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Coercive Combined Field Integral Equations

R. Hiptmair and A. Buffa*

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Abstract

Many boundary integral equations for exterior Dirichlet- and Neumann boundary value problems for the Helmholtz equation suffer from a notorious instability for wave numbers related to interior resonances. The so-called combined field integral equations are not affected. However, if the boundary Γ is not smooth, the traditional combined field integral equations for the exterior Dirichlet problem do not give rise to an $L^2(\Gamma)$-coercive variational formulation. This foils attempts to establish asymptotic quasi-optimality of discrete solutions obtained through conforming Galerkin boundary element schemes.

This article presents new combined field integral equations on two-dimensional closed surfaces that possess coercivity in canonical trace spaces. The main idea is to use suitable regularizing operators in the framework of both direct and indirect methods. This permits us to apply the classical convergence theory of conforming Galerkin methods.

Keywords: Acoustic scattering, indirect boundary integral equations, combined field integral equations (CFIE), coercivity, boundary element methods, Galerkin schemes

Subject Classification: 35J05, 65N38, 65N12

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1. Introduction. The propagation of time-harmonic sound waves in a homogeneous isotropic medium that occupies the domain $\Omega \subset \mathbb{R}^3$ is governed by the Helmholtz equation, which, in non-dimensional form, reads

$$-\Delta U - \kappa^2 U = 0 \, . \tag{1.1}$$

Here, U designates the complex amplitude of either the density or of a velocity potential, see [8, Sect. 2.1], and $\kappa > 0$ stands for a fixed wave number. In acoustic scattering Ω is the complement of a bounded scatterer Ω^- and will be denoted by $\Omega^+ := \mathbb{R}^3 \setminus \bar{\Omega}^-$. In this case Sommerfeld radiation conditions, see [17, Def. 9.5],

$$\frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) = o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty \tag{1.2}$$

have to be imposed "at ∞", whereas on $\Gamma := \partial\Omega^-$ we prescribe either Dirichlet boundary conditions

$$U = g \quad \text{on } \Gamma \text{ for some } g \in H^{\frac{1}{2}}(\Gamma) \, , \tag{1.3}$$

or Neumann boundary condition

$$\operatorname{grad} U \cdot \mathbf{n} = \varphi \quad \text{on } \Gamma \text{ for some } \varphi \in H^{-\frac{1}{2}}(\Gamma) \, . \tag{1.4}$$

We take for granted that the boundary Γ is Lipschitz continuous. Thus, it will possess an exterior unit normal vectorfield $\mathbf{n} \in L^\infty(\Gamma)$ pointing from Ω^- into Ω^+ . Numerical approximation in mind, we will even assume that Γ is a curvilinear Lipschitz polyhedron in the parlance of [10]. This will cover most geometric arrangements that occur in practical simulations. We emphasize that non-smooth geometries are the main focus of this paper.

It is well known that the above exterior boundary value problems possess unique solutions, see [17, Thm. 9.10]:

THEOREM 1.1. *The exterior Dirichlet problem* (1.1) *and* (1.3)*, and the exterior Neumann problem* (1.1) *and* (1.4)*, respectively, for the Helmholtz equation have at most one solution satisfying the Sommerfeld radiation conditions* (1.2)*.*

Integral equation methods are particularly suited for the numerical treatment of exterior scattering problems, because they reduce the problem to equations on the bounded surface Γ. A variety of schemes is conceivable, among them direct and indirect methods. However, those that can be derived from an integral representation formula for Helmholtz solutions in a straightforward fashion display a worrisome instability: if κ^2 agrees with a Dirichlet or Neumann eigenvalue (resonant frequency) of the Laplacian in Ω^-, then the integral equations may fail to possess a unique solution. In light of Thm. 1.1 this has been dubbed a spurious resonance phenomenon.

Spurious resonances are particularly distressing for numerical procedures based on the integral equations, because whenever κ^2 is close to an interior resonant frequency the resulting linear systems of equations will be extremely ill-conditioned. A wonderful remedy is offered by the combined field integral equations (CFIE), which owe their name to the typical complex linear combination of different boundary integral operators on the left hand side of the final boundary integral equation. In the case of indirect schemes this trick was independently be discovered by Brakhage and Werner [1], Leis [16], and Panich [18] in 1965. In 1971 Burton and Miller used the same idea to obtain direct boundary integral equations without spurious resonances

[6]. Meanwhile, CFIEs have become the foundation for numerous numerical methods in direct and inverse acoustic and electromagnetic scattering [8, Ch. 3 & 6].

In terms of mathematical analysis many combined field integral equations are challenging. This is particularly true for non-smooth surfaces, for which the double layer integral operator is no longer a compact perturbation of the identity in $L^2(\Gamma)$. Thus, in the case of the exterior Dirichlet problem, Fredholm theory can no longer be used to settle the issue of existence and uniqueness of solutions of the traditional CFIE. Hence, modified CFIE involving a regularizing operator have been suggested for theoretical purposes [8, 18].

Many options are available for the discretization of combined field integral equations. We will only consider Galerkin schemes, because they seem to be the only approach amenable to a rigorous theoretical treatment so far. However, the very lack of coercivity of combined field integral equations mentioned above turns out to be a major obstacle to obtaining convergence results for Galerkin methods.

Hence, in this paper we take the cue from the idea to introduce regularizing operators. We derive new variational formulations that are coercive in natural trace spaces, which guarantees asymptotically quasi-optimal convergence of Galerkin boundary element solutions. For the indirect approach two regularizing approaches will be examined, which differ in which potential is targeted by regularization. One of these approaches could also be successfully applied to electromagnetic boundary integral equations [5]. We will also demonstrate how it can be adapted to Burton and Miller's direct formulation.

2. Coercivity. In this section we briefly review the abstract theory of coercive bi-linear forms and its implications for Galerkin discretization. In general these results are well known, *cf.* [17, Ch. 2], but they will be supplied for the sake of completeness. Below V stands for a reflexive Banach space over the field $\mathbb{C}$. This space has to support an isometric, involutory, anti-linear mapping $\bar{}: V \mapsto V$ (related to complex conjugation). By V' we denote the dual space, and by $\langle \cdot, \cdot \rangle_{V' \times V}$ the duality pairing.

Let $d : V \times V \mapsto \mathbb{C}$ be a bi-linear form, which is supposed to feature

- continuity, that is $\exists C > 0 : \quad |d(u, v)| \leq C \, ||u||_V \, ||v||_V \quad$ for all $u, v \in V$, (2.1)
- V-ellipticity, that is $\exists c > 0 : \quad |d(u, \bar{u})| \geq c \, ||u||_V^2 \quad$ for all $u \in V$. (2.2)

Therefore, we can associate a bounded operator $D : V \mapsto V'$ to $d(\cdot, \cdot)$ by

$$\langle Du, v \rangle_{V' \times V} := d(u, v) \quad \forall u, v \in V \; .$$

THEOREM 2.1. *Given the above properties* (2.1) *and* (2.2) *of* $d(\cdot, \cdot)$*, the operator* D *is an isomorphism.*

Proof. By the definition of the norm in V' we have

$$||Du||_{V'} = \sup_{v \neq 0} \frac{|d(u, v)|}{||v||_V} \geq \frac{|d(u, \bar{u})|}{||\bar{u}||_V} \geq c \, ||u||_V \quad \forall u \in V \; .$$

This implies that D is injective and has closed range. Assume that $D(V) \neq V'$. Since $D(V) \subset V'$ is closed, the Hahn-Banach theorem confirms the existence of $v^* \in V'' = V$, $v^* \neq 0$, such that $\langle Du, v^* \rangle_{V' \times V} = 0$ for all $u \in V$. In particular $d(\bar{v}^*, v^*) = 0$, which yields a contradiction. Altogether, D has to be surjective. $\Box$

DEFINITION 2.2. *A bi-linear form* $a : V \times V \mapsto \mathbb{C}$ *is called coercive, if it satisfies a Gårding-type inequality*

$$\exists c > 0 : \quad |a(u, \bar{u}) + \langle Ku, \bar{u} \rangle_{V' \times V} \,| \geq c \, ||u||_V^2 \quad \forall u \in V \; ,$$

with a compact operator $K : V \mapsto V'$.

THEOREM 2.3. *The operator $A : V \mapsto V'$ associated with a continuous bi-linear form $a : V \times V \mapsto \mathbb{C}$ through $\langle Au, v\rangle_{V'\times V} = a(u, v)$, $u, v \in V$, is Fredholm of index zero.*

Proof. Set

$$d(u, v) := a(u, v) + \langle Ku, v\rangle_{V'\times V} \ , \quad u, v \in V \ .$$

It is clear that the bi-linear form d is continuous. By Theorem 2.1 and (1.1) its associated operator $D : V \mapsto V'$ is an isomorphism. By definition of d we have

$$D = A + K \quad \Longleftrightarrow A = D - K \ .$$

Hence, A is a compact perturbation of an isomorphism. According to [17, Thm. 2.26] This implies that A is Fredholm of index 0. □

LEMMA 2.4. *If $a : V \times V \mapsto \mathbb{C}$ is a continuous coercive bi-linear form for which $a(u, v) = 0$ for all $v \in V$ implies $u = 0$, then there is $c_s > 0$ such that*

$$\sup_{v\in V} \frac{|a(u, v)|}{\|v\|_V} \geq c_s \|u\|_V \quad \text{and} \quad \sup_{v\in V} \frac{|a(v, u)|}{\|v\|_V} \geq c_s \|u\|_V \quad \forall u \in V \ .$$

Proof. The assumption of the theorem means that the operator $A : V \mapsto V$ related to $a(\cdot, \cdot)$ is injective. By Thm. 2.3 A is bijective and the inf-sup conditions are a consequence of the open mapping theorem and of the fact that the norms of an operator and of its adjoint agree [19, Thm. 4.15]. □

Next, we consider a sequence of closed subspaces $V_n \subset V$, $n \in \mathbb{N}$. The V_n must be stable under conjugation. We assume that there is an associated sequence of bounded linear operators $P_n : V \mapsto V_n$ that converges to zero strongly, i.e.,

$$\forall u \in V : \quad \lim_{n\to\infty} \|u - P_n u\|_V = 0 \ . \tag{2.3}$$

If V is a Hilbert space and $\{V_n\}_{n\in\mathbb{N}}$ is a family of nested finite-dimensional subspaces such that $\cup_n V_n \subset V$ is dense, then P_n can be chosen as orthogonal projection onto V_n.

Now, we consider the variational problem

$$u \in V : \quad a(u, v) = \langle \varphi, v\rangle_{V'\times V} \quad \forall v \in V \ , \tag{2.4}$$

with $\varphi \in V'$. For the remainder of this section, u will always stand for its solution.

The following theorem is the main tool in proving convergence for conforming Galerkin approximations of coercive variational problems. A first version was discovered by A. Schatz [20], see also [23].

