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Current and Voltage Excitations for the Eddy Current Model

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Research Report No. 2003-07
July 2003

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Abstract

We present a systematic study of how to take into account external excitation in the eddy current model. Emphasis is put on mathematically sound variational formulations and on lumped parameter excitation through prescribed currents and voltages. We distinguish between local excitation at known contacts, known generator current distributions and non-local variants that rely on topological concepts. The latter case entails the violation of Faraday's law at so-called cuts and prevents us from reconstructing a meaningful electric field.

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1 Introduction

In order to obtain computationally tractable models of electric devices often one considers the full field equations only for parts, whereas a lumped circuit description is used for the remainder. This makes it necessary to couple both models. In other words, we have to figure out how to link the quantities used in circuit models, currents and voltages, with electromagnetic fields.

We investigate this issue in the case of (weak) variational formulations of the magneto-quasistatic eddy current model. We can distinguish between two basically different weak formulations, voltage or current excitation, and local or non-local excitation. We aim to give a systematic treatment of all these cases leading to stable variational formulations that can serve as the basis for finite element discretization schemes.

Our presentation is part of brisk recent research in this area: without even pretending to give an exhaustive list of references, we would like to mention the pioneering contributions of P. Dular [DGL99, DLN98, DHL99] and, in particular, [Dul01]. Other researchers have considered the field equations after discretization, viewing them as small-scale circuit equations, see e.g. [RTV02]. Moreover, it has been realized that non-local excitation is intimately linked to topological issues [Ket01, Bos00].

It is important to note, that a true lumped parameter excitation is not feasible for the eddy current model, e.g. there is no canonical way to impose a current or voltage without additional modeling information on the *distribution of the sources*. Even in the case of the so-called non-local excitations, that rely on topological concepts, additional information will be necessary if a coupling to a circuit model shall be established.

1.1 Eddy Current Model

In the sequel we consider an open and connected "computational domain" $\Omega \subset \mathbb{R}^3$ with exterior unit normal vector field $\mathbf{n}$ on its boundary $\partial\Omega$. Let $\overline{\Omega}$ be partitioned into two disjoint open subsets, that is $\overline{\Omega} = \overline{\Omega}_C \cup \overline{\Omega}_I$, where Ω_C represents the conductors and Ω_I the insulating air region. Generically, we have $\Omega = \mathbb{R}^3$, but introducing an artificial cut-off boundary will render it bounded. Thus, throughout the remainder of the paper, we can restrict ourselves to bounded Ω. We adopt the notation $\Gamma_C = \partial\Omega_C \cap \partial\Omega_I$ and write Γ_i for the boundaries of the connected components of Ω_C, $\Gamma_C = \cup_i \Gamma_i$.

The complete eddy current model reads [AFV01]

$$
\begin{aligned}
\mathbf{curl}\,\mathbf{e} &= -\tfrac{d}{dt}\mathbf{b} && \text{in } \Omega && \text{(Faraday's law)}\,, && \text{(1a)}\\
\mathbf{curl}\,\mathbf{h} &= \mathbf{j} && \text{in } \Omega && \text{(Ampere's law)}\,, && \text{(1b)}\\
\operatorname{div}\mathbf{b} &= 0 && \text{in } \Omega && \text{(magnetic Gauß' law)}\,, && \text{(1c)}\\
\mathbf{b} &= \mu\mathbf{h} && \text{in } \Omega\,, && && \text{(1d)}\\
\mathbf{j} &= \mathbf{j}_G + \sigma\,\mathbf{e} && \text{in } \Omega && \text{(Ohm's law)}\,, && \text{(1e)}
\end{aligned}
$$

where $\mathbf{e}$ and $\mathbf{h}$ stand for the electric and magnetic field, respectively, $\mathbf{b}$ denotes the magnetic induction, $\mathbf{j}$ the current density, and $\mathbf{j}_G = \mathbf{j}_G(\mathbf{x}, t)$ is an imposed current density (generator current). We assume that $\operatorname{supp}\mathbf{j}_G$ is bounded and that $\mathbf{j}_G$ depends continuously on time. The material parameter $\sigma \in L^\infty(\Omega)$ describes the electric conductivity, and satisfies $\sigma > \sigma_0 > 0$ in Ω_C, $\sigma \equiv 0$ in Ω_I. The coefficient $\mu \in L^\infty(\Omega)$, $\mu > \mu_0 > 0$ everywhere in Ω, is the magnetic permeability. We will also make use of the dielectric constant $\epsilon \in L^\infty(\Omega)$, $\epsilon > \epsilon_0 > 0$ everywhere.

These equations have to be supplemented by suitable boundary conditions on the (artificial) boundary $\partial\Omega$. Below we impose electric boundary conditions

$$\mathbf{n} \times \mathbf{e} = \mathbf{f}$$

on the part $\partial\Omega_e \subset \partial\Omega$ of the boundary, and magnetic boundary conditions

$$\mathbf{n} \times \mathbf{h} = \mathbf{g}$$

on $\partial\Omega_h \subset \partial\Omega$, where $\partial\Omega = \partial\Omega_e \cup \partial\Omega_h$, $\partial\Omega_e \cap \partial\Omega_h = \emptyset$. Both $\mathbf{f}$ and $\mathbf{g}$ depend on space and time and are assumed to be eligible tangential traces of fields.

We point out that in the case of unbounded Ω we have to demand a decay of the fields according to $\mathbf{h}(\mathbf{x}) = O(|\mathbf{x}|^{-2})$, $\mathbf{e}(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \longrightarrow \infty$. In variational formulations we need to use suitable weighted spaces [Hip02b, Sect. 2]. Then all arguments employed in the case of bounded Ω remain valid.

We also stress, that non-linear, in particular ferromagnetic, materials are admitted, that is, μ is allowed to depend on $\mathbf{b}$. Though not incorporated in the above formulation, tensorial material parameters can be easily accommodated, too. We opted for a treatment in the time domain, but most results carry over to the frequency domain after replacing $\frac{d}{dt}$ with $i\omega$, $\omega > 0$ the angular frequency.

Equation (1b) implies a compatibility condition for for $\mathbf{j}_G$ in Ω_I: For any sub-domain $V \subset \overline{\Omega}_I$ must hold

$$\int_{\partial V} \mathbf{j}_G \cdot \mathbf{n} \, dS = 0 \, .$$

This implies $\operatorname{div} \mathbf{j}_G = 0$ in Ω_I. Note that (1c) has to be enforced at initial time only, since $\frac{d}{dt} \operatorname{div} \mathbf{b} = 0$ follows from (1a).

The eddy current model represents a magneto-quasistatic approximation to Maxwell's equations: it is reasonably accurate for *slowly varying* fields, for which the change in magnetic field energy is dominant, see [ABN00, Dir96]. Slowly varying, means that

$$\sqrt{\epsilon \, \mu} \frac{l}{\delta t} \ll 1 \tag{2}$$

where l is the characteristic size of the region of interest and δt the smallest relevant time-scale. This means that Ω_C has to be small compared to the wavelength of electromagnetic waves, which makes it possible to ignore wave propagation. If this cannot be done, the issue of voltage and current excitation becomes far more tricky and might not make much sense any more. This is beyond the scope of this paper.

There is a second condition for the validity of the eddy current approximation, which requires that the typical time-scale is long compared to the relaxation time for space charges, that is, the conductivity must be large enough to make

$$\frac{\epsilon}{\sigma \, \delta t} \ll 1 \tag{3}$$

hold true. This implies that no space charges need to be taken into account.

We point out an important difference between the full Maxwell equations and the eddy current equations (1). Whereas in the former there is a perfect symmetry between electric and magnetic quantities, this symmetry is broken in the case of the eddy current model. This is a consequence of dropping the displacement currents.

It is important to note that the equations (1) do not completely determine the electric field in Ω_I. There it is unique only up to a **curl**-free "electrostatic component". In order to restore uniqueness of $\mathbf{e}$ everywhere in Ω we may demand the gauge condition

$$\operatorname{div}(\epsilon\, \mathbf{e}) = 0 \text{ in } \Omega_I \,,$$

and that all total conductor charges will vanish. For an in-depth discussion in the time-harmonic case and with homogeneous electric ($\mathbf{n}\times\mathbf{e} = 0$ on $\partial\Omega$) and magnetic ($\mathbf{n}\times\mathbf{h} = 0$ on $\partial\Omega$) boundary conditions on $\partial\Omega$ we refer to [AFV01].

However, the electric field in Ω_I is of little interest in many applications. It might not even make much physical sense, for instance close to inductor coils that are described by a given current density distribution. In this case an ungauged formulation makes perfect sense.

