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High-dimensional finite elements for elliptic problems with multiple scales*

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Abstract

Elliptic homogenization problems in a domain $\Omega \subset \mathbb{R}^d$ with $n+1$ separated scales are reduced to elliptic one-scale problems in dimension $(n+1)d$. These one-scale problems are discretized by a sparse tensor product finite element method (FEM). We prove that this sparse FEM has accuracy, work and memory requirement comparable to standard FEM for single scale problems in Ω while it gives numerical approximations of the correct homogenized limit as well as of all first order correctors, throughout the physical domain with performance independent of the physical problem's scale parameters. Numerical examples for model diffusion problems with two and three scales confirm our results.

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1 Introduction

The numerical solution of problems with multiple scales has attracted increasing attention in recent years. Roughly speaking, we are dealing with problems where significant physical phenomena occur on length scales which differ by several orders of magnitude so that their complete resolution within a single numerical simulation is either impossible or extremely costly: if in a physical domain $\Omega \subset \mathbb{R}^d$ of unit diameter the fine scale of the solution has length ε, its resolution by e.g. a Finite Element Method (FEM) with uniform mesh requires $N \geq O(\varepsilon^{-d})$ degrees of freedom.

Most approaches to obtain computationally tractable models for such problems are based on *scale separation*: the ratio between fine and coarse scales tends to zero asymptotically. Classical homogenization uses scale separation for *a-priori scale elimination*, i.e. the analytical elimination of fine scales from the mathematical model [7, 6]. This involves the derivation of the fine scales' effect on the macroscopic, coarse scales of the solution and the analytic derivation of an 'effective' or 'upscaled' homogenized model equation prior to numerical solution. Numerous mathematical tools to achieve this have been developed, starting with the classical two-scale asymptotic expansion technique [5, 7, 6, 16, 24] for linear elliptic problems in divergence form to weak and Gamma convergence techniques for fully nonlinear variational problems with multiple scales (see [17, 11, 16] and the references there). The homogenized limiting problem on the coarse scale can be solved numerically by standard methods. Numerical solutions of homogenized models do not allow directly to extract fine scale information on the physical solution. This is only possible if additional 'corrector' problems which are again of multiscale type are solved numerically. Their numerical solution requires full resolution of the fine scales and is as complex as the direct solution of the original problem.

One approach to avoid analytical homogenization is the direct FEM simulation of the multiscale problem with a coarse mesh and to numerically probe the fine scales of the data only in the set up of the element stiffness matrices. If scales are separated and only solution behaviour on coarse scales is of interest, this is achieved by the so-called Hierarchical Multiscale Methods proposed recently in [12]. For elliptic homogenization problems, these methods use a discretization with $N << O(\varepsilon^{-d})$ "macro" degrees of freedom, say, on the coarse scale. "Probing" the fine scales in the FEM amounts to the numerical solution of one elliptic "cell" problem *for each macro degree of freedom*. This is done again by a suitable FE discretization of these cell problems. Sufficiently accurate numerical "micro" (i.e. cell) solves then require again work algebraic in N *per cell*. This gives an overall superlinear (in N) complexity of this approach.

If scale resolution is required, one can incorporate the solution's fine scale behaviour into the trial spaces of a coarse scale FEM, as proposed for example in [14]. Here, problem adapted shape functions which incorporate the solution's fine scale behaviour must be precomputed. This allows for parallelization, but still has complexity equal to that of a full scale resolution computation. An error analysis is available for example in [14] for periodic two-scale problems – its extension to diffusion problems with multiple scales remains yet to be done.

In the present paper, we propose and analyze an approach to the numerical solution of elliptic multiple scale problems that is of log-linear complexity in N, the number of degrees of freedom in the discretization with respect to the "slow" variable and which converges robustly in $H^1(\Omega)$, i.e. the rate of convergence is independent of ε. It is based on sparse FE discretization of an "unfolded", high-dimensional limit problem derived e.g. in [1, 2] or, more recently, in [9]. As above, derivation of this limiting problem requires *scale separation* in the physical problem. Contrary to analytical homogenization, however, this approach,

proposed first in [20], does not require the a-priori determination coefficients for the homogenized macroscopic problem. Like the homogenized limiting problem, the high-dimensional limit of [1, 2] is independent of the scale parameters, and gives the correct homogenized solution. Moreover, this limit problem provides, to leading order, the physical solution's oscillations in Ω on all scales. Due to the ellipticity of the limit problem, its FE solution is, in principle, straightforward, except for its high dimension: if the physical problem in $\Omega \subset \mathbb{R}^d$ has $n+1$ length scales, the limit problem is elliptic and independent of the scale parameters, but posed on a tensorized domain in $\mathbb{R}^{(n+1)d}$. Therefore, it can in principle be solved robustly with a tensor product FEM in $\mathbb{R}^{(n+1)d}$; in practice, however, this is unrealistic due to the high complexity of FEM in dimension > 3.

As we show, this can be avoided by the use of so-called *sparse tensor products* of FE-spaces also known in approximation theory as *hyperbolic cross approximations*. Such spaces were introduced into the FEM, after earlier work in numerical integration [23] and approximation theory [25], by Zenger in [26].

Apart from a new approach to the numerical solution of elliptic multiscale problems in divergence form, solving the high-dimensional limit problem numerically is sometimes the only possible approach. For example, in the double porosity problem considered in [4], a simple effective equation in $\mathbb{R}^3$ cannot be deduced and the limit problem must be formulated in $\mathbb{R}^6$. This limit problem was numerically solved by T. Arbogast in [3]. There, equations depending on x and y in $\mathbb{R}^3$ and the time variable t are solved first as equations for y by fixing the macro variable $x \in \mathbb{R}^3$ and then as equations for x. The high cost of this approach was handled in [3] by massively parallel solution techniques, since, like in the hierarchical multiscale methods, for each macroscropic degree of freedom, a microscopic solution must be computed.

Using here sparse tensor products of standard FE spaces in each variable gives, to leading order, a description of the solution's fine scales: interpreting e.g. a 2-scale solution $u(x, x/\varepsilon)$ as a map from the slow variable $x \in \Omega$ into the microscopic, or fast, variable $y = x/\varepsilon \in Y$, sparse tensor product Finite Elements resolve the x and y dependence of $u(x, y)$ throughout $\Omega \times Y$ *and* acount for all scale interactions. As is well known from the approximation theory of sparse tensor product spaces, this strategy is only successful if $u(x, y)$ is, *as map* $\Omega \to Y$ *from the slow into the fast scale*, sufficiently smooth (see also [18]). We show here (for scale-separated problems with any finite number of scales) that *for the high-dimensional one-scale problems arising in homogenization, this regularity holds*. A wavelet based preconditioner then implies numerical solvability in (memory and CPU) complexity comparable to that of fast elliptic one-scale solvers in $\mathbb{R}^d$ and independent of the scale parameters (wavelet preconditioning could be replaced by use of a hierarchical basis and a multilevel preconditioner as e.g. in [15]).

We consider here only linear, scalar elliptic problems in divergence form and assume full elliptic regularity; the idea of a numerically sparse full scale resolution, however, is applicable to elliptic systems, problems with corners (see [21] for sparse tensor product spaces of FEM with mesh refinement) – this shall be dealt with elsewhere.

Throughout the paper, we employ standard notations: by $H^k(\Omega)$ we denote the space of generalized functions on Ω whose derivatives of orders from 0 to k belong to $L^2(\Omega)$; $H^k_\#(Y)$ denotes the subspace of functions in $H^k_{\mathrm{loc}}(\mathbb{R}^d)$ that are periodic with respect to the unit cell Y, and $C^1_\#(Y)$ denotes the space of Y-periodic C^1 functions.

The outline of the paper is as follows. In the next Section we state the elliptic multiscale problem and we introduce the high dimensional limit equation, based on notations and results on multiscale convergence from Allaire and Briane

[2]. In Section 3, we define the hierarchic FE spaces in Subsection 3.1 and introduce the tensor product FEM in Subsection 3.2. We prove that sparse tensor product FEM can achieve convergence rates comparable to that of the full tensor product FEM with a number N of degrees of freedom that equals that of a one-scale FEM in $\mathbb{R}^d$, under the assumption of a certain multiscale regularity. In Section 4 we prove that this multiscale regularity (which is necessary to obtain optimal convergence rates with the reduced number of degrees of freedom of the sparse FEM) typically holds for the multiscale limit equations, under mild smoothness assumptions on the coefficients and on the domain. In Section 5, we address the implementation of the sparse FEM. Due to the structure of the basis, the stiffness matrix of the sparse FEM is rather densely populated and its explicit formation and storage would require $O(N^2)$ memory. We show how to generate this matrix in a factored form that requires essentially[1] $O(N)$ memory and that allows to realize a matrix-vector multiplication in essentially $O(N)$ operations. Accordingly, the linear systems for the high dimensional limit problem are solved iteratively, e.g. by cg or GMRES; we present a preconditioner that gives uniformly bounded condition numbers and a solution algorithm for the multidimensional problem of log-linear complexity. In Section 6 we report simple numerical experiments confirming our theory.

2 Homogenization problem

We formulate the elliptic homogenization problem with $n+1$ scales and state the limiting problem as well as an error estimate between the solution of the limiting problem and the solution of the physical problem.

2.1 Problem formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let

$$A(x, y_1, \ldots, y_n) \in L^\infty(\Omega, C(\mathbb{R}^{nd}))^{d\times d}$$

be a matrix function depending on $n+1$ variables taking values in the space $\mathbb{R}^{d\times d}_{sym}$; A is assumed periodic with respect to y_i with period $Y = [0,1]^d$ for each $i = 1, \ldots, n$ **(a different cell Y_i on each scale would not cause any difficulties)**. We assume that A is bounded and uniformly positive definite, i.e there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^d$

$$\gamma|\xi|^2 \le \xi^{\mathrm{T}} A(x, y_1, ..., y_n)\xi \le \gamma^{-1}|\xi|^2, \tag{1}$$

for all $x \in \Omega$ and $y_i \in Y$, $i = 1, \ldots, n$. For a scale parameter $\varepsilon > 0$ we consider the Dirichlet problem

$$-\mathrm{div} A^\varepsilon \nabla u^\varepsilon = f \quad \text{in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega, \tag{2}$$

with $f \in L^2(\Omega)$. The $d \times d$ matrix A^ε is assumed to depend on ε with multiple scales in the following sense: there are n positive functions $\varepsilon_1, \ldots, \varepsilon_n$ of ε that converge to 0 when $\varepsilon \to 0$ and are *scale separated* in the sense that

$$\lim_{\varepsilon\to 0} \varepsilon_{i+1}/\varepsilon_i = 0 \tag{3}$$

for all $i = 1, \ldots, n-1$ and for all $x \in \Omega$

$$A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right).$$

[1] Throughout, "essentially" means up to powers of $\log N$ or of $\log h$ which will be made precise in the proofs

When $n = 1$ we have the classical two-scale homogenization problem

$$-\mathrm{div} A\left(x, \frac{x}{\varepsilon}\right)\nabla u^\varepsilon = f, \tag{4}$$

which is dealt with thoroughly in the book by Bensoussan, Lions and Papanicolaou [7]. The purpose of homogenization is to study the limit of u^ε when ε converges to 0 and to get an asymptotic expansion of u^ε with respect to ε. The original procedure presented in [7] is to perform a formal multiscale (two-scale for the problem (4)) asymptotic expansion and then to justify the convergence using the energy method due to L. Tartar [24]. This method has been widely used for many homogenization problems involving two scales but for multiple scales it can only solve a very narrow class of problems and becomes very complicated (see [2] for details). The two steps, performing the asymptotic expansion and using the energy method, can be incorporated into one single and quite elegant method of two-scale convergence (for two-scale problems) originally due to Nguetseng [19] and developed further by Allaire [1] and generalized to multiscale convergence by Allaire and Briane [2]. The multidimensional limiting equation derived in [2] contains, to leading order, complete information on the physical solution's oscillations on all length scales. We remark that recently Cioranescu et al. [9] introduced the notion of *unfolding homogenization* that can also be used to deduce the limiting equation and corrector results.

We review next some known results on this limiting equation for which we develop below an efficient FEM.

2.2 High-dimensional Limit Problem

A key ingredient of homogenization of elliptic problems with multiple scales is the notion of multiscale convergence. We present the definition here, based on Allaire and Briane [2]. To this end, by $Y_1, \ldots, Y_n$ we denote n unit cells for the n fast scales, and by $C_\#(Y_1 \times \ldots \times Y_n)$ the space of continuous functions $\phi(y_1, \ldots, y_n)$, which are Y_k periodic with respect to y_k for $k = 1, \ldots, n$. **For simplicity of notation only, we assume here** $Y_j = Y = (0,1)^d$**,** $j = 1, ..., n$.