THEOREM 2.5. *If the bi-linear form $a : V \times V \mapsto \mathbb{C}$ is coercive, continuous, and injective (i.e. $a(u, v) = 0$ for all $v \in V$ implies $u = 0$), then there is an $N \in \mathbb{N}$ such that the variational problems*

$$u_h \in V_n : \quad a(u_h, v_h) = \langle \varphi, v_h\rangle_{V'\times V} \quad \forall v_h \in V_n \ ,$$

have unique solutions $u_h \in V_n$ for all $n > N$. Those are asymptotically quasi-optimal in the sense that there is a constant $C > 0$ such that

$$\|u - u_h\|_V \leq C \inf_{v_h\in V_n} \|u - v_h\|_V \ .$$

Proof. We define the operator $S : V \mapsto V$ by

$$a(v, S\bar{w}) = \langle Kw, \bar{v} \rangle_{V' \times V} \quad \forall v \in V \ .$$

Please note that Lemma 2.4 guarantees the existence of A^{-1}. Also by Lemma 2.4 S is continuous and we find $S = (A^*)^{-1}\bar{K}$. Hence, S inherits compactness from K. Remember that compact operators convert strong convergence into uniform convergence, see [15, Cor. 10.4], which means

$$\lim_{n\to\infty} \|(P_n - I)S\|_V = 0 \ . \tag{2.5}$$

Pick some $u_h \in V_n$ and estimate

$$\begin{aligned} |a(u_h, (Id + P_n S)\bar{u}_h)| &\geq |a(u_h, (Id + S)\bar{u}_h)| - |a(u_h, (P_n - Id)S\bar{u}_h)| \\ &\geq |a(u_h, \bar{u}_h) + \langle Ku_h, \bar{u}_h \rangle_{V' \times V}| - \|a\| \, \|(P_n - Id)S\|_V \, \|u_h\|_V^2 \\ &\geq (c_G - \|a\| \, \|(P_n - Id)S\|_V) \, \|u_h\|_V^2 \ . \end{aligned}$$

Thanks to (2.5) it is possible to choose $N \in \mathbb{N}$ such that $\|a\| \, \|(P_n - Id)S\|_V < \frac{1}{2}c_G$ for all $n > N$. Then, with $v_h := (Id + P_n S)\bar{u}_h \in V_n$,

$$|a(u_h, v_h)| \geq \tfrac{1}{2}c_G \, \|u_h\|_V^2 \ .$$

Making use of the (uniform) continuity of P_n and S, this yields the inf-sup condition

$$\sup_{v_h \in V_n} \frac{|a(u_h, v_h)|}{\|v_h\|_V} \geq c_d \, \|u_h\|_V \quad \forall u_h \in V_n, \, n > N \ . \tag{2.6}$$

Using (2.6) and Galerkin orthogonality we get for any $v_h \in V_n$, $n > N$,

$$\begin{aligned} \|u - u_h\|_V &\leq \|u - v_h\|_V + \|v_h - u_h\|_V \\ &\leq \|u - v_h\|_V + \frac{1}{c_d} \sup_{w_h \in V_n} \frac{|a(v_h - u_h, w_h)|}{\|v_h\|_V} \\ &\leq (1 + \frac{\|a\|}{c_d}) \, \|u - v_h\|_V \ . \end{aligned}$$

This is the asserted asymptotic quasi-optimality with $C := 1 + \|a\|/c_d$. □

3. Boundary integral operators. In this section we review important properties of boundary integral operators related to Helmholtz' equation. The main reference is the textbook [17] and the pioneering work by M. Costabel [9].

Without further explanation we will use Sobolev spaces H^s, $s \in \mathbb{R}$, on domains and boundaries, in particular $H^1(\Omega)$, $H^{\frac{1}{2}}(\Gamma)$, and $H^{-\frac{1}{2}}(\Gamma)$, *cf.* [17, Ch. 2]. Here, we merely recall the definition of the Sobolev-Slobodeckij norm

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + |u|^2_{H^{\frac{1}{2}}(\Gamma)} \quad , \quad |u|^2_{H^{\frac{1}{2}}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{(u(\mathbf{x}) - u(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^2} \, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ .$$

The corresponding Frechet spaces on unbounded domains will be tagged by a subscript loc, e.g. $H^1_{\text{loc}}(\Omega)$. Their associated dual spaces will carry the subscript "comp" to illustrate that they contain compactly supported distributions.

Writing

$$H(\Delta, \Omega) := \{U \in H^1_{\mathrm{loc}}(\Omega),\, \Delta U \in L^2_{\mathrm{loc}}(\Omega)\}\ .$$

for the domain of the Laplacian, we have continuous and surjective trace operators, *cf.* [9, Lemma 3.2],

$$\begin{aligned} \text{Dirichlet trace} \quad & \gamma_D : H^1_{\mathrm{loc}}(\Omega) \mapsto H^{\frac{1}{2}}(\Gamma)\ , \\ \text{Neumann trace} \quad & \gamma_N : H(\Delta, \Omega) \mapsto H^{-\frac{1}{2}}(\Gamma) \end{aligned}$$

that generalize the following pointwise traces of smooth $U \in C^\infty(\bar{\Omega})$,

$$(\gamma_D U)(\mathbf{x}) := U(\mathbf{x}) \quad \text{and} \quad (\gamma_N U)(\mathbf{x}) := \operatorname{grad} U(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\ , \quad \mathbf{x} \in \Gamma\ ,$$

respectively.

So far $\Omega \subset \mathbb{R}^3$ has been a generic domain. Returning to our particular setting, superscripts $+$ and $-$ will tag traces from Ω^-/Ω^+. Jumps are defined as

$$[\gamma_D U]_\Gamma = \gamma_D^+ U - \gamma_D^- U \quad , \quad [\gamma_N U]_\Gamma = \gamma_N^+ U - \gamma_N^- U\ .$$

Averages are denoted by

$$\{\gamma_D U\}_\Gamma = \tfrac{1}{2}(\gamma_D^+ U + \gamma_D^- U) \quad , \quad \{\gamma_N U\}_\Gamma = \tfrac{1}{2}(\gamma_N^+ U + \gamma_N^- U)\ .$$

We recall that the bi-linear symmetric pairing

$$\langle \varphi, v \rangle_\Gamma := \int_\Gamma uv\, \mathrm{d}S\ , \quad \varphi, v \in L^2(\Gamma)\ ,$$

can be extended to the duality pairing on $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$. Thanks to the definition of the Neumann trace we have the integration by parts formulas

$$\int_{\Omega^-} \operatorname{grad} U \cdot \operatorname{grad} V + \Delta U\, V\, \mathrm{d}\mathbf{x} = \left\langle \gamma_N^- U, \gamma_D^- V \right\rangle_\Gamma , \tag{3.1}$$

$$-\int_{\Omega^+} \operatorname{grad} U \cdot \operatorname{grad} V + \Delta U\, V\, \mathrm{d}\mathbf{x} = \left\langle \gamma_N^+ U, \gamma_D^+ V \right\rangle_\Gamma\ , \tag{3.2}$$

for $U \in H_{\mathrm{loc}}(\Delta, \Omega^\pm)$, $V \in H^1_{\mathrm{loc}}(\Omega^\pm)$. We will also need spaces with "vanishing average"

$$\begin{aligned} H_*^{\frac{1}{2}}(\Gamma) :=& \{u \in H^{\frac{1}{2}}(\Gamma),\, \langle \mathbf{1}, u \rangle_\Gamma = 0\}\ , \\ H_*^{-\frac{1}{2}}(\Gamma) :=& \{\varphi \in H^{-\frac{1}{2}}(\Gamma),\, \langle \varphi, \mathbf{1} \rangle_\Gamma = 0\}\ , \end{aligned}$$

where $\mathbf{1} \in H^{\frac{1}{2}}(\Gamma)$ means the constant function $\equiv 1$ on Γ, whereas $\mathbf{1} \in H^{-\frac{1}{2}}(\Gamma)$ refers to the functional $v \mapsto \int_\Gamma v\, \mathrm{d}S$.

LEMMA 3.1. *The spaces* $H_*^{\frac{1}{2}}(\Gamma)$ *and* $H_*^{-\frac{1}{2}}(\Gamma)$ *are dual to each other with respect to the pairing* $\langle \cdot, \cdot \rangle_\Gamma$.

Proof. For $w \in H^{\frac{1}{2}}(\Gamma)$ denote by w^* the average $w^* := \int_\Gamma w\, \mathrm{d}S \cdot \mathbf{1}$. We point out that

$$\|w - w^*\|^2_{H^{\frac{1}{2}}(\Gamma)} = \|w\|^2_{L^2(\Gamma)} - \|w^*\|^2_{L^2(\Gamma)} + |w|^2_{H^{\frac{1}{2}}(\Gamma)} \leq \|w\|^2_{H^{\frac{1}{2}}(\Gamma)}\ .$$

Therefore, for $\varphi \in H_*^{-\frac{1}{2}}(\Gamma)$

$$\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} = \sup_{w\in H^{\frac{1}{2}}(\Gamma)} \frac{|\langle\varphi, w\rangle_\Gamma|}{\|w\|_{H^{\frac{1}{2}}(\Gamma)}} \leq \sup_{w\in H^{\frac{1}{2}}(\Gamma)} \frac{|\langle\varphi, w-w^*\rangle_\Gamma|}{\|w-w^*\|_{H^{\frac{1}{2}}(\Gamma)}} = \sup_{w\in H_*^{\frac{1}{2}}(\Gamma)} \frac{|\langle\varphi, w\rangle_\Gamma|}{\|w\|_{H^{\frac{1}{2}}(\Gamma)}} .$$

This amounts to the assertion of the theorem. □

For fixed wavenumber $\kappa > 0$ a distribution U is called a *radiating Helmholtz solution*, if

$$\begin{aligned} \Delta U + \kappa^2 U = 0 \quad \text{in } \Omega^- \cup \Omega^+ , \\ \frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) = o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty . \end{aligned} \tag{3.3}$$

Based on the Helmholtz kernel

$$\Phi_\kappa(\mathbf{x},\mathbf{y}) := \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} .$$

we can state the transmission formula for radiating Helmholtz solution U [17, Thm. 6.10]

$$U = -\Psi_{\mathrm{SL}}^\kappa([\gamma_N U]_\Gamma) + \Psi_{\mathrm{DL}}^\kappa([\gamma_D U]_\Gamma) \tag{3.4}$$

with potentials

$$\text{single layer potential:} \quad \Psi_{\mathrm{SL}}^\kappa(\lambda)(\mathbf{x}) = \int_\Gamma \Phi_\kappa(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) ,$$

$$\text{double layer potential:} \quad \Psi_{\mathrm{DL}}^\kappa(u)(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi_\kappa(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) .$$

The potentials themselves provide radiating Helmholtz solutions, that is

$$(\Delta+\kappa^2)\Psi_{\mathrm{SL}}^\kappa = 0 \quad , \quad (\Delta+\kappa^2)\Psi_{\mathrm{DL}}^\kappa = 0 \quad \text{in } \Omega^- \cup \Omega^+ . \tag{3.5}$$

Moreover, they describe continuous mappings, see [17, Thm. 6.12],

$$\begin{aligned} \Psi_{\mathrm{SL}}^\kappa &: H^{-\frac{1}{2}}(\Gamma) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3) \cap H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+) , \\ \Psi_{\mathrm{DL}}^\kappa &: H^{\frac{1}{2}}(\Gamma) \mapsto H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+) . \end{aligned}$$

This means that we can apply the trace operators to the potentials. This will yield the following four continuous boundary integral operators, *cf.* [17, Thm. 7.1] and [14].

