Theoretical study of axisymmetrical triple flame

M. Savelieva

Theoretical study of axisymmetrical triple flame

M. Savelieva

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Abstract

We study the axysymmetrical triple flame, that could be observed at the base of the laminar diffusion lifted flame. The study of this flame has been done by S.Ghosal and L.Vervisch, who applied the two-dimensional approach for the flame that axisymmetrical in fact. The experiments reveal us the necessity to investigate the influence of the radius of the flame base on the behaviour (i.e. such important characteristics as stability and velocity) of the whole structure.

Both constant- and variable-density case are considered. Instead of solving a free-boundary problem for the flame surface, we approximate it with a help of parabolic profile, the curvature to be self-consistently determined. This method, called by Ghosal and Vervisch "the parabolic flame path approximation", has shown his validity for planar case, and we would like to expand this result for the axisymmetric flame. The method of matched asymptotic expansions in parabolic-cylinder coordinates will be applied, and the closed expressions for the flame curvature and velocity, as well as the temperature field, will be given. We compare this theoretical results with those J. Boulanger, L. Vervisch, J. Reveillon and S. Ghosal got with by DNS (direct numerical simulations).

1 Introduction

We investigate the dynamic behavior of the so-called "triple" (or edge) flame, forming on the base of the lifted laminar jet flame, near the burner exit. Under this flame we consider the whole structure, consisting of diffusion flame and two branches of premixed flame. The point of intersection of the stoichiometric iso-surface of the mixture fraction, Z, and premixed flame zones, forms a so-called triple point. Due to the curvature effects, the triple flame is able to propagate, that is differentiate it from the planar diffusion flame. In practice we would like to stabilize the lifted flame above the burner exit, in order to avoid the damages by the high temperatures on the burner itself. We'd like to be able to control the propagation velocity (and, consequently, the lift-off height), by given mixture fraction and velocity of the gazes on the burner exit. L. Hartley and J. Dold [3] in there theoretical study of triple flames, found the expression for the flame velocity in the case, when density perturbations are neglected.

Ghosal and Vervisch [1] introduced the density changes in their analysis, where flame front was approximated with the help of parabola. In the limit of high activation energy they found approximative solutions for velocity of the well-developed two-dimensional triple flame.

Boulanger, Vervisch and Ghosal [2] applied these results to the lifted jet flame, that is axisymmetric in fact. The velocity of the triple flame is influenced by its curvature, but the effects of the flame base radius are neglected.

The goal of present paper is to find the expressions for velocity and ... , similar to those of [1], but with jet radius taken in account. We will introduce such a dependence in parametric way. The results we get are compared with those of a planar case, as well as the numerical simulations for round jet. The velocity is increased by the effects of the radius, and the under-prediction of the lift-off height is corrected.

The method of asymptotic expansions in the vicinity of the flame front is applied, but in order to keep the 'non-local' influence of the jet radius, we introduce it as a fixed constant in the two-dimensional axisymmetric equations, and then reduce them to the planar case.

Surprisingly, we get very good agreement with numerical experiments, even though the radial component dependence replaced by its approximation in the vicinity of the triple point.

The flame in our work considered to be a parabola, following Ghosal and Vervisch [1], where this approximation showed its validity.

The closed expressions for temperature and velocity in dependence with flame curvature and jet radius are got.

2   Problem formulation

We study the axisymmetric triple flame, forming on the base of lifted laminar jet flame. The fuel is issuing from a nozzle of radius a, then mixing with oxidizer is taking place.

We suppose the chemical process to be described by a one-step combustion: ν_0 molecules of fuel (molecular mass m_0) react with ν_1 molecules of an oxidizer (molecular mass m_1) to form ν_p molecules of product (molecular mass m_p). We suppose also the reaction to be infinitely fast, that allows us to think about a flame front as a surface.

The partial premixing taking place, the triple flamelet forms. It consists of two premixed (lean and rich) brunches and a trailing diffusion flame. This flamelet propagates along the stoichiometric iso-surface of the mixture fraction. At the point, where velocity of the outcoming jet is equal to the propagation triple flame velocity, the flame is stabilized.

3   Mathematical modeling

We consider the reactive flow problem in the law Mach number limit, i.e. concentrating us essentially on the advection-diffusion affects.

The transport equations for species and mixture fraction, as well as energy equation are written. For the velocity components only the continuity equation is kept.

Further, we follow the general lines of the analysis, done in [1], with some modifications, due to the axisymmetric nature of the problem. We right down the equations in cylindrical coordinates, and after pass to the local analysis, introducing the local parabolic-cylinder coordinates, matching the form of the flame (supposed to be parabola of a curvature to be determined), in the flame vicinity.

The solutions for the mixture fractions and temperature are developed in the asymptotic series, and only leading terms are considered.

3.1   Governing equations

The equations, describing reactive flow, contain multiple scales, and because of there nature cannot be solved analytically. We concentrate our attention on the processes, that we suppose to dominate in case we are interesting in, namely, advection and diffusion. We write down all the equations in cylinder coordinates (r, x), and taking into account the axisymmetrical nature of the problem, get following equations:

- Time evolution of the mass fractions Y_0(fuel) and Y_1 (oxidizer):

$$\rho \frac{DY_i}{Dt} = D_i \nabla^2 Y_i - \nu_i m_i w, \quad i = 0, 1 \tag{3.1}$$

here w — reaction rate, D_i—mass diffusivity of species i.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{3.2}$$

with

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial x} \right) \tag{3.3}$$

and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2} \tag{3.4}$$

are the gradient and Laplace operator in cylinder coordinates.