Definition 2.1 *([2] Definition 2.3) A sequence* $\{u^\varepsilon\}_\varepsilon \subset L^2(\Omega)$ $(n+1)-$*scale converges to* $u_0(x, y_1, \ldots, y_n) \in L^2(\Omega \times Y_1 \times \ldots Y_n)$ *if*

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon \phi(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n})dx = \int_\Omega \int_{Y_1} \ldots \int_{Y_n} u_0(x, y_1, \ldots, y_n)\phi(x, y_1, \ldots, y_n)dxdy_1 \ldots dy_n,$$

for any function $\phi \in L^2(\Omega, C_\#(Y_1 \times \ldots \times Y_n))$.

The relevance of this definition lies in the following compactness result.

Theorem 2.1 *([2] Theorem 2.5) Under the assumption (3) of scale separation, from each bounded sequence in* $L^2(\Omega)$ *we can extract a subsequence which* $(n+1)-$*scale converges, as* $\varepsilon \to 0$*, to a function* $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$.

For the variational formulation of the $n+1$-scale limit, we define the space

$$\mathbf{V} = \{(\phi, \{\phi_i\}) : \phi \in H^1_0(\Omega), \phi_i \in L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1_\#(Y_i)/\mathbb{R}), i = 1, ..., n\}$$

endowed with the norm

$$|||(\phi, \{\phi_i\})||| = \|\nabla\phi\|_{L^2(\Omega)} + \sum_{i=1}^n \|\nabla_{y_i}\phi_i\|_{L^2(\Omega\times Y_1\times\ldots\times Y_i)} \tag{5}$$

for $(\phi, \phi_1, \ldots, \phi_i) \in \mathbf{V}$. Then for the problem (2) we have the following result.

Theorem 2.2 *The solution u^ε of the problem (2) converges weakly to a function u in $H_0^1(\Omega)$ and the gradient ∇u^ε $(n+1)-$scale converges to the limit*

$$\nabla u(x) + \sum_{i=1}^{n} \nabla_{y_i} u_k(x, y_1, \ldots, y_i),$$

where $(u, u_1, \ldots, u_n)$ is the unique solution in the space $\mathbf{V}$ of the variational problem

$$\begin{aligned} & B(u, \{u_i\}; \phi, \{\phi_i\}) \\ &= \int_\Omega \int_{Y_1} \ldots \int_{Y_n} A\Big(\nabla_x u + \sum_{i=1}^{n} \nabla_{y_i} u_i\Big).\Big(\nabla_x \phi + \sum_{i=1}^{n} \nabla_{y_i} \phi_i\Big) dx dy_1 \ldots dy_n \quad (6) \\ &= \int_\Omega f\phi dx \quad \forall (\phi, \{\phi_i\}) \in \mathbf{V}. \end{aligned}$$

The bilinear form B is continuous and coercive in $\mathbf{V}$: there are $c_1, c_2 > 0$ independent of ε such that

$$\forall (\phi, \{\phi_i\}) \in \mathbf{V}: \quad B(\phi, \{\phi_i\}; \phi, \{\phi_i\}) \geq c_1 |||(\phi, \{\phi_i\})|||^2, \quad (7)$$

$$\forall (u, \{u_i\}), (v, \{v_i\}) \in \mathbf{V}: \quad B(u, \{u_i\}; v, \{v_i\}) \leq c_2 |||(u, \{u_i\})|||.|||(v, \{v_i\})|||. \quad (8)$$

For a proof, we refer to ([2] Theorem 2.11, equation (2.9)).

Remark 2.1 The limit problem (6) is independent of the scale parameter ε, and formulated in terms of the coefficient A of the physical problem (2). We can also deduce a limiting equation for $u(x)$ only and express u_i in terms of u and of the solutions of appropriate cell problems. This will be done in Section 4 ahead. For example, for a two-scale problem, $n = 1$ and equation (6) becomes

$$\int_\Omega \int_Y A(x, y)\big(\nabla_x u + \nabla_y u_1\big) \cdot \big(\nabla_x \phi + \nabla_y \phi_1\big) dx dy = \int_\Omega f\phi dx \quad (9)$$

for all $\phi \in H_0^1(\Omega)$ and $\phi_1 \in L^2(\Omega, H^1_\#(Y)/\mathbb{R})$. This limit problem is derived in [1].

Next we mention a corrector result to illustrate how u^ε can be approximated in the physical domain Ω in terms of $u_0, u_1, \ldots, u_n$.

Theorem 2.3 *([2] Theorem 2.14) Assume that the solution $(u, u_1, \ldots, u_n)$ of problem (6) is sufficiently smooth, say $u \in C^1(\overline{\Omega})$ and $u_i \in C^1(\overline{\Omega}, C^1_\#(Y_1 \times \ldots \times Y_i))$ for all $i = 1, \ldots, n$. Then, as $\varepsilon \to 0$,*

$$u^\varepsilon(x) - \left[u(x) + \sum_{i=1}^{n} \varepsilon_i u_i\Big(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_i}\Big)\right] \to 0$$

strongly in $H^1(\Omega)$.

Remark 2.2 If the data A, Ω, f are smooth then the functions u, u_i are all smooth. We can also pass to the limit for certain classes of nonsmooth matrices A. This leads to lower regularity of u_k but nevertheless a corrector can always be found using the 'inverse unfolding operator' due to Cioranescu et al. [9]. Details about which matrices A are "admissible" may be found in [2] and [9].

3 Finite Element Discretization

We present FE discretizations of the limit problem (6). Since this problem is elliptic and posed in the product domain $\Omega \times Y_1 \times \ldots \times Y_n$, it is natural to discretize (6) with tensor product finite elements. We treat the full tensor product and the sparse tensor product space separately to highlight their differences. Error estimates follow then straightforwardly from approximation properties of FE spaces in Ω and in the unit cells Y_k.

3.1 Hierarchic FE spaces

For the construction of the sparse tensor product FE spaces in $\Omega \times Y_1 \times ... \times Y_n$, we require hierarchic sequences of FE spaces in the component domains. We sketch exemplarily two constructions of such subspace sequences. Consider a sequence $\{V^l\}_{l=0}^{\infty}$ of subspaces of $H^1(\Omega)$ and a sequence $\{V_\#^l\}_{l=0}^{\infty}$ of subspaces of $H^1_\#(Y)$ such that $V^l \subset V^{l+1}$ and $V_\#^l \subset V_\#^{l+1}$ for all $l = 0, 1, \ldots$. Assuming that Ω is a bounded Lipschitz polyhedron with plane sides, we divide Ω into simplices (triangles in dimension $d = 2$ and tetrahedra in dimension $d = 3$). For each l each element of V^l restricted to each of these simplices is a polynomial. Similarly Y is divided to simplices; and each element of $V_\#^l$ restricted to each of these simplices is a polynomial. We consider the following cases:

Example 3.1 *(h-FEM)* Let $\{\mathcal{T}^l\}_{l=0}^{\infty}$ be a nested sequence of regular simplices of Ω: Ω is divided into a regular family $\mathcal{T}^0 = \{\mathcal{T}\}$ of simplices; then we define $\mathcal{T}^l$ recursively by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles for $d = 2$ and 8 congruent tedrahedra for $d = 3$. The nested sequence $\{\mathcal{T}_\#^l\}_{l=0}^{\infty}$ of regular simplices distributed periodically in Y is defined similarly. For $p \geq 1$ a polynomial degree, and $l \geq 0$ a refinement level, we define the finite element spaces

$$V^l = \{u \in H^1(\Omega) : u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}^l\},$$
$$V_\#^l = \{u \in H^1_\#(Y) : u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}_\#^l\},$$
$$V_0^l = \{u \in H^1_0(\Omega) : u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}^l\}$$

where $\mathcal{P}_p(K)$ denotes the set of polynomials of total degree at most p in K.

For these spaces, we recall the approximation properties. Let $h_l = O(2^{-l})$ denote the meshwidth. Then, for all $t \geq 0$ we have (e.g. [8])

$$\inf_{v \in V^l} \|u - v\|_{H^1(\Omega)} \leq c h_l^{\min(t,p)} \|u\|_{H^{t+1}(\Omega)}, \tag{10}$$

$$\inf_{v \in V^l} \|u - v\|_{L^2(\Omega)} \leq c h_l^{\min(t,p)+1} \|u\|_{H^{t+1}(\Omega)}, \tag{11}$$

for all $u \in H^{t+1}(\Omega)$ and

$$\inf_{v \in V_\#^l} \|u - v\|_{H^1(Y)} \leq c h_l^{\min(t,p)} \|u\|_{H^{t+1}(Y)}, \tag{12}$$

$$\inf_{v \in V_\#^l} \|u - v\|_{L^2(Y)} \leq c h_l^{\min(t,p)+1} \|u\|_{H^{t+1}(Y)}. \tag{13}$$

for all $u \in H_\#^{t+1}(Y)$. Here, $c = c(t, p)$.

Example 3.2 *(p-FEM)* Let Ω and Y be divided into simplices $\mathcal{T}$ and a set of periodically distributed simplices $\mathcal{T}_\#$, respectively. Let $p = 1, 2, \ldots$ be the polynomial degree. We define

$$V^p = \{u \in H^1(\Omega) :\ u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}\},$$
$$V_\#^p = \{u \in H^1_\#(Y) :\ u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}_\#\},$$
$$V_0^p = \{u \in H^1_0(\Omega) :\ u|_K \in \mathcal{P}_p(K)\ \forall K \in \mathcal{T}\}.$$

Then we also have similar approximation properties as in (10), (11), (12) and (13).

3.2 Tensor product FEM

To approximate $u_i(x, y_1, \ldots, y_i)$ we need FE subspaces of $L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1_\#(Y_i))$. Since

$$L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1_\#(Y_i)) \cong L^2(\Omega) \otimes L^2(Y_1) \otimes \ldots \otimes L^2(Y_{i-1}) o \times H^1_\#(Y_i),$$

a natural choice of a FE space is the tensor product FE space

$$\mathbf{V}^L = \{(u^L, \{u_i^L\}) : u^L \in V_0^L, u_i^L \in V_i^L, i = 1, \ldots, n\} \tag{14}$$

where

$$V_i^L = V^L \otimes \underbrace{V_\#^L \otimes \ldots \otimes V_\#^L}_{i \text{ times}}. \tag{15}$$

To investigate the convergence rate of the tensor product FE spaces V_i^L, we quantify the solution's regularity. To describe it, we introduce the space $\mathcal{H}_i^t$ of functions $w = w(x, y_1, \ldots, y_i) \in L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^{1+t}(Y_i))$ that are periodic in y_j for $j = 1, \ldots, i$ such that $w \in L^2(Y_1 \times \ldots \times Y_i, H^t(\Omega))$ and $w \in L^2(\Omega \times \Pi_{j \neq k} Y_j, H^t(Y_k))$ for all $k = 1, \ldots, i-1$. We equip $\mathcal{H}_i^t$ with the norm

$$\|w\|_{\mathcal{H}_i^t} = \|w\|_{L^2(Y_1 \times \ldots \times Y_i, H^t(\Omega))} + \|w\|_{L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^{1+t}(Y_i))} + \sum_{k=1}^{i-1} \|w\|_{L^2(\Omega \times \Pi_{j \neq k} Y_j, H^t(Y_k))}.$$

Lemma 3.1 *If $w \in \mathcal{H}_i^t$ then*

$$\inf_{v \in V_i^L} \|w - v\|_{L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1(Y_i))} \leq c h_l^{\min(t,p)} \|w\|_{\mathcal{H}_i^t}.$$

Proof We first define the following orthogonal projections in the norm of $H^1(Y)$, $L^2(Y)$ and $L^2(\Omega)$ respectively

$$\begin{aligned} P_\#^{l1} &: H^1_\#(Y) \longrightarrow V_\#^l, \\ P_\#^{l0} &: L^2(Y) \longrightarrow V_\#^l, \\ P^{l0} &: L^2(\Omega) \longrightarrow V^l. \end{aligned}$$

For simplicity of notation, in the following inequalities we denote by L_i^2 the space $L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1(Y_i))$. We then have

$$\begin{aligned} \inf_{v \in V_i^L} \|w - v\|_{L_i^2} &\leq \|w - id \otimes P_\#^{l1} w\|_{L_i^2} + \|id \otimes P_\#^{l1} w - id \otimes P_\#^{l0} \otimes P_\#^{l1} w\|_{L_i^2} + \\ \ldots &+ \|id \otimes P_\#^{l0} \otimes \ldots \otimes P_\#^{l0} \otimes P_\#^{l1} w - P^{l0} \otimes P_\#^{l0} \otimes \ldots \otimes P_\#^{l1} \omega\|_{L_i^2} \\ &\leq c h_l^{\min(t,p)} \|w\|_{\mathcal{H}_i^t} \end{aligned}$$

due to the approximation properties (10) - (13) and to the boundedness of the projection operators.□

The finite element approximation of $(u, \{u_i\})$ reads: find $(u^L, \{u_i^L\}) \in \mathbf{V}^L$ such that

$$B(u^L, \{u_i^L\}; \phi^L, \{\phi_i^L\}) = \int_\Omega f \phi^L dx \quad \forall (\phi^L, \{\phi_i^L\}) \in \mathbf{V}^L. \tag{16}$$

By (7), (8), it satisfies the quasi optimal error estimate

$$|||(u - u^L, \{u_i - u_i^L\})||| \leq c \inf_{(v^L, \{v_i^L\}) \in \mathbf{V}^L} |||(u - v^L, \{u_i - v_i^L\})|||. \tag{17}$$

We have immediately from Lemma 3.1

Proposition 3.1 *If $u \in H^2(\Omega)$ and $u_i \in \mathcal{H}_i^1$ for $i = 1, .., n$, then with the full tensor product finite element spaces in Example 1*

$$|||(u - u^L, \{u_i - u_i^L\})||| \leq c h_L \left(\|u\|_{H^2(\Omega)} + \sum_{i=1}^{n} \|u_i\|_{\mathcal{H}_i^1} \right) \tag{18}$$

Remark 3.1 If the solution has regularity $u \in H^{t+1}(\Omega)$ and $u_i \in \mathcal{H}_i^t$ for $t \geq 0$,

$$|||(u - u^L, \{u_i - u_i^L\})||| \leq c h_L^{\min(t,p)} \left(\|u\|_{H^{t+1}(\Omega)} + \sum_{i=1}^{n} \|u_i\|_{\mathcal{H}_i^t} \right). \tag{19}$$

Remark 3.2 For all $p \geq 1$ holds $N_L = \dim \mathbf{V}^L = O(h_L^{-(n+1)d})$.