$$\begin{aligned} \mathsf{V}_\kappa &: H^s(\Gamma) \mapsto H^{s+1}(\Gamma),\ -1 \leq s \leq 0 \quad , \quad \mathsf{V}_\kappa := \{\gamma_D \Psi_{\mathrm{SL}}^\kappa\}_\Gamma , \\ \mathsf{K}_\kappa &: H^s(\Gamma) \mapsto H^s(\Gamma),\ 0 \leq s \leq 1 \quad , \quad \mathsf{K}_\kappa := \{\gamma_D \Psi_{\mathrm{DL}}^\kappa\}_\Gamma , \\ \mathsf{K}_\kappa^* &: H^s(\Gamma) \mapsto H^s(\Gamma),\ -1 \leq s \leq 0 \quad , \quad \mathsf{K}_\kappa^* := \{\gamma_N \Psi_{\mathrm{SL}}^\kappa\}_\Gamma , \\ \mathsf{D}_\kappa &: H^s(\Gamma) \mapsto H^{s-1}(\Gamma),\ 0 \leq s \leq 1 \quad , \quad \mathsf{D}_\kappa := -\{\gamma_N \Psi_{\mathrm{DL}}^\kappa\}_\Gamma . \end{aligned}$$

By the *jump relations* [17, Thm. 6.11]

$$\begin{aligned} [\gamma_D \Psi_{\mathrm{SL}}^\kappa(\lambda)]_\Gamma = 0 \quad &, \quad [\gamma_N \Psi_{\mathrm{SL}}^\kappa(\lambda)]_\Gamma = -\lambda \quad , \forall\lambda \in H^{-\frac{1}{2}}(\Gamma) , \\ [\gamma_D \Psi_{\mathrm{DL}}^\kappa(u)]_\Gamma = u \quad &, \quad [\gamma_N \Psi_{\mathrm{DL}}^\kappa(u)]_\Gamma = 0 \quad , \forall u \in H^{\frac{1}{2}}(\Gamma) . \end{aligned}$$

we find

$$\begin{array}{ll} \gamma_D^- \Psi_{\mathrm{DL}}^\kappa = \mathsf{K}_\kappa - \tfrac{1}{2} Id \quad , & \gamma_D^+ \Psi_{\mathrm{DL}}^\kappa = \mathsf{K}_\kappa + \tfrac{1}{2} Id \\ \gamma_N^- \Psi_{\mathrm{SL}}^\kappa = \mathsf{K}_\kappa^* + \tfrac{1}{2} Id \quad , & \gamma_N^+ \Psi_{\mathrm{SL}}^\kappa = \mathsf{K}_\kappa^* - \tfrac{1}{2} Id \end{array} \tag{3.6}$$

Besides, the Newton potential

$$(\mathsf{N}_\kappa f)(\mathbf{x}) = \int_{\mathbb{R}^3} \phi_\kappa(\mathbf{x},\mathbf{y})\, f(\mathbf{y})\, d\mathbf{y} \quad , \quad \mathsf{N}_\kappa : H^{-1}_{\mathrm{comp}}(\mathbb{R}^3) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3) \, .$$

can be used to get the concise representations

$$\mathsf{V}_\kappa = \gamma_D \circ \mathsf{N}_\kappa \circ \gamma_D^* \, , \tag{3.7}$$

$$\mathsf{K}_\kappa = \{\gamma_D\}_\Gamma \circ \mathsf{N}_\kappa \circ \gamma_N^* \, , \tag{3.8}$$

$$\mathsf{K}_\kappa^* = \{\gamma_N\}_\Gamma \circ \mathsf{N}_\kappa \circ \gamma_D^* \, , \tag{3.9}$$

$$\mathsf{D}_\kappa = \gamma_N \circ \mathsf{N}_\kappa \circ \gamma_N^* \, . \tag{3.10}$$

Here, an $*$ labels the dual adjoint operator. These expressions immediately show the symmetries [17, Thm. 6.15,6.17]

$$\langle \psi, \mathsf{V}_\kappa \varphi \rangle_\Gamma = \langle \varphi, \mathsf{V}_\kappa \psi \rangle_\Gamma \quad \forall \varphi, \psi \in H^{-\frac{1}{2}}(\Gamma) \, , \tag{3.11}$$

$$\langle \varphi, \mathsf{K}_\kappa u \rangle_\Gamma = \langle \mathsf{K}_\kappa^* \varphi, u \rangle_\Gamma \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma),\, u \in H^{\frac{1}{2}}(\Gamma) \, , \tag{3.12}$$

$$\langle \mathsf{D}_\kappa u, v \rangle_\Gamma = \langle \mathsf{D}_\kappa v, u \rangle_\Gamma \quad \forall u, v \in H^{\frac{1}{2}}(\Gamma) \, . \tag{3.13}$$

Crucial will be the ellipticity of boundary integral operators in the natural trace norms [17, Cor. 8.13, Thm. 8.21]

$$\langle \bar{\varphi}, \mathsf{V}_0 \varphi \rangle_\Gamma \geq c_V \, \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \, , \tag{3.14}$$

$$\langle \mathsf{D}_0 v, \bar{v} \rangle_\Gamma \geq c_D \, \|v\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^{\frac{1}{2}}_*(\Gamma) \, . \tag{3.15}$$

Therefore, $\mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and $\mathsf{D}_0 : H^{\frac{1}{2}}_*(\Gamma) \mapsto H^{-\frac{1}{2}}_*(\Gamma)$ are isomorphisms and we conclude that for all $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $v \in H^{\frac{1}{2}}(\Gamma)$

$$\|\mathsf{V}_0 \varphi\|_{H^{\frac{1}{2}}(\Gamma)} \approx \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \, , \tag{3.16}$$

$$\left\langle \mathsf{V}_0^{-1} v, \overline{v} \right\rangle_\Gamma \geq \widetilde{c}_V \, \|v\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^{\frac{1}{2}}(\Gamma) \, , \tag{3.17}$$

$$\|\mathsf{D}_0 v\|_{H_*^{-\frac{1}{2}}(\Gamma)} \approx \|v\|_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^{\frac{1}{2}}_*(\Gamma) \, , \tag{3.18}$$

$$\left\langle \phi, \mathsf{D}_0^{-1} \overline{\phi} \right\rangle_\Gamma \geq \widetilde{c}_D \, \|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \phi \in H^{-\frac{1}{2}}_*(\Gamma) \, . \tag{3.19}$$

Here, $\approx$ designates equality up to constants that only depend on Γ.

¿From [17, Thm. 9.15] we learn that for $u, v \in H^1(\Gamma)$

$$\langle \mathsf{D}_\kappa u, v \rangle_\Gamma = \langle \mathsf{V}_\kappa \mathbf{curl}_\Gamma u, \mathbf{curl}_\Gamma v \rangle_\Gamma - \kappa^2 \, \langle \mathsf{V}_\kappa (u \cdot \mathbf{n}), v \cdot \mathbf{n} \rangle_\Gamma \, , \tag{3.20}$$

where V_κ has to be read as vectorial single layer potential, and $\mathbf{curl}_\Gamma : H^1(\Gamma) \mapsto \boldsymbol{L}_{\mathbf{t}}(\Gamma)$ is the surface rotation, which agrees with the rotated surface gradient. It can be extended to a mapping $\mathbf{curl}_\Gamma : H^{\frac{1}{2}}(\Gamma) \mapsto (H^{-\frac{1}{2}}(\Gamma))^3$ [4].

LEMMA 3.2. *The operators* $\mathsf{V}_\kappa - \mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$, $\mathsf{K}_\kappa - \mathsf{K}_0 : H^{\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$, *and* $\mathsf{D}_\kappa - \mathsf{D}_0 : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ *are compact.*

Proof. Note that $\widetilde{\Phi}(r) := \frac{\exp(i\kappa r)-1}{4\pi r}$ is an analytic function on $\mathbb{R}$. Therefore the integral operator

$$(\widetilde{\mathsf{N}}_\kappa f)(\mathbf{x}) := \int_{\mathbb{R}^3} \widetilde{\Phi}(|\mathbf{x}-\mathbf{y}|) f(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

has a continuous kernel with bounded derivatives and weakly singular second derivatives. This means that $\widetilde{\mathsf{N}}_\kappa$ is an operator of order $+4$, continuous $\widetilde{\mathsf{N}} : H^{-2}_{\mathrm{comp}}(\mathbb{R}^3) \mapsto H^2_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore we conclude the continuity of

$$\mathsf{V}_\kappa - \mathsf{V}_0 = \gamma_D \circ \widetilde{\mathsf{N}}_\kappa \circ \gamma_D^* : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1(\Gamma) \; .$$

The compact embedding $H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$ finishes the proof of the first assertion.

To confirm the second, we point out that

$$\gamma_N^* : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-2}_{\mathrm{comp}}(\Omega^- \cup \Omega^+)$$

is continuous due to the continuous embedding $H^2_{\mathrm{loc}}(\Omega^- \cup \Omega^+) \subset H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+)$. Then, the identity

$$\mathsf{K}_\kappa - \mathsf{K}_0 = \{\gamma_D\}_\Gamma \circ \widetilde{\mathsf{N}}_\kappa \circ \gamma_N^* .$$

Thus, $\mathsf{K}_\kappa - \mathsf{K}_0 : H^{\frac{1}{2}}(\Gamma) \to H^1(\Gamma)$. This, combined with the compactness of the embedding $H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$, gives the result.

To confirm the assertion for the hypersingular operator, we appeal to the formula (3.20) and the compactness of $\mathsf{V}_\kappa - \mathsf{V}_0$ that carries over to the vectorial single layer potential operator. Further the multiplication with $\mathbf{n}$ is an isometry $L^2(\Gamma) \mapsto \boldsymbol{L}^2(\Gamma)$ such that the second term in (3.20) is readily seen to be a compact perturbation. □

4. Indirect boundary integral equations. We recall that indirect methods are based on a potential representation for (exterior) radiating Helmholtz solutions in Ω^+. By virtue of (3.5) we may set

$$U = \Psi^\kappa_{\mathrm{SL}}(\phi),\ \phi \in H^{-\frac{1}{2}}(\Gamma) \quad \text{or} \quad U = \Psi^\kappa_{\mathrm{DL}}(u),\ u \in H^{\frac{1}{2}}(\Gamma) \; . \tag{4.1}$$

Applying γ_D^+ to (3.6) we obtain the following integral equations for the exterior Dirichlet problem:

$$\mathsf{V}_\kappa(\phi) = g \quad \text{or} \quad (\mathsf{K}_\kappa + \tfrac{1}{2} Id) u = g \; . \tag{4.2}$$

Similarly, the resulting boundary integral equations for the Neumann problem are

$$(\mathsf{K}^*_\kappa - \tfrac{1}{2} Id)\phi = \varphi \quad \text{or} \quad -\mathsf{D}_\kappa \phi = \varphi \; . \tag{4.3}$$

However, these boundary integral equations are haunted by the problem of "resonant frequencies" [7, Sect. 7.7]: if κ^2 is a Dirichlet eigenvalue of $-\Delta$ in Ω^-, then the Neumann traces of the corresponding eigenfunctions will belong to the kernel of V_κ and $\mathsf{K}^*_\kappa - \frac{1}{2} Id$. Conversely, if κ^2 is a Neumann eigenvalue, the Dirichlet traces of the eigenfunctions form the kernel of D_κ and $\mathsf{K}_\kappa + \frac{1}{2} Id$. This fact destroys injectivity of the operators in the boundary integral equations and bars us from applying the powerful Fredholm theory outlined in Sect. 2.

