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hp Finite Element Approximations on Non-Matching Grids for the Stokes Problem[*]

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Abstract

We propose and analyze a domain decomposition method on non-matching grids for hp-finite element approximations of the Stokes problem in two dimensions. No weak or strong continuity of the discrete velocities, is imposed across the boundaries of the subdomains. Instead, we employ suitable bilinear forms defined on the common interfaces, typical of discontinuous Galerkin approximations. Our main result is the divergence stability of some finite element approximations on geometrically conforming and non-conforming subdomain partitions. Our lower bound for the inf-sup constant depends on the stability constants of the local problems and the subdomain partition. Our bounds show a slight degradation with the polynomial degree for non-conforming partitions.

Keywords: Mixed problems, hp Finite Element Method, non-matching grids, discontinuous Galerkin approximations

1 Introduction

In this paper we consider the Stokes problem in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

This system of differential equations describes the motion of an incompressible viscous fluid with no convection. Here, $\nu > 0$ is the viscosity of the fluid and $\mathbf{f} : \Omega \to \mathbb{R}^2$ is an external force. The unknown fields are the velocity $\mathbf{u}$ and the pressure p. The second equation represents the incompressibility condition. For simplicity we only consider homogeneous Dirichlet boundary conditions.

The computational domain Ω is supposed to be partitioned into a finite number of subdomains. We wish to employ different and independent conforming hp finite element approximations on each subdomain. As opposed to the mortar method, where weak continuity conditions are imposed on the velocity across the subdomain boundary, we employ a discontinuous Galerkin (DG) approach here. No kind of continuity is imposed across the interface between the subdomains but suitable bilinear forms defined on the interface are added to the variational formulation of the problem in order to ensure the consistency and the well-posedness of the discrete problem.

DG methods have a long history and have recently become more and more popular. They have been heavily tested and studied, and they present considerable advantages for certain types of problems, especially those modelling phenomena where convection is moderate or strong; see the monograph [9]. In addition, more general meshes can be employed than in the case of conforming approximations and thus simpler adaptive strategies are possible.

The main result of this paper is the divergence stability of some finite element approximations obtained using a DG approach for the case of a fixed subdomain partition and is given in Theorem 4.1. Roughly speaking, the lower bound found for the inf-sup constant depends on the inf-sup constants of the local problems and the subdomain partition. If the partition is geometrically conforming, the constant exhibits the same dependence in the polynomial degree as the local ones, and only depends on the topology of the partition, but not on the number of subdomains or their size. If the partition is not conforming our bounds show a slight degradation with the polynomial degree. Only numerical results will be able to show if our bounds are sharp, and to compare our approach with a mortar one. We note in particular that a similar approach as in [2] can be also employed using a DG approach for spectral element approximations. We do not consider this case in this work.

Some work has already been done for the approximation of the Stokes problem on non-matching grids using a mortar approach:
In [1] hp finite element approximations are considered with conforming subdomain partitions. The analysis of the divergence stability of our DG method borrows some techniques originally employed there; see in particular the con-

struction of interface functions in Lemma 5.1 and in [1, Lemma 3.1], and the use of the connectivity matrix of the subdomain partition. In [2], a mortar method for spectral approximations is proposed. The techniques employed for the divergence stability of the corresponding approximations are similar to those in [1], but they rely on different technical tools, a fact that does not allow to combine them with those in [1] in a straightforward way in order to analyse approximations where finite and spectral elements are coupled.

Ours is not the first method where a DG approached is employed with approximations on non-matching grids for the Stokes problem. In [10], a similar approach is proposed and analysed for the Stokes and Navier-Stokes problems. There however the analysis is carried out only for the case of two subdomains and severe restrictions are imposed on the local meshes. In particular, on the interface between the subdomains, a mesh must be a refinement of the other. Moreover, in the discrete problem, extra jump terms are added to the interface bilinear forms, which do not appear to be necessary in our approach.

The remainder of this paper is organised as follows:
In section 2 we present the continuous Stokes problem. In section 3 we define the finite element spaces and make precise assumptions on the subdomain partition. The discrete problem is derived in section 4, where, in particular, we define discrete bilinear forms and norms. In section 5 we derive some technical tools needed in section 6, where the divergence stability is proven for the case of a fixed decomposition. Finally, the analysis of the well-posedness of the discrete problem and a priori estimates are presented in section 7.

2 Problem Setting

Let Ω be a bounded polyhedral domain in $\mathbb{R}^2$. For $\mathcal{D} \subseteq \mathbb{R}^2$ we introduce the following spaces

$$
\begin{aligned}
L^2(\mathcal{D}) &= \left\{ v : \mathcal{D} \to \mathbb{R} \mid \int_{\mathcal{D}} |v|^2 \, dx < \infty \right\}, \\
L_0^2(\mathcal{D}) &= \left\{ v \in L^2(\mathcal{D}) \mid \int_{\mathcal{D}} v \, dx = 0 \right\}, \\
H^m(\mathcal{D}) &= \left\{ v \in L^2(\mathcal{D}) \mid \partial^\alpha v \in L^2(\mathcal{D}), \quad |\alpha| \le m \right\}, \quad m \in \mathbb{N}_0, \\
H_0^1(\mathcal{D}) &= \left\{ v \in H^1(\mathcal{D}) \mid v = 0 \text{ on } \partial\mathcal{D} \right\}.
\end{aligned}
$$

In the following, $(u, v)_\mathcal{D}$, $(\mathbf{u}, \mathbf{v})_\mathcal{D}$, and $(\tau, \epsilon)_\mathcal{D}$ denote the scalar products in $L^2(\mathcal{D})$, $L^2(\mathcal{D})^2$, and $L^2(\mathcal{D})^{2\times 2}$, respectively, with $\|u\|_\mathcal{D}$, $\|\mathbf{u}\|_\mathcal{D}$, and $\|\tau\|_\mathcal{D}$ the corresponding norms. We denote the norm of $H^s(\mathcal{D})$ or $H^s(\mathcal{D})^n$, $s \in \mathbb{R}$, by $\| \cdot \|_{s,\mathcal{D}}$. Analogous notations are employed for the corresponding semi-norms for $s > 0$. In case $\mathcal{D} = \Omega$, we drop the subscript Ω and, in case $s = 0$, we also drop the subscript 0. We recall that the semi-norm $|\mathbf{u}|_{1,\Omega} = \|\nabla \mathbf{u}\|_{0,\Omega}$ is a norm

in $H_0^1(\Omega)^2$. For $\mathcal{D} \subseteq \mathbb{R}^2$ we denote by $|\mathcal{D}|$ the area of $\mathcal{D}$.

For a vector $\mathbf{u}$, the tensor $\nabla \mathbf{u}$ is defined by

$$(\nabla \mathbf{u})_{ij} = u_{i/j} = \frac{\partial u_i}{\partial x_j},$$

with u_i the i-th component of $\mathbf{u}$.

Given $\mathbf{f} \in L^2(\Omega)^2$ and $\nu > 0$, the Stokes problem (1) can be written in variational form as: Find $\mathbf{u} \in H_0^1(\Omega)^2, p \in L_0^2(\Omega)$, such that

$$\begin{cases} \nu\,(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega, & \mathbf{v} \in H_0^1(\Omega)^2, \\ (\nabla \cdot \mathbf{u}, q)_\Omega = 0, & q \in L_0^2(\Omega). \end{cases} \tag{2}$$

The well-posedness of this problem is ensured by the two stability conditions

$$\nu\,(\nabla \mathbf{u}, \nabla \mathbf{v}) \leq \nu\,|\mathbf{u}|_1\,|\mathbf{v}|_1, \tag{3}$$

$$(\nabla \cdot \mathbf{u}, p) \leq \sqrt{2}\,|\mathbf{u}|_1\,\|p\|, \tag{4}$$

the coercivity condition

$$\nu\,(\nabla \mathbf{u}, \nabla \mathbf{u}) \geq \nu\,|\mathbf{u}|_1^2, \quad \mathbf{u} \in H_0^1(\Omega)^2, \tag{5}$$

and the divergence stability condition

$$\sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)^2} \frac{(\nabla \cdot \mathbf{v}, p)}{|\mathbf{v}|_1} \geq \gamma\,\|p\|, \quad p \in L_0^2(\Omega), \quad \gamma > 0, \tag{6}$$

see, e.g., [7, Chapter II] for a comprehensive analysis.