3.3 Sparse tensor product FEM

The full tensor product FE space is natural for the numerical solution of the limiting problem (6). However, since its number of degrees of freedom is $O(h_L^{-(n+1)d})$, its use is prohibitive in practice, especially in dimension $d = 3$ and for more than two scales. In this section we develop the sparse FEM, and prove that its convergence rate is essentially the same as that of the full tensor product FEM but its number of degrees of freedom is essentially that of a one scale problem in $\mathbb{R}^d$.

To do so, we define the following increment, or "detail" spaces

$$W^l = (P^{l0} - P^{(l-1)0})V^l, \; W_\#^{l0} = (P_\#^{l0} - P_\#^{(l-1)0})V_\#^l, \; W_\#^{l1} = (P^{l1} - P^{(l-1)1})V_\#^l;$$

Then

$$V^l = \bigoplus_{0 \leq i \leq l} W^i, \quad V_\#^l = \bigoplus_{0 \leq i \leq l} W_\#^{i0} = \bigoplus_{0 \leq i \leq l} W_\#^{i1}.$$

The full tensor product space V_i^L defined in (15) can then be written as

$$V_i^L = \bigoplus_{\substack{0 \leq j_k \leq L \\ k=0,1,\ldots,i}} W^{j_0} \otimes W_\#^{j_1 0} \otimes \ldots \otimes W_\#^{j_{i-1} 0} \otimes W_\#^{j_i 1}.$$

and the corresponding sparse tensor product FE spaces are defined by

$$\hat{V}_i^L = \bigoplus_{0 \leq j_0 + \ldots + j_i \leq L} W^{j_0} \otimes W_\#^{j_1 0} \otimes \ldots \otimes W_\#^{j_{i-1} 0} \otimes W_\#^{j_i 1}.$$

The sparse FE space for the approximation of (6) is then given by

$$\hat{\mathbf{V}}^L = \{(\hat{u}^L, \{\hat{u}_i^L\}) : \hat{u}^L \in V_0^L, \hat{u}_i^L \in \hat{V}_i^L\}. \tag{20}$$

Using $\hat{\mathbf{V}}^L$, we approximate the solution $(u, \{u_i\})$ of the problem (6) by: find $(\hat{u}^L, \{\hat{u}_i^L\}) \in \hat{\mathbf{V}}^L$ such that

$$B(\hat{u}^L, \{\hat{u}_i^L\}; \hat{\phi}^L, \{\hat{\phi}_i^L\}) = \int_\Omega f \hat{\phi}^L \quad \forall (\hat{\phi}^L, \{\hat{\phi}_i^L\}) \in \hat{\mathbf{V}}^L. \tag{21}$$

By (8), we have here the quasi optimal error estimate

$$|||(u - \hat{u}^L, \{u_i - \hat{u}_i^L\})||| \leq c \inf_{(\hat{\phi}^L, \{\hat{\phi}_i^L\}) \in \hat{\mathbf{V}}^L} |||(u - \hat{\phi}^L, \{u_i - \hat{\phi}_i^L\})|||, \tag{22}$$

where c is independent of ε and of L.

To derive an error estimate for $\hat{\mathbf{V}}^L$ in terms of h_L, we need suitable approximation results. To state them, we denote for a positive integer t by $\hat{\mathcal{H}}_i^t$ the

space of functions $w(x, y_1, ..., y_i)$ that are Y-periodic in y_j for $j = 1, \ldots, i$ and that are such that for all $\alpha = (\alpha_0, \ldots, \alpha_i)$ with $0 \le \alpha_j \le t$ for $j = 0, \ldots, i-1$ and $0 \le \alpha_i \le t+1$ holds $\partial^{|\alpha|}/(\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i) \in L^2(\Omega \times Y_1 \times \ldots \times Y_i)$. We equip $\hat{\mathcal{H}}_i^t$ with the norm

$$\|w\|_{\hat{\mathcal{H}}_i^t} = \sum_{\substack{0 \le \alpha_i \le t+1 \\ 0 \le \alpha_j \le t \ (0 \le j \le i-1)}} \| \frac{\partial^{|\alpha|}}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i} w \|_{L^2(\Omega \times Y_1 \times \ldots \times Y_i)}.$$

The definition for non integer t is as usual by interpolation.

Lemma 3.2 *If $w \in \hat{\mathcal{H}}_i^t$ then*

$$\inf_{v \in \hat{V}_i^L} \|w - v\|_{L_i^2} \le c(L+1)^{i/2} h_L^{\min(t,p)} \|w\|_{\hat{\mathcal{H}}_i^t} \quad ,$$

where we denote, as in Lemma 3.1, by L_i^2 the space $L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1(Y_i))$.

Proof We define the sparse interpolant $\hat{w}^L$ of w by

$$\hat{w}^L = \sum_{0 \le j_0 + \ldots + j_i \le L} (P^{j_0} - P^{j_0 - 1}) \otimes (P_\#^{j_1 0} - P_\#^{(j_1 - 1)0}) \otimes \ldots \otimes (P_\#^{j_{i-1} 0} - P_\#^{(j_{i-1}-1)0}) \otimes (P_\#^{j_i 1} - P_\#^{(j_i - 1)1}) w,$$

where $P^{-1} = P_\#^{(-1)0} = P_\#^{(-1)1} = 0$. Then we have

$$\begin{aligned}
\|w - \hat{w}^L\|_{L_i^2}^2 &= \| \sum_{j_0 + \ldots + j_i \ge L+1} (P^{j_0} - P^{j_0 - 1}) \otimes \ldots \otimes (P_\#^{j_i 1} - P_\#^{(j_i - 1)1}) w \|_{L_i^2}^2 \\
&\le \sum_{j_0 + \ldots + j_i \ge L+1} \| (P^{j_0} - P^{j_0 - 1}) \otimes \ldots \otimes (P_\#^{j_i 1} - P_\#^{(j_i - 1)1}) w \|_{L_i^2}^2 \\
&\le c \sum_{j_0 + \ldots + j_i \ge L+1} (h_{j_0 - 1} \ldots h_{j_i - 1})^{2 \min(t,p)} \|w\|_{\hat{\mathcal{H}}_i^t}^2 \\
&\le c 4^{i \min(t,p)} \sum_{j_0 + \ldots + j_i \ge L+1} 2^{-2(j_0 + \ldots + j_i) \min(t,p)} \|w\|_{\hat{\mathcal{H}}_i^t}^2 \\
&\le c(L+1)^i 2^{-2L \min(t,p)} \|w\|_{\hat{\mathcal{H}}_i^t}^2.
\end{aligned}$$

Note that $h_l = c 2^{-l}$; we set $h_{-1} = 1$. The last inequality is proved as follows:

$$\sum_{j_0 + \ldots + j_i \ge L+1} 2^{-2(j_0 + \ldots + j_i) \min(t,p)} = \sum_{j_0, \ldots, j_{i-1} \ge 0} 2^{-2(j_0 + \ldots + j_{i-1}) \min(t,p)} \sum_{\substack{j_i \ge \max(L+1 - \\ (j_0 + \ldots + j_{i-1}), 0)}} 2^{-2 j_i \min(t,p)}$$

$$= \frac{1}{1 - 2^{-2\min(t,p)}} \sum_{j_0, \ldots, j_{i-1} \ge 0} 2^{-2 \max((L+1), j_0 + \ldots + j_{i-1}) \min(t,p)}$$

$$= \frac{1}{1 - 2^{-2\min(t,p)}} \sum_{j_0 + \ldots + j_{i-1} \le L+1} 2^{-2(L+1)\min(t,p)} +$$

$$\frac{1}{1 - 2^{-2\min(t,p)}} \sum_{j_0 + \ldots + j_{i-1} \ge L+1} 2^{-2(j_0 + \ldots + j_{i-1}) \min(t,p)}$$

$$\le \frac{(L+1)^i 2^{-2(L+1)\min(t,p)}}{1 - 2^{-2\min(t,p)}} + \frac{1}{1 - 2^{-2\min(t,p)}} \sum_{j_0 + \ldots + j_{i-1} \ge L+1} 2^{-2(j_0 + \ldots + j_{i-1}) \min(t,p)}$$

$$\le \frac{(L+1)^i 2^{-2(L+1)\min(t,p)}}{1 - 2^{-2\min(t,p)}} + \frac{(L+1)^{i-1} 2^{-2(L+1)\min(t,p)}}{(1 - 2^{-2\min(t,p)})^2} +$$

$$\frac{1}{(1 - 2^{-2\min(t,p)})^2} \sum_{j_0 + \ldots + j_{i-2} \ge L+1} 2^{-2(j_0 + \ldots + j_{i-2}) \min(t,p)}$$

$$\ldots$$

$$\le ci(L+1)^i 2^{-2(L+1)\min(t,p)}.$$

where the constant c may depend on t but does not depend on L. $\square$

From this we have the following estimate for the error.

Proposition 3.2 *If the solution $(u, \{u_i\})$ of the problem (6) satisfies $u \in H^2(\Omega)$ and $u_i \in \hat{\mathcal{H}}_i^1$ for $i = 1, \ldots, n$ then*

$$|||(u - \hat{u}^L, \{u_i - \hat{u}_i^L\})||| \leq c(L+1)^{n/2} h_L \left(\|u\|_{H^2(\Omega)} + \sum_{i=1}^{n} \|u_i\|_{\hat{\mathcal{H}}_i^1} \right).$$

Remark 3.3 If $u, \{u_i\}$ possess higher regularity i.e. $u \in H^{1+t}(\Omega)$ and $u_i \in \hat{\mathcal{H}}_i^t$ then we will have $h_L^{\min(t,p)}$ in place of h_L in the previous estimate.

Remark 3.4 The number of degrees of freedom of the spaces V^l and $V_\#^l$ are $O(h_l^d) = O(2^{ld})$ so the numbers of degrees of freedom of W^l, $W_\#^{l0}$ and $W_\#^{l1}$ are $O(2^{ld})$. The number of combinations $(j_0, \ldots, j_i)$ such that $j_0 + \ldots + j_i \leq L$ is $O(L^i)$. From this we deduce that the number of degrees of freedom of the sparse FE space $\hat{\mathbf{V}}^L$ is $O(2^{ld} L^n) = O(h_L^{-d} |\log h_L|^n)$ which equals, up to logarithmic terms, the number of degrees of freedom for the one-scale problem when $n = 0$.

3.4 Convergence in physical variables for two scales

We estimate the error between the solution u^ε of the physical problem (2) in terms of the FE approximations of the limit problem (6). We base this on an explicit error estimate between u^ε and the correctors for the two scale case (i.e. $n = 1$). To this end, we recall

Proposition 3.3 *Assume that $A(x, y) \in C^\infty(\bar{\Omega}, C_\#^\infty(Y))_{sym}^{d \times d}$ and that the homogenized solution $u(x)$ belongs to $H^2(\Omega)$. Then*

$$\|u^\varepsilon - (u(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}))\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}. \tag{23}$$

The constant C is independent of ε but depends on u and u_1.

Remark 3.5 This two-scale result is well known. Most of the existing literature ([7, 16]) presents error estimates for the case where A depends only on y. Section 1.4 of [16] presents a concise proof under the assumption that the coefficients are smooth, the homogenized solution u is in $C^2(\bar{\Omega})$ and the correctors w_k are in $W^{1,\infty}(Y)$ (this will be shown in the next section). The error estimate for the case where A also depends on x can be shown simply by following this proof line by line. We need A to depend on x in a smooth manner. A close look at the proof of [16] shows that once the coefficients are assumed to be smooth, it is enough to consider $u \in H^2(\Omega)$ as in Chapter 1 of [7]; this holds as long as $f \in L^2(\Omega)$ and Ω is a convex polyhedron. However for Theorems 3.1 and 3.2 we will need sufficient smoothness of the function $u(x)$ so we will assume f to be smooth. The smoothness conditions imposed in this section can be reduced (with a consequence that the estimates we obtain may be weaker) but for simplicity we do not present this here. The order of convergence in $L^2(\Omega)$ is $O(\varepsilon)$.