- Evolution of temperature T:

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = D_T \nabla^2 T + Qw. \tag{3.5}$$

In our work we suppose the Lewis numbers, $Le = \frac{D_T}{c_p D_i}$, to be equal to unity (this will give us further a possibility to get a linear dependence between Z and H). The characteristic length k in this case can be defined as:

$$k = \frac{D_0}{\rho_\infty} = \frac{D_1}{\rho_\infty} = \frac{D_T}{\rho_\infty c_p} \tag{3.6}$$

(here ρ_∞ – density of unburnt gas far upstream).

- Equation of state for the ideal gas:

$$p = \frac{\rho k_B T}{m} \tag{3.7}$$

with Boltzmann constant k_B and mean molecular height m.

- Arrhenius law [6] for reaction rate:

$$w = A \rho^{\nu_0 + \nu_1} Y_0^{\nu_0} Y_1^{\nu_1} \exp(-T_a/T) \tag{3.8}$$

(here T_a is the reaction temperature, A - pre-exponential factor, suppose to be constant for simplicity).

- Density changes coupled to the velocity field through the equation of mass conservation).

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3.9}$$

everything here is in cylinder coordinates, i.e. the equation in coordinates (r, x) is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial(\rho u_x)}{\partial x} = 0 \,. \tag{3.10}$$

3

From the momentum equation we can get that $(p - p_\infty)/p_\infty \backsim M^2$, here M—Mach number, $M \ll 1$ and this means that we can approximate $\rho \approx \frac{mp_\infty}{k_B T}$. So, now we can re-write equation (3.5):

$$\rho c_p \frac{DT}{Dt} = k\rho_\infty c_p \nabla^2 T + Qw \,. \tag{3.11}$$

In our model problem we suppose axis x to be parallel to the iso-lines of the mixture fraction, Z, the iso-line $Z = Z_s$ is fixed at the distance $r = r_0$ from coordinate origin. Axe r is a tangent to the contour lines of the reaction rate, and perpendicular to x (see figure 1). We suppose this frame to move together with our triple flamelet, that allows us to consider the problem to be stationary. A uniform flow issues from infinity $(x = -\infty)$ with velocity U_∞.

We specify the values for a chemical species at $x = -\infty$:

$Y_0 = 1$ in the fuel stream $(r \to 0)$
$Y_0 = 0$ in the oxidizer stream $(r \to +\infty)$
$Y_1 = 1$ in the fuel stream $(r \to +\infty)$
$Y_1 = 0$ in the oxidizer stream $(r \to 0)$,

and

$$Y_0(r, -\infty) + Y_1(r, -\infty) = 1 \tag{3.12}$$

$T = T_\infty$ far upstream for fuel and reactant,
$\rho = \rho_\infty$ far upstream.

Under this conditions the fuel mixture fraction Z will be defined:

$$Z = \frac{\tilde{r}Y_0 - Y_1 + 1}{1 + \tilde{r}} \quad \left(\tilde{r} = \frac{\nu_1 m_1}{\nu_0 m_0}\right) \,. \tag{3.13}$$

From this formulation it's clear, that:

$Z = 1$ for fuel stream,

$Z = 0$ for oxidizer stream

$Z = Z_s = \frac{1}{1+\tilde{r}}$ under stoichiometric conditions.

From (3.1) and (3.13) follows that Z evolves as a passive scalar:

$$\rho \frac{DZ}{Dt} = k\rho_\infty \nabla^2 Z \,. \tag{3.14}$$

For the pair of equations (3.11) and (3.1) we get that "Shvab-Zeldovich variables" $H_i = T + QY_i/(\nu_i m_i c_p)$ also evolve as a passive scalar:

$$\rho \frac{DH_i}{Dt} = k\rho_\infty \nabla^2 H_i \,. \tag{3.15}$$

4

Under the condition $T = T_\infty$ far upstream it's clear, that H_i is a linear combination of Z, i.e. $H_i = A_i + B_i Z$. We determine constants A_i and B_i we use the boundary conditions for Y_0 and Y_1 and get:

$$Y_0 = (T_\infty - T)\frac{m_0 \nu_0 c_p}{Q} + Z, \tag{3.16}$$

$$Y_1 = (T_\infty - T)\frac{m_1 \nu_1 c_p}{Q} + Z + 1. \tag{3.17}$$

3.2 Dimensionless variables

We introduce the dimensionless variables: $\tilde{w} = \rho/\tilde{\rho}$ — dimensionless density $(u, v) = (u_r/U_\infty, u_x/U_\infty)$ — dimensionless velocity.

With the appropriate length scale: diffusion length k/U_∞ we get the dimensionless coordinates: $(r, X) = (rU_\infty/k, xU_\infty/k)$.

In order to get an appropriate temperature scale, we need to find some measure of changing across the flame. Because the maximum temperature T_s is reached just behind the flame surface, along the stoichiometric line, where the combustion is complete. We can find this T_s, putting $Z = Z_s = 1/(1 + \tilde{r})$ and $Y_0 = 0$ in (3.16)

$$T_s = T_\infty + \frac{Q}{m_0 \nu_0 c_p}\frac{1}{1 + r} = T_\infty + \frac{Q}{(m_0 \nu_0 + m_1 \nu_1)c_p} \tag{3.18}$$

The dimensionless parameter

$$\alpha = \frac{T_s - T_\infty}{T_s} = \left[1 + \frac{(m_0 \nu_0 + m_1 \nu_1)c_p T_\infty}{Q}\right]^{-1} \tag{3.19}$$

characterizes the amount of heat, released to the flame.

The dimensionless temperature, Θ:

$$\Theta = \frac{T - T_\infty}{T_s - T_\infty} \tag{3.20}$$

Zeldovich number β:

$$\beta = \alpha\frac{T_a}{T_s} \tag{3.21}$$

measures the sensitivity of the reaction rate to the temperature.