3   Finite Element Spaces

We partition Ω into N non-overlapping, shape-regular polygonal subdomains Ω_i, $i = 1, \ldots, N$, of diameter H_i, with $H := \max\{H_i\}$. We assume that our partition is shape-regular, i.e., the aspect ratio of subdomains is bounded. In this paper we do not assume that this partition is geometrically conforming (regular), i.e., that the intersections between two different subdomains are either empty, or a vertex or an edge that is common to both subdomains, but we also consider non-conforming (irregular) partitions. We make the following assumption.

Assumption 3.1. *The subdomain partition is shape-regular and the length of the sides of each polygon Ω_i is comparable to its diameter H_i.*

On each Ω_i we then introduce a conforming, shape-regular affine quadrilateral mesh $\mathcal{T}_i$ of maximum diameter h_i; see, e.g., [14]. These meshes are independent and they do not need to match across the subdomain interfaces. In each subdomain we then introduce a conforming and divergence stable approximation for the Stokes problem:

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- $\mathbb{Q}_{k+2}$ - $\mathbb{Q}_k$ with discontinuous pressures; see, e.g., [17].

- $\mathbb{Q}_{k+1}$ - $\mathbb{Q}_k$ with continuous pressures, also known as Taylor-Hood elements: see, e.g., [6, 7].

Other choices are also possible; see, e.g., [4].

More precisely, on each subdomain we choose one of the following velocity/pressure pairs for $k_i \geq 0$:

$$V_{k_i}(\Omega_i) = \left\{ \mathbf{u} \in H^1(\Omega_i)^2 \mid \mathbf{u}_{|_\kappa} \in \mathbb{Q}_{k_i+2}(\kappa)^2 \quad \kappa \in \mathcal{T}_i, \mathbf{u}_{|_{\partial\Omega \cap \partial\Omega_i}} = 0 \right\}, \quad (7)$$

$$M_{k_i}(\Omega_i) = \left\{ p \in L^2(\Omega_i) \mid p_{|_\kappa} \in \mathbb{Q}_{k_i}(\kappa) \quad \kappa \in \mathcal{T}_i \right\}, \quad (8)$$

or

$$V_{k_i}(\Omega_i) = \left\{ \mathbf{u} \in H^1(\Omega_i)^2 \mid \mathbf{u}_{|_\kappa} \in \mathbb{Q}_{k_i+2}(\kappa)^2 \quad \kappa \in \mathcal{T}_i, \mathbf{u}_{|_{\partial\Omega \cap \partial\Omega_i}} = 0 \right\}, \quad (9)$$

$$M_{k_i}(\Omega_i) = \left\{ p \in H^1(\Omega_i) \mid p_{|_\kappa} \in \mathbb{Q}_{k_i+1}(\kappa) \quad \kappa \in \mathcal{T}_i \right\}, \quad (10)$$

where $\mathbb{Q}_k(\kappa)$ is the space of the polynomials of maximum degree k in each variable on κ. We define the N-vector $\mathbf{k} := \{k_1, k_2, \ldots, k_N\}$ and we take $k := \max\{\mathbf{k}\}$. The global approximation spaces are defined as

$$V_k = V_k(\Omega) \quad := \quad \prod_{i=1}^N V_{k_i}(\Omega_i), \quad (11)$$

$$M_k = M_k(\Omega) \quad := \quad L_0^2(\Omega) \cap \prod_{i=1}^N M_{k_i}(\Omega_i). \quad (12)$$

Given a vector $\mathbf{w}$ or a function v, we denote by $\mathbf{w}_i$ and v_i respectively, their restrictions to Ω_i. We next define the intersections

$$E_{ij} = \partial\Omega_i \cap \partial\Omega_j,$$

the set

$$M = \{(i,j) \mid \text{lenght}\, (E_{ij}) \neq 0, \quad i \neq j\},$$

and the skeleton

$$\Gamma = \bigcup_{(i,j) \in M} E_{ij}.$$

We note that one edge $E = E_{ij} = E_{ij}$ corresponds to two couples in M and, since the subdomain partition may not be geometrical conforming, it may not coincide with an entire side of the polygons Ω_i and Ω_j.

Given an interior edge $E \in \Gamma$, there are two subdomains, Ω_i and Ω_j, with, e.g., $i < j$, that share this edge. We define the jump $[v]$ and the average $< v >$ on E as

$$[v]_E = v_{i|_E} - v_{j|_E}, \quad < v >_E = \frac{1}{2} \left(v_{i|_E} + v_{j|_E} \right),$$

and $\mathbf{n}$ as the unit normal which points from Ω_i to Ω_j, i.e., $\mathbf{n} = \mathbf{n}_i$.

The following local stability result holds.

Lemma 3.1. *There exist constants γ_{k_i} independent of $\mathcal{T}_i$ such that:*

$$\sup_{\mathbf{v}_i \in V_{k_i}(\Omega_i) \cap H_0^1(\Omega_i)} \frac{-\displaystyle\int_{\Omega_i} \operatorname{div} \mathbf{v}_i \, p_i \, dx}{|\mathbf{v}_i|_{1,\Omega_i}} \geq \gamma_{k_i} \|p_i\|_{0,\Omega_i}, \quad p_i \in M_{k_i}(\Omega_i) \cap L_0^2(\Omega_i).$$

For the case of $\mathbb{Q}_{k+2} - \mathbb{Q}_k$ elements, the inf-sup constant depends only on k: $\gamma_k \geq ck^{-1/2}$; see [17]. We recall that this bound is sharp; see [3, Remark 25.2]. For the case $\mathbb{Q}_{k+2} - \mathbb{Q}_{k+1}$ elements, we know of no theoretical sharp bound explicit in k, but numerical evidence shows that $\gamma_k \sim ck^{-\alpha}$, with $\alpha = 1/2$ and c independent of the local mesh size; see [18].

The local meshes are required to satisfy the following property:

Assumption 3.2. *There exists constants such that for $(i,j) \in M$:*

$$c\,h_j \leq h_i \leq C\,h_j$$

We define $\underline{i}$ as the set of the indices j so that the pair $(i,j) \in M$. To a given decomposition we associate a connectivity matrix $A = (a_{ij})_{1 \leq i,j \leq N}$, the entries of which are defined by:

$$a_{ij} = \begin{cases} \operatorname{card}(\underline{i}), & \text{if } j = i, \\ -1, & \text{if } j \in \underline{i}, \\ 0, & \text{otherwise.} \end{cases}$$

The symbol $\operatorname{card}(\underline{i})$ denotes the cardinality of the set $\underline{i}$, or, in other words, the number of the neighbours of Ω_i. This connectivity matrix describes the topology of the decomposition of Ω and does not depend on the size of the subdomains. We remark that $\operatorname{card}(\underline{i})$ gives an upper bound for the number of sides of the polygon Ω_i.
We make the following assumption:

Assumption 3.3. *For each subdomain Ω_i, the number of neighbours is uniformly bounded, i.e., there exists a constant C such that*

$$\operatorname{card}(\underline{i}) \leq C, \quad i = 1, \ldots, N.$$

Before proceeding, we recall some definitions and properties.
An $N \times N$ matrix $B = (b_{ij})_{1 \leq i,j \leq N}$ is an L-matrix if

$$b_{ii} > 0 \qquad \text{and} \qquad b_{ij} \leq 0, \quad i \neq j.$$

In addition, B is said to be irreducible if, for any pair i,j $(1 \leq i,j \leq N)$, there exists a sequence $i_1, i_2, \ldots, i_n$ such that

$$b_{ii_1} \cdot b_{i_1 i_2} \cdot \ldots \cdot b_{i_n j} \neq 0$$

Since Ω is connected, it is then easy to check that the connectivity matrix A is an irreducible L-matrix. The proof of the following property can be found in [1, Lemma 4.1].

Lemma 3.2. *Let A be a symmetric, irreducible L-matrix that satisfies*

$$\sum_{j=1}^{N} a_{ij} = 0 \qquad i = 1, \dots, N.$$

Then, its eigenvalues $(\lambda_i)_{1 \le i \le N}$ are all nonnegative and, if in increasing order, the first eigenvalue $\lambda_1 = 0$ is simple.