1.2 Variational Formulations

Corresponding to primal and dual ways to cast second-order elliptic boundary value problems into weak form, we can distinguish between two different variational formulations of (1), which use one of the equations (1a) or (1b) in strong form ("pointwise") and the other in weak form ("averaged") [Bos85].

h-based formulation

This formulation retains the magnetic field $\mathbf{h}$ as primary unknown. The variational problem relies on the function spaces

$$\boldsymbol{\mathcal{V}}(\mathbf{j}_g, \mathbf{g}) := \{\mathbf{h}' \in \boldsymbol{H}(\mathbf{curl};\Omega), \mathbf{curl}\,\mathbf{h}' = \mathbf{j}_g \text{ in } \Omega_I, \mathbf{n}\times\mathbf{h}' = \mathbf{g} \text{ on } \partial\Omega_h\} \tag{4}$$

and

$$\boldsymbol{\mathcal{V}}_0 := \boldsymbol{\mathcal{V}}(0,0)\,. \tag{5}$$

For definitions and properties of function spaces like $\boldsymbol{H}(\mathbf{curl};\Omega)$ the reader is referred to [GR86, Ch. 1]. Then, the variational formulation describing the evolution of the magnetic field from $t = 0$ to $t = T$, $T > 0$ fixed, reads:

Find $\mathbf{h} \in C^1([0,T], \boldsymbol{\mathcal{V}}(\mathbf{j}_g, \mathbf{g}))$ [1], $\mathbf{h}(0) = \mathbf{h}_0 \in \boldsymbol{\mathcal{V}}(\mathbf{j}_g(0), \mathbf{g}(0))$, such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$

$$\int_{\Omega_C} \frac{1}{\sigma}\mathbf{curl}\,\mathbf{h}\cdot\mathbf{curl}\,\mathbf{h}'\,d\mathbf{x} + \int_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}'\,d\mathbf{x} = \int_{\Omega_C} \frac{1}{\sigma}\mathbf{j}_G\cdot\mathbf{curl}\,\mathbf{h}'\,d\mathbf{x} + \int_{\partial\Omega_e} \underbrace{(\mathbf{n}\times\mathbf{e})}_{=\mathbf{f}}\cdot\mathbf{h}'\,dS\,. \tag{6}$$

This equation contains Ampere's law (1b) in strong form, Faraday's law (1a) in weak form.

a-based formulation

Since (1c) is true in all of $\mathbb{R}^3$, we can write $\mathbf{b} = \mathbf{curl}\,\mathbf{a}$. Then $\mathbf{e} = -\frac{d}{dt}\mathbf{a} - \mathbf{grad}\,v$ with v standing for a scalar potential. To fix $\mathbf{a}$ we can rely on the so-called "temporal gauge", which sets $v = 0$, that is $\mathbf{e} = -\frac{d}{dt}\mathbf{a}$. We introduce the space

$$\boldsymbol{\mathcal{W}}(\mathbf{f}) := \{\mathbf{a}' \in \boldsymbol{H}(\mathbf{curl};\Omega), \mathbf{n}\times\mathbf{a}' = -\int \mathbf{f}\,dt \text{ on } \partial\Omega_e\}$$

[1] The requirement that $\mathbf{h}$ is continuously differentiable w.r.t. time can be relaxed, *cf.* [RR93, Sect. 10.1]. Then excitations that jump in time can be accommodated.

and arrive at the following variational formulation:

Find $\mathbf{a} \in C^1([0,T], \boldsymbol{\mathcal{W}}(\mathbf{f}))$, $\mathbf{a}(0) = \mathbf{a}_0 \in \boldsymbol{\mathcal{W}}(\mathbf{f}(0))$, such that for all $\mathbf{a}' \in \boldsymbol{\mathcal{W}}(0)$

$$\int_{\Omega} \frac{1}{\mu} \operatorname{\mathbf{curl}} \mathbf{a} \cdot \operatorname{\mathbf{curl}} \mathbf{a}' \, d\mathbf{x} + \int_{\Omega_C} \sigma \tfrac{d}{dt} \mathbf{a} \cdot \mathbf{a}' \, d\mathbf{x} = \int_{\Omega} \mathbf{j}_G \cdot \mathbf{a}' \, d\mathbf{x} - \int_{\partial\Omega_h} \underbrace{(\mathbf{n} \times \mathbf{h})}_{=\mathbf{g}} \cdot \mathbf{a}' \, dS \,. \tag{7}$$

The reader should be aware that this is an *ungauged* formulation, since extra conditions are needed to determine $\mathbf{a}$ in Ω_I uniquely. Such ungauged formulations will be used throughout this paper. Coulomb type gauges enforcing the orthogonality of $\mathbf{a}$ to all gradients in Ω_I are straightforward.

It turns out that these two variational formulations possess vastly different properties as far as lumped parameter excitation is concerned. Therefore, we will discuss different kinds of current and voltage excitation separately for the **h**- and **a**-based equations. To begin with, we will ruminate about the meaning of "current" and "voltage" in the next section. Then, we treat different cases of excitations and we will show how to incorporate them into the variational equations (6) and (7).

As soon as a variational formulation is established a conforming Galerkin finite element discretization based on a given mesh of Ω is readily available: $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming edge elements [Hip02a, Sect. 3] can be used to approximate both $\mathbf{a}$ and $\mathbf{h}$ and the corresponding test spaces, whereas piecewise polynomial globally continuous Lagrangian finite elements have to be used for scalar potentials. Skirting issues of boundary fitting and numerical quadrature, this instantly yields systems of equations that represent the discretized problem. In light of this canonical procedure, we will not address discretization below, unless special provisions have to be taken.

2 Fundamentals of Excitation

It is straightforward how to obtain the global current I flowing in a loop of the conductor or through a contact from the eddy current solution: the current through an oriented surface Σ is readily available from

$$I = \int_{\Sigma} \mathbf{j} \cdot \mathbf{n} \, dS \,.$$

Conversely, it is difficult to give a meaning to the concept of voltage. To begin with, it is only meaningful related to an oriented path γ via the formula

$$U_\gamma = \int_{\gamma} \mathbf{e} \cdot d\vec{s} \,. \tag{8}$$

Obviously, U is not only a function of the endpoints of γ, because $\operatorname{\mathbf{curl}} \mathbf{e} \neq 0$. On the other hand, circuit modeling assumes a unique voltage between two nodes. Using (8), coupling of fields and circuits cannot be accomplished.

We recall, that the voltage between two nodes of a circuit measures the difference in potential energy of charge carriers. This suggests that we *define* voltage based on power flux. Seen from the circuit the field domain is a network component with one or more ports. If there is only one port and P denotes the total power fed into the field domain, we can use the formula

$$P := U \cdot I$$

to *define* the voltage drop U between the two contacts. Summing up, we rely on the *conservation of current and power* to establish the coupling of the eddy current model with circuit equations.

If there are several ports, the situation is not so clear, because it will be only the total power consumed in the field region that is accessible. However, it will turn out that the methods devised to realize voltage excitation possess some inherent locality, which makes it easy to relate a voltage to contacts on inductor loops. Using a pragmatic superposition principle, we give a meaning to individual voltages for each of the several ports of the field region.

From (1) we deduce the following power balance, *cf.* Poynting's theorem,

$$P_{mag} + P_{Ohm} = P = P_{\Omega} + P_{\partial\Omega} \tag{9}$$

where

$$P_{mag} := \int_{\Omega} \tfrac{d}{dt}\mathbf{b}\cdot\mathbf{h}\,d\mathbf{x}\,, \qquad P_{Ohm} := \int_{\Omega_C} \sigma\,|\mathbf{e}|^2\,d\mathbf{x}$$

$$P_{\Omega} := -\int_{\Omega} \mathbf{e}\cdot\mathbf{j}_G\,d\mathbf{x}\,, \qquad P_{\partial\Omega} = -\int_{\partial\Omega} (\mathbf{e}\times\mathbf{h})\cdot\mathbf{n}\,dS\,.$$

We see that two types of power sources occur in the eddy current model, which compensate the Ohmic losses P_{Ohm} and the magnetic power P_{mag}, that includes a change in the magnetic energy and magnetic losses: Firstly, sources P_{Ω} due to impressed currents, which push charges against the electric field $\mathbf{e}$. Secondly, sources $P_{\partial\Omega}$ due to boundary conditions. We have chosen the signs in a way that renders $P_{\Omega}, P_{\partial\Omega}$ positive, if the sources fed power into the eddy current model.

Remark 1. *In the frequency domain power balance* (9) *has to be stated as*

$$\tfrac{i\omega}{2}\int_{\Omega} \mu\,|\mathbf{h}|^2\,d\mathbf{x} + \tfrac{1}{2}\int_{\Omega_C} \sigma\,|\mathbf{e}|^2\,d\mathbf{x} = -\tfrac{1}{2}\int_{\Omega} \mathbf{e}\cdot\mathbf{j}_G^*\,d\mathbf{x} - \tfrac{1}{2}\int_{\partial\Omega} (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{n}\,, \tag{10}$$

where all fields are complex amplitudes (phasors) and $*$ labels complex conjugates. The real part of eq. (10) *states the balance of the so called* active power*, which is the mean value of the power with respect to one period that compensates losses. The imaginary part of eq.* (10) *states the balance of the so called* reactive power *which is the maximum value of the change in the magnetic energy.*

Topological preliminaries

Here, we give a cursory description of topological concepts indispensable for the understanding of lumped parameter excitation. More information can be found in [Kot87, Bos98a, Bos00, Ket01] and textbooks about homology theory.

First, we recall the notion of an orientable l-dimensional piecewise (p.w.) smooth manifold in a generic open bounded domain $\Omega \subset \mathbb{R}^3$. For $l = 1$ these are directed paths, for $l = 2$ the reader may think of surfaces equipped with a crossing direction. We appeal to geometric intuiting to introduce the concept of a boundary $\partial\Sigma$ of an l-dimensional piecewise smooth manifold $\Sigma \subset \Omega$. The boundary carries an induced orientation and is itself an $l-1$-dimensional p.w. smooth manifold.