4.1. Classical CFIE. As pointed out in the introduction, the awkward potential lack of uniquness of solutions of (4.2) and (4.3) led to the development of the classical combined field integral equations [8, Sect. 3.2]. They can be obtained by an indirect approach starting from the trial expression

$$U = \Psi^{\kappa}_{\mathrm{DL}}(u) + i\eta\Psi^{\kappa}_{\mathrm{SL}}(u) \ , \tag{4.4}$$

with real $\eta \neq 0$. Applying the exterior Dirichlet trace results in the boundary integral equation

$$g = (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u + i\eta\mathsf{V}_\kappa u \ , \tag{4.5}$$

whereas the exterior Neumann problem leads to

$$\varphi = -\mathsf{D}_\kappa u + i\eta(\mathsf{K}^*_\kappa - \tfrac{1}{2}Id)u \ . \tag{4.6}$$

To begin with, we discuss (4.6) and set

$$\mathsf{C}_\kappa := -\mathsf{D}_\kappa + i\eta(\mathsf{K}^*_\kappa - \tfrac{1}{2}Id) \ .$$

LEMMA 4.1. *The operator* $\mathsf{C}_\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ *is injective.*

Proof. Let $u \in H^{\frac{1}{2}}(\Gamma)$ be a solution of $\mathsf{C}_\kappa u = 0$. Then U given by (4.4) is a Helmholtz solution that satisfies $\gamma^+_N U = 0$. Thus, the unique solvability of the exterior Neumann problem according to Thm. 1.1 enforces $U = 0$ in Ω^+.

By the jump conditions we conclude

$$\gamma^-_D U = -u \quad \text{and} \quad \gamma^-_N U = i\eta u \ .$$

As a consequence of the integration by parts formula

$$i\eta \int\limits_\Gamma |u|^2 \,\mathrm{d}S = \langle \gamma^-_N U, \gamma^-_D \bar{U} \rangle_\Gamma = \int\limits_{\Omega^-} |\operatorname{grad} U|^2 - \kappa^2 |U|^2 \,\mathrm{d}\mathbf{x} \ .$$

Since $\eta \in \mathbb{R} \setminus \{0\}$, this involves $u = 0$. □

The equation (4.6) is set in the space $H^{-\frac{1}{2}}(\Gamma)$. Hence, the natural test space is $H^{\frac{1}{2}}(\Gamma)$, which perfectly matches the space for the unknown u. We arrive at the variational problem: find $u \in H^{\frac{1}{2}}(\gamma)$ with

$$\langle \mathsf{C}_\kappa u, v \rangle_\Gamma = \langle \varphi, v \rangle_\Gamma \quad \forall v \in H^{\frac{1}{2}}(\Gamma) \ . \tag{4.7}$$

The next result shows that the assumptions of the abstract theory of Sect. 2 is satisfied for (4.7).

LEMMA 4.2. *The bilinear form* $\langle \mathsf{C}_\kappa \cdot, \cdot \rangle_\Gamma : H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \mapsto \mathbb{C}$ *is coercive.*

Proof. We can split

$$\langle \mathsf{C}_\kappa u, v \rangle_\Gamma = -\langle \mathsf{D}_0 u, v \rangle_\Gamma + \langle (\mathsf{D}_0 - \mathsf{D}_\kappa) u, v \rangle_\Gamma + i\eta \left\langle (\mathsf{K}^*_\kappa - \tfrac{1}{2}Id)u, v \right\rangle_\Gamma \ .$$

The last term is compact since $\mathsf{K}^*_\kappa - \frac{1}{2}Id : L^2(\Gamma) \mapsto L^2(\Gamma)$ is continuous and the embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact. The second term is compact by Lemma 3.2. The $H^{\frac{1}{2}}_*(\Gamma)$-ellipticity of the first term according to (3.15) finishes the proof. □

Summing up, we conclude existence and uniqueness of solutions of (4.6). In addition we get asymptotic quasi-optimality for any conforming Galerkin boundary element discretization. The discussion of actual convergence will be postponed until Sect. 6.

The situation is much worse in the case of the exterior Dirichlet problem and the associated CFIE (4.5). Actually, the equation is set in $H^{\frac{1}{2}}(\Gamma)$ and the density u should be sought in $H^{-\frac{1}{2}}(\Gamma)$. For obvious reasons, this is not possible, unless we use a pairing in $H^{-\frac{1}{2}}(\Gamma)$ to convert the equation into weak form. Yet, this will introduce products of non-local operators, which render the equations unsuitable for numerical purposes. The fundamental difficulty is that, unlike in the case of the exterior Neumann problem, we cannot use matching trial and test spaces, because the potentials involved in (4.1) require arguments with different regularity. What remains is to lift the equation (4.5) into $L^2(\Gamma)$ and seek the unknown density u in $L^2(\Gamma)$, too.

A key argument in the theoretical treatment of (4.5) in $L^2(\Gamma)$ is the compactness of the double layer potential operator $\mathsf{K}_\kappa : L^2(\Gamma) \mapsto L^2(\Gamma)$ on smooth surfaces, which renders the boundary integral operator associated with (4.5) a compact perturbation of the identity. On non-smooth surfaces this argument is not available. This prompted us to explore the regularized formulations presented in the next two sections.

4.2. Single layer regularization. The idea is to introduce the regularizing operator into the argument of the single layer potential in the trial expression (4.4). However, this operator has to be chosen carefully in order to permit us to prove uniqueness of solutions along the lines of the proof of Lemma 4.1. Crucial is the following result, *cf.* [22, Sect. 5]:

LEMMA 4.3. *With a constant* $c_1 > 0$ *we have*

$$\left\langle \mathsf{D}_0 v, (\tfrac{1}{2}Id + \mathsf{K}_0)\overline{v} \right\rangle_\Gamma \geq c_1 \left\| v \right\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^{\frac{1}{2}}_*(\Gamma) \; .$$

Proof. Using integration by parts (3.2) and $\Delta \Psi^0_{\mathrm{DL}} = 0$ in Ω^+, we get for $v \in H^{\frac{1}{2}}_*(\Gamma)$

$$\begin{aligned} \left\langle \mathsf{D}_0 v, (\tfrac{1}{2}Id + \mathsf{K}_0)\overline{v} \right\rangle_\Gamma &= - \left\langle \gamma_N^+ \Psi^0_{\mathrm{DL}}(v), \gamma_D^+ \Psi^0_{\mathrm{DL}}(\overline{v}) \right\rangle_\Gamma \\ &= \left\| \operatorname{grad} \Psi^0_{\mathrm{DL}}(v) \right\|^2_{\boldsymbol{L}^2(\Omega^+)} \geq c \left\| \gamma_N^+ \Psi^0_{\mathrm{DL}}(v) \right\|^2_{H_*^{-\frac{1}{2}}(\Gamma)} \\ &\geq c \left\| \mathsf{D}_0(v) \right\|^2_{H_*^{-\frac{1}{2}}(\Gamma)} \geq c \left\| v \right\|^2_{H^{\frac{1}{2}}(\Gamma)} \; . \end{aligned}$$

Here, we have also used the continuity of γ_N, the estimate (3.18), and the ellipticity of D_0. □

Setting $v := \mathsf{D}_0^{-1}\phi$, using (3.18) and the symmetry properties of the boundary integral operators, we conclude from Lemma 4.3 that there is $c_N > 0$ such that

$$\left\langle \phi, \mathsf{D}_0^{-1}(\tfrac{1}{2}Id + \mathsf{K}_0^*)\overline{\phi} \right\rangle_\Gamma \geq c_N \left\| \phi \right\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \phi \in H_*^{-\frac{1}{2}}(\Gamma) \; .$$

Note that $(\frac{1}{2}Id + \mathsf{K}_0^*)\varphi \in H_*^{-\frac{1}{2}}(\Gamma)$ for all $\varphi \in H^{-\frac{1}{2}}(\Gamma)$. Thus, owing to Thm. 2.1 and Lemma 3.1, the operator

$$\mathsf{R} := \mathsf{D}_0^{-1}(\tfrac{1}{2}Id + \mathsf{K}_0^*) : H_*^{-\frac{1}{2}}(\Gamma) \mapsto H_*^{\frac{1}{2}}(\Gamma)$$

is an isomorphism.

We still have to deal with the constant functions that are in the kernel of D_0.

LEMMA 4.4. *We have* $\|\mathbf{1}\|_{H^{\frac{1}{2}}(\Gamma)} = |\Gamma|^{1/2}$ *and* $\|\mathbf{1}\|_{H^{-\frac{1}{2}}(\Gamma)} = |\Gamma|^{1/2}$.

Proof. Using the definition of the Sobolev-Slobodeckij norm $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma)}$, the statement about $\|\mathbf{1}\|_{H^{\frac{1}{2}}(\Gamma)}$ is trivial. To compute $\|\mathbf{1}\|_{H^{-\frac{1}{2}}(\Gamma)}$ consider the variational problem

$$\inf\{\tfrac{1}{2}\|v\|^2_{H^{\frac{1}{2}}(\Gamma)},\ \int_\Gamma v\,\mathrm{d}S = 1\},$$

which gives rise to the saddle point problem: seek $v \in H^{\frac{1}{2}}(\Gamma)$

$$\begin{array}{lclcl} (v,q)_{H^{\frac{1}{2}}(\Gamma)} & + & \lambda\int_\Gamma q\,\mathrm{d}S & = & 0 \qquad \forall q \in H^{\frac{1}{2}}(\Gamma)\,, \\ \int_\Gamma v\,\mathrm{d}S & & & = & 1\,. \end{array}$$

Its unique solution is $v \equiv |\Gamma|^{-1}$. Then

$$\|\mathbf{1}\|_{H^{-\frac{1}{2}}(\Gamma)} = \sup_{v\in H^{\frac{1}{2}}(\Gamma)} \frac{\int_\Gamma v\,\mathrm{d}S}{\|v\|_{H^{\frac{1}{2}}(\Gamma)}} = \frac{\int_\Gamma \mathbf{1}\,\mathrm{d}S}{\|\mathbf{1}\|_{H^{\frac{1}{2}}(\Gamma)}} = |\Gamma|^{1/2}\,,$$

where we have used the definition of the dual norm. ☐

For $\nu > 0$ we define

$$\widetilde{\mathsf{R}}\varphi := \mathsf{R}(\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|) + \nu\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,\mathbf{1} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)\,. \tag{4.8}$$

Since R maps into $H^{\frac{1}{2}}_*(\Gamma)$, this implies that for all $\varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{aligned} \left\langle\overline{\varphi},\widetilde{\mathsf{R}}\varphi\right\rangle_\Gamma &= \langle\overline{\varphi},\mathsf{R}(\varphi-\varphi(\mathbf{1})\mathbf{1}/|\Gamma|) + \nu\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,\mathbf{1}\rangle_\Gamma \\ &= \langle\overline{\varphi}-\overline{\varphi}(\mathbf{1})\mathbf{1}/|\Gamma|,\mathsf{R}(\varphi-\varphi(\mathbf{1})\mathbf{1}/|\Gamma|)\rangle_\Gamma + \nu|\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,|^2 \\ &\geq c_N\,\|\varphi-\varphi(\mathbf{1})\mathbf{1}/|\Gamma|\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \nu|\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,|^2 \\ &\geq c_N\left(\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} - |\varphi(\mathbf{1})|/|\Gamma|\,\|\mathbf{1}\|_{H^{-\frac{1}{2}}(\Gamma)}\right)^2 + \nu|\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,|^2 \\ &\geq c_N\left(\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} - |\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,||\Gamma|^{-1/2}\right)^2 + \nu|\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,|^2 \\ &\geq \tfrac{1}{2}c_N\,\|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} + (\nu - 2c_N/|\Gamma|)|\,\langle\varphi,\mathbf{1}\rangle_\Gamma\,|^2\,. \end{aligned}$$

