h \cdot \mathbf{n}_\Sigma\,\mathrm{d}S = \int_\Sigma \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{n}_\Sigma\,\mathrm{d}S = \int_\gamma \mathbf{u}_h \cdot \mathrm{d}\vec{s} = 0\ .$$

After choosing a basis for this space, it is clear that $\boldsymbol{\eta}_h$ can be extended to $\mathbf{h}_h \in \mathcal{W}_0^1(\mathcal{M}_h) \cap \boldsymbol{H}(\mathbf{curl}\,0;\Omega)$ such that, with $C = C(\mathcal{M}_h) > 0$,

$$\mathbf{h}_h \times \mathbf{n} = \boldsymbol{\eta}_h \text{ on } \partial\Omega\ , \quad \|\mathbf{h}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C\,\|\boldsymbol{\eta}_h\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\,\|\widehat{\mathbf{u}}_h \times \mathbf{n}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\ . \tag{9.5}$$

As far as φ_h is concerned, we recall the proof of Lemma 9.2. If confirms the existence of $\widetilde{\varphi}_h \in \mathcal{W}_p^0(\mathcal{M}_h)$ with

$$\widetilde{\varphi}_h = \varphi_h \text{ on } \partial\Omega\ , \quad \|\widetilde{\varphi}_h\|_{H^1(\Omega)} \leq C\,\|\varphi_h\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\,\|\widehat{\mathbf{u}}_h \times \mathbf{n}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\ , \tag{9.6}$$

where $C = C(\mathcal{M}_h) > 0$ does not depend on p. Moreover, we recall the trace theorem for $\boldsymbol{H}(\mathbf{curl};\Omega)$ from [10] that gives

$$\exists C = C(\Omega) > 0 : \quad \|\widehat{\mathbf{u}}_h \times \mathbf{n}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C \|\widehat{\mathbf{u}}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \ . \tag{9.7}$$

Then, we find that

$$\widetilde{\mathbf{u}}_h := \widehat{\mathbf{u}}_h - \mathbf{h}_h - \mathbf{grad}\,\widetilde{\varphi}_h$$

satisfies

$$\mathbf{curl}\,\widetilde{\mathbf{u}}_h = \mathbf{curl}\,\widehat{\mathbf{u}}_h = \mathbf{curl}\,\mathbf{u}_h \quad \text{and} \quad \widetilde{\mathbf{u}}_h \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \ , \tag{9.8}$$

and, by taking into account (9.5), (9.6), and (9.7) we see that

$$\|\widetilde{\mathbf{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \|\widehat{\mathbf{u}}_h\|_{L^2(\Omega)} + \|\mathbf{h}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\widetilde{\varphi}_h\|_{H^1(\Omega)} \leq C \|\mathbf{curl}\,\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \ , \tag{9.9}$$

and $C > 0$ is independent of p.

As $\widetilde{\mathbf{u}}_h - \mathbf{u}_h \in \mathcal{W}^1_{p,0}(\mathcal{M}_h)$ is **curl**-free, the definition of $\boldsymbol{Z}_{p,0}(\boldsymbol{\epsilon}, \mathcal{M}_h)$ implies

$$(\boldsymbol{\epsilon}\mathbf{u}_h, \widetilde{\mathbf{u}}_h - \mathbf{u}_h)_{L^2(\Omega)} = 0 \ .$$

From this we infer

$$\|\mathbf{u}_h\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\,(\boldsymbol{\epsilon}\mathbf{u}_h, \mathbf{u}_h)_{L^2(\Omega)} = C\,(\boldsymbol{\epsilon}\mathbf{u}_h, \widetilde{\mathbf{u}}_h)_{L^2(\Omega)} \leq C \|\mathbf{u}_h\|_{\boldsymbol{L}^2(\Omega)} \ \|\widetilde{\mathbf{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \ .$$

Using (9.8) and (9.9), the assertion of the theorem follows. □

Remark. In the case of a mesh $\mathcal{M}_h$ with non-zero second Betti number like the hollow sphere discussed above, it is possible to "fill up the cavities" by adding tetrahedra to $\mathcal{M}_h$. Subsequently, $\mathbf{u}_h$ can be extended by zero to the new parts of the mesh and, as above, a discrete vector potential $\widetilde{\mathbf{u}}_h$ can then be found for the extended vectorfield. Finally, zero boundary values can be restored as in the proof of Thm. 9.5. By this trick a topological obstruction to a mesh being fragmentable can be overcome.

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