We call an l-dimensional piecewise smooth oriented manifold $\subset \Omega$ an *l-cycle*, $l = 1, 2$, if its boundary is empty. Two l-cycles are said to be homologous, if their union (after a possibly change of orientation) is the boundary of an $l+1$-dimensional manifold, which is a proper subset of Ω. This defines an equivalence relation on the set $Z_l(\Omega)$ of l-cycles. An l-dimensional manifold $\Sigma \subset \Omega$ is called a *relative l-cycle*, if $\partial\Sigma \subset \partial\Omega$. Two relative l-cycles are homologous, if their union supplemented by a part of $\partial\Omega$ bounds an $l+1$-dimensional manifold $\subset \overline{\Omega}$. This introduces an equivalence relation on the set $Z_l(\Omega, \partial\Omega)$ of relative l-cycles.

The equivalence classes of homologous, non-bounding 1-cycles in Ω are called *loops*. There are $\beta_1(\Omega)$ such loops, where $\beta_1(\Omega)$ denotes the first Betti number of Ω, which is a fundamental topological invariant. Crudely speaking, $\beta_1(\Omega)$ is equal to the number of holes piercing Ω. A loop can be described by a representative in the form of a closed non-bounding oriented path. Moreover, by *cuts* of Ω we refer to the homology equivalence classes in $Z_2(\Omega, \partial\Omega)$, which can be represented by $\beta_1(\Omega)$ disjoint p.w. smooth oriented surfaces. These cuts are also known as *Seifert surfaces*. We remark that cuts and loops are dual to each other (Poincaré duality): we can find representatives for loops and cuts such that they form pairs of paths/surfaces that intersect each other.

If $\Sigma_1, \dots, \Sigma_N$, $N := \beta_1(\Omega)$ stands for a complete set of cuts, then all 1-cycles in $\Omega \setminus (\Sigma_1 \cup \ldots \cup \Sigma_N)$ are bounding. In particular, every **curl**-free vectorfield in $\Omega \setminus (\Sigma_1 \cup \ldots \cup \Sigma_N)$ agrees with a gradient of a scalar function.

If we consider a **curl**-free vectorfield in all of Ω, it need not be a gradient. However there is a finite-dimensional co-homology space $\mathcal{H}^1(\Omega) \subset \boldsymbol{H}(\mathbf{curl}; \Omega)$ that satisfies $\dim \mathcal{H}^1(\Omega) = \beta_1(\Omega)$ such that

$$\{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}; \Omega),\ \mathbf{curl}\,\mathbf{u} = 0\} = \mathbf{grad}\, H^1(\Omega) \oplus \mathcal{H}^1(\Omega)\ .$$

3 Excitation by Generator Current Distributions

The generator currents $\mathbf{j}_G$ play the role of right hand sides in the eddy current equations. As such they are natural candidates for realizing an excitation.

Prescribed generator currents $\mathbf{j}_G$ can be put into two different categories:

(a) (see Fig. 1, left) Closed current loops (stranded inductors in the parlance of [Dul01]) in Ω_I, that is $\operatorname{supp} \mathbf{j}_g \subset \Omega_I$ and $\beta_1(\operatorname{supp} \mathbf{j}_g) = 1$, which model coils with known currents.

(b) (see Fig. 1, right) Current sources adjacent to conductors, i.e., $\operatorname{supp} \mathbf{j}_g \cap \overline{\Omega_C} \neq \emptyset$. Note that $\operatorname{supp} \mathbf{j}_g \cup \overline{\Omega}_C$ should have an additional loop compared to $\overline{\Omega}_C$.

In both cases we end up with the same variational formulations.

We mainly restrict ourselves to the case of one port so that we have to take into account either one total current I or one voltage U. In addition, we make the assumption $\operatorname{supp} \mathbf{j}_g \subset \overline{\Omega_I}$ and use homogeneous electric boundary conditions $\mathbf{n} \times \mathbf{e} = 0$ on $\partial\Omega$. The case $\operatorname{supp} \mathbf{j}_G \subset \Omega_C$ is not completely irrelevant, as it represents a source with internal resistivity, but it can be easily modeled on the circuit level.

3.1 h-based formulation

Current excitation

A prescribed total current is implied by the choice of $\mathbf{j}_G$. In the situation (b), see Fig. 1, this fixes the total current in the conducting loop as a consequence of Ampere's law, which is present

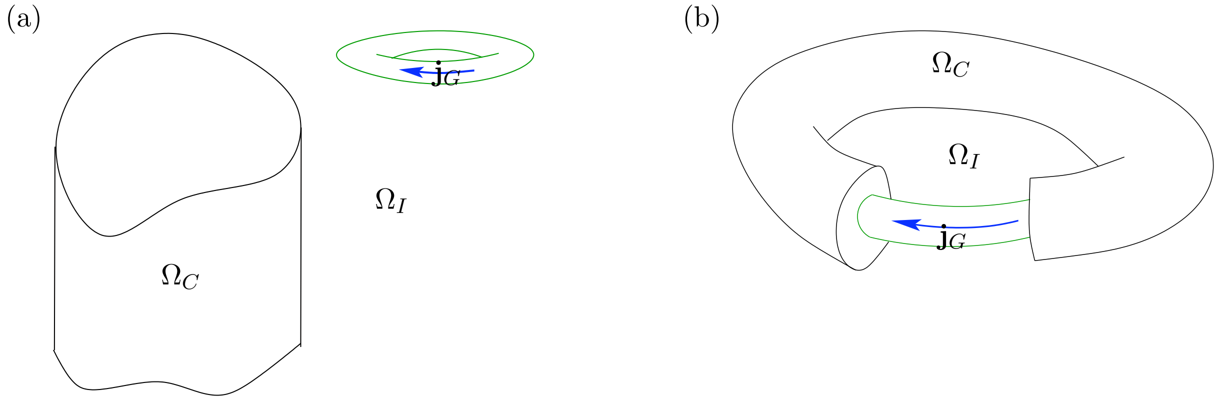

Figure 1: Two typical situations encountered in the case of prescribed $\mathbf{j}_G$: left (a): closed inductor loop detached from $\overline{\Omega_C}$; right (b): current source adjacent to $\overline{\Omega_C}$

.

in the variational formulation in strong form.

Choose $\mathbf{h}_G \in C^1([0,T], \boldsymbol{H}(\mathbf{curl};\Omega))$ such that $\mathbf{curl}\,\mathbf{h}_G = \mathbf{j}_G$ in Ω_I for all times. In situation (a) we could use the Biot-Savart law to compute such a $\mathbf{h}_G$, but for practical computations the support of $\mathbf{h}_G$ should be as small as possible, which can be achieved by techniques, presented, for instance, in [DHR+97]. Please note that Ampere's law usually rules out $\operatorname{supp}\mathbf{h}_G = \operatorname{supp}\mathbf{j}_G$.

Given $\mathbf{h}_G$ we get:

Seek $\mathbf{h} \in \mathbf{h}_G + C^1([0,T], \boldsymbol{\mathcal{V}}_0)$[2] such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$

$$\int\limits_{\Omega_C} \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h}\cdot\mathbf{curl}\,\mathbf{h}'\,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}'\,d\mathbf{x} = 0\,. \tag{11}$$

Obviously the choice of $\mathbf{h}_G$ has no impact on the solution $\mathbf{h}$, because for two different excitation fields $\mathbf{h}_G$ and $\mathbf{h}'_G$ we have $\mathbf{h}_G - \mathbf{h}'_G \in \boldsymbol{\mathcal{V}}_0$.

As explained above, the voltage is defined through power. According to (9) the power injected into the eddy current domain is equal to

$$\begin{aligned} P &= \int\limits_{\Omega_C} \frac{1}{\sigma}|\,\mathbf{curl}\,\mathbf{h}|^2\,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}\,d\mathbf{x} \\ &= \int\limits_{\Omega_C} \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h}\cdot\mathbf{curl}(\mathbf{h}_G+\mathbf{h}')\,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot(\mathbf{h}_G+\mathbf{h}')\,d\mathbf{x} \\ &= \int\limits_{\Omega_C} \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h}\cdot\mathbf{j}_G\,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}_G\,d\mathbf{x} = \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}_G\,d\mathbf{x}\,, \end{aligned}$$

where $\mathbf{h}' := \mathbf{h} - \mathbf{h}_G$. The bottom equality is justified, because we assume $\operatorname{supp}\mathbf{j}_G \cap \Omega_C = \emptyset$.

Thus, the voltage is given by

$$U \cdot I = \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}_G\,d\mathbf{x}\,. \tag{12}$$

[2]For the sake of brevity initial conditions will be dropped in the sequel.

Let Σ be a cut associated with a loop in the conductor in the sense of Poincaré duality. Using a normalized generator current scaled by the yet unknown total current

$$\mathbf{j}_G = I\,\mathbf{j}_0\,, \qquad \int\limits_{\Sigma} \mathbf{j}_0 \cdot \mathbf{n}\, dS = 1\,, \tag{13}$$

we can write (12)

$$U = \int\limits_{\Omega} \tfrac{d}{dt}(\mu \mathbf{h}) \cdot \mathbf{h}_0\, d\mathbf{x}\,, \tag{14}$$

provided that $\mathbf{curl}\,\mathbf{h}_0 = \mathbf{j}_0$. Here, the normal of Σ has to be oriented in such a way that we get $I\,U > 0$ if power is fed into the eddy current model.