In terms of these dimensionless variables, (3.11) may be written as

$$\tilde{w}(u\frac{\partial \Theta}{\partial r} + v\frac{\partial \Theta}{\partial X}) =$$
$$= \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2} + \frac{\partial^2 \Theta}{\partial X^2} + \lambda \tilde{w}^{\nu_0 + \nu_1}\Sigma(Z/Z_s, \Theta)\exp\left[-\frac{\beta(1 - \Theta)}{1 - \alpha(1 - \Theta)}\right] \tag{3.22}$$

5

where
$$\Sigma(x, y) = (x - y)^{\nu_0} \left[(1 - x) + r(1 - y)\right]^{\nu_1} \tag{3.23}$$

and λ is defined by
$$\lambda = \frac{kQA}{c_p T_s \alpha} \frac{\rho_\infty^{\nu_0 + \nu_1 - 1}}{(1 + r)^{\nu_0 + \nu_1}} \exp(-\beta/\alpha) \frac{1}{U_\infty^2}. \tag{3.24}$$

The equation for the mixture fraction, Z, becomes:
$$\tilde{w}\left(u\frac{\partial Z}{\partial r} + v\frac{\partial Z}{\partial X}\right) = \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{\partial Z^2}{\partial r^2} + \frac{\partial^2 Z}{\partial X^2} \tag{3.25}$$

and the density is related to the temperature through $\rho = \frac{mp_\infty}{k_B T}$, which may be re-expressed in dimensionless form as
$$\tilde{w} = \frac{1 - \alpha}{1 - \alpha(1 - \Theta)} \tag{3.26}$$

(3.10) in dimensionless variables:
$$\frac{\partial}{\partial r}\left(\tilde{w}ur\right) + \frac{\partial(\tilde{w}v)}{\partial X} = 0. \tag{3.27}$$

First we suppose the constant density case, i.e. $\tilde{w} = 1$, and in this case velocity decouples form density. We take the trivial solution $u = 0, v = 1$.

We also consider the approximation of the "law heat release", by which we mean $\alpha \ll 1$. The simplified set of equations is:
$$\frac{\partial\Theta}{\partial X} = \frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{\partial^2\Theta}{\partial r^2} + \frac{\partial^2\Theta}{\partial X^2} + \lambda\Sigma(Z/Z_s, \Theta)\exp(\beta(1 - \Theta)) \tag{3.28}$$

and
$$\frac{\partial Z}{\partial X} = \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{\partial^2 Z}{\partial r^2} + \frac{\partial^2 Z}{\partial X^2}. \tag{3.29}$$

Due to the coordinate choice and problem formulation, the only "surviving" influences in the equation for mixture fraction are:
$$0 = \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{\partial^2 Z}{\partial r^2}. \tag{3.30}$$

We develop solution of this equation in the vicinity of $r = r_0$ (here r_0 is the radius of the flame base) in Taylor series, and get, prior to boundary conditions $r \to +\infty, Z \to 0$, and $r = r_0, Z = 1/2$:
$$Z = \frac{1}{2}\left[1 + \mu(r - r_0) + O(r - r_0)^2\right] \tag{3.31}$$

6

where

$$\mu = \frac{1}{Z_s} \left.\frac{\partial Z}{\partial r}\right|_{r=r_0} \tag{3.32}$$

is the value of the mixture fraction gradient, prescribed on the inlet. Here we have to note, that the linear approximation of the mixture fraction is also an exact solution in the case, when we suppress all non-local effects, i.e. the first term on the right hand side of (3.30). Further we will consider, that

$$Z = \frac{1}{2} \left[1 + \mu(r - r_0)\right]. \tag{3.33}$$

Since the gases far upstream are at the uniform temperature, we have the boundary condition:

$$\Theta(r, -\infty) = 0. \tag{3.34}$$

$$\Theta = \begin{cases} (1 - Z)/(1 - Z_s), & r \leq r_0 \\ Z/Z_s, & r \geq r_0 \end{cases}. \tag{3.35}$$

Together with these boundary conditions, equation (3.28) defines an eigenvalue problem in λ (related to the flame speed via (3.24).

4 Activation-energy asymptotics

We study the basic equations in the so-called "activation-energy asymptotics" (AEA), $\beta \to \infty$. Only "well-developed" triple flames are considered, i.e. the flames with $\beta\mu \sim O(1)$.

In this limit the source term vanishes except in the immediate vicinity of the flame front, ($\Theta = 1$). The problem we get is a singular perturbative problem, the method of matched asymptotic expansions is applied to find the solution.

4.1 Solution

The solution consists of two parts: the "outer solution", valid everywhere except in the vicinity of the flame front, and the "inner solution", that connects smoothly two branches (one in front and other behind the flame), of the outer solution.

4.1.1 The outer solution

While we consider the reaction zone to be infinitely small, the temperature just behind the flame front may be obtained by setting $Y_1 = 0$ (in the fuel stream,

7

$r < r_0$), or $Y_0 = 0$ (in the oxidizer stream, $r > r_0$) in (3.17) and (3.16), since the component that is deficient, is completely consumed. We get from (3.20) and (3.18):

$$\Theta = 2(1 - Z) = 1 - \mu(r - r_0).$$
(4.1)

Ahead the premixed flame:

$$\frac{\partial \Theta}{\partial X} = \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2} + \frac{\partial^2 \Theta}{\partial X^2}.$$
(4.2)

After two coordinate changes,

$$\Theta = \exp(X/2)F,$$

and

$$F = \frac{1}{\sqrt{r}}f$$

we get following equation:

$$\frac{\partial^2 f}{\partial r^2} - \frac{1}{4}f\left(-\frac{1}{r^2} + 1\right) + \frac{\partial^2 f}{\partial X^2} = 0.$$
(4.3)

In this form the "non-local" forces are described as potential through $-\frac{1}{r^2} + 1$. We'd like to keep only parametric dependence on this force, reducing afterwards the equations to the two-dimensional case in the vicinity of the flame. That's exactly what we get, keeping only leading term of the potential in the vicinity of $r = r_0$, i.e. replacing potential force by constant $1 - \frac{1}{r_0^2}$.