In the sequel we assume $\nu > 2c_N/|\Gamma|$. Then, $\widetilde{\mathsf{R}}$ turns out to be $H^{-\frac{1}{2}}(\Gamma)$-elliptic. Thus, according to Thm. 2.1, $\widetilde{\mathsf{R}} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is an isomorphism and for some $\tilde{c} > 0$

$$\left\langle\widetilde{\mathsf{R}}^{-1}v,\overline{v}\right\rangle_\Gamma \geq \tilde{c}\,\|v\|^2_{H^{\frac{1}{2}}(\Gamma)} \qquad \forall v \in H^{\frac{1}{2}}(\Gamma)\,. \tag{4.9}$$

The new combined field integral equation (CFIE) arises from an indirect boundary integral approach to the exterior Dirichlet problem (1.1) and (1.3) using the special trial expression

$$U = \Psi^\kappa_{\mathrm{DL}}(u) + i\eta\Psi^0_{\mathrm{SL}}(\widetilde{\mathsf{R}}^{-1}u) \quad u \in H^{\frac{1}{2}}(\Gamma)\,. \tag{4.10}$$

By (3.5), this is a radiating Helmholtz solution in $\Omega^- \cup \Omega^+$. As before, applying the Dirichlet trace to (4.10) yields the boundary integral equation

$$g = (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u + i\eta(\mathsf{V}_\kappa \circ \widetilde{\mathsf{R}}^{-1})(u) \quad \text{in } H^{\frac{1}{2}}(\Gamma)\,. \tag{4.11}$$

For the sake of brevity, we introduce the boundary integral operator

$$\mathsf{B}_\kappa := (\tfrac{1}{2}Id + \mathsf{K}_\kappa) + i\eta \mathsf{V}_\kappa \circ \widetilde{\mathsf{R}}^{-1} : H^{\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)\ .$$

LEMMA 4.5. *The boundary integral operator* B_κ *is injective.*

Proof. We adapt the proof of Lemma 4.1. Let $v \in H^{\frac{1}{2}}(\Gamma)$ be a solution of $\mathsf{B}_\kappa u = 0$. Set $U := \Psi^\kappa_{\mathrm{DL}}(u) + i\eta\Psi^0_{\mathrm{SL}}(\widetilde{\mathsf{R}}^{-1}u)$, whose restriction to Ω^+ is a radiating exterior Helmholtz solution with $\gamma_D^+ U = 0$. From Thm. 1.1 we conclude $U = 0$ in Ω^+. Thus, by the jump relations,

$$-\gamma_D^- U = [\gamma_D U]_\Gamma = u \quad , \quad \gamma_N^- U = -[\gamma_N U]_\Gamma = -i\eta\widetilde{\mathsf{R}}^{-1}u\ .$$

the integration by parts formula (3.1) yields

$$i\eta \left\langle \widetilde{\mathsf{R}}^{-1}u, \overline{u}\right\rangle_\Gamma = \langle \gamma_N^{-1} U, \gamma_D^{-1}\overline{U}\rangle_\Gamma = \|\mathrm{grad}\, U\|^2_{\boldsymbol{L}^2(\Omega^-)} - \kappa^2 \|U\|^2_{L^2(\Omega^-)}\ .$$

Thanks to (4.9) and $\eta > 0$ the left hand side is purely imaginary, whereas the right hand side is real. Necessarily, $\left\langle \widetilde{\mathsf{R}}u, \overline{u}\right\rangle_\Gamma = 0$, which, by (4.9), implies $u = 0$. ☐

A Galerkin discretization cannot deal with the products of boundary integral operators occurring in the definition of B_κ. The usual trick to avoid operator products is to switch to a mixed formulation. Here, this is done by introducing the new unknown $\lambda := \widetilde{\mathsf{R}}^{-1}u \in H^{-\frac{1}{2}}(\Gamma)$ and gives us

$$\begin{array}{rcccll} i\eta\mathsf{V}_\kappa(\lambda) & + & (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u & = & g & \text{in } H^{\frac{1}{2}}(\Gamma)\ , \\ \widetilde{\mathsf{R}}\lambda & - & u & = & 0 & \text{in } H^{\frac{1}{2}}(\Gamma)\ . \end{array} \tag{4.12}$$

These equations are equivalent to (4.11), as $\widetilde{\mathsf{R}}$ is an isomorphism. However, a product of integral operators is still concealed in the definition of $\widetilde{\mathsf{R}}$. Fortunately, it involves the inverse of the boundary integral operator D_0, which suggests plain multiplication of the second equation of (4.12) with D_0. Yet, D_0 is not an isomorphism and this simple approach is not feasible, unless we take care of the kernel of D_0: for $\xi > 0$ define

$$\widetilde{\mathsf{D}}_0 v := \mathsf{D}_0(v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma|) + \xi\, \langle \mathbf{1}, v\rangle_\Gamma\, \mathbf{1}\ , \quad v \in H^{\frac{1}{2}}(\Gamma)\ ,$$

which, due to (3.15) and Lemma 4.4, satisfies

$$\begin{aligned} \left\langle \widetilde{\mathsf{D}}_0 v, \overline{v}\right\rangle_\Gamma &= \langle \mathsf{D}_0(v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma|), \overline{v} - \langle \mathbf{1}, \overline{v}\rangle_\Gamma \mathbf{1}/|\Gamma|\rangle_\Gamma + \xi |\langle \mathbf{1}, v\rangle_\Gamma|^2 \\ &\geq c_D \|v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma|\|^2_{H^{\frac{1}{2}}(\Gamma)} + \xi |\langle \mathbf{1}, v\rangle_\Gamma|^2 \\ &\geq \tfrac{1}{2}\|v\|^2_{H^{\frac{1}{2}}(\Gamma)} + (\xi - 2c_D/|\Gamma|) |\langle \mathbf{1}, v\rangle_\Gamma|^2\ . \end{aligned}$$

If $\xi > 2c_D/|\Gamma|$, then $\widetilde{\mathsf{D}}_0$ is $H^{\frac{1}{2}}(\Gamma)$-elliptic and gives rise to an isomorphism $\widetilde{\mathsf{D}}_0 : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$. This choice of the parameter will be assumed, henceforth.

As illustrated by the following lemma, we can now get rid of all products of integral operators by multiplying the second equation of (4.12) with $\widetilde{\mathsf{D}}_0$.

LEMMA 4.6. *We have*

$$\widetilde{\mathsf{D}}_0\widetilde{\mathsf{R}}\varphi = (\tfrac{1}{2}Id + \mathsf{K}_0^*)(\varphi) + \mathsf{T}\varphi\ ,$$

where

$$\mathsf{T}\varphi := -\langle\varphi,\mathbf{1}\rangle_\Gamma/|\Gamma|(\tfrac{1}{2}Id+\mathsf{K}_0^*)(\mathbf{1})+\nu\xi|\Gamma|\langle\varphi,\mathbf{1}\rangle_\Gamma\mathbf{1}\,.$$

Proof. Obviously $\widetilde{\mathsf{D}}_0\mathbf{1}=\xi|\Gamma|\mathbf{1}$ and $\widetilde{\mathsf{D}}_0 v=\mathsf{D}_0 v$, if $v\in H_*^{\frac{1}{2}}(\Gamma)$. This means

$$\begin{aligned}\widetilde{\mathsf{D}}_0\widetilde{\mathsf{R}}\varphi &= \widetilde{\mathsf{D}}_0\mathsf{R}(\varphi-\varphi(\mathbf{1})\mathbf{1}/|\Gamma|)+\nu\xi|\Gamma|\langle\varphi,\mathbf{1}\rangle_\Gamma\mathbf{1}\\ &=(\tfrac{1}{2}Id+\mathsf{K}_0^*)(\varphi)-\langle\varphi,\mathbf{1}\rangle_\Gamma/|\Gamma|(\tfrac{1}{2}Id+\mathsf{K}_0^*)(\mathbf{1})+\nu\xi|\Gamma|\langle\varphi,\mathbf{1}\rangle_\Gamma\mathbf{1}\,,\end{aligned}$$

where (4.8) has been employed. □

Hence, applying the isomorphism $\widetilde{\mathsf{D}}_0$ to the second line of (4.12) gives

$$\begin{array}{rcccl} i\eta\mathsf{V}_\kappa\lambda & + & (\tfrac{1}{2}Id+\mathsf{K}_\kappa)u & = & g\,,\\ (\tfrac{1}{2}Id+\mathsf{K}_0^*)\lambda+\mathsf{T}\lambda & - & \widetilde{\mathsf{D}}_0 u & = & 0\,.\end{array}\tag{4.13}$$

We remark that the u-component of any solution of (4.13) instantly yields a solution of $\mathsf{B}_\kappa u=g$. Therefore, Lemma 4.5 also asserts the uniqueness of solutions of (4.13).

The first equation of (4.13) is set in $H^{\frac{1}{2}}(\Gamma)$, the second in $H^{-\frac{1}{2}}(\Gamma)$. Thus the duality of these spaces gives rise to the natural weak form of (4.13): seek $\lambda\in H^{-\frac{1}{2}}(\Gamma)$, $u\in H^{\frac{1}{2}}(\Gamma)$ such that for all $\mu\in H^{-\frac{1}{2}}(\Gamma)$, $v\in H^{\frac{1}{2}}(\Gamma)$

$$\begin{array}{rcccl} i\eta\langle\mu,\mathsf{V}_\kappa(\lambda)\rangle_\Gamma & + & \langle\mu,(\tfrac{1}{2}Id+\mathsf{K}_\kappa)u\rangle_\Gamma & = & \langle g,\mu\rangle_\Gamma\,,\\ -\langle(\tfrac{1}{2}Id+\mathsf{K}_0^*)\lambda,v\rangle_\Gamma-\langle\mathsf{T}\lambda,v\rangle_\Gamma & + & \left\langle\widetilde{\mathsf{D}}_0 u,v\right\rangle_\Gamma & = & 0\,.\end{array}\tag{4.14}$$

The bi-linear form $a:(H^{-\frac{1}{2}}(\Gamma)\times(H^{\frac{1}{2}}(\Gamma))\times(H^{-\frac{1}{2}}(\Gamma)\times(H^{\frac{1}{2}}(\Gamma))\mapsto\mathbb{C})$ associated with (4.14) reads

$$\begin{aligned}a(\begin{pmatrix}\lambda\\u\end{pmatrix},\begin{pmatrix}\mu\\v\end{pmatrix}):= &\; i\eta\langle\mu,\mathsf{V}_\kappa\lambda\rangle_\Gamma+\langle\mu,(\tfrac{1}{2}Id+\mathsf{K}_\kappa)u\rangle_\Gamma-\\ &-\overline{\langle(\tfrac{1}{2}Id+\mathsf{K}_0^*)\lambda,v\rangle_\Gamma}-\overline{\langle\mathsf{T}\lambda,v\rangle_\Gamma}+\overline{\left\langle\widetilde{\mathsf{D}}_0 u,v\right\rangle_\Gamma}\,.\end{aligned}$$

Now, we alter this bi-linear form by adding compact terms. First, we drop $\langle\mathsf{T}\lambda,v\rangle_\Gamma$, which is obviously compact since the range of T has dimension two. Next, we invoke Lemma 3.2 to replace V_κ and K_κ with V_0 and K_0, respectively. Ultimately, we end up with the perturbed bi-linear form

$$\begin{aligned}\widetilde{a}(\begin{pmatrix}\lambda\\u\end{pmatrix},\begin{pmatrix}\mu\\v\end{pmatrix}):= &\; i\eta\langle\mu,\mathsf{V}_0\lambda\rangle_\Gamma+\langle\mu,(\tfrac{1}{2}Id+\mathsf{K}_0)u\rangle_\Gamma-\\ &-\overline{\langle(\tfrac{1}{2}Id+\mathsf{K}_0^*)\lambda,v\rangle_\Gamma}+\overline{\left\langle\widetilde{\mathsf{D}}_0 u,v\right\rangle_\Gamma}\,.\end{aligned}$$

The symmetry (3.12) permits us to cancel cross terms and confirms $H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$-ellipticity

$$\begin{aligned}|a(\begin{pmatrix}\lambda\\u\end{pmatrix},\begin{pmatrix}\overline{\lambda}\\\overline{u}\end{pmatrix})| &=|i\eta\langle\overline{\lambda},\mathsf{V}_0\lambda)\rangle_\Gamma+\left\langle\widetilde{\mathsf{D}}_0 u,\overline{u}\right\rangle_\Gamma|\\ &\geq\frac{1}{\sqrt{2}}\left(\eta c_V\,||\lambda||^2_{H^{-\frac{1}{2}}(\Gamma)}+c_D\,||u||^2_{H^{\frac{1}{2}}(\Gamma)}\right)\,.\end{aligned}$$

This means that the bilinear form associated with 4.14) is coercive in $H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$. In addition we have established uniqueness of solutions. Therefore, we have verified all assumptions of Lemma 2.4 and Thm. 2.5 and reap all the desirable consequences for Galerkin discretization discussed at the end of the previous section.