Remark 2. *In the case of multiple ports, we assume that the associated generator currents have disjoint supports. Thus,* (14) *gives a voltage for each port.*

Voltage Excitation

Voltage excitation can be achieved by imposing (14) as constraint and using the total current as Lagrangian multiplier, which yields the variational formulation:

Seek $\mathbf{h} \in C^1([0,T], \boldsymbol{\mathcal{V}}_0)$ and $I \in C^1([0,T])$ such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$

$$\begin{aligned}
\int\limits_{\Omega_C} \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h} \cdot \mathbf{curl}\,\mathbf{h}'\, d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu(\mathbf{h} + I\,\mathbf{h}_0)) \cdot \mathbf{h}'\, d\mathbf{x} &= 0 \\
\int\limits_{\Omega} \tfrac{d}{dt}(\mu \mathbf{h}) \cdot \mathbf{h}_0\, d\mathbf{x} \qquad &= U\,.
\end{aligned} \tag{15}$$

Here $\mathbf{h}_0 \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ has to satisfy $\mathbf{curl}\,\mathbf{h}_0 = \mathbf{j}_0$ with $\mathbf{j}_0$ as in (13). In the case of multiple ports localization is performed as in Rem. 2 above.

3.2 $\mathbf{a}$-based formulation

Current Excitation

As in the case of the $\mathbf{h}$-based formulations, the total current I is fixed by specifying $\mathbf{j}_G$. In situation (b), see Fig. 1, Ampere's law, which is weakly incorporated into the $\mathbf{a}$-based formulation, ensures that the total current I will flow in the entire conducting loop. The ungauged variational equation is straightforward:

Seek $\mathbf{a} \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\int\limits_{\Omega} \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\mathbf{a}'\, d\mathbf{x} + \int\limits_{\Omega_C} \sigma\, \tfrac{d}{dt}\mathbf{a} \cdot \mathbf{a}'\, d\mathbf{x} = \int\limits_{\Omega} \mathbf{j}_G \cdot \mathbf{a}'\, d\mathbf{x}\,. \tag{16}$$

Again, we use the power

$$P = \int\limits_{\Omega} \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\tfrac{d}{dt}\mathbf{a}\, d\mathbf{x} + \int\limits_{\Omega_C} \sigma\, \tfrac{d}{dt}\mathbf{a} \cdot \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x} = \int\limits_{\Omega} \mathbf{j}_G \cdot \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x} = I \int\limits_{\Omega} \mathbf{j}_0 \cdot \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x}$$

to define the voltage

$$U = P/I = \int\limits_{\Omega} \mathbf{j}_0 \cdot \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x}\,. \tag{17}$$

Voltage Excitation

Again, a constraint equation can be used to impose a voltage:

Seek $\mathbf{a} \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ and $I \in C^1([0,T])$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\begin{aligned} \int_\Omega \frac{1}{\mu} \mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\mathbf{a}' \,d\mathbf{x} + \int_{\Omega_C} \sigma \tfrac{d}{dt}\mathbf{a} \cdot \mathbf{a}' \,d\mathbf{x} - I \int_\Omega \mathbf{j}_0 \cdot \mathbf{a}' \,d\mathbf{x} &= 0 \\ \int_\Omega \mathbf{j}_0 \cdot \tfrac{d}{dt}\mathbf{a} \,d\mathbf{x} &= U\,. \end{aligned} \tag{18}$$

4 Excitation through Boundary Conditions (Contacts at $\partial\Omega$)

Another kind of excitation is supplied by currents imposed at contacts $\Sigma = \overline{\Omega_C} \cap \partial\Omega$ on the boundary $\partial\Omega$. For the sake of simplicity we will only consider two contacts $\Sigma = \Sigma^+ \cup \Sigma^-$ on $\partial\Omega$, as sketched in Fig. 2. Two situations can be distinguished.

(a) The contacts are located where Ω_C meets the exterior artificial boundary, see Fig. 2, left.

(b) The contacts Σ^+, Σ^- and Θ bound a volume Ω^*, the electromotive region, in which the laws of electrodynamics are not in effect, *cf.* [Dul01, Sect. 3.3]. Adding Ω^* to Ω_C creates a new loop and both Σ^+ and Σ^- are cuts for this loop, which are supposed to be disjoint. In particular $\Sigma^+ \cup \Sigma^-$ is the relative boundary of Ω^* in $\overline{\Omega}_C \cup \Omega^*$.

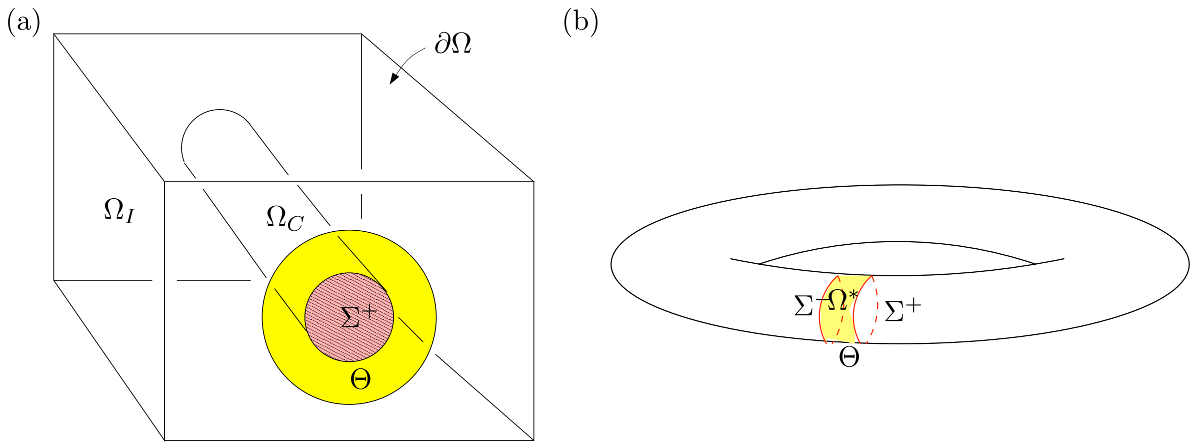

Figure 2: Situation (a): excitation through "exterior" boundary conditions. Situation (b) Boundary conditions on surface of "hole in the universe" Ω^*, $\Omega = \mathbb{R}^3 \setminus \Omega^*, \Theta = \partial\Omega^* \cap \partial\Omega_I$

4.1 h-based formulation

Voltage excitation

Voltage excitation is realized by means of boundary conditions for the electric field

$$\mathbf{n} \times \mathbf{e} = 0 \qquad \text{on } \partial\Omega \setminus \Theta \tag{19}$$

$$\mathbf{n} \times \mathbf{e} = -U(t)\,\mathbf{grad}_\Gamma\, v \qquad \text{on } \Theta\,, \tag{20}$$

where

$$v|_{\Sigma^+} = 1\,, \quad v|_{\partial\Omega\setminus(\overline{\Theta\cup\Sigma^+})} = 0\,, \quad v \in H^{\frac{1}{2}}(\partial\Omega)\,. \tag{21}$$

Hence, the zone $\Theta \subset (\partial\Omega \cap \overline{\Omega_I})$ is the surface through which power is injected. Incorporating conditions (19)–(21) into the boundary term in (6) yields

$$\int_{\partial\Omega} (\mathbf{n}\times\mathbf{e})\cdot\mathbf{h}'\,dS = U\int_{\partial\Omega} \mathbf{grad}_\Gamma\, v\cdot(\mathbf{n}\times\mathbf{h}')\,dS =$$

$$= U\int_{\Theta} v\,\mathbf{curl}\,\mathbf{h}'\cdot\mathbf{n}\,dS + U\int_{\partial\Theta} v\,\mathbf{h}'\cdot d\vec{s} = U\int_{\gamma^+} \mathbf{h}'\cdot d\vec{s}\,,$$

where $\gamma^+ = \partial\Sigma^+$. Here we have integration by parts on the surface. The variational formulation taking into account voltage excitation reads:

Seek $\mathbf{h} \in C^1([0,T], \boldsymbol{\mathcal{V}}_0)$ such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$

$$\int_{\Omega_C} \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h}\cdot\mathbf{curl}\,\mathbf{h}'\,d\mathbf{x} + \int_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h})\cdot\mathbf{h}'\,d\mathbf{x} = U\int_{\gamma^+} \mathbf{h}'\cdot d\vec{s}\,. \tag{22}$$

Note that by Ampere's law we have

$$I = -\int_{\gamma^+} \mathbf{h}\cdot d\vec{s}\,,$$

as the conductor is penetrating the loop γ^+. Moreover, $\gamma^+ \subset \overline{\Omega_I}$ and $\mathbf{curl}\,\mathbf{h}' = 0$ in Ω_I, which renders $\mathbf{h}' \mapsto \int_{\gamma^+} \mathbf{h}'\cdot d\vec{s}$ a continuous functional on $\boldsymbol{\mathcal{V}}_0$.

Now, replacing $\mathbf{h}'$ in (22) with $\mathbf{h}$ reveals that $P = U\,I$. This means that the voltage U introduced via the electric boundary conditions (19) matches the definition of voltage based on power, see Sect. 2.

Remark 3. *The right hand side of* (22) *is of the form $U\cdot f(\mathbf{h}')$, where f is a continuous functional on $\boldsymbol{\mathcal{V}}_0$ measuring the total current through a contact.*

Remark 4. *Note that in situation (a) of Fig. 2 the geometry of Θ does not enter the variational formulation at all: it has no impact on $\mathbf{h}$. Yet, the electric field $\mathbf{e}$ in Ω_I will depend on Θ.*

Another observation is that in situation (b), cf. Fig. 2, the path γ^+ can be any closed path winding around that loop of the conductor that is cut by Σ^+/Σ^-. In a sense, the excitation possesses a non-local character. On the other hand, the location of the contacts still enters the variational formulation via the domain Ω.