Once the parabolic-cylinder coordinates

$$\begin{aligned} X &= \frac{\sqrt{c}}{2}\left[\xi^2 - \eta^2 + \eta_0^2\right], \\ r - r_0 &= \sqrt{c}\xi\eta \end{aligned}$$
(4.4)

are introduced, the problem is completely reduced to the local one, keeping nevertheless the dependence on radius of the flame base $r = r_0$. Here $\eta = \eta_0$ corresponds to the flame surface, and $c = 1 - \frac{1}{r_0^2}$. The solution prior to boundary condition $X \to \infty, \Theta \to 0$ is:

$$\Theta = \exp\left[\frac{\sqrt{c}}{4}(\xi^2 + \eta_0^2 - \eta^2)\right]\exp\left(-\frac{1}{4}c\xi^2\right)\sum_{n=0}^{\infty} a_n H_n\left(\frac{\sqrt{c}}{\sqrt{2}}\xi\right)U(n + 1/2, \sqrt{c}\eta).$$
(4.5)

The boundary condition

$$\Theta(\xi, \eta_0) = 1 - \mu\xi\eta_0 + O(\mu^2).$$
(4.6)

leads us to the following solution for temperature:

$$\Theta^{(0)} = \frac{\text{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta\right)}{\text{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta_0\right)} \exp\left[\left(\frac{\sqrt{c}}{4} - \frac{c}{4}\right)\left(\eta_0{}^2 - \eta^2\right)\right] + O(1/\beta). \qquad (4.7)$$

The factor $\exp\left[\left(\frac{\sqrt{c}}{4} - \frac{c}{4}\right)\left(\eta_0{}^2 - \eta^2\right)\right]$ decaying from 1 much slowly, then those of erfc-function, that's why we finally write the first approximation for temperature in the form:

$$\Theta^{(0)} = \frac{\text{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta\right)}{\text{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta_0\right)} + O(1/\beta). \qquad (4.8)$$

Behind the flame temperature is approximated by

$$\Theta(r, X) = 1 + O(1/\beta). \qquad (4.9)$$

4.1.2 The inner solution and the asymptotic matching

We now determine the solution to (3.28) in the vicinity of the flame zone. Equation for the temperature could be re-written in parabolic-cylinder coordinates:

$$\left(\xi - \frac{\eta}{\xi\eta + r_0}\right)\frac{\partial\Theta}{\partial\xi} - \left(\eta - \frac{\xi}{\xi\eta + r_0}\right)\frac{\partial\Theta}{\partial\eta} =$$
$$= \frac{\partial^2\Theta}{\partial\xi^2} + \frac{\partial^2\Theta}{\partial\eta^2} + \lambda c(\xi^2 + \eta^2)\Sigma(1 + \mu\xi\eta, \Theta)\exp(-\beta(1 - \Theta)). \qquad (4.10)$$

Since the flame sickness is $\sim 1/\beta$, an appropriate inner variable is $\tau = \beta(\eta - \eta_0)$. We also define the new variable θ which is a deviation of the temperature from those on the flame surface, scaled to order of unity:

$$\Theta = 1 - \mu\eta_0|\xi| - \frac{\theta}{\beta}. \qquad (4.11)$$

We re-write (3.7) in terms of these scaled variables, expand θ in asymptotic series in $1/\beta$, and get at the leading order:

$$\theta_{\tau\tau} = c\Lambda_0(\xi^2 + \eta_0^2)\theta^\nu(\nu + 2B\eta_0|\xi|)^\nu \exp(-\theta)\exp(-B\eta_0|\xi|), \qquad (4.12)$$

here Λ_0 is the first term in the expansion $\lambda = \beta^{2\nu+1}(\Lambda_0 + \dots)$, $B = \mu\beta = O(1)$. We integrate this equation prior to boundary conditions $\theta_\tau(\tau = 0) = 0$, $\theta_\tau(\tau = \infty) = 0$, and get:

$$\theta_\tau^2(\tau \to +\infty) = 2c\Lambda_0(\xi^2 + \eta_0^2)\exp(-B\eta_0|\xi|)F_\nu(2B\eta_0|\xi|). \qquad (4.13)$$

9

Here the function F_ν is defined:

$$F_\nu(\alpha) = \int_0^\infty \theta^\nu (\theta + \alpha)^\nu \exp(-\theta) d\theta. \tag{4.14}$$

To determine Λ_0, we reenforce the asymptotic matching condition:

$$\Theta_\eta(\xi, \eta \to \eta_0) = -\theta_\tau(\xi, \tau \to \infty), \tag{4.15}$$

where Θ_η corresponds to the outer, and θ_τ — to the inner solution. Equating terms of right and left hand sides by ξ^0 and ξ, we get:

$$\sqrt{2\Lambda_0 \Gamma(2\nu + 1)} = \sqrt{\frac{2}{\pi}} \frac{\exp(-c\eta_0^2/2)}{\eta_0 \mathrm{erfc}(\sqrt{c}\eta_0/\sqrt{2})}. \tag{4.16}$$