Remark 4.1. The reader has to be aware that the choice of the regularizing operator $\widetilde{\mathsf{R}}$ is tightly constrained by the essential cancellation of the cross terms of $\widetilde{a}$. This forces us to incorporate $\frac{1}{2}Id + \mathsf{K}_0^*$ into $\widetilde{\mathsf{R}}$. In addition, $\widetilde{\mathsf{R}}$ has to be $H^{-\frac{1}{2}}(\Gamma)$-elliptic, see (4.9), and it is by no means obvious, how a choice different from (4.8) can comply with both requirements. △

Remark 4.2. The product of the parameters ν and ξ will enter the final variational formulation (4.14). It is important to note that uniqueness of solutions will be squandered, if $\nu\xi$ is chosen too small. Conversely, a large value for $\nu\xi$ might delay the onset of asymptotic phase, that is, in terms of the statement of Thm. (2.6) N will become very large. The necessity to pick parameters is definitely a drawback of this regularized formulation. △

4.3. Double layer regularization. In the previous subsection we incorporated the regularizing operator into the single layer potential. In light of (3.14) it is also an option to target the double layer potential in an attempt to get a compact perturbation of the single layer boundary integral operator.

The advantage of this is that we have much more freedom in choosing the regularizing operator, *cf.* Rem. 4.1. Now, we make explicit use of Ω^- being a (curvilinear) Lipschitz polyhedron: denote by $\Gamma_1,\dots,\Gamma_p$, $p\in\mathbb{N}$, its smooth (curved) polygonal faces and introduce the space

$$H^1_{\mathrm{pw},0}(\Gamma) := H^1_0(\Gamma_1)\times\cdots\times H^1_0(\Gamma_p)\subset H^1(\Gamma)\ . \tag{4.15}$$

Then define the regularizing operator $\mathsf{M}: H^{-1}(\Gamma)\mapsto H^1_{\mathrm{pw},0}(\Gamma)$ by

$$\langle \mathrm{grad}_\Gamma \mathsf{M}\varphi, \mathrm{grad}_\Gamma v\rangle_\Gamma = \langle\varphi, v\rangle_\Gamma \quad \forall v\in H^1_{\mathrm{pw},0}(\Gamma)\ . \tag{4.16}$$

In words, M is a combination of inverse Laplace–Beltrami operators on the individual faces Γ_i, $i=1,\dots,p$. Continuity of M is straightforward. The next lemma shows that M is even injective, when restricted to $H^{-\frac{1}{2}}(\Gamma)$.

LEMMA 4.7. *The space $H^1_{\mathrm{pw},0}(\Gamma)$ is dense in $H^{\frac{1}{2}}(\Gamma)$.*

Proof. Denote by Σ the union of closed edges of Ω^-. We can rely on [11, Lemma 2.6] that claims that the embedding

$$C^\infty_\Sigma := \{u\in C^\infty(\bar{\Omega}^-),\ \mathrm{supp}\, u\cap\Sigma=\emptyset\}\subset H^1(\Omega^-)$$

is dense. Obviously, $\gamma_D^-(C^\infty_\Sigma)\subset H^1_{\mathrm{pw},0}(\Gamma)$ and the continuity of $\gamma_D^-: H^1(\Omega^-)\mapsto H^{\frac{1}{2}}(\Gamma)$ finishes the proof. □

We conclude that for $\varphi\in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{aligned}
\mathsf{M}\varphi = 0 \quad &\Rightarrow \quad \langle\varphi, v\rangle_\Gamma = 0\ \forall v\in H^1_{\mathrm{pw},0}(\Gamma) && \text{by (4.16)}\\
&\Rightarrow \quad \langle\varphi, v\rangle_\Gamma = 0\ \forall v\in H^{\frac{1}{2}}(\Gamma) && \text{by Lemma 4.7}\\
&\Rightarrow \quad \varphi = 0 && \text{by duality of } H^{\frac{1}{2}}(\Gamma) \text{ and } H^{-\frac{1}{2}}(\Gamma)\ .
\end{aligned}$$

In particular, this involves

$$\langle\varphi, \mathsf{M}\overline{\varphi}\rangle_\Gamma = |\mathsf{M}\varphi|^2_{H^1(\Gamma)} > 0 \quad \forall\varphi\in H^{-\frac{1}{2}}(\Gamma)\setminus\{0\}\ . \tag{4.17}$$

The new regularized indirect method is based on the trial expression

$$U = \Psi_{\mathrm{DL}}^{\kappa}(\mathsf{M}\varphi) + i\eta\Psi_{\mathrm{SL}}^{\kappa}(\varphi)\ , \quad \varphi \in H^{-\frac{1}{2}}(\Gamma)\ . \tag{4.18}$$

As above, U is a radiating Helmholtz solution in $\Omega^- \cup \Omega^+$. If we apply the Dirichlet trace, we arrive at the boundary integral equation

$$g = ((\tfrac{1}{2}Id + \mathsf{K}_\kappa) \circ \mathsf{M})(\varphi) + i\eta\mathsf{V}_\kappa\varphi \quad \text{in } H^{\frac{1}{2}}(\Gamma)\ . \tag{4.19}$$

This prompts us to introduce the boundary integral operator

$$\mathsf{S}_\kappa := (\tfrac{1}{2}Id + \mathsf{K}_\kappa) \circ \mathsf{M} + i\eta\mathsf{V}_\kappa\ .$$

LEMMA 4.8. *The boundary integral operator* $\mathsf{S}_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ *is injective.*

Proof. Again, the idea of the proof of Lemma 4.1 can be applied: we assume that $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ solves

$$((\tfrac{1}{2}Id + \mathsf{K}_\kappa) \circ \mathsf{M})(\varphi) + i\eta\mathsf{V}_\kappa\varphi = 0\ .$$

It is immediate from the jump relations that U given by (4.18) is a Helmholtz solution with $\gamma_D^+ U = 0$, which, by Thm. 1.1, implies $U = 0$ in Ω^+. Hence, the jump relations confirm that

$$\gamma_D^- U = -\mathsf{M}\varphi \quad , \quad \gamma_N^- U = i\eta\varphi\ .$$

Next, we use the integration by parts formula (3.1) and get

$$-i\eta\,\langle \mathsf{M}\varphi, \overline{\varphi}\rangle_\Gamma = \langle \gamma_N^- U, \gamma_D^- \overline{U}\rangle_\Gamma = \int_{\Omega^-} |\operatorname{grad} U|^2 - \kappa^2|U|^2\, d\mathbf{x}$$

Necessarily, $\langle \mathsf{M}\varphi, \overline{\varphi}\rangle_\Gamma = 0$, which can only be satisfied, if $\varphi = 0$, *cf.* (4.17). □

Now, regard M as an operator $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1_{\mathrm{pw},0}(\Gamma)$. As such it inherits compactness from the embeddings $H^{-\frac{1}{2}}(\Gamma) \hookrightarrow H^{-1}(\Gamma)$. Thus, Lemma 4.8 allows to deduce existence of solutions of (4.19) by means of a Fredholm argument, *cf.* Thm. 2.3.

As the equation (4.19) is set in the space $H^{\frac{1}{2}}(\Gamma)$, a natural weak formulation can be obtained by testing with functions in $H^{-\frac{1}{2}}(\Gamma)$. In order to avoid undesirable products of integral operators a mixed formulation comes handy, again. We introduce the new unknown $u := \mathsf{M}\varphi \in H^1_{\mathrm{pw},0}(\Gamma)$ and the definition of M is used as second variational equation, which leads to the following saddle point problem: seek $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $u \in H^1_{\mathrm{pw},0}(\Gamma)$, such that

$$\begin{array}{rcccll} i\eta\,\langle \mathsf{V}_\kappa\varphi, \xi\rangle_\Gamma & + & \left\langle (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u, \xi\right\rangle_\Gamma & = & \langle g, \xi\rangle_\Gamma & \forall \xi \in H^{-\frac{1}{2}}(\Gamma)\ , \\ -\langle \varphi, v\rangle_\Gamma & + & \langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma v\rangle_\Gamma & = & 0 & \forall v \in H^1_{\mathrm{pw},0}(\Gamma)\ . \end{array} \tag{4.20}$$

It goes without saying that the first components of a solution (φ, u) of (4.20) will give us a solution of (4.19). Thus, Lemma 4.8 also asserts the uniqueness of solutions of (4.20).

Next, we aim to identify compact perturbations of the bi-linear form $a : (H^{-\frac{1}{2}}(\Gamma) \times H^1_{\mathrm{pw},0}(\Gamma)) \times (H^{-\frac{1}{2}}(\Gamma) \times H^1_{\mathrm{pw},0}(\Gamma)) \mapsto \mathbb{C}$ associated with (4.20).

LEMMA 4.9. *The bilinear form* $\langle \cdot, \cdot \rangle_\Gamma : H^1_{\mathrm{pw},0}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \mapsto \mathbb{C}$ *is compact.*

Proof. Compactness of a continuous bilinear form $b(\cdot,\cdot) : X \times Y \mapsto \mathbb{C}$, X, Y Banach spaces, is equivalent to the compactness of the associated operator $B : X \mapsto Y'$. In this particular case, it boils down to the compactness of the embedding operator $H^1_{\mathrm{pw},0}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$, which is a well-known embedding result. □

Moreover, recall the continuity $\mathsf{K}_\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and take into account Lemma 3.2. Hence, up to compact perturbations we need only examine the modified bi-linear form

$$\tilde{a}\left(\begin{pmatrix}\lambda\\u\end{pmatrix},\begin{pmatrix}\mu\\v\end{pmatrix}\right) := i\eta\,\langle \mathsf{V}_0\lambda,\mu\rangle_\Gamma + \langle \mathrm{grad}_\Gamma\, u, \mathrm{grad}_\Gamma\, v\rangle_\Gamma \ , \tag{4.21}$$

$\lambda,\mu \in H^{-\frac{1}{2}}(\Gamma)$, $u, v \in H^1_{\mathrm{pw},0}(\Gamma)$. By (3.14) and the Poincaré-Friedrichs inequalities on the faces Γ_i, $i = 1,\dots,p$, it is obvious that $\tilde{a}$ is elliptic on $H^{-\frac{1}{2}}(\Gamma) \times H^1_{\mathrm{pw},0}(\Gamma)$. This permits us to conclude that the bilinear form belonging to (4.20) is coercive. By virtue of Thm. 2.5 we get asymptotic quasi-optimality of approximate Galerkin solutions.