Remark 5. *Consider situation (b) and assume that Σ^+ and Σ^- are parallel with distance δ. We point out that a "limit" of the magnetic field when $\delta \to 0$ will not agree with the solution of the variational problem when Ω^* is ignored altogether. To see this note that even for a general geometry of the loop there is never a magnetic flux through $\Sigma^\pm$, no matter how close they are. It is also worth noting that the energy of the electric field solution will blow up as $\delta \to 0$.*

Remark 6. *The generalization of* (22) *to the multiport setting is obvious, because all occurring surfaces and paths are clearly associated with a particular contacts. In situation (a) one contact would be assigned the role of ground and the others would be treated like Σ^+ in the previous considerations.*

Remark 7. *As has become clear, topological concepts pervade the discussion. A comprehensive discussion of excitation through boundary conditions from a topological point of view is given in the landmark paper [Bos00]. It is worth mentioning that also in this paper a consistent energy balance is a key point.*

Current excitation

In the situations depicted in Fig. 2 a total current $I \in C^1([0,T])$ is imposed by prescribing the normal component of a suitable current density $\mathbf{j}$ on the contacts

$$I_+ = \int\limits_{\Sigma^+} \mathbf{j} \cdot \mathbf{n} \, dS \quad , \quad I_- = \int\limits_{\Sigma^-} \mathbf{j} \cdot \mathbf{n} \, dS \, .$$

Charge conservation requires

$$I_+ = -I_- \, .$$

The orientation of Σ_+ and Σ_- is induced by the orientation of $\partial\Omega$. To obtain $I\,U > 0$ if power is fed into the eddy current model we set

$$I := I_- = -I_+ \, .$$

Furthermore, we chose $\mathbf{h}_{j_n} \in C^1([0,T], \boldsymbol{H}(\mathbf{curl};\Omega))$ such that

$$\mathrm{div}_\Gamma(\mathbf{h}_{j_n} \times \mathbf{n}) = \mathbf{curl}\, \mathbf{h}_{j_n} \cdot \mathbf{n} = (\mathbf{j} \cdot \mathbf{n})/I \text{ on } \partial\Omega, \quad \mathbf{curl}\, \mathbf{h}_{j_n} = 0 \text{ in } \Omega_I$$

and define

$$\boldsymbol{\mathcal{V}}_0^+ := \{\mathbf{h}' \in \boldsymbol{H}(\mathbf{curl};\Omega);\ \mathbf{curl}\, \mathbf{h}' = 0 \text{ in } \Omega_I,\ \mathrm{div}_\Gamma(\mathbf{h}' \times \mathbf{n}) = 0 \text{ on } \partial\Omega\} \, .$$

This leads to the following variational formulation:

Seek $\mathbf{h} \in I\, \mathbf{h}_{j_n} + C^1([0,T], \boldsymbol{\mathcal{V}}_0^+)$ such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0^+$

$$\int\limits_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \mathbf{h} \cdot \mathbf{curl}\, \mathbf{h}' \, d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu \mathbf{h}) \cdot \mathbf{h}' \, d\mathbf{x} = 0 \, . \tag{23}$$

Note that power is injected through $\Theta \cup \Sigma$.

Besides the prescribed normal components of the current density on the contact, the variational formulation implies the boundary condition

$$\tfrac{d}{dt}\mathbf{b} \cdot \mathbf{n} = 0 \, .$$

This can be seen by testing (23) with gradients.

Parallel to (14) voltage has to be defined through power. Thus, we get

$$U = \frac{P}{I} = \int\limits_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \mathbf{h} \cdot \mathbf{curl}\, \mathbf{h}_{j_n} d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu \mathbf{h}) \cdot \mathbf{h}_{j_n} d\mathbf{x} \, . \tag{24}$$

Remark 8. *Another option for enforcing a particular total current through the contacts is by means of a constraint. This approach could be pursued in the case of the* **h***-based formulation, but we will elucidate it for the* **a***-based scheme in the next section.*

4.2 a-based formulation

Voltage excitation

Again, we use the electric boundary conditions from (19) based on the potential v from (21). We denote by $\widetilde{v}$ a $H^1(\Omega)$-extension of v and point out that $\mathbf{grad}\,\widetilde{v} \notin \boldsymbol{H}_0(\mathbf{curl};\Omega)$. These boundary conditions lead to the following variational formulation:

Seek $\mathbf{a} \in \int U\,dt\,\mathbf{grad}\,\widetilde{v} + C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\mathbf{a}'\,d\mathbf{x} + \int_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a} \cdot \mathbf{a}'\,d\mathbf{x} = 0\,. \tag{25}$$

Looking at the power balance, we find that the voltage satisfies $P = U\,I$:

$$\begin{aligned} P &= \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \tfrac{d}{dt}\,\mathbf{curl}\,\mathbf{a}\,d\mathbf{x} + \int_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a} \cdot \tfrac{d}{dt}\mathbf{a}\,d\mathbf{x} \\ &= U\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\mathbf{grad}\,\widetilde{v}\,d\mathbf{x} + U\int_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a} \cdot \mathbf{grad}\,\widetilde{v}\,d\mathbf{x} = U\int_{\Sigma^+} \sigma\,\tfrac{d}{dt}\mathbf{a} \cdot \mathbf{n}\,dS = U\,I\,, \end{aligned}$$

because the functional $F(\mathbf{a}) = \int_{\Sigma^+} \sigma\,\frac{d}{dt}\mathbf{a} \cdot \mathbf{n}\,dS = I$ actually provides the total current through the contact Σ^+, *cf.* Rem. 3.

Current Excitation

We first study prescribed normal components $\mathbf{j} \cdot \mathbf{n} = j_n$ of the current density at contacts. We introduce the space

$$\boldsymbol{\mathcal{W}}_0^+ := \{\mathbf{a}' \in \boldsymbol{H}(\mathbf{curl};\Omega);\ \mathrm{div}_\Gamma(\mathbf{a}' \times \mathbf{n}) = 0 \text{ on } \partial\Omega\}\,.$$

Then, any $\mathbf{a}' \in \boldsymbol{\mathcal{W}}_0^+$ can be written as

$$\mathbf{a}' = \mathbf{a}_0' + \mathbf{grad}\,v' \text{ with } \mathbf{a}_0' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\,, v' \in H^1(\Omega)\,, \tag{26}$$

since $\partial\Omega$ has been assumed to be simply connected. In contrast to the $\mathbf{h}$-based formulation, the boundary condition $\mathbf{b} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is incorporated strongly by virtue of the construction of $\boldsymbol{\mathcal{W}}_0^+$, because $\mathbf{curl}\,\mathbf{a} = \mathbf{b}$. On the other hand the normal component $j_n = \mathbf{j} \cdot \mathbf{n}$ of the current density is imposed only weakly: using (26), the surface integral on the right hand side of the variational formulation (7) can be expressed by

$$-\int_{\partial\Omega} (\mathbf{n} \times \mathbf{h}) \cdot \mathbf{a}'\,dS = \int_{\partial\Omega} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{grad}_\Gamma\,v'\,dS = \int_{\partial\Omega} v'\mathrm{div}_\Gamma(\mathbf{h} \times \mathbf{n})\,dS = \int_{\partial\Omega} v'\,j_n\,dS\,.$$

Hence, an $\mathbf{a}$-based variational formulation using prescribed $\mathbf{j}\cdot\mathbf{n}$ to effect current excitation reads:

Seek $\mathbf{a} \in C^1([0,T], \boldsymbol{\mathcal{W}}_0^+)$ such that for all $\mathbf{a}_0' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and all $v' \in H^1(\Omega)$

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a} \cdot \mathbf{curl}\,\mathbf{a}'\,d\mathbf{x} + \int_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a} \cdot \mathbf{a}'\,d\mathbf{x} = \int_{\partial\Omega} v'\,j_n\,dS\,. \tag{27}$$

In the spirit of Sect. 2 voltage is defined through power. To this end, consider a splitting of the solution of (27) $\mathbf{a} = \mathbf{a}_0 + \mathbf{grad}\, v$ according to (26). This will be plugged into the expression for the power.

$$P = \int\limits_{\Omega} \frac{1}{\mu} \mathbf{curl}\, \mathbf{a} \cdot \mathbf{curl}\, \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x} + \int\limits_{\Omega_C} \sigma\, \tfrac{d}{dt}\mathbf{a} \cdot \tfrac{d}{dt}\mathbf{a}\, d\mathbf{x} = \int\limits_{\partial\Omega} v\, j_n\, dS\,,$$

which means

$$U = \frac{P}{I} = \frac{\int\limits_{\partial\Omega} v\, j_n\, dS}{I}\,.$$

As far as finite element discretization of (27) is concerned, we recommend that the decomposition (26) is taken into account by discretizing $\mathbf{a}_0'$ by means of edge elements in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, whereas v' is approximated in the space of continuous scalar shape functions associated with nodes on the boundary. This will yield a direct splitting in the semi-discrete setting.