The equation for the curvature of the parabola, η_0, at leading order in $1/\beta$:

$$\eta_0 = \frac{(4\nu - 2)^{1/4}}{\sqrt{B}}. \tag{4.17}$$

Formulae for the normalized velocity, expressed in terms of the mixture fraction gradient:

$$\frac{U_\infty}{U_\infty{}^s} = \sqrt{\frac{\pi}{2}} \eta_0 \exp\left(\frac{c\eta_0{}^2}{2}\right) \mathrm{erfc}\left(\sqrt{\frac{c}{2}}\eta_0\right), \tag{4.18}$$

$$\frac{k}{U_\infty{}^s} \frac{1}{Z_s} \frac{\partial Z}{\partial r} = \frac{\sqrt{4\nu - 2}}{\beta\eta_0{}^2} \frac{U_\infty}{U_\infty{}^s}, \tag{4.19}$$

$$\frac{k}{U_\infty{}^s} \kappa = \frac{1}{\eta_0{}^2} \frac{U_\infty}{U_\infty{}^s}. \tag{4.20}$$

Here U_∞—flame velocity in physical units, U_∞^S— stoichiometric planar flame speed, κ—the flame curvature. We note the dependence of the velocity on the global characteristics, r_0, through the coefficient $c = 1 - 1/r_0^2$. The limit $r_0 \to \infty$ corresponds to the planar case, and our formulae reduce to those of planar case.

5 Perturbative solutions, effects of heat release

With the help of approximations for the pressure and density changes in the vorticity equation, and due to the low Mach number approximation, we can conclude, that the velocity perturbation could be described in a potential form, namely:

$$u = \alpha\psi_r + \ldots, \quad v = 1 + \alpha\psi_X + \ldots \tag{5.1}$$

10

the velocity perturbation due to the heat release. We introduce these expressions for the velocity into the temperature equation (3.28), and get at leading order in α:

$$\frac{1}{r}\psi_r + \psi_{rr} + \psi_{XX} = \Theta_X^{(0)}. \tag{5.2}$$

re-written in local parabolic-cylinder coordinates (ξ, η), these equations take form:

$$\frac{1}{r_0}\left(\eta\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial \eta}\right)\psi + \psi_{\xi\xi} + \psi_{\eta\eta} = -\eta\Theta_\eta^{(0)}, \tag{5.3}$$

behind the premixed front the right hand side in (5.3) is 0, and above the premixed front is equal to

$$\sigma\eta\exp\left(\frac{c(\eta_0^2 - \eta^2)}{2}\right), \quad \text{here} \quad \sigma = \frac{\sqrt{c}}{\sqrt{2\pi}}\exp\left(-\frac{c}{2}\eta_0^2\right)/\operatorname{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta_0\right). \tag{5.4}$$

The boundary conditions are:

$$\psi_\xi(0, \eta) = 0, \quad \psi_\eta(\xi, 0) = 0. \tag{5.5}$$

We are looking for solution in the vicinity of $r = r_0$, that means $\xi = 0$ or $\eta = 0$. Together with boundary conditions, introduced above, and due to the fact, that velocity perturbation stays limited even at infinity, we conclude, that the first term on the left hand side of (5.3) can be suppressed. Further, it's easy to see that the only physically relevant solution of (5.3) is a function $\psi_0 = \psi(\eta)$.

We are in 2D situation again, and the solution can directly be written:

$$\psi_0(\eta) = \frac{\operatorname{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta\right)}{\operatorname{erfc}\left(\frac{\sqrt{c}}{\sqrt{2}}\eta_0\right)} + \sigma\sqrt{c}(\eta - \eta_0) - 1 \tag{5.6}$$

for $\eta > \eta_0$, and

$$\psi_0(\eta) = 0 \tag{5.7}$$

for $\eta < \eta_0$.

The local expressions for velocity components along axes $r = r_0, X = 0$ are:

$$u(r, 0) = \frac{\alpha\sigma\xi}{\eta_0^2 + 2\xi^2}\left[1 - \exp\left(-\frac{c}{2}\xi^2\right)\right] \tag{5.8}$$

$$v(X, r = r_0) = \begin{cases} 1 - \dfrac{\alpha\sigma}{\sqrt{\eta_0^2 - 2X}}[1 - \exp(\sqrt{c}X)], & X < 0, \\ 1, & \text{otherwise} \end{cases} \tag{5.9}$$

11

5.1 Reduction of the mixture fraction gradient near the flame tip

We perturb the linear approximation for Z, and present this new solution as a serie in α:

$$Z = \frac{1}{2}(1 + \mu(r - r_0)) + \alpha\zeta + \ldots \tag{5.10}$$

with ζ— mixture fraction perturbation, due to the density changes.

We put this expression into the equation for mixture fraction gradient (3.25), develop everything in series in α, and, on the leading order, get:

$$\frac{1}{r}\frac{\partial\zeta}{\partial r} - \frac{\partial\zeta}{\partial X} + \frac{\partial^2\zeta}{\partial r^2} + \frac{\partial^2\zeta}{\partial X^2} = \frac{1}{2}\mu\psi_r. \tag{5.11}$$

We transform this equation into local parabolic-cylinder coordinates (ξ, η), and get:

$$\frac{1}{\xi\eta + r_0}\left(\xi\frac{\partial\zeta}{\partial\eta} + \eta\frac{\partial\zeta}{\partial\xi}\right) - \xi\frac{\partial\zeta}{\partial\xi} + \eta\frac{\partial\zeta}{\partial\eta} + \frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} =$$

$$= \frac{1}{2}\left(\eta\frac{\partial\psi}{\partial\xi} + \xi\frac{\partial\psi}{\partial\eta}\right). \tag{5.12}$$