An error estimate like (23) for more than two scales is still to be established; since this is not the main focus of this paper, we confine ourselves in this section to the two scale case.

The norms of the FE solutions u_1^L and $\hat{u}_1^L$ can be unbounded in $C^{0,1}(\Omega, W^{1,\infty}(Y))$ when the mesh size $h_L \to 0$, so an estimate of the form (23) in terms of these functions is not possible in general. We therefore "postprocess" the FE approximation of u_1 using the "folding" operator $\mathcal{U}^\varepsilon$ introduced in [9].

Definition 3.1 *For $\Phi \in L^1(\Omega \times Y)$ we define*

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}\Big) dz$$

where $[x/\varepsilon]$ denotes the "integer" part of x/ε with respect to Y and where $\{x/\varepsilon\} := x/\varepsilon - [x/\varepsilon]$.

We have the following properties of $\mathcal{U}^\varepsilon$ from [9]:

Lemma 3.3 *For $\Phi \in L^1(\Omega \times Y)$,*

$$\int_\Omega \mathcal{U}^\varepsilon(\Phi)(x)dx = \int_{\Omega\times Y} \Phi(x,y)dxdy.$$

Proof Let I be a subset of $\mathbb{Z}^d$ such that $\Omega \subset \bigcup_{m\in I} \varepsilon(m + \overline{Y})$ and extend the function Φ by 0 outside Ω. Then

$$\int_\Omega \int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}\Big) dzdx = \sum_{m\in I} \int_Y \int_{\varepsilon m + \varepsilon Y} \Phi\left(\varepsilon m + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}\right) dxdz$$
$$= \varepsilon^d \sum_{m\in I} \int_Y \int_Y \Phi\left(\varepsilon m + \varepsilon z, y\right) dzdy = \int_{\Omega\times Y} \Phi(x,y)dxdy. \quad \square$$

Lemma 3.4 *If $\Phi \in C^1(\bar{\Omega}, C_\#(Y))$ then there is $C(\Phi)$ such that*

$$\int_\Omega |\Phi(x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\Phi)(x)|^2 dx \le C\varepsilon^2 \quad \textit{holds for} \quad 0 < \varepsilon \le 1.$$

Proof Since $\Phi \in C^1(\bar{\Omega}, C_\#(Y))$ we have for all $x \in \Omega$ and $z \in Y$

$$\Big|\Phi\Big(x, \frac{x}{\varepsilon}\Big) - \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \frac{x}{\varepsilon}\Big)\Big| \le C\varepsilon$$

where C depends on Φ. Then

$$\int_\Omega \Big| \int_Y \Big(\Phi(x, \frac{x}{\varepsilon}) - \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \frac{x}{\varepsilon}\Big)\Big) dz|^2 dx \le C\varepsilon^2. \quad \square$$

We can now prove an estimate between the physical solution u^ε and the "folded" FE solution of the multidimensional limit problem.

Theorem 3.1 *Assume that $A(x,y)$ and f are smooth and Ω is a convex polyhedron. With the full tensor product of the FE spaces in Example 1 with degree $p = 1$, we have the asymptotic error estimate*

$$\|\nabla_x u^\varepsilon(x) - \left[\nabla_x u^L(x) + \mathcal{U}^\varepsilon(\nabla_y u_1^L)(x)\right]\|_{L^2(\Omega)} \le c(\varepsilon^{1/2} + h_L). \qquad (24)$$

Proof Since A and f are smooth and Ω is convex, $u(x)$ and $u_1(x,y)$ are smooth (see the next section). From Lemma 3.4 and (23) we have

$$\|\nabla_x u^\varepsilon(x) - \nabla_x u(x) - \mathcal{U}^\varepsilon(\nabla_y u_1)(x)\|_{L^2(\Omega)} \le c\varepsilon^{1/2}.$$

For all the function $\Phi \in L^2(\Omega \times Y)$ we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^2 \le \mathcal{U}^\varepsilon(\Phi^2)(x).$$

From Lemma 3.3 we have then

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1)(x) - \mathcal{U}^\varepsilon(\nabla_y u_1^L)(x)\|_{L^2(\Omega)} \le \|\nabla_y u_1(x,y) - \nabla u_1^L(x,y)\|_{L^2(\Omega\times Y)} \le ch_L.$$

The result then follows. $\square$

One main result of the paper is that for the sparse FE space we have a completely analogous error estimate:

Theorem 3.2 *Assume that A and f are smooth and that the homogenized solution $u(x)$ belongs to $H^2(\Omega)$. Then with the sparse tensor product FE space from Example 1 it holds*

$$\|\nabla_x u^\varepsilon(x) - \big[\nabla_x \hat{u}^L(x) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(x)\big]\|_{L^2(\Omega)} \leq c(\varepsilon^{1/2} + L^{1/2} h_L). \tag{25}$$

The proof is analogous to that of Theorem 3.1 if we use that the solution $(u, \{u_i\})$ has $H^2(\Omega), \hat{\mathcal{H}}_i^1$-regularity. Sufficient conditions for this will be given in the next section.

Remark 3.6 Theorem 3.2 was stated only for elements of degree $p = 1$. Under suitably strong regularity assumptions for the limiting problem (6), e.g. $u \in H^{p+1}(\Omega)$, we get for $p > 1$ the asymptotic error bound

$$\|\nabla_x u^\varepsilon(x) - \big[\nabla_x \hat{u}^L(x) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(x)\big]\|_{L^2(\Omega)} \leq c(\varepsilon^{1/2} + L^{1/2} h_L^p).$$

Remark 3.7 The proofs of Theorems 3.1 and 3.2 are based on the asymptotic regularity (23) of the exact solution; use of (23) is the source of the term $\varepsilon^{1/2}$ in the error bounds (24) and (25). With a more refined two-scale regularity theory which is not based on formal asymptotics (as e.g. in [18] for $A = A(y)$), this term can likely be avoided.

3.5 Convergence in physical variables for multiple scales

For problems with more than two scales, an error estimate in the form (23) appears not to be available so we can not obtain an error bound which is explicit in ε like (24) or (25). However we can construct a numerical corrector for the case where $\varepsilon_i/\varepsilon_{i+1}$ is an integer for all $i = 1, \ldots, n-1$ and establish convergence in $H^1(\Omega)$. Generalizing the notation of [9], we define:

Definition 3.2 *The "unfolding" $\mathcal{T}_n^\varepsilon$: $L^1(\Omega) \to L^1(\Omega \times Y_1 \times \ldots \times Y_n)$ of a function ϕ at the scales $\varepsilon_1, ..., \varepsilon_n$ is defined as*

$$\mathcal{T}_n^\varepsilon(\phi)(x, y_1, \ldots, y_n) = \phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_2\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big] + \ldots + \varepsilon_n\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big] + \varepsilon_n y_n\Big)$$

where ϕ is extended by zero outside Ω.

It is easy to see that

$$\int_\Omega \phi dx = \int_{\Omega^{\varepsilon_1}} \int_{Y_1^{\varepsilon_2/\varepsilon_1}} \ldots \int_{Y_{n-1}^{\varepsilon_n/\varepsilon_{n-1}}} \int_{Y_n} \mathcal{T}^\varepsilon(\phi) dy_n \ldots dy_1 dx, \tag{26}$$

where Ω^{ε_1} is the $2\varepsilon_1$ neighbourhood of Ω and $Y_i^{\varepsilon_{i+1}/\varepsilon_i}$ is the $2\varepsilon_{i+1}/\varepsilon_i$ neighbourhood of Y_i. As in [9], one can show that, as $\varepsilon \to 0$, for the solution u^ε of the multiscale problem (2) with the scale separation (3), it holds

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla_x u + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n \quad \text{in } L^2(\Omega \times Y_1 \times \ldots \times Y_n). \tag{27}$$

We next define the operator $\mathcal{U}_n^\varepsilon$ which "folds" a function Φ of several separated scales.

Definition 3.3 *Assume (3). For $\Phi \in L^1(\Omega \times Y_1 \times \ldots \times Y_n)$ define*

$$\mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \ldots \int_{Y_n} \Phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 z_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{\varepsilon_1}{\varepsilon_2}\Big\{\frac{x}{\varepsilon_1}\Big\}\Big] + \frac{\varepsilon_2}{\varepsilon_1} z_2, \ldots, \frac{\varepsilon_n}{\varepsilon_{n-1}}\Big[\frac{\varepsilon_{n-1}}{\varepsilon_n}\Big\{\frac{x}{\varepsilon_{n-1}}\Big\}\Big] + \frac{\varepsilon_n}{\varepsilon_{n-1}} z_n, \Big\{\frac{x}{\varepsilon_n}\Big\}\Big) dz_n \ldots dz_1.$$

With the folding operator $\mathcal{U}_n^\varepsilon$ we can state a multiscale analog to (23) which shows that the functions $u_1, ..., u_n$ in (27) describe, to leading order, oscillations of the sequence $\{u^\varepsilon\}$ as it approaches its weak limit $u(x)$.

Lemma 3.5 *For the multiscale problem (2) with scale separation (3) holds*

$$\lim_{\varepsilon\to 0} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla_x u + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n)\|_{L^2(\Omega)} = 0.$$

Proof Consider

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} \mathcal{T}^\varepsilon(A^\varepsilon)(\mathcal{T}^\varepsilon(\nabla_x u^\varepsilon) - (\nabla_x u + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n)).$$
$$(\mathcal{T}^\varepsilon(\nabla_x u^\varepsilon) - (\nabla_x u + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n))dy_n \ldots dy_1 dx.$$

Using (2), (6), (26) and (27) it is easy to see that this expression converges to 0 when $\varepsilon \to 0$ so the convergence (27) is indeed strong. A similar argument as in the proof of Lemma 3.3 shows that for all functions $\Phi \in L^1(\Omega \times Y_1 \times \ldots \times Y_n)$ holds

$$\int_\Omega \mathcal{U}_n^\varepsilon(\Phi)(x)dx = \int_\Omega \int_{Y_1} \ldots \int_{Y_n} \Phi dy_n \ldots dy_1 dx.$$

Furthermore, for all functions $\Phi \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$ we have

$$\int_\Omega |\mathcal{U}_n^\varepsilon(\Phi)(x)|^2 dx \le \int_\Omega \mathcal{U}_n^\varepsilon(\Phi^2)(x) = \int_\Omega \int_{Y_1} \ldots \int_{Y_n} \Phi^2 dy_n \ldots dy_1 dx.$$

From this we deduce, as $\varepsilon \to 0$, that

$$\|\mathcal{U}_n^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon)) - \mathcal{U}_n^\varepsilon(\nabla_x u + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n)\|_{L^2(\Omega)}$$
$$\le \|\mathcal{T}_n^\varepsilon(\nabla_x u^\varepsilon) - (\nabla_x u + \ldots + \nabla_{y_n} u_n)\|_{L^2(\Omega\times Y_1\times\ldots\times Y_n)} \to 0.$$

The conclusion follows as $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\Phi)) = \Phi$ for all functions $\Phi(x)$. □

Remark 3.8 When the solution $(u, u_1, \ldots, u_n)$ is smooth, this result can be inferred from the corrector result in Theorem 2.3.

From these facts, we have the following convergence result for the "folded" tensor product finite element solutions. As the corrector result Lemma 3.5 does not give a rate in ε, we abandon also the FE convergence rates in the next results. They hold without any additional regularity assumptions on the solutions of the limiting problem (6).