4   Discrete Problem

In this section we introduce a DG formulation. Unlike the mortar finite element method, where the continuity of the velocities between subdomains is imposed through suitable matching conditions, here we take independent discrete velocity spaces on the subdomains. As in DG approximations on conforming meshes, the idea is to consider Problem (1) on each subdomain Ω_i and impose Dirichlet conditions weakly on the boundary $\partial\Omega_i$ using the value on the boundary of the neighbouring subdomains. We then choose suitable numerical fluxes on the interface Γ. Finally an interface term penalising the jumps of the velocity is added, as for similar DG approximations of second order problems. This is a standard procedure in the derivation of DG formulations; see, e.g., [13, 8, 12]. Here the penalization term is chosen as

$$\int_{\Gamma} \sigma[\mathbf{u}] \cdot [\mathbf{v}] \, ds = \frac{1}{2} \sum_{(i,j) \in M} \int_{E_{ij}} \sigma[\mathbf{u}] \cdot [\mathbf{v}] \, ds = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \underline{i}} \int_{E_{ij}} \sigma[\mathbf{u}] \cdot [\mathbf{v}] \, ds, \quad (13)$$

where the penalization coefficient for the velocity space is

$$\sigma(x) = \sigma_0 \frac{\mathbf{k}(x)^2}{\mathbf{h}(x)}, \qquad x \in \Gamma, \tag{14}$$

with σ_0 a positive constant, and

$$\mathbf{k}(x) = \begin{cases} \max\{k_i, k_j\}, & \text{if } x \in \partial\Omega_i \cap \partial\Omega_j, \\ k_i, & \text{if } x \in \partial\Omega_i \cap \partial\Omega, \end{cases} \tag{15}$$

and

$$\mathbf{h}(x) = \begin{cases} \min\{h_i, h_j\}, & \text{if } x \in \partial\Omega_i \cap \partial\Omega_j, \\ h_i, & \text{if } x \in \partial\Omega_i \cap \partial\Omega; \end{cases} \tag{16}$$

see [15].

Following [18], we introduce the following bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$:

$$a(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{N} \int_{\Omega_i} \nu \, \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx + \int_{\Gamma} \sigma \, \nu \, [\mathbf{u}] \cdot [\mathbf{v}] \, ds +$$

$$+ \int_{\Gamma} ([\mathbf{u}] < \nu \, \nabla \mathbf{v} : \mathbf{n} > - [\mathbf{v}] < \nu \, \nabla \mathbf{u} : \mathbf{n} >) \, ds, \tag{17}$$

$$b(\mathbf{u}, q) := - \sum_{i=1}^{N} \int_{\Omega_i} \operatorname{div} \mathbf{u} \cdot q \, dx + \int_{\Gamma} < q > [\mathbf{u} \cdot \mathbf{n}] \, ds, \tag{18}$$

and define the following discrete problem:
Find $(\mathbf{u}, p) \in V_k \times M_k$ such that:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \displaystyle\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, & \mathbf{v} \in V_k, \\[2mm] b(\mathbf{u}, q) = 0, & q \in M_k. \end{cases} \tag{19}$$

We note that, by integrating by parts, we can also write $b(\cdot, \cdot)$ as

$$b(\mathbf{v}, p) = \sum_{i=1}^{N} \int_{\Omega_i} \mathbf{v} \cdot \nabla p \, dx - \int_{\Gamma} [p] < \mathbf{v} \cdot \mathbf{n} > \, ds. \tag{20}$$

For discrete velocities we define the norm

$$|\mathbf{u}|_h^2 := \sum_{i=1}^{N} \|\nabla \mathbf{u}\|_{0,\Omega_i}^2 + \int_{\Gamma} \sigma \, |[\mathbf{u}]|^2 ds, \quad \mathbf{u} \in V_k. \tag{21}$$

The main result of this paper is the following divergence stability property.

Theorem 4.1. *There exists a positive constant β_k such that*

$$\sup_{0 \neq \mathbf{v} \in V_k} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \beta_k \, \|p\|, \quad p \in M_k, \tag{22}$$

where β_k depends on the constants γ_{k_i} of Lemma 3.1 and the partition of Ω.

The precise form of β_k is given in section 6.1.

5 Technical Tools

In this section we develop some tools needed for the proof of Theorem 4.1. We have the following important property.

Lemma 5.1. *Let $E_{ij} = \partial \Omega_i \cap \partial \Omega_j$. Then there exist functions $\mathbf{w}^{ij} \in V_k$ such that*

$$\mathbf{w}^{ij} = 0 \ \text{on} \ \overline{\Omega} \setminus (\Omega_i \cup \Omega_j \cup E_{ij}), \tag{23}$$

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$$\int\limits_{E_{ij}} \mathbf{w}_i^{ij} \cdot \mathbf{n}\, ds = \int\limits_{E_{ij}} \mathbf{w}_j^{ij} \cdot \mathbf{n}\, ds = 1, \tag{24}$$

$$|\mathbf{w}^{ij}|_h \leq \alpha_{ij} = \widehat{\alpha}_{ij} \,/\, |E_{ij}|, \tag{25}$$

$$\mathbf{w}^{ij} = \mathbf{w}^{ji}. \tag{26}$$

with $\mathbf{w}_i^{ij}$ and $\mathbf{w}_j^{ij}$ the restrictions of $\mathbf{w}^{ij}$ to Ω_i and Ω_j, respectively.

Here we have $\widehat{\alpha}_{ij} \leq C$ for a conforming partition and $\widehat{\alpha}_{ij} \leq C \max(k_i, k_j)$ for a non-conforming partition, where the constant C depends only on the topology of the partition.

The remainder of this section is devoted to the proof of Lemma 5.1. This proof is carried out separately for the cases of a conforming and a non-conforming partition.

5.1 Proof for conforming partitions

Let $\overline{E}_{ij} = \partial\Omega_i \cap \partial\Omega_j = [z_1, z_2]$, where z_1 and z_2 are the endpoints of our edge. We suppose for simplicity that the edge E_{ij} is parallel to the x-axis. We define

$$w^{ij}(x) := \frac{6\,\varphi(x)}{(z_2 - z_1)}, \tag{27}$$

with the quadratic bubble

$$\varphi(x) := \frac{(x - z_1)(z_2 - x)}{(z_2 - z_1)^2}. \tag{28}$$

We note that

$$w^{ij}(z_1) = w^{ij}(z_2) = 0, \quad \int\limits_{E_{ij}} w^{ij}\, ds = 1.$$

This trace can be then extended by zero on the rest of $\partial\Omega_i$ (resp. $\partial\Omega_j$), in order to give a piecewise quadratic function defined on the boundary $\partial\Omega_i$ (resp. $\partial\Omega_j$). We take the extension $R_{\Omega_i} w^{ij}$ to Ω_i, as the piecewise quadratic, discrete harmonic extension of $\mathbf{w}^{ij}$ on the whole subdomain Ω_i. In order to find a bound for $\left|R_{\Omega_i} w^{ij}\right|_{1,\Omega_i}$ we use a scaling argument. We first consider a dilation $\widehat{x} \mapsto x$ that maps a reference domain $\widehat{\Omega}$ into Ω_i. We suppose that the edge E_{ij} is the image of the reference interval $\widehat{E} = (-1, 1)$. We can write

$$\left|R_{\Omega_i}\varphi\right|_{1,\Omega_i}^2 \leq \widehat{C}\,\left|R_{\widehat{\Omega}}\widehat{\varphi}\right|_{1,\widehat{\Omega}}^2 \leq \widehat{C}C_{\widehat{\Omega}}\,\|\widehat{\varphi}\|_{H_{00}^{1/2}(\widehat{E})}^2 \leq C,$$

where $\widehat{\varphi}(\widehat{x}) = (1 - \widehat{x}^2)/4$ and the constants only depend on the shape of Ω_i. We recall that $H_{00}^{1/2}(\widehat{E})$ is the largest subspace of $H^{1/2}(\widehat{E})$ for which the extension by zero from $\widehat{E}$ to the whole of $\partial\widehat{\Omega}$ is contained in $H^{1/2}(\partial\widehat{\Omega})$; see, e.g., [11]. Using (27) then yields

$$\left|R_{\Omega_i} w^{ij}\right|_{1,\Omega_i}^2 \leq C/|E_{ij}|^2. \tag{29}$$

Our velocities $\mathbf{w}_i^{ij} \in V_k(\Omega_i)$ and $\mathbf{w}_j^{ij} \in V_k(\Omega_j)$ are taken equal to the vectors $(0, R_{\Omega_i} w^{ij})$, $(0, R_{\Omega_j} w^{ij})$. We then have

$$\int\limits_{E_{ij}} \mathbf{w}_i^{ij} \cdot \mathbf{n}\, ds = \int\limits_{E_{ij}} \mathbf{w}_j^{ij} \cdot \mathbf{n}\, ds = \int\limits_{E_{ij}} w^{ij}\, ds = 1.$$