The right-hand side is given by function $\psi_0(\eta)$, calculated above. We are looking only for factor by $\xi\eta$, so we are looking for solution in form:

$$\zeta(\xi, \eta) = \xi F(\eta) + \phi(\xi, \eta). \tag{5.13}$$

When we put everything in the original equation, we see, that the equation for function F, we are interesting in, is:

$$F'' - F + \eta F' = \frac{1}{2}\mu\psi_0'(\eta), \tag{5.14}$$

that is, the same equation, as in two-dimensional case, and the solution is written directly, following Ghosal and Vervisch:

$$Z = 1/2(1 + \mu')(r - r_0), \tag{5.15}$$

with modified mixture fraction gradient

$$\mu' = \mu + 2\alpha a, \quad a = \frac{\mu}{4}\left(c\eta_0^2 - \sigma\sqrt{c}\eta_0 - 1\right). \tag{5.16}$$

We see, that the gradient on the top of the edge flame will be reduced firstly by premixed flame front curvature and secondly, by the radius of the flame base. If we right down the expressions for the velocity components along the axes, we see, that propagation velocity is increased, as awaited, not only by local effects, but also by global one, such as flame base radius.

5.2 Modified solution for temperature

The perturbated equations for the temperature are:

$$\psi_r \Theta_R{}^{(0)} + \frac{1}{r}\psi_r \Theta^{(0)} - \Theta^{(0)}\Theta_X{}^{(0)} + \psi_X \Theta_X{}^{(0)}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\Theta^{(1)} + \frac{\partial^2}{\partial X^2}\Theta^{(1)} - \Theta_X{}^{(1)}. \tag{5.17}$$

Proceeding in the similar way, as before, (i.e. suppressing all the influences that of not - 2D nature), we reduce the equation to 2D case, and afterwards follow [1]. In fact, only the appropriate coordinate change ($\eta \mapsto \eta\sqrt{c}$) is necessary in order to write down the result:

$$\Theta_\eta{}^{(1)}(\xi, \eta_0) = \sigma L(\eta_0) + O(1/\beta), \tag{5.18}$$

where

$$L(\eta_0) = \sigma^2 \left[\pi \exp(c\eta_0{}^2)\int_{\sqrt{c/2}\eta_0}^{\infty} x\,\mathrm{erfc}^2 x\,dx - \frac{\sqrt{c}\eta_0}{\sigma} + 1 - \frac{1}{2\sigma^2}\right]. \tag{5.19}$$

5.3 Expression for velocity in dependence on mixture fraction gradient, effects of heat release included

Results for velocity and mixture fraction gradient with constant c, depending on radius r_0 of the flame base as $c = 1 - \frac{1}{r_0^2}$:

$$\frac{U_\infty}{U_\infty^s(\alpha)} = \frac{\eta_0}{\sigma}[1 + \alpha L(\eta_0)], \tag{5.20}$$

where

$$L(\eta_0) = \sigma^2 \left[\pi \exp(c\eta_0{}^2)\int_{\sqrt{c/2}\eta_0}^{\infty} x\,\mathrm{erfc}^2 x\,dx - \frac{\sqrt{c}\eta_0}{\sigma} + 1 - \frac{1}{2\sigma^2}\right] \tag{5.21}$$

and

$$\sigma^{-1} = \sqrt{\pi/2}\exp(-\eta_0^2 c/2)\mathrm{erfc}(\sqrt{c/2}\eta_0). \tag{5.22}$$

$$\frac{k}{U_\infty^s(\alpha)}\frac{1}{Z_s}\left(\frac{\partial Z}{\partial r}\right)_0 = \frac{\sqrt{4\nu - 2}}{\beta}\frac{1}{\eta_0^2}\frac{U_\infty}{U_\infty{}^s(\alpha)}, \tag{5.23}$$

$$\left(\frac{\partial Z}{\partial r}\right)_0 = \left(\frac{\partial Z}{\partial r}\right)_\infty \left[1 + \frac{\alpha}{2}(1 + \sqrt{c}\sigma\eta_0 - c\eta_0^2)\right]^{-1}, \tag{5.24}$$

$$\frac{k}{U_\infty^s(\alpha)}\kappa = \frac{1}{\sqrt{c}\eta_0^2}\frac{U_\infty}{U_\infty^s(\alpha)}. \tag{5.25}$$

13

6 Comparison with DNS results; blow-out limit, lift-off height

In their paper Boulanger, Vervisch, Reveillon and Ghosal [2] applied three different approximations for triple flame velocity, in order to estimate the lift-off height. They compared the results with numerical simulations of full compressible axisymmetrical Navier-Stokes equations. The results show, that the best theoretical description is naturally those taking into consideration the heat release effects.

We'll see, that the new expressions for the triple flame velocity, including the dependence on the jet radius, will ameliorate the situation considerably.

The edge flame supposed to propagate with the local velocity, defined above. We find the lift-off height, equating this local velocity to those of the jet. As a solution for round jet we consider those proposed by Landau and Lifshitz (see [5]). The Reynolds number, Re, is proportional to the opening angle of the jet, θ_0:

$$Re = \frac{32}{3}\theta_0^{-2}.$$

We calculate the lift-off heights for different values of Re, till the blow-out is reached.

We'd like to apply the new approximation we've got in formulae (5.25) - (5.27). In our model problem stoichiometric line supposed to be parallel to the X-axis. In real situation the radial component is not fixed along stoichiometric line, but varies, when the flame changes its position. In fact, the r_0 we've chosen to characterize the problem, is the mean value of the displacement of the triple point on the interval $(1, r_{max})$ (for values of r smaller then one, our approximation is not correct, by definition of the constant c). The r_{max} is easily found from the Landau solution, so for each concrete problem r_0 is defined. If we keep the same notations, as in [2], the new approximation for velocity takes form:

$$U^{IV} = S_L{}^0 \frac{(1+\alpha)}{\sqrt{c}} - \frac{1}{\sqrt{c}}\mathcal{F}_\alpha(\chi_s). \tag{6.1}$$

(here $S_L{}^0$ is a laminar velocity of a planar flame), with

$$\mathcal{F}_\alpha(\chi_s) = \mathcal{A}_\alpha \chi_s{}^{1/2},$$

correction, due to the curvature of the flame. Here

$$\mathcal{A}_\alpha = \beta/(Z_s\sqrt{4\nu - 2})(\lambda/\rho C_P)^{1/2}/(1+\alpha),$$

and χ_s is a mixture fraction dissipation rate.