Remark 4.3. It is also possible to use $\mathsf{M} := -\Delta_\Gamma^{-1}$, where $\Delta_\Gamma : H^1(\Gamma) \mapsto H^{-1}(\Gamma)$ is the Laplace-Beltrami operator on all of Γ. The rationale why we opted for a localized operator M is explained in the next section. △

5. Direct boundary integral equations. The direct approach to the derivation of boundary integral equations uses the representation formula

$$U = \Psi^\kappa_{\mathrm{DL}}(\gamma_D^+ U) - \Psi^\kappa_{\mathrm{SL}}(\gamma_N^+ U) \ , \tag{5.1}$$

valid for any exterior Helmholtz solution. Applying both the Dirichlet and Neumann trace operator to (5.1), we obtain the formulas of the *Calderón projector*

$$\gamma_D^+ U = (\mathsf{K}_\kappa + \tfrac{1}{2} Id)(\gamma_D^+ U) - \mathsf{V}_\kappa(\gamma_N^+ U) \ , \tag{5.2}$$

$$\gamma_N^+ U = -\mathsf{D}_\kappa(\gamma_D^+ U) - (\mathsf{K}^*_\kappa - \tfrac{1}{2} Id)(\gamma_N^+ U) \ . \tag{5.3}$$

¿From these equations we can extract two boundary integral equations for both the exterior Dirichlet and Neumann problem for the Helmholtz equation (1.1). Since, the boundary operators applied to the unknown Cauchy datum are the same as in (4.2) and (4.3), respectively, these boundary integral equations will also be affected by spurious resonances.

5.1. Classical CFIE. It was the idea of Burton and Miller in [6] to consider the following complex linear combination of the two equations (5.2) and (5.3)

$$(i\eta(\mathsf{K}_\kappa - \tfrac{1}{2} Id) - \mathsf{D}_\kappa)(\gamma_D^+ U) - (i\eta \mathsf{V}_\kappa + \tfrac{1}{2} Id + \mathsf{K}^*_\kappa)(\gamma_N^+ U) = 0 \ , \tag{5.4}$$

where $\eta \neq 0$ is a real parameter. Then, for the exterior Neumann problem (1.1), (1.2), and (1.4) we seek the unknown Dirichlet datum $u \in H^{\frac{1}{2}}(\Gamma)$ that satisfies

$$\mathsf{C}_\kappa(u) = (i\eta \mathsf{V}_\kappa + \tfrac{1}{2} Id + \mathsf{K}^*_\kappa)(\varphi) \ , \tag{5.5}$$

where

$$\mathsf{C}_\kappa := i\eta(\mathsf{K}_\kappa - \tfrac{1}{2} Id) - \mathsf{D}_\kappa \ .$$

First we establish a counterpart of Lemma 4.1.

Lemma 5.1. *The boundary integral operator* $\mathsf{C}_\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ *is injective.*

Proof. Pick $u \in H^{\frac{1}{2}}(\Gamma)$ such that $\mathsf{C}_\kappa u = 0$ and set $U := \Psi^\kappa_{\mathrm{SL}}(u)$. Then $U_{|\Omega^-}$ is an interior Helmholtz solution that complies with the Robin boundary conditions

$$(i\eta\gamma_D^- + \gamma_N^-)U = (i\eta(\mathsf{K}_\kappa - \tfrac{1}{2}Id) - \mathsf{D}_\kappa)u = \mathsf{C}_\kappa u = 0 \;.$$

This is an immediate consequence of the jump relations. As simple argument based on the weak formulation of the Robin boundary value problem shows $U_{|\Omega^-} \equiv 0$. Again appealing to the jump relations, we see $\gamma_N^+ U = 0$. Owing to 1.1 this implies $U \equiv 0$ everywhere. Eventually we get $u = [\gamma_D U]_\Gamma = 0$. □

Now, the arguments used for the analysis of the boundary integral equation (4.6) can be copied verbatim and give us the existence of unique weak solutions of (5.5), *cf.* Sect. 4.1.

As before, the exterior Dirichlet problem described by (1.1), (1.2), and (1.3), eludes this simple analysis, because the boundary integral equation

$$\mathsf{F}_\kappa(\varphi) = (i\eta(\mathsf{K}_\kappa - \tfrac{1}{2}Id) - \mathsf{D}_\kappa)(\gamma_D^+ U) \;,$$

where $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ and

$$\mathsf{F}_\kappa := i\eta\mathsf{V}_\kappa + \tfrac{1}{2}Id + \mathsf{K}_\kappa^* \;,$$

cannot be considered in $H^{\frac{1}{2}}(\Gamma)$. Lifting it to $L^2(\Gamma)$ is a remedy only on smooth surfaces, *cf.* Sect. 4.1. Again, it takes regularization to get a coercive variational formulation.

5.2. Regularized formulation. The strategy for regularization closely follows the "double layer regularization" elaborated in Sect. 4.3. It relies on a regularizing operator $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ that has to satisfy

1. M is compact.
2. $\langle \varphi, \mathsf{M}\overline{\varphi}\rangle_\Gamma > 0$ for all $\varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\}$.

The trick is to apply M to (5.3) before adding it to $i\eta\cdot$(5.2). Doing so is strongly suggested by the fact that Dirichlet traces and Neumann traces belong to different spaces so that $\gamma_N^+ U$ should be lifted into $H^{\frac{1}{2}}(\Gamma)$ before adding it to $i\eta\gamma_D^+ U$. This yields the following boundary integral equation for the exterior Dirichlet problem

$$\mathsf{S}_\kappa(\varphi) = (i\eta(\mathsf{K}_\kappa - \tfrac{1}{2}Id) - \mathsf{M} \circ \mathsf{D}_\kappa)g \;, \tag{5.6}$$

where $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ is the unknown Neumann datum and

$$\mathsf{S}_\kappa := \mathsf{M} \circ (\mathsf{K}_\kappa^* + \tfrac{1}{2}Id) + i\eta\mathsf{V}_\kappa \;.$$

The first result corresponds to Lemma 4.8.

Lemma 5.2. *The boundary integral operator* $\mathsf{S}_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ *is injective.*

Proof. We consider $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ with $\mathsf{S}_\kappa\varphi = 0$ and set $U = \Psi^\kappa_{\mathrm{SL}}(\varphi)$. Thanks to (3.5) $U_{|\Omega^-}$ is a solution of

$$\begin{aligned} -\Delta U - \kappa^2 U &= 0 \quad \text{in } \Omega^- \;, \\ \mathsf{M}(\gamma_N^- U) + i\eta\,\gamma_D^- U &= 0 \quad \text{on } \Gamma \;. \end{aligned}$$

This is clear from the jump relations for the single layer potential. As a Helmholtz solution $U \in H^1(\Omega^-)$ satisfies

$$\int_{\Omega^-} \operatorname{grad} U \cdot \operatorname{grad} V - \kappa^2 U\, V \, d\mathbf{x} - \left\langle \gamma_N^- U, \gamma_D^- V \right\rangle_\Gamma = 0 \quad \forall V \in H^1(\Omega^-) \;.$$

Now, use $V = \bar{U}$ and use the boundary conditions to express $\gamma_D^- U$

$$\int_{\Omega^-} |\operatorname{grad} U|^2 - \kappa^2 |U|^2 \, d\mathbf{x} + \frac{i}{\eta} \left\langle \gamma_N^- U, \mathsf{M}(\gamma_N^- \bar{U}) \right\rangle_\Gamma = 0 \, .$$

Equating imaginary parts and using the assumptions on M we find $\gamma_N^- U = 0$, which implies $\gamma_D^- U = 0$. Hence, $U_{|\Omega^-} \equiv 0$. The jump relations for the single layer potential involve $\gamma_D^+ U = 0$, which, by Thm. 1.1, means $U_{|\Omega^+} = 0$. As a consequence, $\varphi = -[\gamma_N U]_\Gamma = 0$. □

This paves the way for applying the Fredholm alternative to (5.6): since

$$\mathsf{S}_\kappa = i\eta \mathsf{V}_0 + i\eta(\mathsf{V}_\kappa - \mathsf{V}_0) + \mathsf{M} \circ (\mathsf{K}_\kappa^* + \tfrac{1}{2} Id) \, ,$$

we can invoke Lemma 3.2, the continuity of K_κ^*, and the compactness of M to conclude that S_κ is bijective.

In the sequel, let us use the same operator M as in Sect. 4.3, namely the one given by (4.16). This makes it possible to switch to a simple mixed formulation by introducing the new unknown

$$u := \mathsf{M}((\tfrac{1}{2} Id + \mathsf{K}_\kappa^*)\varphi + \mathsf{D}_\kappa g) \, . \tag{5.7}$$

Actually, u is mislabelled, because it is by no means an unknown: recalling (5.3) we quickly realize that $u = 0$, if φ is the exact Neumann trace. What is the point of introducing u, nevertheless? The reason is that we aim to get a variational formulation suitable for Galerkin discretization and in the discrete setting the approximation of u does not necessarily vanish. The concrete variational problem reads: seek $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $u \in H^1_{\mathrm{pw},0}(\Gamma)$ such that for all $\xi \in H^{-\frac{1}{2}}(\Gamma)$, $v \in H^1_{\mathrm{pw},0}(\Gamma)$

$$\begin{array}{rcccl} i\eta \left\langle \xi, \mathsf{V}_\kappa \varphi \right\rangle_\Gamma & + & \left\langle \xi, u \right\rangle_\Gamma & = & i\eta \left\langle \xi, (\mathsf{K}_\kappa - \tfrac{1}{2} Id) g \right\rangle_\Gamma \, , \\ -\left\langle (\tfrac{1}{2} Id + \mathsf{K}_\kappa^*)\varphi, v \right\rangle_\Gamma & + & \left\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma v \right\rangle_\Gamma & = & \left\langle \mathsf{D}_\kappa g, v \right\rangle_\Gamma \, . \end{array} \tag{5.8}$$

It is clear that $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ solves (5.6) if and only if (φ, u), u given by (5.7), solves (5.8). Hence, Lemma 5.2 also implies uniqueness of solutions of (5.8).

As in Sect. 4.3 we can appeal to Lemma 4.9 to see that the off-diagonal terms in (5.8) are compact. Eventually, up to compact perturbations, it turns out that the bilinear form associated with (5.8) equals the $H^{-\frac{1}{2}}(\Gamma) \times H^1_{\mathrm{pw},0}(\Gamma)$-elliptic bi-linear form $\widetilde{a}$ defined in (4.21). Hence, it is a coercive bi-linear form that underlies the variational problem (5.8).