Finally, we define $\mathbf{w}^{ij} \in V_k$ as

$$\mathbf{w}^{ij}(x) = \begin{cases} \mathbf{w}_i^{ij}(x), & \text{if } x \in \Omega_i, \\ \mathbf{w}_j^{ij}(x), & \text{if } x \in \Omega_j, \\ 0, & \text{otherwise.} \end{cases}$$

We note that those functions are continuous across Γ.
Inequality (29) yields that

$$\begin{aligned} \left| \mathbf{w}^{ij} \right|_h &\leq \nu \left| \mathbf{w}^{ij} \right|_{1,\Omega_i} + \nu \left| \mathbf{w}^{ij} \right|_{1,\Omega_j} \leq \\ &\leq \frac{C}{|E_{ij}|} + \frac{C}{|E_{ij}|} \leq \frac{C}{|E_{ij}|} := \frac{\widehat{\alpha}_{ij}}{|E_{ij}|}, \end{aligned}$$

which proves (25).
Properties (23) and (24) follow directly from the construction of $\mathbf{w}^{ij}$, while (26) follows from the symmetry of the problem.

5.2 Proof for non-conforming partitions

We fix $(i, j) \in M$. As in the previous subsection, we suppose for simplicity that the edge E_{ij} is parallel to the x-axis. We define $\mathscr{N}$ as the set of the vertices of all the subdomains Ω_i.
For each pair $(i, j) \in M$ there exists two points z_1 and z_2 contained in $\mathscr{N}$ and lying on $\overline{E}_{ij}$ such that $E_{ij} = E_{ji} = (z_1, z_2)$. We note that, if the subdomain partition is not conforming, z_1 and z_2 may not be vertices of both subdomains; see Figure 1. We also define the following points:

- $z_1^{(i)}$ is the nearest mesh point to z_1 that belongs to $[z_1, z_2)$ and is a node of the triangulation of $\mathcal{T}_i$.

- $z_2^{(i)}$ is the nearest mesh point to z_2 that belongs to $(z_1, z_2]$ and is a node of the triangulation of $\mathcal{T}_i$.

An analogous definition holds for $z_1^{(j)}$ and $z_2^{(j)}$. We assume that the above points are defined for each edge. In particular, this is true if, e.g., the following Assumption is satisfied.

Assumption 5.1. *Let $E_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Then*

$$h_i \leq \frac{1}{2}|E_{ij}|, \quad h_j \leq \frac{1}{2}|E_{ij}|.$$
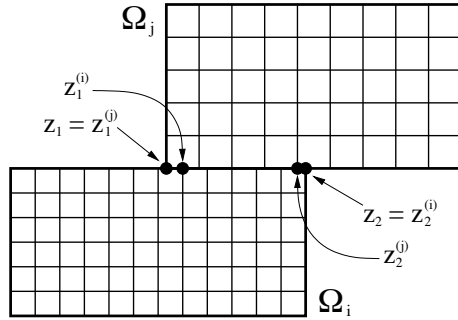
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Figure 1: Intersection of two substructures of a non-conforming partition.

We consider the following functions defined on E_{ij} for l equal to i or j, as showed in Figure 5.2:

$$
\varphi_l(x) = \begin{cases} \dfrac{\left(z_2^{(l)} - x\right)\left(x - z_1^{(l)}\right)}{\left(z_2^{(l)} - z_1^{(l)}\right)^2}, & \text{if } x \in \left[z_1^{(l)}, z_2^{(l)}\right], \\ 0, & \text{if } x \in E_{ij} \setminus \left[z_1^{(l)}, z_2^{(l)}\right]. \end{cases}
$$
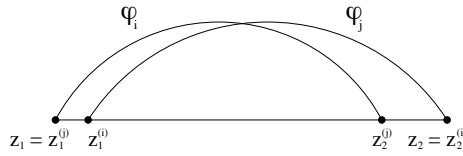
Figure 2:

We then define

$$
w_l(x) := \frac{\varphi_l(x)}{\beta_l},
$$

where

$$
\beta_l := \int_{z_1^{(l)}}^{z_2^{(l)}} \varphi_l(x)\, dx = \int_{E_{ij}} \varphi_l(x)\, dx = z_2^{(l)} - z_1^{(l)}.
$$

As for the case of a conforming decomposition, we extend these functions by zero to the rest of $\partial \Omega_i$ (resp. $\partial \Omega_j$) and we take $R_{\Omega_i} w_i$ (resp. $R_{\Omega_i} w_j$) as the discrete harmonic extension of w_i (resp. w_j) on the whole subdomain Ω_i (resp.

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Ω_i). We then define our velocity $\mathbf{w}^{ij} \in V_k$ as

$$\mathbf{w}^{ij}(x) = \begin{cases} (0, R_{\Omega_i} w_i), & \text{if } x \in \Omega_i, \\ (0, R_{\Omega_j} w_j), & \text{if } x \in \Omega_j, \\ 0, & \text{otherwise.} \end{cases}$$

Our first purpose is prove property (25), i.e., to find a bound for:

$$\left| \mathbf{w}^{ij} \right|_h^2 := \sum_{i=1}^N \left\| \nabla \mathbf{w}^{ij} \right\|_{0,\Omega_i}^2 + \int_\Gamma \sigma \left| \left[\mathbf{w}^{ij} \right] \right|^2 ds \qquad (30)$$

We note that, as opposed to the case of a conforming partition, this velocity is not continuous across E_{ij}. For the first term we proceed exactly as in the case of a conforming partition and, as for (29), we obtain

$$\left| \mathbf{w}^{ij} \right|_{1,\Omega_i}^2 \leq \frac{C}{|E_{ij}|^2}, \qquad \left| \mathbf{w}^{ij} \right|_{1,\Omega_j}^2 \leq \frac{C}{|E_{ij}|^2} \qquad (31)$$

For the second term we proceed in the following way.
We first map the interval $(z_1^{(l)}, z_2^{(l)})$ into the reference interval:

$$\widehat{t}_l : (z_1^{(l)}, z_2^{(l)}) \longrightarrow (-1,1), \qquad x \longmapsto \widehat{x} = \widetilde{t}_l(x) = \frac{2\left(x - Z^{(l)}\right)}{\Delta Z^{(l)}},$$

with $Z^{(l)} := \dfrac{z_1^{(l)} + z_2^{(l)}}{2}$ and $\Delta Z^{(l)} := z_2^{(l)} - z_1^{(l)}$. We can then write our functions φ_l as

$$\varphi_l(x) = \widehat{\varphi}(\widehat{t}_l(x)),$$

where

$$\widehat{\varphi}(x) = \frac{1 - x^2}{4}.$$

The second term in (30) can then be written as

$$\begin{aligned}
\int_\Gamma \sigma \left| \left[\mathbf{w}^{ij} \right] \right|^2 ds &= \int_{E_{ij}} \sigma \left| \left[\mathbf{w}^{ij} \right] \right|^2 ds = \int_{E_{ij}} \sigma \, (w_i - w_j)^2 \, ds = \\
&= \int_{E_{ij}} \sigma \left(\frac{\varphi_i(x)}{\beta_i} - \frac{\varphi_j(x)}{\beta_j} \right)^2 dx \leq \\
&\leq 2 \frac{\sigma}{\beta_j^2} \int_{E_{ij}} (\varphi_i(x) - \varphi_j(x))^2 \, dx + 2 \, \sigma \left(\frac{1}{\beta_i} - \frac{1}{\beta_j} \right)^2 \int_{E_{ij}} \varphi_i(x)^2 \, dx = \\
&=: A + B.
\end{aligned}$$