14

Equating this velocity with those of round jet, we get the following transcendental equation in variable $x = \frac{\theta}{\theta_0}$:

$$(1 + x^2)^{2Sc-2}[1 + Ax(1 + x^2)] = B, \qquad (6.2)$$

with

$$A = \frac{\beta\theta_0}{2\sqrt{4\nu - 2}(1 + \alpha)\sqrt{c}}, \qquad (6.3)$$

$$B = \frac{\theta_0 a(1 + \alpha)}{8Z_s S_c k\sqrt{c}}S_L{}^0. \qquad (6.4)$$

We solve this equation with a help of Newton method, and once we've got the solution x, the lift-off height is calculated:

$$h = r(x\theta_0)_s \cos(x\theta_0) - \frac{a}{\theta_0}. \qquad (6.5)$$

Here the subscript s denotes the value on the stoichiometric line.

The lift-off heights for different Reynolds numbers (or different opening angles of the jet) are calculated. We've done this for two values of heat release: $\alpha = 0.3$, and $\alpha = 0.8$. The comparison with DNS (by Boulanger, Vervisch, Reveillon and Ghosal) shows us, that in both cases an important amelioration in theoretical prediction of the lift-off heights is achieved (see figures 2 and 3).

7   Conclusions

The asymptotic analysis of the triple "ring" flame in the limit $\beta \to \infty$ has been done. Introduction of the dependence on the non-local parameter r_0 allowed us to get new expressions for triple flame velocity and lift-off heights of round jet. In fact, the dependence on the radial component in our results is simply a stretching with a factor $\sqrt{c}$ of the coordinate η, matching the form of the parabolic flame profile.

The comparison with high-order DNS results showed the validity of the this new laminar triple ring velocity approximation, even though the expressions we've derived are not valid in the whole domain of the triple flame existence (the approximation of the mixture fraction by a linear profile valid only if we are not too close to the burner exit; the result we've got is valid only in zone $r > 1$).

References

[1] S. Ghosal, L. Vervisch. 2000 Theoretical and numerical study of a symmetrical triple flame using the parabolic flame path approximation. J. Fluid Mech., vol. 415, pp. 227-260.

[2] J. Boulanger, L. Vervisch, J. Reveillon, S. Ghosal. 2002 Effects of heat release in laminar diffusion flames lifted on round jets. Submitted to Combustion and Flame.

[3] L. Hartley, J. Dold. 1991 Flame propagation in a non-uniform mixture: analysis of a propagating triple flame. Combustion Sci. and Tech., 80(1/3), pp. 23-46.

[4] C. M. Bender, S. A. Orszag. 1978 Advanced mathematical methods for scientists and engineers. International series in pure and applied mathematics.

[5] L. Landau, E. Lifshitz. 1987 Fluid Mechanics. Pergamon Press, New York.

[6] A. Linan, F. A. Williams. 1993 Fundamental aspects of combustion. Oxford University press.
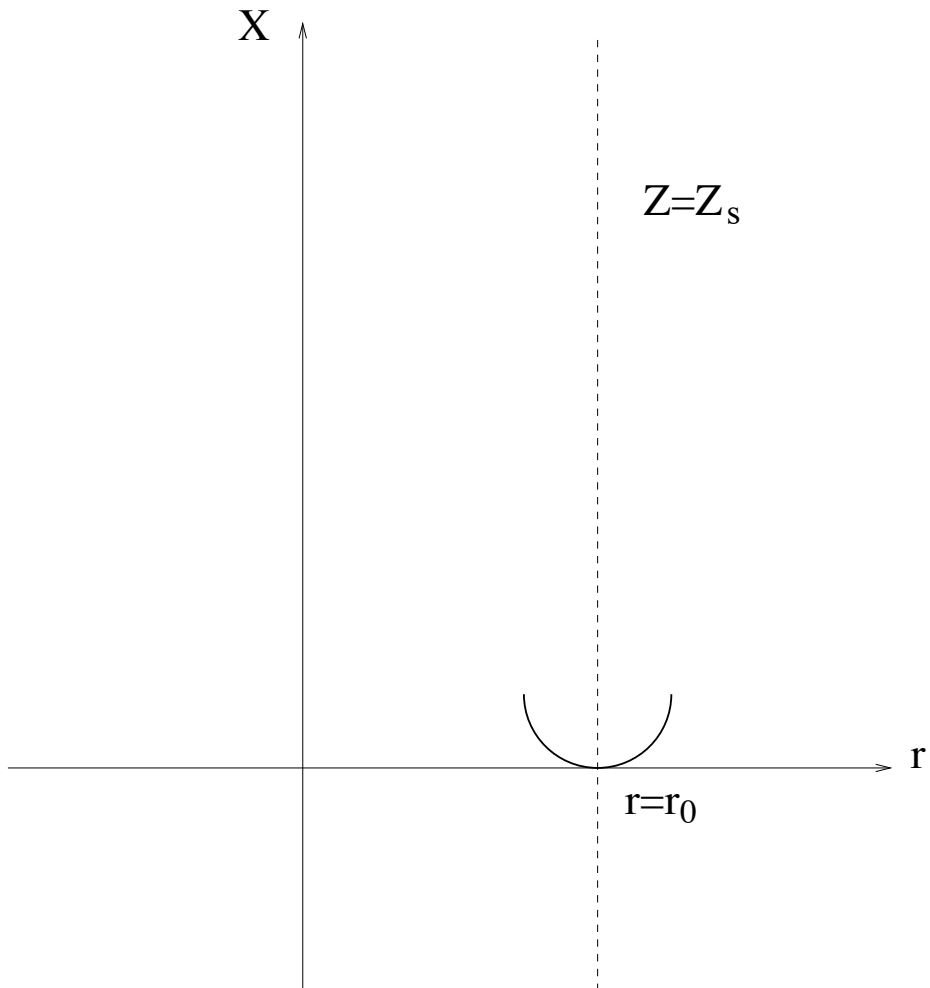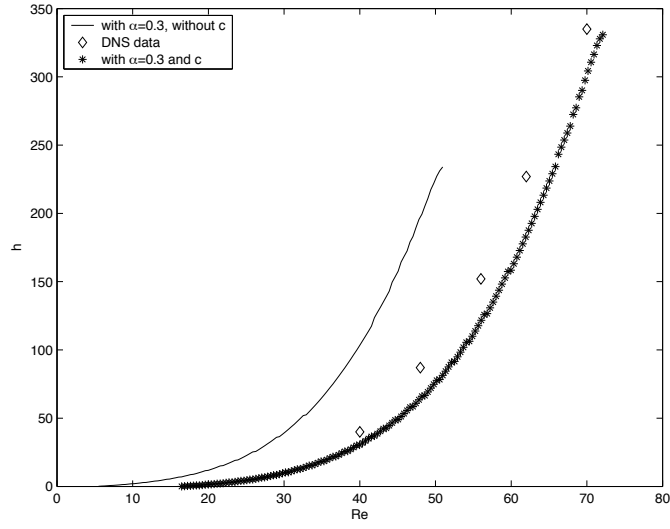
Figures