Second, we consider a contact touching an exterior PEC boundary and want to prescribe a total current I. Temporarily, we abandon the temporal gauge and set for the electric field

$$\mathbf{e} = -\tfrac{d}{dt}\mathbf{a} - U\, \mathbf{grad}\, \widetilde{v}\,,$$

where $\widetilde{v}$ is an arbitrary $H^1(\Omega)$-extension of $v \in H^{\frac{1}{2}}(\partial\Omega)$. As in the case of voltage excitation, we demand that $v = 1$ on Σ^+, $v = 0$ on Σ^-, and $\mathbf{a} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$. This means that the current through a suitably oriented cut Σ' is given by

$$I = \int\limits_{\Sigma'} \mathbf{j} \cdot \mathbf{n}\, dS = -\int\limits_{\Sigma^+} \widetilde{v}\, \mathbf{j} \cdot \mathbf{n}\, dS\,.$$

Since

$$\mathbf{j} \cdot \mathbf{n} = 0 \text{ on } \Gamma_C \quad \text{and} \quad \operatorname{div} \mathbf{j} = 0 \text{ in } \Omega_C$$

holds in a weak sense, we arrive at

$$I = -\int\limits_{\partial\Omega_C} \widetilde{v}\, \mathbf{j} \cdot \mathbf{n}\, dS = -\int\limits_{\Omega_C} \mathbf{j} \cdot \mathbf{grad}\, \widetilde{v}\, d\mathbf{x} = \int\limits_{\Omega_C} \sigma(\tfrac{d}{dt}\mathbf{a} + U\, \mathbf{grad}\, \widetilde{v}) \cdot \mathbf{grad}\, \widetilde{v}\, d\mathbf{x}\,.$$

Using this expression for the current, an $\mathbf{a}$-based variational formulation with current excitation can be recast into a formulation with unknown voltage U and a constraint enforcing the current:

Seek $\mathbf{a} \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ and $U \in C^1([0,T])$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\begin{aligned} \int\limits_{\Omega} \frac{1}{\mu} \mathbf{curl}\, \mathbf{a} \cdot \mathbf{curl}\, \mathbf{a}'\, d\mathbf{x} + \int\limits_{\Omega_C} \sigma\, \tfrac{d}{dt}\mathbf{a} \cdot \mathbf{a}'\, d\mathbf{x} + U \int\limits_{\Omega_C} \sigma\, \mathbf{grad}\, \widetilde{v} \cdot \mathbf{a}'\, d\mathbf{x} &= 0 \\ \int\limits_{\Omega_C} \sigma\, \tfrac{d}{dt}\mathbf{a} \cdot \mathbf{grad}\, \widetilde{v}\, d\mathbf{x} \qquad + U \int\limits_{\Omega_C} \sigma\, |\mathbf{grad}\, \widetilde{v}|^2\, d\mathbf{x} &= I\,. \end{aligned} \tag{28}$$

Checking the power balance confirms that U gives the voltage as defined in Sect. 2.

Theorem 9. *For the variational problems* (25) *and* (28) *we find* $\mathbf{b} = \mathbf{curl}\, \mathbf{a}$ *and, thus,* $\mathbf{e}|_{\Omega_C}$ *independently of the choice of* $\widetilde{v}$.

Proof. Plug $\widetilde{v} = \widetilde{v_1}$ and $\widetilde{v} = \widetilde{v_2}$ into (25) and (28), respectively, and set

$$\delta v := \int U\, dt\, (\widetilde{v_1} - \widetilde{v_2}) \quad \text{in } \Omega_C\,.$$

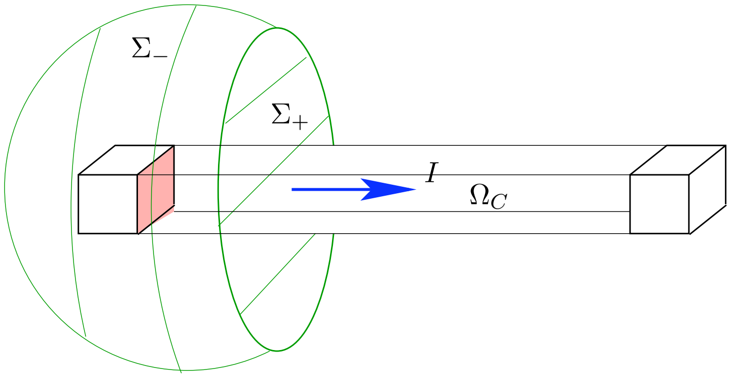

Figure 3: A nonzero current through a separated contact violates Ampere's law.

Then there is $\delta\mathbf{a} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, such that

$$\delta\mathbf{a}|_{\Omega_C} = \mathbf{grad}\,\delta v\,.$$

As $\delta v|_{\Sigma^+} = \delta v|_{\Sigma^-} = 0$, we find an $H^1(\Omega)$-extension $\widetilde{\delta v}$ of $\delta v|_{\Omega_C}$. Setting

$$\delta\mathbf{a} := \mathbf{grad}\,\widetilde{\delta v}$$

we can conclude: if $\mathbf{a}_1$ is a solution belonging to $\widetilde{v}_1$, then $\mathbf{a}_2 = \mathbf{a}_1 + \delta\mathbf{a}$ is a solution for the choice $\widetilde{v}_2$. This shows

$$\mathbf{curl}\,\mathbf{a}_1 = \mathbf{curl}\,\mathbf{a}_2\,.$$

$\square$

Remark 10. *It is tempting to remove the contacts from $\partial\Omega$ and put "contact layers" on parts of $\partial\Omega_C$. However, the eddy current model cannot accommodate a current flowing out of Ω_C into Ω_I as in Fig. 3. The reason is that this will violate Ampere's law: The assumption $I > 0$ leads to a contradiction as can be seen by integrating* (1b) *over Σ^+. In short, there is no meaningful way to take into account separate contacts inside Ω in eddy current computations.*

$$0 \neq I = \int_{\Sigma_+} \mathbf{j}\cdot\mathbf{n}\,dS = \int_{\partial\Sigma^+} \mathbf{h}\cdot d\vec{s} = \int_{\partial\Sigma^-} \mathbf{h}\cdot d\vec{s} = \int_{\Sigma_-} \mathbf{j}\cdot\mathbf{n}\,dS = 0 \qquad \square$$

5 Non-Local Excitations

In the case of eddy current simulation it is not unusual that the geometric CAD model does not contain information on the location of contacts on the surface of massive inductors. Nevertheless, there are ways to impose total currents and voltages.

5.1 h-based formulation

Voltage Excitation

Examining voltage excitation in Sect. 4.1 we saw, that in situation (a) (contacts on $\partial\Omega$, see Fig. 2) the magnetic field solution $\mathbf{h}$ turned out to be independent of the location of the surface

Θ through which exciting power was supplied (see Fig. 2). Also in situation (b) excitation gave rise to a right hand side functional that did no longer contain geometric information about the electromotive region Ω^*.

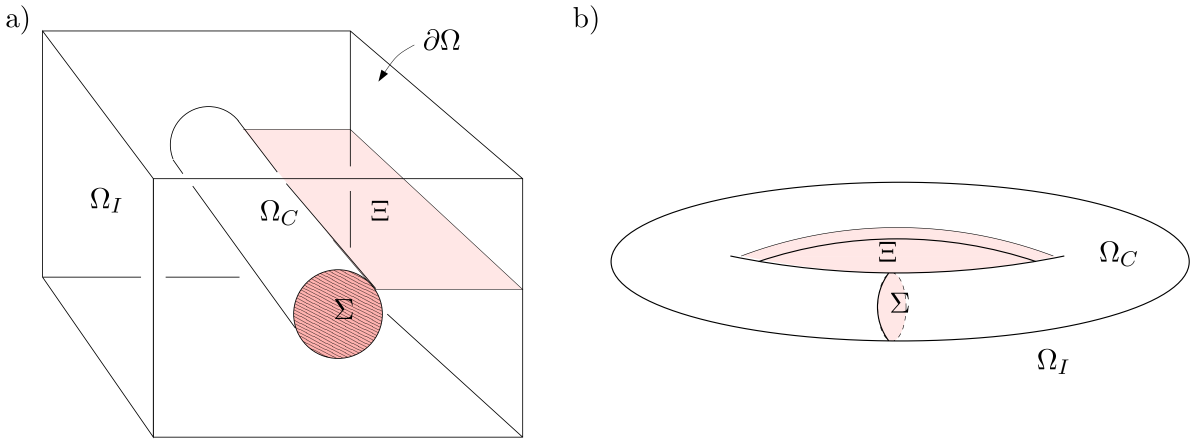

Figure 4: Non-local excitation: (a) Conductor touching $\partial\Omega$. Cutting surface Ξ in Ω_I is depicted. (b) Conducting loop away from $\partial\Omega$, closed by Seifert surface Ξ in Ω_I and cut by surface Σ inside Ω_C.

This suggests that in both situations (a) and (b) Θ is completely removed from the variational formulation (22). In situation (b) of Fig. 2 this means that we incorporate Ω^* into Ω_C, that is, we alter the computational domain Ω (see Fig. 4). Thus, we arrive at the following variational formulation that is compliant with Rem. 3.

Seek $\mathbf{h} \in C^1([0,T], \boldsymbol{\mathcal{V}}_0)$ such that for all $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$

$$\int\limits_{\Omega_C} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \mathbf{h} \cdot \operatorname{\mathbf{curl}} \mathbf{h}' \, d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu \mathbf{h}) \cdot \mathbf{h}' \, d\mathbf{x} = U \int\limits_{\gamma^+} \mathbf{h}' \cdot d\vec{s}. \tag{29}$$

Note that U plays the role of a circulation voltage, as will be explained in Sect. 5.3. An extension to the multiport case is straightforward, if ports are associated with loops of Ω_C.