6. Galerkin discretization. Conforming boundary element spaces for the approximation of functions in $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, respectively, are standard. First, we equip Γ with a family $\{\mathcal{T}_h\}_h$ of triangulations comprising (curved) triangles and/or quadrilaterals. The meshes $\mathcal{T}_h$ have to resolve the shape of the curvilinear polyhedron Ω^- in the sense that none of their elements may reach across an edge of Ω^-. Then, the boundary element spaces $\mathcal{S}_h \subset H^{\frac{1}{2}}(\Gamma)$ and $\mathcal{Q}_h \subset H^{-\frac{1}{2}}(\Gamma)$ will contain piecewise polynomials of total/maximal degree k, $k \in \mathbb{N}_0$. Further, functions in $\mathcal{S}_h$ have to be continuous so that $k \geq 1$ is required in this case.

For the Galerkin discretization of (4.20) we will need boundary element subspaces $\mathcal{S}_h^{\mathrm{pw}}$ of $H^1_{\mathrm{pw},0}$. They can be constructed as subspaces of $\mathcal{S}_h$ by setting all degrees of freedom associated with edges of Γ to zero.

Let h denote the meshwidth of $\mathcal{T}_h$ and assume uniform shape-regularity, which, sloppily speaking, imposes a uniform bound on the distortion of the elements. Then we can find constants $C_s, C_q > 0$ such that for all $0 \leq t \leq k+1$ [2, Sect. 4.4]

$$\inf_{\phi_h \in \mathcal{Q}_h} \|\phi - \phi_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C_q h^{t+\frac{1}{2}} \|\phi\|_{H^t(\Gamma)} \quad \forall \varphi \in H^t(\Gamma), \forall h \,, \tag{6.1}$$

$$\inf_{v_h \in \mathcal{S}_h} \|v - v_h\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_s h^{t-\frac{1}{2}} \|v\|_{H^t(\Gamma)} \quad \forall v \in H^t(\Gamma), \forall h \,. \tag{6.2}$$

Thus, the quantitative investigation of convergence boils down to establishing the Sobolev regularity of the continuous solutions. We will embark on this for the variational boundary integral equations (4.7), (4.14), (4.20), and (5.8).

It is useful to characterize the lifting properties of Neumann-to-Dirichlet maps for the interior/exterior Helmholtz problem by means of two real numbers α^+/α^-. In particular, let α^-/α^+ be the largest real number such that for an interior/exterior Helmholtz solution $\gamma_N^\pm U \in H^{s-\frac{1}{2}}(\Gamma)$ implies $\gamma_D^\pm U \in H^{s+\frac{1}{2}}(\Gamma)$ for all $s \leq \alpha^\pm$ and vice-versa. It is known that for mere Lipschitz domains $\alpha^-, \alpha^+ \geq \frac{1}{2}$ [17, Thm. 4.24].

We first examine (4.7) and assume that the Neumann data φ belong to $H^{-\frac{1}{2}+\sigma}(\Gamma)$, $\sigma > 0$. According to the definition of α^+ this implies $\gamma_D^+ U \in H^{\frac{1}{2}+\min\{\sigma,\alpha^+\}}(\Gamma)$. Now, let $u \in H^{\frac{1}{2}}(\Gamma)$ stand for the unique solution of (4.7) and let the Helmholtz solution U be given by (4.4). By the jump relations

$$[\gamma_N U]_\Gamma = -i\eta u \quad , \quad [\gamma_D U]_\Gamma = u \,. \tag{6.3}$$

we conclude that $U_{|\Omega^-}$ satisfies the inhomogeneous Robin-type boundary conditions

$$\gamma_N^- U - i\eta\gamma_D^- U = \varphi - i\eta\gamma_D^+ U \,. \tag{6.4}$$

This will endow the Neumann data with extra regularity and we can crank up the machine of a bootstrap argument that confirms higher and higher regularity for Neumann and Dirichlet data in turns. A limit will be set by the lifting exponents α^-, α^+: the best we can get is

$$u \in H^{\frac{1}{2}+\min\{\sigma,\alpha^-,\alpha^+\}}(\Gamma) \,.$$

For piecewise linear continuous boundary elements on a sequence of shape regular surface meshes this will mean $O(h^{\min\{\sigma,\alpha^+,\alpha^-\}})$ convergence in $H^{\frac{1}{2}}(\Gamma)$.

In the case of the single layer regularization (4.14) of Sect. 4.2 the lifting arguments will fail. Please note that for U from (4.10), where $u \in H^{\frac{1}{2}}(\Gamma)$ is the solution of (4.11), the following interior Robin-type boundary conditions hold:

$$\gamma_N^- U - i\eta\widetilde{\mathsf{R}}^{-1}(\gamma_D^- U) = \gamma_N^- U - i\eta\widetilde{\mathsf{R}}^{-1} g \,. \tag{6.5}$$

In contrast to (6.4), we cannot infer any enhanced regularity of either $\gamma_N^- U$ or $\gamma_D^- U$ from (6.5). Hence, no quantitative rate of convergence can be obtained for a Galerkin boundary element discretization of (4.14). Due to the density of the boundary element spaces on infinite sequences of ever finer meshes in $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, respectively, the method will converge for $h \to 0$, but can be arbitrarily slow.

A similar reasoning as for (4.7) applies to the regularized formulation introduced in Sect. 4.3. If $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ is the solution of (4.19) and the Helmholtz solution U is given by (4.18), the jump relations give us

$$[\gamma_D U]_\Gamma = \mathsf{M}\varphi \quad , \quad [\gamma_N U]_\Gamma = -i\eta\varphi \,. \tag{6.6}$$

It is clear that the regularizing properties of M will come into play. To measure them define for $s \geq 1$

$$H^s_{\mathrm{pw},0}(\Gamma) := \{v \in H^1_{\mathrm{pw},0}(\Gamma),\ v_{|\Gamma_i} \in H^s(\Gamma_i), i = 1, \dots, p\} .$$

We will write $0 < \beta \leq 1$ for the largest real number such that $\mathsf{M}v \in H^{s-1}(\Gamma)$ implies $v \in H^{s+1}_{\mathrm{pw},0}(\Gamma)$ for all $s \leq \beta$. From [13] we know that $\beta > \frac{1}{2}$, and that $\beta \geq 1$ can be choosen, if all Γ_i are diffeomorphic images of convex polygons.

Assume that the Dirichlet boundary values g belong to $H^{\sigma+\frac{1}{2}}(\Gamma)$. This means that $\gamma_N^+ U \in H^{-\frac{1}{2}+\min\{\sigma,\alpha^+\}}(\Gamma)$. In addition, $U_{|\Omega^-}$ satisfies the inhomogeneous boundary conditions

$$i\eta\mathsf{M}(\gamma_N^- U) - \gamma_D^- U = i\eta\mathsf{M}(\gamma_N^+ U) - g . \tag{6.7}$$

Since $\gamma_N^+ U \in H^{-\frac{1}{2}+\min\{\sigma,\alpha^+\}}(\Gamma)$, using the mapping property of M, we deduce that the right hand side of (6.7) belongs to $H^r(\Gamma)$, with $r = \min\{\frac{3}{2}+\alpha^+, 1+\beta, \frac{1}{2}+\sigma\}$.

We first have that $\gamma_D^- U \in H^{\min\{1,r\}}(\Gamma)$, thus $\gamma_N^- U \in H^{\min\{0,r-1,-\frac{1}{2}+\alpha^-\}}(\Gamma)$. Now, a bootstrap argument can be used. By the shift theorem for M, we obtained an improved regularity for $\mathsf{M}\gamma_N^- U \in H^{\min\{2,r+1,\frac{3}{2}+\alpha^-,1+\beta\}}(\Gamma)$.

Using again (6.7), we then have $\gamma_D^- U \in H^{\min\{2,r,\frac{3}{2}+\alpha^-,1+\beta\}}(\Gamma)$. Thus finally, recalling the definition of r, we have $\gamma_N^- U \in H^{\min\{1,-\frac{1}{2}+\alpha^-,\beta,-\frac{1}{2}+\sigma\}}(\Gamma)$.

By (6.6), this involves

$$\varphi \in H^{\min\{1,\beta,-\frac{1}{2}+\sigma,-\frac{1}{2}+\alpha^+,-\frac{1}{2}+\alpha^-\}}(\Gamma) . \tag{6.8}$$

Note that, since Ω is a polyhedron, either α^+ or α^- is smaller than 1. Without loss of generality, we can then reduce (6.8) to:

$$\varphi \in H^{-\frac{1}{2}+\min\{\sigma,\alpha^+,\alpha^-\}}(\Gamma) , \tag{6.9}$$

which means that the regularity of φ depends only on the regularity of the Dirichlet datum and of the interior and exterior Dirichlet-to-Neumann maps.

For the mixed variational problem (4.20) convergence will also hinge on the regularity of the auxiliary variable $u := \mathsf{M}\varphi$.

The regularity (6.9) of φ will directly translate into the regularity $u \in H^{\min\{\frac{3}{2}+\sigma,\frac{3}{2}+\alpha^+,\frac{3}{2}+\alpha^-\}}_{\mathrm{pw},0}(\Gamma)$. We point out that the approximation estimate (6.2) remains true when we replace $\mathcal{S}_h$ with $\mathcal{S}_h^{\mathrm{pw}}$ and $H^t(\Gamma)$ with $H^t_{\mathrm{pw},0}(\Gamma)$. Therefore, u is smooth enough not to impair the convergence estimate that can be gained from combining (6.9) with (6.1).

Remark 6.1. The ease with which we get good regularity for u motivated the concrete choice of M in Sect. 4.3. If we had opted for $\mathsf{M} = \Delta_\Gamma^{-1}$, *cf.* Rem. 4.3, Thm. 5.3 of [3] tells us that u may be only slightly more regular than merely belonging to $H^1(\Gamma)$. This could make u limit the overall convergence. △

The simplest case is that of the direct formulation (5.8) for the exterior Dirichlet problem. The variational problem features the Cauchy datum $\varphi := \gamma_N^+ U$ as the principal unknown. Assuming $g \in H^{\frac{1}{2}+\sigma}(\Gamma)$, $\sigma \geq 0$, we conclude $\varphi \in H^{\min\{-\frac{1}{2}+\sigma,-\frac{1}{2}+\alpha^+\}}(\Gamma)$. Moreover, the exact solution for the auxiliary unknown u will be $u = 0$, which means that it does not affect the asymptotic convergence of the

Galerkin scheme. Summing up, we find an a priori estimate $O(h^{\min\{\sigma,\alpha^+,k+1\}})$ for the rate of convergence.

Remark 6.2. The behavior of an exterior Helmholtz solution at edges and corners of Γ is well known [12]. This gives a lot of information about the local behavior of the Neumann trace $\gamma_N^+ U$. This knowledge can be exploited to construct more efficient locally adapted approximation by means of local anisotropic refinement in conjunction with *hp*-adaptivity [21]. △

7. Conclusion. We found that the classical combined field integral equation for the exterior Neumann problem for Helmholtz' equation leads to a $H^{\frac{1}{2}}(\Gamma)$-coercive variational problem. Satisfactory rates of convergence can be deduced for conforming Galerkin BEM schemes. Conversely, the analysis of the CFIE for the exterior Dirichlet problem has to rely on special regularizing operators. For the indirect method two approaches to regularization have been pursued and both ensure coercivity of the final mixed variational problems. However, only regularization aimed at the single layer potential part of the trial expression yields information about extra regularity of the unknown surface density. Hence, quantitative estimates of the convergence of Galerkin BEM solutions could only be achieved in this case. On the other hand the regularized direct CFIE instantly yields a priori error estimates that only depend on the regularity of the Neumann data: the auxiliary unknown has no influence. In this respect the direct method is definitely superior to the indirect approach.

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