We start with the term A.
Using the mean-value theorem of differential calculus we can write the following bound:

$$|\varphi_i(x) - \varphi_j(x)| = \left|\widehat{\varphi}\left(\widehat{t}_i(x)\right) - \widehat{\varphi}\left(\widehat{t}_j(x)\right)\right| \leq |\widehat{\varphi}|_{1,\infty} \cdot \left|\widehat{t}_i(x) - \widehat{t}_j(x)\right|. \qquad (32)$$

The last term on the right-hand side can be further decomposed as

$$
\begin{aligned}
\widehat{t}_i(x) - \widehat{t}_j(x) &= \frac{2\left(x - Z^{(i)}\right)}{\Delta Z^{(i)}} - \frac{2\left(x - Z^{(j)}\right)}{\Delta Z^{(j)}} = \\
&= \frac{2\left(x - Z^{(i)}\right) - 2\left(x - Z^{(j)}\right)}{\Delta Z^{(i)}} + 2\left(x - Z^{(j)}\right)\left(\frac{1}{\Delta Z^{(i)}} - \frac{1}{\Delta Z^{(j)}}\right) = \\
&=: I + II,
\end{aligned}
\qquad (33)
$$

and, since

$$|\widehat{\varphi}|_{1,\infty}^2 \leq C, \qquad |I| \leq \frac{2\,\max(h_i, h_j)}{|E_{ij}|}, \qquad |II| \leq \frac{4\,\max(h_i, h_j)}{|E_{ij}|^2}\,|x - Z^{(j)}|,$$

it follows from the definition of σ and Assumption 3.2 that

$$
\begin{aligned}
A &\leq 4 \int\limits_{E_{ij}} \frac{\sigma}{\beta_j^2}\,|\widehat{\varphi}|_{1,\infty}^2\left(|I|^2 + |II|^2\right)\,dx \leq \\
&\leq \frac{C\,\max(k_i^2, k_j^2)}{\beta_j^2} + \frac{C\,\max(k_i^2, k_j^2)}{\beta_j^2} \leq \\
&\leq \frac{C\,\max(k_i^2, k_j^2)}{|E_{ij}|^2}.
\end{aligned}
\qquad (34)
$$

We now consider the term B.
Combining inequality (32) and property (33) and using similar argument as before, we obtain

$$
\begin{aligned}
|\beta_i - \beta_j| &\leq \int\limits_{E_{ij}} |\varphi_i(x) - \varphi_j(x)|\,dx \leq \\
&\leq |\widehat{\varphi}|_{1,\infty} \int\limits_{E_{ij}} \left|\widehat{t}_i(x) - \widehat{t}_j(x)\right|\,dx \leq \\
&\leq C\,\max(h_i, h_j),
\end{aligned}
$$

and thus, using the definition of σ and Assumption 3.2,

$$B = 2\,\sigma\left(\frac{1}{\beta_i} - \frac{1}{\beta_j}\right)^2 \int\limits_{E_{ij}} \varphi_i(x)^2\,dx \leq \frac{C\,\max(k_i^2, k_j^2)}{|E_{ij}|^2}, \qquad (35)$$

where we have used

$$\int\limits_{E_{ij}} \varphi(x)^2\,dx \leq C\,|E_{ij}|$$

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Combining the bounds for A and B in (34) and (35), we then obtain

$$\int_\Gamma \sigma \left|\left[\mathbf{w}^{ij}\right]\right|^2 ds \le \frac{C \max(k_i^2, k_j^2)}{|E_{ij}|^2}.$$
(36)

Finally, combining (31) and (36) yields

$$|\mathbf{w}^{ij}|_h^2 \le \frac{C}{|E_{ij}|^2} + \frac{C \max(k_i^2, k_j^2)}{|E_{ij}|^2} := \frac{\widehat{\alpha}_{ij}^2}{|E_{ij}|^2},$$

which proves (25). We note that here, as opposed to the conforming case, the constant α_{ij} also depends on the degrees $\mathbf{k}$. This is due to the fact that the functions $\mathbf{w}^{ij}$ are not continuous on the whole domain and the penalization term depends on the degrees $\mathbf{k}$.

The other properties of Lemma 5.1 follow directly from the definition of $\mathbf{w}^{ij}$.

6 Proof of Theorem 4.1

We define the set of all piecewise constant pressures

$$M_0 := \left\{ q \in L_0^2(\Omega) \mid q_{|\Omega_i} \in \mathbb{Q}_0(\Omega_i),\ i = 1, \dots, N \right\},$$
(37)

and the space defined by the velocities founded in the previous section

$$X := \operatorname{span}\left\{\mathbf{w}^{ij} \mid (i,j) \in M\right\} \subset V_k.$$

We use an argument which was originally proposed by Boland & Nicolaides; see [5]. Any $p \in M_k$ can be decomposed into two functions, one with zero mean value in each Ω_i, the other constant on each subdomain:

$$p = \widetilde{p} + \overline{p},$$
(38)

with

$$\overline{p}_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} p(x) dx.$$
(39)

Is easy to see that $\overline{p}$ belongs to M_0 and that $\widetilde{p}_i \in L_0^2(\Omega_i) \cap M_{k_i}(\Omega_i)$
We note that if $\overline{p} \in M_0$ then

$$\|\overline{p}\|_{0,\Omega}^2 = \sum_{i=1}^N \overline{p}_i^2 |\Omega_i|$$
(40)

$$\sum_{i=1}^N \overline{p}_i |\Omega_i| = 0.$$
(41)

We proceed by proving two stability results for $\widetilde{p}$ and $\overline{p}$.

Lemma 6.1. *For each $\widetilde{p}_i \in L_0^2(\Omega_i) \cap M_{k_i}(\Omega_i)$ there exists a velocity $\widetilde{\mathbf{v}}_i \in V_k(\Omega_i) \cap H_0^1(\Omega_i)$ such that*

$$
\begin{cases}
- \displaystyle\int_{\Omega_i} \operatorname{div} \widetilde{\mathbf{v}}_i \, \widetilde{p}_i \, dx = \|\widetilde{p}_i\|_{0,\Omega_i}^2, \\[2ex]
|\widetilde{\mathbf{v}}_i|_{1,\Omega_i} \leq \dfrac{1}{\gamma_{k_i}} \|\widetilde{p}_i\|_{0,\Omega_i}.
\end{cases}
$$

Proof. Since in each Ω_i we have chosen conforming and stable finite element spaces, Lemma 3.1 ensures the existence of this velocity. $\qquad\square$

We will then define $\widetilde{\mathbf{v}} \in V_k$ by

$$
\widetilde{\mathbf{v}}_{|\Omega_i} = \widetilde{\mathbf{v}}_i, \quad i = 1, \dots, N.
$$

Since $\widetilde{\mathbf{v}}$ vanishes on Γ, is continuous on Ω. Consequently we have

$$
b(\widetilde{\mathbf{v}}, \widetilde{p}) = \sum_{i=1}^{N} \|\widetilde{p}_i\|_{0,\Omega_i}^2 = \|\widetilde{p}\|_{0,\Omega}^2 \tag{42}
$$

and

$$
|\widetilde{\mathbf{v}}|_h^2 = \sum_{i=1}^{N} \|\nabla\widetilde{\mathbf{v}}\|_{0,\Omega_i}^2 \leq \sum_{i=1}^{N} \frac{C}{\gamma_{k_i}^2} \|\widetilde{p}_i\|_{0,\Omega_i}^2,
$$

i.e.,

$$
|\widetilde{v}|_h \leq \frac{C}{\gamma_k} \|\widetilde{p}\|_{0,\Omega} = \frac{1}{\widetilde{\beta}} \|\widetilde{p}\|_{0,\Omega}, \tag{43}
$$

where

$$
\gamma_k := \min_{1 \leq i \leq N} \gamma_{k_i}, \qquad \widetilde{\beta} := \gamma_k / C.
$$