Current excitation

The constraint that $\operatorname{\mathbf{curl}} \mathbf{h}' = 0$ in Ω_I built into the space $\boldsymbol{\mathcal{V}}_0$ defined in (5) can be taken into account by means of scalar potentials together with so-called co-homology vector fields:

$$\boldsymbol{\mathcal{V}}_0 \ni \mathbf{h}'|_{\Omega_I} = \operatorname{\mathbf{grad}} \phi + \sum_{i=1}^{L} \mathbf{q}_i \,, \qquad \phi \in H^1(\Omega_I)\,, \tag{30}$$

where $\mathbf{q}_i$ is a basis of the first co-homology space $\mathcal{H}^1(\Omega_I)$, and L stands for the first Betti number of Ω_I, see Sect. 2. Using the Seifert surfaces of Ω_I we can easily construct the fields $\mathbf{q}_i$ [ABDG98]. For the situation illustrated in Fig. 4, that is, for $L = 1$, a possible co-homology field $\mathbf{q} := \mathbf{q}_1$ can be obtained as the (generalized) gradient of a function in $H^1(\Omega_I \setminus \Sigma)$ that has a jump of height 1 across the cut Ξ

$$\mathbf{q} := \widetilde{\operatorname{\mathbf{grad}}}\, \theta \,, \qquad \theta \in H^1(\Omega_I \setminus \Xi)\,, \qquad [\theta]_\Xi = 1\,. \tag{31}$$

Let $\widetilde{\mathbf{q}}$ be an extension of $\mathbf{q} \in \boldsymbol{H}(\mathbf{curl};\Omega_I)$ to a function in $\boldsymbol{H}(\mathbf{curl};\Omega)$. Then, based on the variational formulation (6), the total current through Σ^+ can be prescribed by fixing the contribution to $\boldsymbol{\mathcal{V}}_0$ from $\widetilde{\mathbf{q}}$. Accordingly, we define

$$\widetilde{\boldsymbol{\mathcal{V}}}_0 := \{\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0 \,, \int\limits_{\gamma} \mathbf{h}' \cdot d\vec{s} = 0\} \,,$$

where $\gamma := \partial\Sigma$ (see Fig. 4). Then we get the following variational formulation that includes non-local current excitation by a temporally varying total current I:

Seek $\mathbf{h} \in I\widetilde{\mathbf{q}} + C^1([0,T], \widetilde{\boldsymbol{\mathcal{V}}}_0)$ such that for all $\mathbf{h}' \in \widetilde{\boldsymbol{\mathcal{V}}}_0$

$$\int\limits_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\,\mathbf{h} \cdot \mathbf{curl}\,\mathbf{h}' \,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h}) \cdot \mathbf{h}' \,d\mathbf{x} = 0 \,. \tag{32}$$

Recovery of voltages can formally be done according to (24), where $\mathbf{h}_{j_n}$ has to be replaced with the extended co-homology vector field $\widetilde{\mathbf{q}}$, that is,

$$U = \frac{P}{I} = \int\limits_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\,\mathbf{h} \cdot \mathbf{curl}\,\widetilde{\mathbf{q}} \,d\mathbf{x} + \int\limits_{\Omega} \tfrac{d}{dt}(\mu\mathbf{h}) \cdot \widetilde{\mathbf{q}} \,d\mathbf{x} \,. \tag{33}$$

Again, U plays the role of a circulation voltage, see 5.3.

If a current should be imposed on more than one loop, the contributions of other basis functions of the co-homology space to $\mathbf{h}$ have to be fixed. Assignment of voltages to loop is straightforward then by plugging different extended co-homology vector fields into (33).

Inconsistencies in the case of non-local excitations

In Sect. 2 we pointed out that power can be delivered to the eddy current domain either by a generator current $\mathbf{j}_G$ or by non-homogeneous boundary conditions on $\partial\Omega$. In (29) and (32) none of these possibilities is implemented. The question is where the power necessary to sustain currents in Ω_C comes from. This motivates some doubts on the physical relevance of non-local excitation.

It turns out that the variational problems (29) and (32) are not completely consistent with the eddy current model: they do not allow the recovery of a valid electric field in Ω_I that fits $\mathbf{h}$. In other words, there is no electric field $\mathbf{e}$ that solves

$$\mathbf{n} \times \mathbf{e} = \mathbf{n} \times \mathbf{e}_C \qquad \text{on } \Gamma_C \,, \tag{34a}$$

$$\mathbf{n} \times \mathbf{e} = 0 \qquad \text{on } \partial\Omega \setminus \Gamma_C \quad \text{(only situation (a))} \,, \tag{34b}$$

$$\mathbf{curl}\,\mathbf{e} = -\tfrac{d}{dt}(\mu\,\mathbf{h}) \qquad \text{in } \Omega_I \,, \tag{34c}$$

$$\operatorname{div}(\epsilon\,\mathbf{e}) = 0 \qquad \text{in } \Omega_I \,, \tag{34d}$$

$$\int\limits_{\Gamma_i} \epsilon\,\mathbf{e} \cdot \mathbf{n}\,dS = 0 \,, \tag{34e}$$

where $\mathbf{e}_C = \frac{1}{\sigma}\mathbf{curl}\,\mathbf{h}$ in Ω_C. The reason is that, by Stokes' theorem applied to $\partial\Xi$, the right hand sides in (34a), (34b), and (34c) have to satisfy the compatibility condition

$$\int\limits_{\partial\Xi} \mathbf{e}_C \cdot d\vec{s} = -\int\limits_{\Xi} \tfrac{d}{dt}(\mu\,\mathbf{h}) \cdot \mathbf{n}\,dS \,. \tag{35}$$

This is Faraday's law on the Seifert surface Ξ. However, the fields used in the variational formulation are not regular enough ($\mathbf{e}$ is merely in $\boldsymbol{L}^2(\Omega)$) to allow the evaluation of the path integral (35). Hence, we have to consider (35) in weak form (see [AFV01]):

$$\int\limits_{\partial\Omega_C} \frac{1}{\sigma}(\mathbf{curl}\,\mathbf{h}\times\mathbf{n})\cdot\widetilde{\mathbf{q}}\,dS + \int\limits_{\Omega_I} \tfrac{d}{dt}(\mu\,\mathbf{h})\cdot\widetilde{\mathbf{q}}\,d\mathbf{x} = 0 \tag{36}$$

Theorem 11. *Solutions of* (29) *or* (32) *violate* (36).

Proof. Testing (29) with $\mathbf{h}' \in \boldsymbol{\mathcal{V}}_0$ and (32) with $\mathbf{h}' \in \widetilde{\boldsymbol{\mathcal{V}}}_0$, respectively, where

$$\mathbf{h}' = \begin{cases} \mathbf{h}'|_{\Omega_C} \in \boldsymbol{C}_0^\infty(\Omega_C) \\ \mathbf{h}'|_{\Omega_I} \equiv 0 \end{cases} ,$$

we see that every solution $\mathbf{h}$ satisfies

$$\mathbf{curl}\,\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{h} + \tfrac{d}{dt}(\mu\mathbf{h}) = 0 \quad \text{in } \Omega_C$$

in the sense of distributions. Integration by parts applied to (32) yields

$$\int\limits_{\partial\Omega_C} \frac{1}{\sigma}(\mathbf{curl}\,\mathbf{h}\times\mathbf{n})\cdot\mathbf{h}'\,dS + \int\limits_{\Omega_I} \tfrac{d}{dt}(\mu\,\mathbf{h})\cdot\mathbf{h}'\,d\mathbf{x} = 0\,.$$

Since $\widetilde{\mathbf{q}} \notin \widetilde{\boldsymbol{\mathcal{V}}}_0$ the equality (36) is not implied! For (29) we may use $\mathbf{h}' = \widetilde{\mathbf{q}} \in \boldsymbol{\mathcal{V}}_0$ and obtain

$$\int\limits_{\partial\Omega_C} \frac{1}{\sigma}(\mathbf{curl}\,\mathbf{h}\times\mathbf{n})\cdot\widetilde{\mathbf{q}}\,dS + \int\limits_{\Omega_I} \tfrac{d}{dt}(\mu\,\mathbf{h})\cdot\widetilde{\mathbf{q}}\,d\mathbf{x} = U \neq 0\,,$$

which contradicts (36). $\square$

The bottom line is that we have to sacrifice some of the conditions in (34) in order to obtain a candidate for the "electric field" in Ω_I. For instance, we can give up the continuity $[\mathbf{n}\times\mathbf{e}]_{\Gamma_C} = 0$, which means that we condone $\mathbf{e} \notin \boldsymbol{H}(\mathbf{curl};\Omega)$.

If we do so, it will become clear where the energy comes from that feeds the eddy current problem: The jump $[\mathbf{n}\times\mathbf{e}]_{\Gamma_C}$ results in a jump of the normal component of the Poynting vector $S := \mathbf{e}\times\mathbf{h}$ along Γ_C. The right hand side of the power balance (9) has to be augmented by the term

$$P_{\Gamma_C} := \int\limits_{\Gamma_C} \mathbf{h}\cdot[\mathbf{n}\times\mathbf{e}]_{\Gamma_C}\,dS\,.$$

5.2 a-based formulation

Again, we consider the arrangement of Fig. 4. As explained above, non-local excitations may rule out $\mathbf{e} \in \boldsymbol{H}(\mathbf{curl};\Omega)$. Therefore, with a function $U : [0,T] \mapsto \mathbb{R}$, we set

$$\mathbf{e} = -\tfrac{d}{dt}\mathbf{a} - U\widetilde{\mathbf{p}}\,, \tag{37}$$

where we have use an extension by zero

$$\widetilde{\mathbf{p}} := \begin{cases} \mathbf{p} & \text{in } \Omega_C\,, \\ 0 & \text{in } \Omega_I\,. \end{cases}$$

In situation (a) $\mathbf{p}$ is given by

$$\mathbf{p} := \mathbf{grad}\,\theta\,, \qquad \theta \in H^1(\Omega_C)\,, \qquad \theta|_{\Sigma^+} = 1\,, \qquad \theta|_{\Sigma^-} = 0\,. \tag{38}$$

Please note that there is no **curl**-free extension of $\mathbf{p}$ to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$.