Theorem 3.3 *For the multiscale problem (2), with the full tensor product FE approximation in Example 1 of the limit problem (6), we have*

$$\lim_{\substack{\varepsilon\to 0\\ L\to\infty}} \|\nabla_x u^\varepsilon(x) - \mathcal{U}_n^\varepsilon(\nabla_x u^L + \nabla_{y_1} u_1^L + \ldots + \nabla_{y_n} u_n^L)\|_{L^2(\Omega)} = 0. \tag{28}$$

Proof As the FE spaces are dense in $\mathbf{V}$, we find from (17)

$$\|\mathcal{U}_n^\varepsilon(\nabla_x u + \ldots + \nabla_{y_n} u_n) - \mathcal{U}_n^\varepsilon(\nabla_x u^L + \ldots + \nabla_{y_n} u_n^L)\|_{L^2(\Omega)}$$
$$\le |||(\nabla_x u + \ldots + \nabla_{y_n} u_n) - (\nabla_x u^L + \ldots + \nabla_{y_n} u_n^L)||| = o(1) \quad \text{as} \quad L \to \infty,$$

so from Lemma 3.5 we get the conclusion. □

Since $V_n^{[L/n]} \subseteq \hat{V}_n^L$, the sequence of sparse tensor product FE spaces is, as $L \to \infty$, also dense in $\mathbf{V}$. Hence, the same argument as before establishes also

Theorem 3.4 *For the multiscale problem (2), with the sparse tensor product FE approximation of the high dimensional limit problem (6), we have*

$$\lim_{\substack{\varepsilon\to 0\\ L\to\infty}} \|\nabla_x u^\varepsilon(x) - \mathcal{U}_n^\varepsilon(\nabla_x \hat{u}^L + \nabla_{y_1} \hat{u}_1^L + \ldots + \nabla_{y_n} \hat{u}_n^L)\|_{L^2(\Omega)} = 0. \tag{29}$$

4 Multiscale Regularity

From Proposition 3.2 and Theorem 3.2 we see that it is possible to approximate the solution of the high dimensional limit problem (6) at an essentially optimal convergence rate, provided that the functions u_i exhibit joint regularity in each fast variable y_k, i.e. $u_i \in \hat{\mathcal{H}}_i^p$. In this section we show that this regularity for $(u, \{u_i\})$ typically holds under mild regularity conditions on the coefficients A.

Theorem 4.1 *Assume that* $A(x, y_1, \ldots, y_n) \in C^{0,1}(\Omega, C^{0,1}_{\#}(Y_1, \ldots, C^{0,1}_{\#}(Y_n)\ldots))$ *i.e* A *is Lipschitz with respect to each* $x, y_1, \ldots, y_n$*, and is symmetric. Assume* Ω *has a smooth boundary and* $f = f(x) \in L^2(\Omega)$*. Then the solution* $(u, \{u_i\})$ *of the limit problem (6) satisfies* $u \in H^2(\Omega)$ *and* $u_i \in \hat{\mathcal{H}}_i^1$ *for each* $i = 1, \ldots, n$*.*

Proof From (6) we have that

$$u_n = w_{nl}\left(\frac{\partial u}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \ldots + \frac{\partial u_{n-1}}{\partial y_{(n-1)l}}\right)$$

(the repeated index l indicates summation from 1 to d) where $w_{nl} \in L^2(\Omega \times Y_1 \times \ldots \times Y_{n-1}, H^1_{\#}(Y_n))$ is the solution of the variational problem

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} A(e_l + \nabla_{y_n} w_{nl}).\nabla_{y_n}\phi_n dx dy_1 \ldots dy_n = 0$$

for $\phi_n \in \mathcal{D}(\Omega \times Y_1 \times \ldots \times Y_n)$ where e_l is the l^{th} unit vector in $\mathbb{R}^d$. Substituting this form of u_n back into (6) we have

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} A(I + \nabla_{y_n} w_n)(\nabla_x u + \sum_{k=1}^{n-1} \nabla_{y_k} u_k).\nabla_{y_{n-1}}\phi_{n-1} dx dy_1 \ldots dy_{n-1} = 0$$

where w_n denotes the vector $(w_{n1}, \ldots, w_{nd})$ and I is the unit matrix. Then

$$u_{n-1} = w_{(n-1)l}\left(\frac{\partial u}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \ldots + \frac{\partial u_{n-2}}{\partial y_{(n-2)l}}\right)$$

where $w_{(n-1)l} \in L^2(\Omega \times Y_1 \times \ldots \times Y_{n-2}, H^1_{\#}(Y_{n-1}))$ is the solution of the variational problem

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_{n-1}} A_{n-1}(e_l + \nabla_{y_{n-1}} w_{(n-1)l}).\nabla_{y_{n-1}}\phi_{n-1} dx dy_1 \ldots dy_{n-1} = 0,$$

where

$$A_{n-1} = \int_{Y_n} A(I + \nabla_{y_n} w_n) dy_n = \int_{Y_n} A(I + \nabla_{y_n} w_n).(I + \nabla_{y_n} w_n) dy_n.$$

With $A_n = A$, this formula allows to define recursively A_i for all i and we have

$$u_i = w_{il}\left(\frac{\partial u}{\partial x_l} + \ldots + \frac{\partial u_{i-1}}{\partial y_{(i-1)l}}\right)$$

where $w_{il} \in L^2(\Omega \times Y_1 \times \ldots \times Y_{i-1}, H^1_{\#}(Y_i))$ is the solution of the problem

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_i} A_i(e_l + \nabla_{y_i} w_{il}).\nabla_{y_i}\phi_i dx dy_1 \ldots dy_i = 0, \tag{30}$$

for all $\phi_i \in \mathcal{D}(\Omega \times Y_1 \times \ldots \times Y_i)$. The matrix A_i is defined as

$$A_i(x, y_1, \ldots, y_i) = \int_{Y_{i+1}} A_{i+1}(I + \nabla_{y_{i+1}} w_{i+1}).(I + \nabla_{y_{i+1}} w_{i+1}) dy_{i+1} \tag{31}$$

for $i < n$. Finally the effective diffusivity is

$$A_0(x) = \int_{Y_1} A_1(I + \nabla_{y_1} w_1).(I + \nabla_{y_1} w_1) dy_1$$

and we have the following "effective" problem for u:

$$\int_\Omega A_0(x) \nabla_x u . \nabla_x \phi = \int_\Omega f\phi,$$

for all $\phi \in \mathcal{D}(\Omega)$. The functions u_i are then determined by

$$u_i = w_i.(I + \nabla_{y_{i-1}} w_{i-1}) \dots (I + \nabla_{y_1} w_1) \nabla_x u. \tag{32}$$

Since the matrix A_0 is Lipschitz with respect to x and the domain Ω is convex, $u \in H^2(\Omega)$. Furthermore since A_i are Lipschitz with respect to $x, y_1, \dots, y_i$, $w_{il} \in C^{0,1}(\Omega, C^{0,1}_{\#}(Y_1, \dots, H^2_{\#}(Y_i) \dots))$ (see Gilbarg and Trudinger [13]). From (32) we have $u_i \in \hat{\mathcal{H}}^1_i$.□

Remark 4.1 If the coefficient A is smooth, we solve the cell problems (30) as parametric equation for y_i in $\mathbb{R}^d$ for many points in $(x, y_1, \dots, y_{i-1}) \in \mathbb{R}^{id}$. The sparse FE scheme developed in the previous section is applicable to solve these cell problems with reduced complexity.

Remark 4.2 If the coeffient A is in $C^{t-1,1}$ with respect to each variable $x, y_1, \dots, y_n$ and $f \in H^{t-1}(\Omega)$ then $u \in H^{t+1}(\Omega)$ and $u_i \in \hat{\mathcal{H}}^t_i$ for $i = 1, \dots, n$.

5 Implementation

The sparse tensor product FE discretization (21) of the limit problem (6) leads to solving a linear system

$$\hat{S} C^L_u = C^L_f, \tag{33}$$

where the stiffness matrix $\hat{S}$ has dimension

$$\hat{N}^L := \sum_{i=0}^{n} d^L_i$$

with $d^L_0 = \dim V^L_0 = O(h_L^{-d})$ and $d^L_i = \dim \hat{V}^L_i = O(h_L^{-d} |\log h_L|^i)$.

Due to the large support of the basis functions in the sparse tensor product space, the matrix $\hat{S}$ is non sparse. Storing it requires memory which grows superlinearly in the number of degrees of freedom. In order to build a solution algorithm of log-linear complexity, we solve (33) iteratively and avoid forming the matrix explicitly. Rather, we form the matrix out of blocks corresponding to one-scale problems which can be written in terms of tensor products of smaller, banded matrices. We will show that this is possible in memory and work that grows essentially linear in the number of degrees of freedom for the one-scale problem, d^L_0.

To simplify the exposition, we consider the case where each entry of the matrix $\hat{S}$ is a finite sum of terms of the form $a_0(x)a_1(y_1) \dots a_n(y_n)$. Specifically, we work with the case where A is isotropic and of the form

$$A = a_0(x)a_1(y_1) \dots a_n(y_n) I, \tag{34}$$

where I is the $d \times d$ identity matrix. Assumption (34) is used below in factoring the stiffness matrix $\hat{S}$ to derive a linear complexity matrix vector multiplication. If the matrix A is smooth, but not of product form, it can be approximated by sums of products (34) and our matrix vector multiplication algorithm as well as our complexity estimates can still be applied.

Remark 5.1 If A is anisotropic then it is necessary to deal with each single entry of A separately and to add up the resulting matrices $\hat{S}$. The result for a single entry of A is achieved when the matrix I in (34) is not the identity matrix but has only one entry as 1 and others are zero. This can be done by a similar procedure as the one we are going to present below. We note that if A is of the form (34) the cell problem (30) reduces to a problem in $\mathbb{R}^d$ without parameters but if A is more complicated, e.g

$$A = (a_0(x)a_1(y_1)\dots a_n(y_n) + b_0(x)b_1(y_1)\dots b_n(y_n))I,$$

this is no longer the case.

To simplify the presentation, we assume in (34)

$$\int_Y a_i(y)dy = 1 \quad \text{for} \quad i = 1, \dots, n.$$

5.1 Multilevel FE space and preconditioning

It is most convenient to use the wavelet preconditioning procedures to establish bases for W^l, $W^{l0}_{\#}$ and $W^{l1}_{\#}$. We make the

Assumption 5.1 *i) For each $j \in {\mathbb{N}_0}^d$ there is a set of indices $I^j \subset {\mathbb{N}_0}^d$ and a family of functions $\psi^{jk} \in H^1(\Omega)$, $k \in I^j$ such that $\{\psi^{jk} : |j|_\infty \le l\}$ is a basis for V^l. There are two constants $0 < c_1 < c_2$ such that if $\psi = \sum_{|j|_\infty \le l, k \in I^j} \psi^{jk} c_{jk} \in V^l$ then*

$$c_1 \sum_{\substack{|j|_\infty \le l \\ k \in I^j}} |c_{jk}|^2 \le \|\psi\|^2_{L^2(\Omega)} \le c_2 \sum_{\substack{|j|_\infty \le l \\ k \in I^j}} |c_{jk}|^2.$$

ii) For each $j \in {\mathbb{N}_0}^d$ there is a set of indices $I^j_0 \subset {\mathbb{N}_0}^d$ and a family of functions $\psi^{jk}_{0\#} \in H^1_{\#}(Y)$, $k \in I^j_0$ such that $\{\psi^{jk}_{0\#} : |j|_\infty \le l\}$ is a basis for the linear space $V^l_{\#}$. There are two constants $0 < c_3 < c_4$ such that if $\psi = \sum_{|j|_\infty \le l, k \in I^j_0} \psi^{jk}_{0\#} c_{jk} \in V^l_{\#}$ then

$$c_3 \sum_{\substack{|j|_\infty \le l \\ k \in I^j_0}} |c_{jk}|^2 \le \|\psi\|^2_{L^2(Y)} \le c_4 \sum_{\substack{|j|_\infty \le l \\ k \in I^j_0}} |c_{jk}|^2.$$

iii) For each $j \in {\mathbb{N}_0}^d$ there is a set of indices $I^j_1 \subset {\mathbb{N}_0}^d$ and a family of functions $\psi^{jk}_{1\#} \in H^1_{\#}(Y)/\mathbb{R}$, $k \in I^j_1$ such that $\{\psi^{jk}_{1\#} : |j|_\infty \le l\}$ is a base for the linear space $V^l_{\#}/\mathbb{R}$. There are two constants $0 < c_5 < c_6$ such that if $\psi = \sum_{|j|_\infty \le l, k \in I^j} c_{jk} \psi^{jk}_{1\#} \in V^l_{\#}/\mathbb{R}$ then

$$c_5 \sum_{\substack{|j|_\infty \le l \\ k \in I^j_1}} |c_{jk}|^2 \le \|\psi\|^2_{H^1(Y)/\mathbb{R}} \le c_6 \sum_{\substack{|j|_\infty \le l \\ k \in I^j_1}} |c_{jk}|^2.$$

We give some examples of hierarchical Finite Element wavelet bases in one dimension that we will use later. For analogous multivariate constructions we refer to [10].