Lemma 6.2. *There exists a constant $\overline{\beta}$, independent of h but dependent on the decomposition of Ω such that*

$$
\sup_{0 \neq \mathbf{v} \in X} \frac{b(\mathbf{v}, \overline{p})}{|\mathbf{v}|_h} \geq \overline{\beta} \, \|\overline{p}\|, \quad \overline{p} \in M_0.
$$

Proof. For every $\overline{p} \in M_0$, we construct a function $\overline{\mathbf{v}} \in V_k$ such that:

$$
\begin{aligned}
b(\overline{\mathbf{v}}, \overline{p}) &\geq \|\overline{p}\|_0^2, \\
|\overline{\mathbf{v}}|_h &\leq \frac{1}{\overline{\beta}} \|\overline{p}\|_0.
\end{aligned} \tag{44}
$$

Thanks to Lemma 5.1, we can find a function $\mathbf{w}^{ij} \in V_k$, supported in $\overline{\Omega}_i \cup \overline{\Omega}_j$ such that:

$$
\int_{E_{ij}} \mathbf{w}_i^{ij} \cdot \mathbf{n} \, ds = \int_{E_{ij}} \mathbf{w}_j^{ij} \cdot \mathbf{n} \, ds = \int_{\partial\Omega_i} \mathbf{w}_i^{ij} \cdot \mathbf{n}_i \, ds = - \int_{\partial\Omega_j} \mathbf{w}_j^{ij} \cdot \mathbf{n}_j \, ds = 1, \quad i > j. \tag{45}
$$

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The divergence theorem ensures that:

$$\int_{\Omega_i} \operatorname{div} \mathbf{w}_{ij}^{(i)} \, dx = - \int_{\Omega_j} \operatorname{div} \mathbf{w}_{ij}^{(j)} \, dx = 1. \tag{46}$$

We can also find a velocity $\overline{\mathbf{w}}^{ij} \in V_k$ supported in $\overline{\Omega}_i \cup \overline{\Omega}_j$, defined as $\overline{\mathbf{w}}^{ij} := |\Omega_i| \, \overline{p}_i \, \mathbf{w}^{ij}$, such that:

$$-\int_{\Omega_i} \operatorname{div} \overline{\mathbf{w}}_i^{ij} \, dx = \int_{\Omega_j} \operatorname{div} \overline{\mathbf{w}}_j^{ij} \, dx = |\Omega_i| \, \overline{p}_i.$$

Unlike the function $\mathbf{w}^{ij}$ defined in lemma (5.1), we see that these functions $\overline{\mathbf{w}}^{ij}$ and $\overline{\mathbf{w}}^{ji}$ are different. We then set

$$\overline{\mathbf{w}} := \sum_{i=1}^{N} \sum_{j \in \underline{i}} \overline{\mathbf{w}}^{ij}. \tag{47}$$

We can easily check that $\overline{\mathbf{w}}$ belongs to X. Indeed, there are two contributions in the sum in (47) for each edge E_{ij}, corresponding to E_{ij} and to $E_{ji} = E_{ij}$. We first note that for $(i,j) \in M$, $i > j$,

$$
\begin{aligned}
\int_{\Gamma} <\overline{p}> [\overline{\mathbf{w}}^{ij} \cdot \mathbf{n}] \, ds &= \sum_{(m,l) \in M} \int_{E_{ml}} <\overline{p}>_{E_{ml}} [\overline{\mathbf{w}}^{ij} \cdot \mathbf{n}]_{E_{ml}} \, ds = \\
&= \sum_{(m,l) \in M} <\overline{p}>_{E_{ml}} \int_{E_{ml}} [\overline{\mathbf{w}}^{ij} \cdot \mathbf{n}]_{E_{ml}} \, ds = \\
&= <\overline{p}>_{E_{ij}} (-|\Omega_i| \, \overline{p}_i + |\Omega_i| \, \overline{p}_i) = 0,
\end{aligned}
$$

where we have used (45). Using (46), we then find

$$
\begin{aligned}
b(\overline{\mathbf{w}}^{ij}, \overline{p}) &= -\sum_{m=1}^{N} \int_{\Omega_m} \operatorname{div} \overline{\mathbf{w}}_m^{ij} \cdot \overline{p}_m \, dx + \int_{\Gamma} <\overline{p}> [\overline{\mathbf{w}}^{ij} \cdot \mathbf{n}] \, ds = \\
&= -\int_{\Omega_i} \operatorname{div} \overline{\mathbf{w}}_i^{ij} \cdot \overline{p}_i \, dx - \int_{\Omega_j} \operatorname{div} \overline{\mathbf{w}}_j^{ij} \cdot \overline{p}_j \, dx = \\
&= -\overline{p}_i \int_{\Omega_i} \operatorname{div} \overline{\mathbf{w}}_i^{ij} \, dx - \overline{p}_j \int_{\Omega_j} \operatorname{div} \overline{\mathbf{w}}_j^{ij} \, dx = \\
&= \overline{p}_i \, |\Omega_i| \, \overline{p}_i - \overline{p}_j \, |\Omega_i| \, \overline{p}_i = \overline{p}_i \, |\Omega_i| \, (\overline{p}_i - \overline{p}_j).
\end{aligned}
$$

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We can then write

$$
\begin{aligned}
b(\overline{\mathbf{w}}, \overline{p}) &= \sum_{i=1}^{N} \sum_{j \in \underline{i}} b(\overline{\mathbf{w}}^{ij}, \overline{p}) = \\
&= \sum_{i=1}^{N} \sum_{j \in \underline{i}} \overline{p}_i \, |\Omega_i| \, (\overline{p}_i - \overline{p}_j) = \\
&= \sum_{i=1}^{N} \overline{p}_i \sum_{j \in \underline{i}} |\Omega_i| \, (\overline{p}_i - \overline{p}_j),
\end{aligned}
$$

or, equivalently,

$$
b(\overline{\mathbf{w}}, \overline{p}) = \overline{p}^T \, B \, \overline{p},
$$

where $B = (b_{ij})_{1 \le i, j \le N}$ is a a sparse matrix defined as

$$
b_{ij} = \begin{cases} | \, \Omega_i \, | \, \mathrm{card} \, (\underline{i}), & \text{if } j = i, \\ - \, | \, \Omega_i \, |, & \text{if } j \in \underline{i}, \\ 0, & \text{otherwise.} \end{cases}
$$

If we introduce the matrix $D = \mathrm{diag} \, (|\Omega_1|, \ldots, |\Omega_N|)$, we see that from (40)

$$
\overline{p}^T \, D \, \overline{p} = \|\overline{p}\|_{0,\Omega}^2, \quad \overline{p} \in M_0,
$$

where we have used the same notation for a function $\overline{p} \in M_0$ and the corresponding vectors of values $\overline{p}_i$. In order to prove the first of (44), we need to show that the minimum

$$
\gamma = \min_{\overline{p} \in P \setminus \{0\}} \frac{\overline{p}^T B \overline{p}}{\overline{p}^T D \overline{p}} \tag{48}
$$

is positive and does not depend on h. We consider the eigenvalue problem

$$
D^{-1} \, B \, \overline{p} = \lambda \, \overline{p}, \qquad \overline{p} \ne 0.
$$

It is easy to check that the matrix $A = D^{-1} B$ is the connectivity matrix defined in section 3. Thanks to Lemma 3.2, its eigenvalues λ_i are all positive except $\lambda_1 = 0$, which is simple. The kernel of A involves only constant vectors (see proof of [1, Lemma 4.1]). Therefore

$$
\gamma = \lambda_2 = \inf_{i \ge 2} \lambda_i.
$$

The choice $\overline{\mathbf{v}} = \gamma^{-1} \, \overline{\mathbf{w}}$ ensures that the first equation of (44) holds.
Using (25), we have

$$
|\overline{\mathbf{w}}^{ij}|_h \le \alpha_{ij} \, |\Omega_i| \, |\overline{p}_i|.
$$

With the definition of the h-norm, see (21), and property (13), we can also write

$$
|\overline{\mathbf{v}}|_h^2 = \sum_{l=1}^{N} |\overline{\mathbf{v}}|_{1,\Omega_l}^2 + \frac{1}{2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \, |[\overline{\mathbf{v}}]|^2 \, ds. \tag{49}
$$