Figure 1: Sketch of coordinate system

Figure 2: Comparison of our theoretical results with DNS and theoretical approximation by J. Boulanger, L. Vervisch, J. Reveillon and S. Ghosal for $\alpha = 0.3$
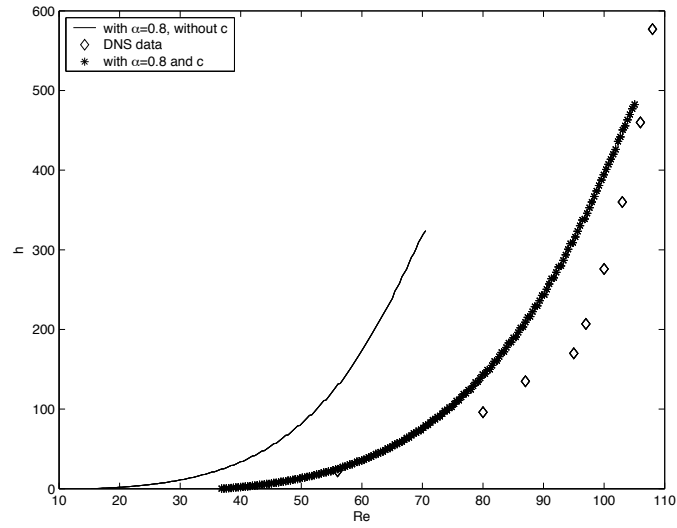

Figure 3: Comparison of our theoretical results with DNS and theoretical approximation by J. Boulanger, L. Vervisch, J. Reveillon and S. Ghosal for $\alpha = 0.8$

18

Research Reports

No.	Authors	Title
02-26	M. Savelieva	Theoretical study of axisummetrical triple flame
02-25	D. Schötzau, C. Schwab, A. Toselli	Mixed hp-DGFEM for incompressible flows III: Pressure stabilization
02-24	F.M. Buchmann, W.P. Petersen	A stochastically generated preconditioner for stable matrices
02-23	A.W. Rüegg, A. Schneebeli, R. Lauper	Generalized hp-FEM for Lattice Structures
02-22	L. Filippini, A. Toselli	hp Finite Element Approximations on Non-Matching Grids for the Stokes Problem
02-21	D. Schötzau, C. Schwab, A. Toselli	Mixed hp-DGFEM for incompressible flows II: Geometric edge meshes
02-20	A. Toselli, X. Vasseur	A numerical study on Neumann-Neumann and FETI methods for hp-approximations on geometrically refined boundary layer meshes in two dimensions
02-19	D. Schötzau, Th.P. Wihler	Exponential convergence of mixed hp-DGFEM for Stokes flow in polygons
02-18	P.-A. Nitsche	Sparse approximation of singularity functions
02-17	S.H. Christiansen	Uniformly stable preconditioned mixed boundary element method for low-frequency electromagnetic scattering
02-16	S.H. Christiansen	Mixed boundary element method for eddy current problems
02-15	A. Toselli, X. Vasseur	Neumann-Neumann and FETI preconditioners for hp-approximations on geometrically refined boundary layer meshes in two dimensions
02-14	Th.P. Wihler	Locking-Free DGFEM for Elasticity Problems in Polygons
02-13	S. Beuchler, R. Schneider, C. Schwab	Multiresolution weighted norm equivalences and applications
02-12	M. Kruzik, A. Prohl	Macroscopic modeling of magnetic hysteresis
02-11	A.-M. Matache, C. Schwab, T. von Petersdorff	Fast deterministic pricing of options on Lévy driven assets
02-10	D. Schötzau, C. Schwab, A. Toselli	Mixed hp-DGFEM for incompressible flows
02-09	Ph. Frauenfelder, Ch. Lage	Concepts - An object-oriented software package for partial differential equations