Example 5.1 i) Consider the case $\Omega = (0,1)$. The hierarchical base for $L^2((0,1))$ can be taken as follows. At level 0, we choose three continuous, piecewise linear functions in $(0,1)$: ψ^{01} takes values (1,0) at (0,1/2) and 0 in (1/2,1), ψ^{02} is the hat function ϕ taking values $0, 1, 0$ at $0, 1/2, 0$ and ψ^{03} takes values (0,1) at (1/2,1) and 0 in (0,1/2). For other level j, we consider the piecewise linear function ψ taking values (0,-1,2,-1,0) at (0,1/2,1,3/2,2) and 0 outside (0,2), the left

boundary piecewise linear function ψ^{left} taking value (-2,2,-1,0) at (0,1/2,1,3/2) and the right boundary piecewise linear function ψ^{right} taking value (0,-1,2,-2) at (1/2,1,3/2,2). We choose the index set $I^j = \{1, 2, \ldots, 2^j\}$, the function $\psi^{j1}(x) = 2^{-j/2}\psi^{\text{left}}(2^j x)$, the functions $\psi^{jk}(x) = 2^{-j/2}\psi(2^j x - k + 3/2)$ for $k = 2, \ldots, 2^j - 1$ and $\psi^{j2^j} = \psi^{\text{right}}(2^j x - 2^j + 2)$. These functions satisfy Assumption 1i)

ii) For $Y = (0, 1)$, the base can be taken as the one defined in i) above excluding the piecewise linear function ψ^{01} and ψ^{02} and also at other levels we consider the left boundary functions taking values (0,2,-1,0) at (0,1/2,1,3/2) and the right boundary function taking values (0,-1,2,0) at (1/2,1,3/2,2) (note that the functions are now defined within an additive constant). We can also consider the piecewise linear hat function ϕ in $\mathbb{R}$ taking values $0, 1, 0$ at $0, 1/2, 0$ and vanishing outside $(0, 1)$. The set of indices $I^j = \{1, 2, \ldots, 2^j\}$. The wavelet functions $\psi^{jk}(x) = 2^{-j/2}\phi(2^j x - i + 1)$ satisfy Assumption 5.1, iii).

From the norm equivalences in Assumption 1 it is obvious that the quantity $(\sum c_{jk}^2)^{1/2}$ defines an equivalent norm in $L^2(\Omega)$, $L^2(Y)$ and $H^1(Y)/\mathbb{R}$ in V^l, $V_\#^l$ and $V_\#^l/\mathbb{R}$ respectively. From the coercivity (7) and the continuity (8) of the form B and the norm equivalences in Assumption 1 we have immediately a result on multilevel preconditioning:

Proposition 5.1 *The condition number of the stiffness matrix $\hat{S}$ for the sparse grid discretization of the one-scale limit problem has uniformly bounded condition number as $L \to \infty$.*

In these multilevel bases, the increment spaces can be explicitly represented as

$$\begin{aligned} W^l &= \text{Span}\{\psi^{jk} : \ |j|_\infty = l\}, \\ W_\#^{l0} &= \text{Span}\{\psi_{0\#}^{jk} : \ |j|_\infty = l\}, \\ W_\#^{l1} &= \text{Span}\{\psi_{1\#}^{jk} : \ |j|_\infty = l\}. \end{aligned}$$

For $i \geq 1$, let $\mathbf{j}$ be an $(i+1) \times d$ index matrix of rows $j_0, \ldots, j_i$ and $\mathbf{k}$ be an $(i+1) \times d$ index matrix of rows $k_0, \ldots, k_i$ where $k_0 \in I^{j_0}$ and $k_\nu \in I_0^{j_\nu}$ for $\nu = 1, \ldots, i-1$ and $k_i \in I_1^{j_i}$. Denoting by $\psi^{\mathbf{jk}} = \psi^{j_0 k_0}\psi_{0\#}^{j_1 k_1} \ldots \psi_{1\#}^{j_i k_i}$ we can write

$$\hat{V}_i^L = \text{Span}\{\psi^{\mathbf{jk}} : \ \sum_{\nu=0}^{i} |j(\nu)|_\infty \leq L\}$$

where $j(\nu)$ denotes the ν^{th} row of the $(i+1) \times d$ matrix $\mathbf{j}$.

The algebraic excess (or detail space) of $\hat{V}_i^L$ is then defined as

$$\hat{W}_i^L = \text{Span}\{\psi^{\mathbf{jk}} : \ \sum_{\nu=0}^{i} |j(\nu)|_\infty = L\} = \bigoplus_{\underline{l} \in \mathbb{N}^{i+1}, |\underline{l}| = L} W_i^{\underline{l}},$$

where

$$W_i^{\underline{l}} = \text{Span}\{\psi^{\mathbf{jk}} : \ |\mathbf{j}(\nu)| = l_\nu\}.$$

We denote by $\Psi_i^{\underline{l}}, \Psi_\#^{l0}, \Psi_\#^{l1}, \Psi^L$ the column vectors containing the bases of $W_i^{\underline{l}}$, $W_\#^{l0}$, $W_\#^{l1}$ and V_0^L, respectively.

To obtain a log-linear complexity solution algorithm, we make the following assumptions.

Assumption 5.2 *There exists a number $p > 0$ such that*

i) For all $k \in I^j$ and $j' \in \mathbb{N}^d$, the set $\text{supp}(\psi^{jk}) \cap \text{supp}(\psi^{j'k'})$ has nonempty interior for at most $p^d . \Pi_{q=1}^d \max(1, 2^{j'_q - j_q})$ values of k'.

ii) For all $k \in I^j$, *the set* $\mathrm{supp}(\psi^{jk}) \cap \mathrm{supp}(\psi^L_{k'})$ *where* $\{\psi^L_{k'}\}$ *is the basis of* V_0^L *has nonempty interior for at most* $p^d \sum_{0 \le |j'| \le L} \Pi_{q=1}^d \max(1, 2^{j'_q - j_q})$ *values of* k'.

iii) For all $k \in I_0^j$ *and* $j' \in \mathbb{N}^d$, *the set* $\mathrm{supp}(\psi^{jk}_{0\#}) \cap \mathrm{supp}(\psi^{j'k'}_{0\#})$ *has nonempty interior for at most* $p^d . \Pi_{q=1}^d \max(1, 2^{j'_q - j_q})$ *values of* k'.

iv) For all $k \in I_0^j$ *and* $j' \in \mathbb{N}^d$, *the set* $\mathrm{supp}(\psi^{jk}_{0\#}) \cap \mathrm{supp}(\psi^{j'k'}_{1\#})$ *has nonempty interior for at most* $p^d . \Pi_{q=1}^d \max(1, 2^{j'_q - j_q})$ *values of* k'.

v) For all $k \in I_0^j$ *and* $j' \in \mathbb{N}^d$, *the set* $\mathrm{supp}(\psi^{jk}_{1\#}) \cap \mathrm{supp}(\psi^{j'k'}_{1\#})$ *has nonempty interior for at most* $p^d . \Pi_{q=1}^d \max(1, 2^{j'_q - j_q})$ *values of* k'.

The assumptions i), iii), iv) and v) hold for the wavelet basis given in the examples above and also for similar wavelet constructions in polygons [10]. Assumption ii) holds when for example we assume a wavelet hierarchical basis for $H_0^1(\Omega)$.

5.2 Matrix-vector multiplication

To achieve a complexity that is essentially linear in d_0^L, the stiffness matrix $\hat{S}$ is never formed explicitly and hence, iterative solution methods are needed for the solution of the linear system (33). In our approach, $\hat{S}$ will be generated in factored form and only certain building blocks of the matrix $\hat{S}$ will be computed and stored.

To this end, we define certain block matrices: we denote by A^0 the stiffness matrix with respect to $a_0(x)$ and the basis Ψ of V_0^L; by $A^i_{l,l'}$ $(i = 1, \ldots, n)$ the stiffness matrix describing the interaction between $W^{l1}_\#$ and $W^{l'1}_\#$ with respect to $a^i(y)$, by $M^0_{l,l'}$ the mass matrix with respect to $a_0(x)$ describing the interaction between W^l and $W^{l'}$, by $M^i_{l,l'}$ $(i = 1, \ldots, n)$ the mass matrix describing the interaction between $W^{l0}_\#$ and $W^{l'0}_\#$ with respect to $a^i(y)$, and by $B^{i\alpha}_l$ $(i = 1, \ldots, n,$ $\alpha = 1, \ldots, d)$ the row matrix defined by

$$(B^{i\alpha}_l)_{1,\nu} = \int_Y a_i(y) \frac{\partial \Psi^{l1}_{\#\nu}}{\partial y_\alpha} dy,$$

where $\Psi^{l1}_{\#\nu}$ is the ν^{th} function in the base of $W^{l1}_\#$. We further require C^i_l $(i = 1, \ldots, n-1)$, the row matrix defined by

$$(C^i_l)_{1,\nu} = \int_Y a_i(y) \Psi^{l0}_{\#\nu} dy,$$

where $\Psi^{l0}_{\#\nu}$ is the ν^{th} function in the base of $W^{l0}_\#$; and $D^{0\alpha}_l$ $(\alpha = 1, \ldots, d)$ the matrix defined by

$$(D^{0\alpha}_l)_{\mu,\nu} = \int_\Omega a_0(x) \frac{\partial \Psi_\mu}{\partial x_\alpha} \Psi^l_\nu(x) dx$$

where Ψ_μ is the μ^{th} function in the base of V_0^L and Ψ^{l0}_ν is the ν^{th} function in the base of W^l; $D^{i\alpha}_{l,l'}$ $(i = 1, \ldots, n-1,\ \alpha = 1, \ldots, d)$ is the matrix defined by

$$(D^{i\alpha}_{l,l'})_{\mu,\nu} = \int_\Omega a_i(y) \frac{\partial \Psi^{l1}_{\#\mu}}{\partial y_\alpha} \Psi^{l'0}_{\#\nu}(x) dx$$

where $\Psi^{l1}_{\#\mu}$ is the μ^{th} function in the base of $V^{l1}_\#$ and $\Psi^{l'0}_{\#\nu}$ is the ν^{th} function in the base of $W^{l'0}_\#$.

From assumption 5.2, it it simple to see that

$$\mathrm{nnz}(M^0_{l,l'}) \le p^d (\min(l, l') + 1)^{d-1} 2^{d \max(l,l')}$$

so the memory needed to store all matrices $M^0_{l,l'}$ for $0 \le l, l' \le L$ is bounded by

$$\sum_{l,l'=0}^{L} (\min(l,l')+1)^{d-1} 2^{d\max(l,l')} \lesssim (L+1)^{d-1} 2^{dL}.$$

Similar results are readily seen to hold also for the other matrices, so that the total memory needed to store the matrices A, B, C, D and M grows as $O(L^{d-1}2^{dL})$.

We proceed to derive the matrix vector multiplication algorithm. To this end, we express the bilinear form $B(\cdot,\cdot)$ on $\hat{\mathbf{V}}^L \times \hat{\mathbf{V}}^L$ in terms of the wavelet basis.

To this end, we write $\hat{u}^L = \underline{x}_{u0} \cdot \Psi^L$, $\hat{u}^L_i = \underline{x}^{\underline{l}}_{ui} \cdot \Psi^{\underline{l}}_i$, $\hat{\phi}^L = \underline{x}_0 \cdot \Psi^L$, $\hat{\phi}^L_{i'} = \underline{x}^{\underline{l}'}_{i'} \cdot \Psi^{\underline{l}'}_{i'}$ where $\underline{l} \in \mathbb{N}_0^{i+1}$ and $\underline{l}' \in \mathbb{N}_0^{i'+1}$ and $|\underline{l}|, |\underline{l}'| \le L$. Then the bilinear form for the limit problem (6) has the matrix representation

$$\begin{aligned}
&B(\hat{u}^L, \{\hat{u}^L_i\}; \hat{\phi}, \{\hat{\phi}^L_i\}) = \\
&\Big(A^0 \underline{x}_{u0} + \sum_{i=1}^{n} \sum_{0\le|\underline{l}|\le L} \sum_{\alpha=1}^{d} D^{0\alpha}_{\underline{l}_0} \bigotimes_{\nu=1}^{i-1} C^{\nu}_{\underline{l}_\nu} \bigotimes B^{i\alpha}_{\underline{l}_i} \underline{x}^{\underline{l}}_{ui}\Big) . \underline{x}_0 + \\
&\sum_{i'=1}^{n} \sum_{0\le|\underline{l}'|\le L} \Bigg(\sum_{\alpha=1}^{d} (D^{0\alpha}_{\underline{l}'})^\top \bigotimes_{\nu=1}^{i'-1} (C^{\nu}_{\underline{l}'_\nu})^\top \bigotimes (B^{i'\alpha}_{\underline{l}'_i})^\top \underline{x}_{u0} + \\
&\quad \sum_{i=1}^{i'-1} \sum_{0\le|\underline{l}|\le L} \sum_{\alpha=1}^{d} \bigotimes_{\nu=0}^{i-1} M^{\nu}_{\underline{l}'_\nu \underline{l}_\nu} \bigotimes (D^{i\alpha}_{\underline{l}_i \underline{l}'_i})^\top \bigotimes_{\nu=i+1}^{i'-1} (C^{\nu}_{\underline{l}'_\nu})^\top \bigotimes (B^{i'\alpha}_{\underline{l}'_{i'}})^\top \underline{x}^{\underline{l}}_{ui} + \\
&\quad \sum_{0\le|\underline{l}|\le L} \bigotimes_{\nu=0}^{i'-1} M^{\nu}_{\underline{l}'_\nu \underline{l}_\nu} \bigotimes A^{i'}_{\underline{l}'_{i'} \underline{l}_{i'}} \underline{x}^{\underline{l}}_{ui'} + \\
&\quad \sum_{i=i'+1}^{n} \sum_{0\le|\underline{l}|\le L} \sum_{\alpha=1}^{d} \bigotimes_{\nu=0}^{i'-1} M^{\nu}_{\underline{l}'_\nu \underline{l}_\nu} \bigotimes D^{i'\alpha}_{\underline{l}'_{i'} \underline{l}_{i'}} \bigotimes_{\nu=i'+1}^{i-1} C^{\nu}_{\underline{l}_\nu} \bigotimes B^{i\alpha}_{\underline{l}_i} \underline{x}^{\underline{l}}_{ui} \Bigg) . \underline{x}^{\underline{l}'}_{i'}.
\end{aligned}$$

We see that $\hat{S}$ is formed from tensor products of the 'elementary' matrices A, B, C, D and M defined above.