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The first term in (49) can be written as

$$
\begin{aligned}
\sum_{l=1}^{N} |\overline{\mathbf{v}}|_{1,\Omega_l}^2 &= \frac{1}{\gamma^2} \sum_{l=1}^{N} \left| \sum_{n \in \underline{l}} \left(\overline{\mathbf{w}}^{nl} + \overline{\mathbf{w}}^{ln} \right) \right|_{1,\Omega_l}^2 \leq \\
&\leq \frac{2}{\gamma^2} \sum_{l=1}^{N} \operatorname{card}(\underline{l}) \sum_{n \in \underline{l}} \left(\left| \overline{\mathbf{w}}^{nl} \right|_{1,\Omega_l}^2 + \left| \overline{\mathbf{w}}^{ln} \right|_{1,\Omega_l}^2 \right) = \\
&= \frac{2}{\gamma^2} \left(\max_{1 \leq l \leq N} \operatorname{card}(\underline{l}) \right) \sum_{l=1}^{N} \sum_{n \in \underline{l}} \left(\left| \overline{\mathbf{w}}^{ln} \right|_{1,\Omega_l}^2 + \left| \overline{\mathbf{w}}^{ln} \right|_{1,\Omega_n}^2 \right).
\end{aligned}
$$

For the second term we find

$$
\begin{aligned}
\frac{1}{2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \left| [\overline{\mathbf{v}}] \right|^2 ds &= \frac{1}{2\gamma^2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \left| [\overline{\mathbf{w}}] \right|^2 ds = \\
&= \frac{1}{2\gamma^2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \left| \left[\overline{\mathbf{w}}^{ln} + \overline{\mathbf{w}}^{nl} \right] \right|^2 ds = \\
&\leq \frac{1}{\gamma^2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \left(\left| \left[\overline{\mathbf{w}}^{ln} \right] \right|^2 + \left| \left[\overline{\mathbf{w}}^{nl} \right] \right|^2 \right) ds = \\
&= \frac{2}{\gamma^2} \sum_{(l,n) \in M} \int_{E_{ln}} \sigma \left| \left[\overline{\mathbf{w}}^{ln} \right] \right|^2 ds = \\
&= \frac{2}{\gamma^2} \sum_{l=1}^{N} \sum_{n \in \underline{l}} \int_{E_{ln}} \sigma \left| \left[\overline{\mathbf{w}}^{ln} \right] \right|^2 ds.
\end{aligned}
$$

Combining these two inequalities, using the definition of the h-norm first and

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of $\overline{\mathbf{w}}^{ln}$ then, we obtain

$$
\begin{aligned}
|\overline{\mathbf{v}}|_h^2 \;\leq\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)\sum_{l=1}^{N}\sum_{n\in\underline{l}}\left(\left|\overline{\mathbf{w}}^{ln}\right|_{1,\Omega_l}^2+\left|\overline{\mathbf{w}}^{ln}\right|_{1,\Omega_n}^2\right)+ \\
& \hspace{4cm} +\frac{2}{\gamma^2}\sum_{l=1}^{N}\sum_{n\in\underline{l}}\int_{E_{ln}}\sigma\left|\left[\overline{\mathbf{w}}^{ln}\right]\right|^2\,ds\;\leq \\
\leq\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)\sum_{l=1}^{N}\sum_{n\in\underline{l}}\left|\overline{\mathbf{w}}^{ln}\right|_h^2\;\leq \\
\leq\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)\sum_{l=1}^{N}\sum_{n\in\underline{l}}\alpha_{ln}^2\,\overline{p}_l^2\,|\Omega_l|^2\;= \\
=\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)\sum_{l=1}^{N}\overline{p}_l^2\,|\Omega_l|^2\sum_{n\in\underline{l}}\alpha_{ln}^2\;\leq \\
\leq\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)^2\sum_{l=1}^{N}\overline{p}_l^2\,|\Omega_l|^2\left(\max_{n\in\underline{l}}\frac{\widehat{\alpha}_{ln}^2}{|E_{ln}|^2}\right)\;\leq \\
\leq\; & \frac{2}{\gamma^2}\left(\max_{1\leq l\leq N}\operatorname{card}(\underline{l})\right)^2\left(\max_{1\leq l\leq N}\eta_l\right)\sum_{l=1}^{N}\overline{p}_l^2\,|\Omega_l|,
\end{aligned}
$$

where

$$
\eta_l := \max_{n\in\underline{l}}\left(\frac{\widehat{\alpha}_{ln}^2}{|E_{ln}|^2}\right)|\Omega_l|.
$$

Using Assumption 3.3 and equation (40) we have also proved the second equation of (44) with

$$
\overline{\beta}=\frac{1}{C\left(\displaystyle\max_{1\leq l\leq N}\eta_l\right)}.
$$

$\square$

Now we are able to check the inf-sup condition (22). The pressure can be decomposed as $p=\widetilde{p}+\overline{p}$. We define also the velocity

$$
\mathbf{v}=\widetilde{\mathbf{v}}+\lambda\,\overline{\mathbf{v}}, \tag{50}
$$

where λ is a real number to be chosen later.
It is easy to check, since $\overline{p}\in P$ and $\widetilde{\mathbf{v}}_i\in H_0^1(\Omega_i)$ $(1\leq i\leq N)$, that $b(\widetilde{\mathbf{v}},\overline{p})=0$.

In addition, since the bilinear form $b(\cdot, \cdot)$ is continuous, we have:

$$
\begin{aligned}
b(\mathbf{v}, p) &= b(\widetilde{\mathbf{v}}, \widetilde{p}) + b(\widetilde{\mathbf{v}}.\overline{p}) + \lambda\, b(\overline{\mathbf{v}}, \widetilde{p}) + \lambda\, b(\overline{\mathbf{v}}, \overline{p}) \geq \\
&\geq \|\widetilde{p}\|_0^2 - \lambda\, |b(\overline{\mathbf{v}}, \widetilde{p})| + \lambda\, \|\overline{p}\|_0^2 \geq \\
&\geq \|\widetilde{p}\|_0^2 - \lambda\, c\, |\overline{\mathbf{v}}|_h\, \|\widetilde{p}\|_0 + \lambda\, \|\overline{p}\|_0^2 \geq \\
&\geq \|\widetilde{p}\|_0^2 - \frac{\lambda\, c}{\overline{\beta}}\, \|\overline{p}\|_0\, \|\widetilde{p}\|_0 + \lambda\, \|\overline{p}\|_0^2 \geq \\
&\geq \frac{1}{2}\, \|\widetilde{p}\|_0^2 + \lambda\, \left(1 - \frac{\lambda\, c^2}{2\, \overline{\beta}^2}\right)\, \|\overline{p}\|_0^2.
\end{aligned}
$$

The choice $\lambda = \dfrac{\overline{\beta}^2}{c^2}$ ensures

$$
\begin{aligned}
b(\mathbf{v}, p) &\geq \frac{1}{2}\, \|\widetilde{p}\|_0^2 + \frac{\overline{\beta}^2}{2\, c^2}\, \|\overline{p}\|_0^2 \geq \\
&\geq \frac{1}{2}\, \min\left\{1, \frac{\overline{\beta}^2}{c^2}\right\}\, \|p\|_0^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{v}|_h &= |\widetilde{\mathbf{v}} + \lambda\, \overline{\mathbf{v}}|_h \leq |\widetilde{\mathbf{v}}|_h + \lambda\, |\overline{\mathbf{v}}|_h \leq \frac{1}{\widetilde{\beta}}\, \|\widetilde{p}\|_0 + \frac{\lambda}{\overline{\beta}}\, \|\overline{p}\|_0 \leq \\
&\leq \sqrt{\frac{1}{\widetilde{\beta}^2} + \frac{\lambda^2}{\overline{\beta}^2}}\, \|p\|_0 = \sqrt{\frac{1}{\widetilde{\beta}^2} + \frac{\overline{\beta}^2}{c^4}}\, \|p\|_0.
\end{aligned}
$$

We also have proved Theorem 4.1 with

$$
\beta_k = \frac{\min\{1, \frac{\overline{\beta}^2}{c^2}\}}{2\sqrt{\frac{1}{\widetilde{\beta}^2} + \frac{\overline{\beta}^2}{c^4}}} = \frac{\widetilde{\beta}\, \overline{\beta}^2}{2\sqrt{c^4 + \widetilde{\beta}^2\, \overline{\beta}^2}} \approx C\, \widetilde{\beta}\, \overline{\beta}^2
$$

6.1   Remarks on the inf-sup constant

In this section we want to analyse the inf-sup constant β_k found previously for the case of conforming and non-conforming partitions. First we have to analyse the constants $\widetilde{\beta}$ and $\overline{\beta}$ in the two cases. In Lemma 6.1 we have seen that there is no distinction for $\widetilde{\beta}$: in both cases this constant depends only on the γ_{k_i} of Lemma 3.1.