In situation (b) we chose $\mathbf{p}$ as a representative of the first co-homology space $\mathcal{H}^1(\Omega_C)$ constructed according to

$$\mathbf{p} := \widetilde{\mathbf{grad}}\,\theta\,, \qquad \theta \in H^1(\Omega_C \setminus \Sigma)\,, \qquad [\theta]_\Sigma = 1\,. \tag{39}$$

Thus, (37) defines a field $\mathbf{e} \notin \boldsymbol{H}(\mathbf{curl};\Omega)$ whose tangential components are discontinuous across Γ_C. After we have given up the continuity of tangential components of $\mathbf{e}$, we can state $\mathbf{a}$-based variational formulations that allow non-local excitations.

Remark 12. *For practical computations it is more appropriate to chose a* $\mathbf{p}$ *with small support, see [Dul01].*

Voltage Excitation

In this case we can state the problem: seek $\mathbf{a} \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\int\limits_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a}\cdot\mathbf{curl}\,\mathbf{a}'\,d\mathbf{x} + \int\limits_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a}\cdot\mathbf{a}'\,d\mathbf{x} = -U\int\limits_{\Omega_C}\sigma\mathbf{p}\cdot\mathbf{a}'\,d\mathbf{x}\,. \tag{40}$$

Current Excitation

The appropriate variational problem reads: seek $\mathbf{a} \in C^1([0,T], \boldsymbol{H}_0(\mathbf{curl};\Omega))$ and $U \in C^1([0,T],\mathbb{R})$ such that for all $\mathbf{a}' \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$

$$\begin{aligned}
\int\limits_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{a}\cdot\mathbf{curl}\,\mathbf{a}'\,d\mathbf{x} + \int\limits_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a}\cdot\mathbf{a}'\,d\mathbf{x} + U\int\limits_{\Omega_C}\sigma\,\mathbf{p}\cdot\mathbf{a}'\,d\mathbf{x} &= 0\\
\int\limits_{\Omega_C} \sigma\,\tfrac{d}{dt}\mathbf{a}\cdot\mathbf{p}\,d\mathbf{x} \qquad\qquad + U\int\limits_{\Omega_C}\sigma\,|\mathbf{p}|^2\,d\mathbf{x} &= I\,.
\end{aligned} \tag{41}$$

The expression for the power fed into the eddy current model in case of voltage as well as current excitation reads $P = U\,I$, and U is a circulation voltage. The (weak) expression for the current in the conductor Ω_C is given in both cases by the second equation in (41).

5.3 Physical Interpretation of Non-Local Excitations

The bottom line is that the non-local excitations discussed above lead to electric fields that do not belong to $\boldsymbol{H}(\mathbf{curl};\Omega)$, since their tangential components have a jump across Γ_C. How can this "flaw" of the mathematical model be reconciled with physics?

We propose the following explanation: the jumps $[\mathbf{n}\times\mathbf{e}]_{\Gamma_C}$ can be regarded as the effect of very thin ideal coils that do not create an exterior magnetic field. What we have in mind is depicted in Fig. 5. Letting the thickness of the coils tend to zero, they will only affect the electric field, because any impact on the magnetic field is ruled out by the design of the coils. The fields inside the coils are completely ignored. This causes inconsistency with Faraday's law, because

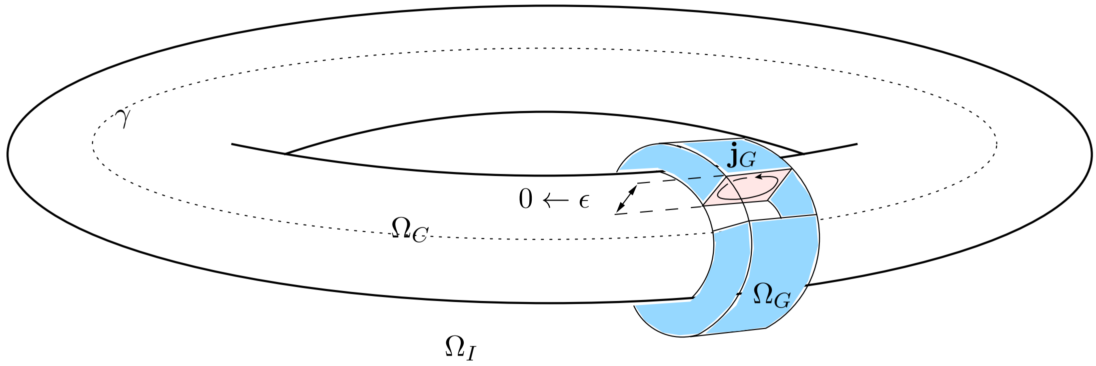

Figure 5: Thin coil endowed with current that does not create a magnetic field outside the coil (To illustrate the current distribution the coil has been cut open.).

the variation of magnetic flux inside the thin coils is neglected. Eventually, it is a given generator current in an extremely thin coil that is responsible for the non-local excitation.

Coils that do not generate any external magnetic field can easily be constructed theoretically: their associated vector potential $\mathbf{a}_G$ is given in analogy to the co-homology field $\widetilde{\mathbf{q}}$ from (32) (see also [Bos98b, Exercise 8.5]): let Ω_G be the domain occupied by the coil and Ω'_G its complement $\Omega'_G := \Omega \setminus \overline{\Omega}_G$. Denote by Ξ the corresponding Seifert surface such that the co-homology space $\mathcal{H}^1(\Omega'_G \setminus \Xi)$ becomes trivial. Define

$$\mathbf{a}_G|_{\Omega'_G} := \widetilde{\mathbf{grad}}\,\theta\,, \qquad \theta \in H^1(\Omega'_G \setminus \Xi)\,, \qquad [\theta]_\Xi = 1$$

and perform a $\boldsymbol{H}(\mathbf{curl};\Omega)$-extension to Ω_G. Then, we have with $\mathbf{b}_G := \mathbf{curl}\,\mathbf{a}_G$

$$\mathbf{b}_G \equiv 0 \quad \text{in } \Omega'_G\,.$$

On the other hand, given a temporally varying $\mathbf{a}_G$, the circulation $\int_\gamma -\frac{d}{dt}\mathbf{a}_G \cdot d\vec{s}$ along the closed path $\gamma \subset \Omega_C$ in Fig. 5 does not vanish. Hence, we get an inductive excitation of currents through a field $\frac{d}{dt}\mathbf{b}_G$ that vanishes outside the coil.

We conclude that the exact location of the thin coil does not affect the magnetic field. This accounts for the term "non-local excitation". However, in order to reconstruct the electric field in the insulating region Ω_I we need to know the location of the coil. If we want to get $\mathbf{e}$, the excitation has to be localized. This highlights the following fact: in order to recover the electric field from the eddy current model everywhere in Ω we need extra information compared to what is required for the computation of the magnetic field. Besides information about possible charge densities in Ω_I (which are usually assumed to be zero) we need to know the position of idealized sources in the case of non-local excitation.

Remark 13. *As in Sect. 3 one might try to start from the generic excitation by a thin coil as described above, and then to derive variational formulations by passing to the limit $\epsilon \to 0$. However, this will lead to a sequence of vector potentials $\mathbf{a}_G(\epsilon)$ that does not converge in $\boldsymbol{H}(\mathbf{curl};\Omega)$.*

Coupling to Circuit Equations

The interpretation of the excitation as an *inductive* one has an important consequence concerning coupling to a circuit model: there is no reason for the terminal current in the circuit model I_{term}

and the current I in the conductor Ω_C to be equal. The circuit current has to represent the current in the coil that is not included in the eddy current model with a non-local excitation. This means that coupling by conservation of current is ruled out here.

However, the change of the magnetic flux Φ hidden behind the jump $[\mathbf{n}\times\mathbf{e}]_{\Gamma_C}$ is a good candidate for a coupling quantity because it can be uniquely defined in both models. For the change of magnetic flux inside a thin coil along any curve $\gamma \subset \Gamma_C$ circulating around the conductor loop we get

$$-\tfrac{d}{dt}\Phi = \int_\gamma [\mathbf{n}\times\mathbf{e}]_{\Gamma_C} \cdot d\vec{s} = U \,,$$

which motivates the name "circulation voltage" for U. The fact that it does not depend on the path γ is a consequence of the very design of the coil and is necessary, since otherwise it would not be meaningful to talk about "*the* circulation voltage". Thus, the induced terminal voltage of the coil will be equal to the circulation voltage U in the eddy current problem (apart from the number of windings N):

$$U_{ind} = -N\,\tfrac{d}{dt}\Phi = N\,U$$

The terminal current I_{term} can be computed by the conservation of power if the self-inductance L of the coil or the change of the magnetic energy W_{coil} inside the coil are known:

$$P_{term} = U_{ind}\,I_{term} = \tfrac{d}{dt}W_{coil} + P = \tfrac{d}{dt}(\tfrac{1}{2}L\,I_{term}^2) + U\,I \,,$$

where P is the (computationally available) power fed into the eddy current problem. We conclude that even for coupling reasons we need additional information, reflecting the fact that we can use different coils inducing the same resulting circulation voltage but with different terminal voltages.

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