The algorithm for applying $\hat{S}$ to a vector with the optimal number of floating point operations is based on [21] and [22].

Step 1: Vector multiplication $D^{0\alpha}_{\underline{l}_0} \bigotimes_{\nu=1}^{i-1} C^{\nu}_{\underline{l}_\nu} \bigotimes B^{i\alpha}_{\underline{l}_i} \underline{x}^{\underline{l}}_{ui}$.
Let R_{ij} be the row matrix $C^1_{\underline{l}_1} \bigotimes \ldots \bigotimes B^{i\alpha}_{\underline{l}_i}$. The number of nonzero entries of this matrix is bounded by

$$(\underline{l}_1+1)^d \ldots (\underline{l}_i+1)^d 2^{d(\underline{l}_1+\ldots+\underline{l}_i)} \lesssim (|\underline{l}|+1)^{id} 2^{d(|\underline{l}|-\underline{l}_0)}.$$

Let $\mathrm{Id}_{\underline{l}_0\underline{l}_0}$ be the identity matrix whose dimension is $\dim W^{\underline{l}_0}$. The multiplication is performed as $D^{0\alpha}_{\underline{l}_0}(\mathrm{Id}_{\underline{l}_0\underline{l}_0} \bigotimes R_{ij})\underline{x}^j_i$, which requires a number of floating point operations bounded by $\mathrm{nnz}(D^{0\alpha}_{\underline{l}_0}) + \mathrm{nnz}(\mathrm{Id}_{\underline{l}_0\underline{l}_0} \bigotimes R_{ij})$. From Assumption 5.2, we get

$$\mathrm{nnz}(D^{0\alpha}_{\underline{l}_0}) \lesssim \sum_{l=0}^{L} (\min(l,\underline{l}_0)+1)^{d-1} . 2^{d\max(\underline{l}_0,l)} \lesssim (|\underline{l}|+1)^{d-1} L 2^{dL}.$$

We also have $\dim W^{\underline{l}_0} \sim (\underline{l}_0+1)^d 2^{d\underline{l}(0)}$. Thus the floating point operations needed to perform this step over all i and $\underline{l}$ is bounded by

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \sum_{|\underline{l}|=l} (l+1)^{(i+1)d} 2^{ld} + (l+1)^{d-1} 2^{Ld} L \lesssim \sum_{i=1}^{n} L^{(i+1)d+i} 2^{Ld} \lesssim L^{(n+1)d+n} 2^{Ld}$$

(note that the number of vectors $\underline{l} \in \mathbb{N}_0^{i+1}$ such that $|\underline{l}| = l$ is at most $O(l^i)$).

Step 2: Multiplication $(D^{0\alpha}_{\underline{l}'_0})^\top \bigotimes_{\nu=1}^{i'-1} (C^{\nu}_{\underline{l}'_\nu})^\top \bigotimes (B^{i'\alpha}_{\underline{l}'_{i'}})^\top \underline{x}_{u0}$.

This multiplication is performed as $((R_{ij})^\top \bigotimes \mathrm{Id}_{\underline{l}'_0\underline{l}'_0})(D^{0\alpha}_{u\underline{l}'_0})^\top \underline{x}_{u0}$. Similarly as above, the number of floating point operations needed to perform this step over all i', $\underline{l}'$ is bounded by $L^{(n+1)d+n} 2^{Ld}$.

Step 3: Matrix vector multiplication

$$\bigotimes_{\nu=0}^{i-1} M^{\nu}_{\underline{l}'_\nu \underline{l}_\nu} \bigotimes (D^{i\alpha}_{\underline{l}_i \underline{l}'_i})^\top \bigotimes_{\nu=i+1}^{i'-1} (C^{\nu}_{\underline{l}'_\nu})^\top \bigotimes (B^{i'\alpha}_{\underline{l}'_{i'}})^\top . \underline{x}^{\underline{l}}_{ui},$$

($1 \le i \le i'-1$ and $\alpha = 1, \dots, d$). We regard $\underline{l}_\nu = 0$ for $\nu = i+1, \dots, i'$. From [22] Lemma 5.8, there is a permutation σ of $(0, \dots, i')$ such that

$$\forall 0 \le q \le i' : \qquad \sum_{\nu=0}^{q} \underline{l}_{\sigma(\nu)} + \sum_{\nu=q+1}^{i'} \underline{l}'_{\sigma(\nu)} \le \max\Big(\sum_{\nu=0}^{i} \underline{l}_\nu + \sum_{\nu=0}^{i'} \underline{l}'_\nu\Big).$$

Let $T_{\underline{l},\underline{l}',q} = \bigotimes_{\nu=1}^{i'} U_\nu(\underline{l}, \underline{l}', q)$ where

$$U_\nu(\underline{l}, \underline{l}', q) = \begin{cases} \mathrm{Id}_{\underline{l}_\nu, \underline{l}_\nu}, & \nu \in \{\sigma(0), \dots, \sigma(q-1)\} \\ M^{\sigma(q)}_{\underline{l}'_{\sigma(q)} \underline{l}_{\sigma(q)}}, & \nu = \sigma(q) \le i-1 \\ (D^{i\alpha}_{\underline{l}_i \underline{l}'_i})^\top, & \nu = \sigma(q) = i \\ (C^{\sigma(q)}_{\underline{l}'_{\sigma(q)}})^\top, & i < \nu = \sigma(q) < i' \\ (B^{i'\alpha}_{\underline{l}'_{i'}})^\top, & \nu = \sigma(q) = i' \\ \mathrm{Id}_{\underline{l}'_\nu, \underline{l}_\nu}, & \nu \in \{\sigma(q+1), \dots, \sigma(i')\} \end{cases}$$

where $\mathrm{Id}_{00} = 1$, $\mathrm{Id}_{\underline{l}_0,\underline{l}_0}$ is the identity matrix of size $\dim W^{\underline{l}_0}$, $\mathrm{Id}_{\underline{l}_\nu \underline{l}_\nu}$ is the identity matrix of size $\dim W_{\#}^{\underline{l}_\nu 0}$ if $0 < \nu < i$ and of size $\dim W_{\#}^{\underline{l}_i 1}$ if $\nu = i$; $\mathrm{Id}_{\underline{l}'_\nu \underline{l}'_\nu}$ is defined similarly (we replace i by i' in the definition). Then

$$\bigotimes_{\nu=0}^{i-1} M^{\nu}_{\underline{l}'_\nu \underline{l}'_\nu} \bigotimes (D^{i\alpha}_{\underline{l}_i \underline{l}'_i})^\top \bigotimes_{\nu=i+1}^{i'-1} (C^{\nu}_{\underline{l}'_\nu})^\top \bigotimes (B^{i'\alpha}_{\underline{l}'_{i'}})^\top . \underline{x}^{\underline{l}}_{ui} = T_{\underline{l},\underline{l}',1} \dots T_{\underline{l},\underline{l}',i'} . \underline{x}^{\underline{l}}_{ui}.$$

The order of multiplication is essential here to get the following estimate on the number of floating points operations needed. We have

$$\begin{aligned} \mathrm{nnz}(T_{\underline{l},\underline{l}',q}) &\lesssim \Pi_{\nu=1}^{q-1} (\underline{l}_{\nu(q)} + 1)^d . (\min(\underline{l}'_{\sigma(q)}, \underline{l}_{\sigma(q)}) + 1)^{d-1} . \Pi_{\nu=q+1}^{i'} (\underline{l}'_{\sigma(\nu)} + 1)^d \\ &\cdot \quad 2^{d(\sum_{\nu=1}^{q-1} \underline{l}_{\sigma(\nu)} + \max(\underline{l}'_{\sigma(q)}, \underline{l}_{\sigma(q)}) + \sum_{\sigma=q+1}^{i'} \underline{l}_{\sigma(\nu)})} \\ &\lesssim (\max(|\underline{l}|, |\underline{l}'|) + 1)^{d(i'+1)} 2^{d \max(|\underline{l}|, |\underline{l}'|)}; \end{aligned}$$

here we used that $\dim W^{l0}$, $\dim W_{\#}^{l0}$ and $\dim W_{\#}^{l1} \simeq (l+1)2^{dl}$. Thus this matrix vector multiplication needs at most

$$\sum_{q=1}^{i'} \mathrm{nnz}(T_{\underline{l},\underline{l}',q}) \lesssim (\max(|\underline{l}|, |\underline{l}'|) + 1)^{d(i'+1)} 2^{d \max(|\underline{l}|, |\underline{l}'|)}$$

floating point operations. The amount of memory needed to store these matrices is bounded by $\lesssim L^{d(n+1)} 2^{dL}$. The number of floating point operations needed to perform this step over all l, l', $|\underline{l}| = l$, $|\underline{l}'| = l'$, i and i' is bounded by

$$\begin{aligned} & \sum_{i'=1}^{n} \sum_{\substack{l=0 \\ l'=0}}^{L} \sum_{|\underline{l}'|=l'} \sum_{i<i'} \sum_{|\underline{l}|=l} (\max(|\underline{l}|, |\underline{l}'|) + 1)^{d(i'+1)} 2^{d \max(|\underline{l}|, |\underline{l}'|)} \\ \lesssim & \sum_{l,l'=1}^{L} l^n l'^n (\max(l, l') + 1)^{d(n+1)} 2^{d \max(l, l')} \lesssim L^{2n+d(n+1)} 2^{dL}. \end{aligned}$$

Step 4: We perform the multiplication $\bigotimes_{\nu=0}^{i'-1} M^{\nu}_{\underline{l}'_\nu,\underline{l}_\nu} \bigotimes A^{i'}_{\underline{l}'_{i'},\underline{l}_{i'}}.\underline{x}^{\underline{l}}_{ui'}$, similarly: we find an appropriate permutation of $(0,1,\ldots,i')$ and we write this multiplication in the form $T_{\underline{l},\underline{l}',1}\ldots T_{\underline{l},\underline{l}',i'}.\underline{x}^{\underline{l}}_{i'}$ to get the same estimate as in Step 3 for the number of required floating point operations.

Step 5: We perform the multiplication $\bigotimes_{\nu=0}^{i'-1} M^{\nu}_{\underline{l}'_\nu,\underline{l}_\nu} \bigotimes D^{i'\alpha}_{\underline{l}'_{i'}\underline{l}_{i'}} \bigotimes_{\nu=i'+1}^{i-1} C^{\nu}_{\underline{l}_\nu} \bigotimes B^{\alpha} i_{\underline{l}_i} x^{\underline{l}}_i$, similarly as in step 3: we find a permutation σ of $(0,1,\ldots,i)$ for the sets of indices $(\underline{l}_0,\ldots,\underline{l}_i)$ and $(\underline{l}'_0,\ldots,\underline{l}'_{i'},0,\ldots,0)$ ($i-i'$ zeros). Then we have the same estimate as in Step 3 for the number of floating point operations needed.