However, for $\overline{\beta}$ we have to separate the cases. With a conforming partition, using Assumption 3.1 and the property

$$
|\Omega_i| \leq C\, H_l^2,
$$

which bound the term $\max \eta_l$, follows that

$$\overline{\beta} \geq c\,\gamma.$$

We have also found that $\overline{\beta}$ depends only on the second eigenvalue $\gamma = \lambda_2$ of our connectivity matrix, i.e., depends only on the topology of the decomposition in subdomains. Likewise we have that

$$\beta_k \geq c\,\gamma^2\,(\min\gamma_{k_i})$$

depends only on the topology of the decomposition and on the γ_{k_i}.
Otherwise, in the case of a non-conforming partition, we recall that

$$\widehat{\alpha}_{ij} \leq C\,\max(k_i, k_j);$$

therefore follows that

$$\overline{\beta} \geq c\,\gamma\,k^{-1},$$

where c depends to the size of the edges E_{ij}. We have also found that our inf-sup constant β_k, besides depending to γ, k and the γ_{k_i}, depends to the partition in subdomains (not only the topology).

7 Well-posedness and a priori estimates

This section is based on [18, section 7], also we will omit the proofs and write only the results.
Before proceeding, we note that our discrete bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are not continuous on the original spaces $H^1(\Omega)$ and $L_0^2(\Omega)$, due to the interface contributions. This makes the analysis more complicated. However two weaker continuity properties hold. We need to define two suitable stronger norms. For a velocity $\mathbf{V}$ we set

$$|||\mathbf{V}|||_v^2 = |\mathbf{V}|_h^2 + \sum_{e \in M} \int_e \sigma\,|[\mathbf{V}]|^2\,ds + \sum_{i=1}^N \int_{\partial\Omega_i} \frac{1}{\sigma}\,|\nabla\mathbf{V}|^2\,ds.$$

We note however that, in case $\mathbf{v} \in V_k$, the inverse estimate [16, Eqq. 4.6.4 and 4.6.5] and the definition of σ ensure that

$$|\mathbf{v}|_h \leq |||\mathbf{v}|||_v \leq C\,|\mathbf{v}|_h, \tag{51}$$

with a constant C that only depends on σ_0. We have the following property.

Lemma 7.1. *Let $\mathbf{V} \in L^2(\Omega)^2$, such that $\mathbf{V} \in H^2(\Omega_i)^2$, for $i = 1, \ldots, N$, and $\mathbf{w} \in V_k$. Then there exist constants independent of $\mathbf{V}, \mathbf{w}, h$ and k such that*

$$|a(\mathbf{V}, \mathbf{w})| \leq \alpha\,\nu\,|||\mathbf{V}|||_v\,|\mathbf{w}|_h,$$

and, in case $\mathbf{V} \in V_k$,

$$|a(\mathbf{V}, \mathbf{w})| \leq \alpha'\,\nu\,|\mathbf{V}|_v\,|\mathbf{w}|_h.$$

Analogously, we define a stronger norm for the pressure:

$$|||Q|||_p^2 = ||Q||_{0,\Omega}^2 + \sum_{i=1}^{N} \int_{\partial\Omega_i} \frac{1}{\sigma} Q^2 \, ds.$$

In case $q \in M_k$, the inverse estimate yields

$$||q||_{0,\Omega} \leq |||q|||_p \leq C \, ||q||_{0,\Omega}, \tag{52}$$

with a constant that depends only on σ_0.

Lemma 7.2. *Let $Q \in L_0^2(\Omega)$ and $\mathbf{v} \in L^2(\Omega)^2$, such that $Q \in H^1(\Omega_i)$ and $\mathbf{v} \in H^1(\Omega_i)$, $i = 1, \ldots, N$. Then there exist constant independent of $Q, \mathbf{v}, h$ and k such that*

$$|b(\mathbf{v}, Q)| \leq \beta \, |\mathbf{v}|_h \, |||Q|||_p,$$

and, in case $Q \in M_k$,

$$|b(\mathbf{v}, Q)| \leq \beta' \, |\mathbf{v}|_h \, ||Q||_0.$$

We finally recall that the bilinear form $a(\cdot, \cdot)$ is coercive, i.e.,

$$a(\mathbf{u}, \mathbf{u}) = \nu \, |\mathbf{u}|_h^2, \qquad \mathbf{u} \in V_k. \tag{53}$$

Existence and uniqueness of the discrete problem (19) are ensured by (53), the continuity properties in Lemmas 7.1 and 7.2, and the discrete inf-sup condition. With the following lemma we will proof the consistency of our methods.

Lemma 7.3. *Let $\{\mathbf{U}, P\} \in H^1(\Omega)^2 \times L_0^2(\Omega)$ be the solution of the continuous problem (1). If $\mathbf{U} \in H_2(\Omega_i)^2$ and $P \in H^1(\Omega_i)$, for $i = 1, \ldots, N$, the $\{\mathbf{U}, P\}$ satisfies the discrete problem*

$$\begin{cases} a(\mathbf{U}, \mathbf{v}) + b(\mathbf{v}, P) = \displaystyle\int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx, & \mathbf{v} \in V_k, \\ b(\mathbf{U}, q) = 0, & q \in M_k. \end{cases}$$

With the following lemmas we want to proof a priori error estimates for the velocity and for the pressures.

Lemma 7.4. *Let the exact solution $\{\mathbf{U}, P\} \in H^1(\Omega)^2 \times L_0^2(\Omega)$ be in $H^{m_i}(\Omega_i)^2 \times H^{n_i}(\Omega_i)$, $i = 1, \ldots, N$, with $m_i \geq 2$ and $n_i \geq 1$. Then there exists a constant C, independent of h and k, but depending on ν and σ_0, such that*

$$|\mathbf{U} - \mathbf{u}|_h \leq C \sum_{i=1}^{N} \left(\frac{1}{\beta_k} \frac{h_i^{s_i-1}}{k_i^{m_i-\frac{3}{2}}} |\mathbf{U}|_{m_i,\Omega_i} + \frac{h_i^{r_i}}{k_i^{n_i}} |P|_{n_i,\Omega_i} \right), \tag{54}$$

with $1 \leq s_i \leq \min\{k_i + 2, m_i\}$, $1 \leq r_i \leq \min\{k_i + 1, n_i\}$ and β_k the inf-sup constant.

Lemma 7.5. *Let the exact solution $\{\mathbf{U}, P\} \in H^1(\Omega)^2 \times L_0^2(\Omega)$ be in $H^{m_i}(\Omega_i)^2 \times H^{n_i}(\Omega_i)$, $i = 1, \ldots, N$, with $m_i \geq 2$ and $n_i \geq 1$. Then there exists a constant C, independent of h and k, but depending on ν and σ_0, such that*

$$\|P - p\|_0 \leq C \sum_{i=1}^{N} \left(\frac{1}{\beta_k^2} \frac{h_i^{s_i-1}}{k_i^{m_i-\frac{3}{2}}} |\mathbf{U}|_{m_i.\Omega_i} + \frac{1}{\beta_k} \frac{h_i^{r_i}}{k_i^{n_i}} |P|_{n_i,\Omega_i} \right), \qquad (55)$$

with $1 \leq s_i \leq \min\{k_i + 2, m_i\}$, $1 \leq r_i \leq \min\{k_i + 1, n_i\}$ and β_k the inf-sup constant.

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