The matrix-vector multiplication is summarized in the following

Algorithm

Store A^0, $A^i_{l,l'}$, $M^0_{l,l'}$, $M^i_{l,l'}$, $B^{i\alpha}_l$ $(i=1,\ldots,n,\ \alpha=1,\ldots,d)$,

C^i_l $(i=1,\ldots,n-1)$, $D^{0\alpha}_l$, $D^{i\alpha}_{ll'}$ (all sparse), $(\underline{x}_0,(x^{\underline{l}}_i)_{\underline{l}_0+\underline{l}_1+\ldots+\underline{l}_i\le L})$.

initialize $(\hat{S}x)_0 = A^0.\underline{x}_0$

for $i=1,\ldots,n$, **for** $\underline{l}:\ \sum_{\nu=0}^{i}\underline{l}_\nu \le L$:

Step 1: $\underline{y}_0 = \sum_{\alpha=1}^{d} D^{0\alpha}_{\underline{l}_0}.(\mathrm{Id}_{\underline{l}_0\underline{l}_0}\bigotimes R_{ij}).\underline{x}^{\underline{l}}_i,\ \ (\hat{S}x)_0 = (\hat{S}x)_0 + \underline{y}_0$

end , **end**

for $i'=1,\ldots,n$, **for** $\underline{l}':\ \sum_{\nu=0}^{i'}\underline{l}'_\nu \le L$

initialize $(\hat{S}x)^{\underline{l}'}_{i'} = 0$

Step 2: $\underline{y}^{\underline{l}'}_{i'} = \sum_{\alpha=0}^{d}(R^{\top}_{i'j'}\bigotimes \mathrm{Id}_{\underline{l}'_0\underline{l}'_0})(D^{0\alpha}_{\underline{l}'_0})^{\top}\underline{x}_0,\ \ (\hat{S}x)^{\underline{l}'}_{i'} = (\hat{S}x)^{\underline{l}'}_{i'} + \underline{y}^{\underline{l}'}_{i'}$

Step 3: **for** $i=1,\ldots,i'-1$, **for** $\underline{l}:\ \sum_{\nu=0}^{i}\underline{l}_i \le L$

$$\underline{y}^{\underline{l}'}_{i'} = \sum_{\alpha=1}^{d} T_{\underline{l},\underline{l}',1}\ldots T_{\underline{l},\underline{l}',i'}.\underline{x}^{i}_{\underline{l}},\ \ (\hat{S}x)^{\underline{l}'}_{i'} = (\hat{S}x)^{\underline{l}'}_{i'} + \underline{y}^{\underline{l}'}_{i'}$$

end, **end**

Step 4: $\underline{y}^{\underline{l}'}_{i'} = \sum_{\alpha=1}^{d} T_{\underline{l},\underline{l}',1}\ldots T_{\underline{l},\underline{l}',i'}.\underline{x}^{i}_{\underline{l}},\ \ (\hat{S}x)^{\underline{l}'}_{i'} = (\hat{S}x)^{\underline{l}'}_{i'} + \underline{y}^{\underline{l}'}_{i'}$

Step 5: **for** $i=i'+1,\ldots,n$, **for** $\underline{l}:\ \sum_{\nu=0}^{i}\underline{l}_i \le L$

$$\underline{y}^{\underline{l}'}_{i'} = \sum_{\alpha=1}^{d} T_{\underline{l},\underline{l}',1}\ldots T_{\underline{l},\underline{l}',i}.\underline{x}^{\underline{l}}_{i},\quad (\hat{S}x)^{\underline{l}'}_{i'} = (\hat{S}x)^{\underline{l}'}_{i'} + \underline{y}^{\underline{l}'}_{i'}$$

end, **end**

end, **end**

We use the conjugate gradient (cg) method to solve the linear problem (33). Due to the boundedness of the condition number of the matrix $\hat{S}$, the number of cg-steps required to compute the solution up to a prescribed accuracy is bounded. The number of degrees of freedom equals $\hat{N}_L = O(2^{dL}L^n)$. Analogous resoning applies also to Finite Elements of polynomial degree $p>1$. Thus we have shown

Theorem 5.1 *Assuming $u \in H^{p+1}(\Omega)$ and $u_i \in \hat{\mathcal{H}}^p_i$, the multidimensional limit problem (6) is numerically solvable by sparse tensor product Finite Elements of degree p with relative accuracy $O(\hat{N}_L^{-p/d}(\log \hat{N}_L)^{n(1/2+1/d)})$ in the norm $|||.|||$ defined in (5) at a cost of at most $O(\hat{N}_L(\log \hat{N}_L)^{n+d(n+1)})$ floating point operations with at most $O(\hat{N}_L(\log \hat{N}_L)^{d(n+1)-n})$ words of memory.*

Remark 5.2 So far, we performed the convergence and complexity analysis of the sparse FEM for the numerical solution of the high dimensional limit problem only under the strong regularity hypothesis that $u \in H^{t+1}(\Omega)$ and that $u_i \in \hat{\mathcal{H}}^t_i$. It is well-known, however, that for problems in polygonal domains the solution u of the limiting problem may not even belong to $H^2(\Omega)$, due to corner singlarities. We emphasize that in such cases our convergence analysis is still valid, if the

quasiuniform mesh in the FE space in Ω is replaced by a mesh with suitable refinement toward the solution's singular supports. Since we constructed the members $\hat{V}_i^L$ of the sparse tensor product space (20) from *full* component FE spaces with hierarchical basis, any type of mesh refinement in these component FE spaces is admissible in order to resolve solution singularities while preserving our error and complexity estimates; the regularity spaces $\hat{\mathcal{H}}_i^t$ have to be replaced by weighted spaces, however. See [21] for sparse grid convergence estimates for component FE spaces with mesh refinement, in the case $n = 1$.

6 Numerical results

In this section, we illustrate the theoretical results presented above with numerical experiments.

6.1 Two scale problem

We start with a simple one dimensional two scale problem where $A(x, y) = a_0(x)a_1(y)$, $a_0(x) = 1 + x$ and $a_1(y) = (2/3)(1 + \cos^2 2\pi y)$, $f = -1$ in the domain $\Omega = (0, 1)$. The multiscale problem (2) becomes

$$\frac{d}{dx}(\frac{2}{3}(1+x)(1+\cos^2(2\pi\frac{x}{\varepsilon}))\frac{du^\varepsilon}{dx}) = 1 \quad \text{in} \quad \Omega, \qquad u^\varepsilon(0) = u^\varepsilon(1) = 0.$$

The two scale limiting equation (9) has the exact (homogenized) solution

$$u(x) = \frac{3}{2\sqrt{2}}(x - \frac{\log(1+x)}{\log 2}),$$

and the scale interaction term

$$u_1(x, y) = \frac{3}{2\sqrt{2}}(1 - \frac{1}{(1+x)\log 2})(\frac{1}{2\pi}\tan^{-1}(\frac{\tan 2\pi y}{\sqrt{2}}) - y + C),$$

(note that fixing x, $u_1(x, y)$ is determined within an additive constant).

In the sparse FEM, we use the hierarchical base in example 3i) for $L^2(\Omega)$ and the hat function base in 3iii) for $H^1_\#(Y)/\mathbb{R}$. The number of degrees of freedom for the sparse FEM is $(L + 3)2^{L+1} - 2$.

In Figure 1, we plot the energy error versus the mesh size h for the full tensor product FE and the sparse tensor product.

We see that the full and sparse tensor product FE - errors, i.e. $|||(u - u^{FE}, u_1 - u_1^{FE})|||^2$ and $|||(u - \hat{u}^{FE}, u_1 - \hat{u}_1^{FE})|||^2$, are roughly the same for the same mesh size, which illustrates our result that the scale interaction functions $u_i(x, y_1, ..., y_i)$ have sufficient regularity so as to allow a sparse approximation without reducing the convergence rate, in terms of the meshwidth.

In Figure 2, we compare the energy error with the number of degrees of freedom N. It is clear that the same energy error is obtained with a much smaller number of degrees of freedom in the sparse tensor product FE case as compared to the full tensor product case. We also see that the energy error for the sparse tensor product FEM decreases of order $O(N^{-2}(\log N)^3)$ where N is the number of degrees of freedom.

6.2 Three scale problem

Next we consider in $\Omega = (0, 1)$ the three scale problem

$$\frac{d}{dx}(\frac{4}{9}(1+x)(1+\cos^2(2\pi\frac{x}{\varepsilon_1}))(1+\cos^2(2\pi\frac{x}{\varepsilon_2}))\frac{du^\varepsilon}{dx}) = 1 \text{ in } \Omega,$$

$$u^\varepsilon(0) = u^\varepsilon(1) = 0.$$

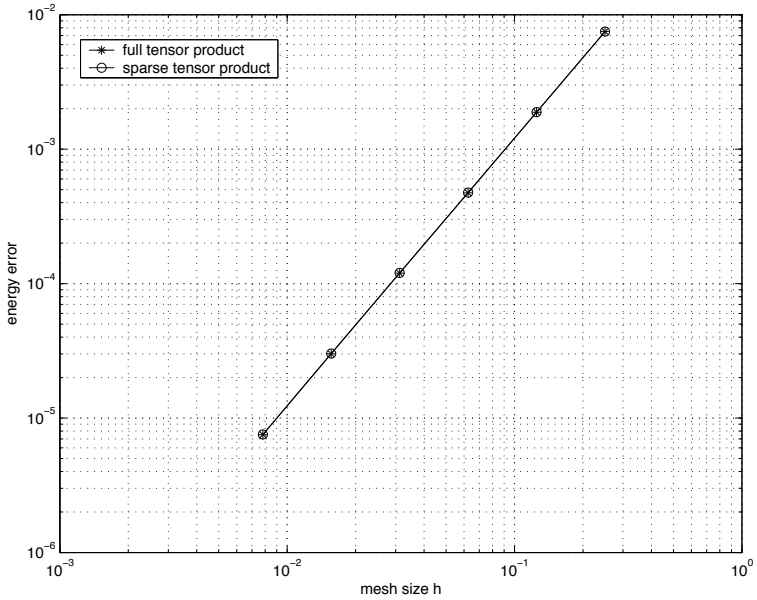

Figure 1: The energy error versus the mesh size h

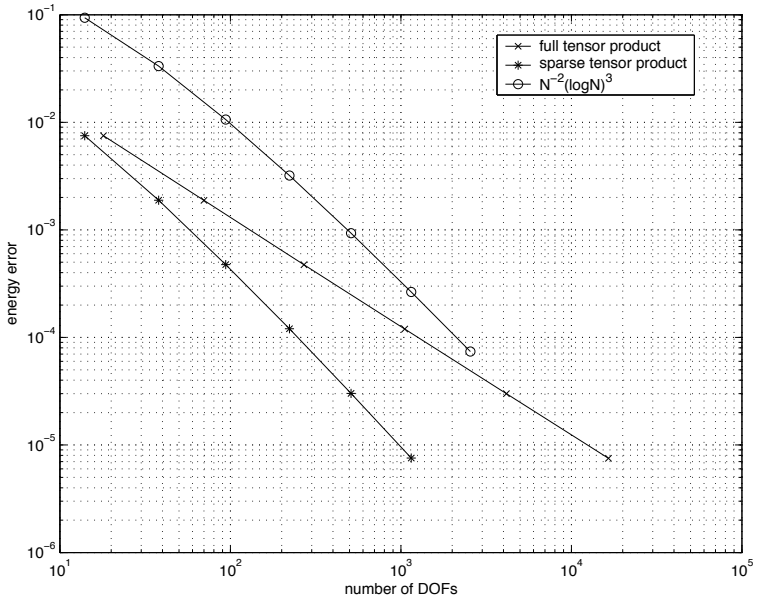

Figure 2: The energy error versus the number of degrees of freedom

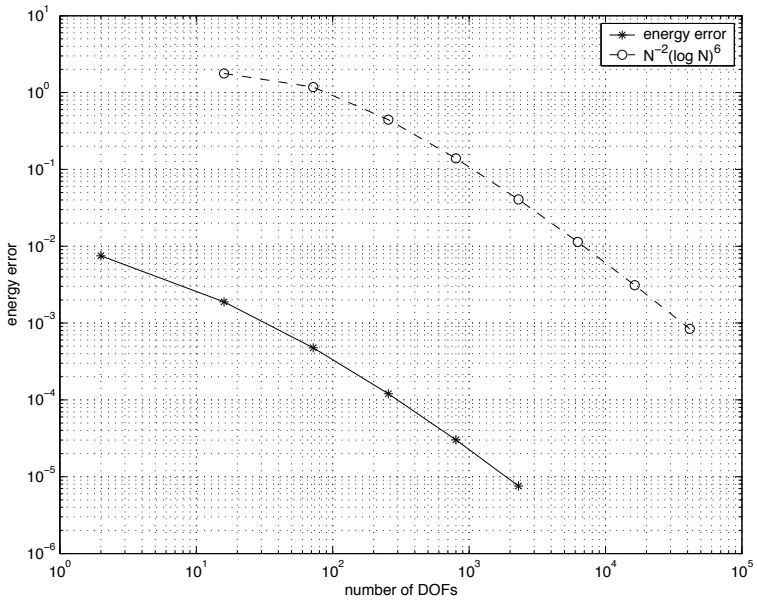

Figure 3: The energy error versus the number of degrees of freedom for three scales

The three scale limiting problem (6) has the exact homogenized solution

$$u(x) = \frac{9}{8}(x - \frac{\log(1+x)}{\log 2}),$$

and the scale interaction terms

$$u_1(x, y_1) = \frac{9}{8}(1 - \frac{1}{(1+x)\log 2})(\frac{1}{2\pi}\tan^{-1}(\frac{\tan 2\pi y_1}{\sqrt{2}}) - y_1 + C_1),$$

$$u_2(x, y_1, y_2) = \frac{9}{8}(1 - \frac{1}{(1+x)\log 2})(\frac{1}{2\pi}\tan^{-1}(\frac{\tan 2\pi y_2}{\sqrt{2}}) - y_2 + C_2)\frac{\sqrt{2}}{1+\cos^2 2\pi y_1}.$$

In Figure 3 the energy error is plotted versus the number of degrees of freedom. The predicted convergence behaviour of $O(N^{-2}(\log N)^6)$ is not visible here since, in the range of N used for the computation, the logarithmic terms in the convergence estimate still dominate the error. From our asymptotic convergence analysis, we expect this effect to be even more pronounced for more than three scales.

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