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Mixed boundary element method for eddy current problems

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Research Report No. 2002-16
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Abstract

We propose a mixed boundary element method for the eddy current problem. It involves two divergence conforming tangent fields on the surface. The zero divergence condition on one of them is enforced with scalar Lagrange multipliers. An LBB Inf-Sup condition is proved for the resulting discrete saddle-point problem, leading to quasi-optimal convergence rates.

Introduction

The simulation of electromagnetic devices operating at low frequencies is of practical interest to many engineering problems such as the design of electric engines. A basic problem is to determine the total electromagnetic field surrounding a conductor subject to a known electromagnetic excitation. At low frequencies the eddy current model provides a satisfactory approximation of the full Maxwell equations (Ammari, Buffa, Nédélec [1]).

The use of the free-space Green's kernel reduces these equations to integral equations on the surface of the conductor (which might not be connected). Among the many possible choices of integral equations we will use the so-called E-based one. It requires the discretization of divergence-free tangent fields on the surface. Since we are interested in objects with arbitrary topology these are not all rotationals. While Hiptmair [17] uses an explicit (but costly) construction of a supplementary of the space of rotationals in the kernel of the divergence operator, we enforce the constraint on the divergence using Lagrange multipliers. Moreover our proof of stability of the discretization is different from Hiptmair's.

Our analysis is based on the use of discrete Hodge decompositions which for the analysis of integral equations on smooth surfaces was suggested in Christiansen [12]. In most industrial applications the surface is only piecewise smooth, therefore we use also a functional framework proposed by Buffa and Ciarlet [8] and Buffa, Costabel and Sheen [9]. These two techniques have already been successfully combined to study the electric field integral equation both on closed and open surfaces, by Hiptmair and Schwab [18] and Buffa and Christiansen [7]. In addition to these techniques we will use a symmetry argument very similar to the one used in Buffa, Hiptmair, Petersdorff and Schwab [11].

The paper is organized as follows. First we recall the functional setting we will use. Then we provide a quick derivation of the integral equations. Then we turn to their variational formulation, as a saddlepoint problem and show that it is Fredholm. Finally we turn to the Galerkin discretization of this saddlepoint problem and prove the dicrete inf-sup condition under natural hypotheses.

1 Preliminaries

Let Ω_- be a bounded Lipschitz domain in $\mathbb{R}^3$. For simplicity we suppose furthermore that Ω_- is a piecewise flat polyhedron. It models the conductor. We denote by Γ its surface, and by Ω_+ the exterior domain. The unit length outward pointing orthogonal vector field on Γ is denoted by n.

For any $k \in \mathbb{C}$, let Φ_k denote the single layer potential associated with the operator $-\Delta + k^2$. That is, for any field u on Γ, $\Phi_k u$ is the field on $\mathbb{R}^3 \setminus \Gamma$ defined by:

$$(\Phi_k u)(y) = \int_\Gamma \frac{e^{-k|x-y|}}{4\pi|x-y|} u(x) \mathrm{ds}_x. \tag{1}$$

Let γ denote the trace operator on Γ, which is defined on smooth functions by:

$$\gamma \ : \ u \mapsto u|_\Gamma. \tag{2}$$

On Γ, Sobolev spaces $\mathrm{H}^s(\Gamma)$ can be defined by local charts and dualization, for $s \in [-1;1]$, see e.g. Costabel [16]. One shows that γ has a unique continuous extension to a surjective map:

$$\gamma : \mathrm{H}^1(\mathbb{R}^3) \to \mathrm{H}^{1/2}(\Gamma). \tag{3}$$

Following Buffa, Ciarlet [8] and Buffa, Costabel and Sheen [9] we recall the functional framework for traces of vector fields. On vector fields we denote by γ_{T} the tagential trace operator:

$$\gamma_{\mathrm{T}} \ : \ u \mapsto u|_\Gamma - (u|_\Gamma \cdot n)n. \tag{4}$$

We define the space $\mathrm{L}^2_{\mathrm{T}}(\Gamma)$ by:

$$\mathrm{L}^2_{\mathrm{T}}(\Gamma) = \{u \in \mathrm{L}^2(\Gamma)^3 \ : \ u \cdot n = 0\}. \tag{5}$$

The operator γ_{T} has a unique extension to a continuous map $\mathrm{H}^1(\mathbb{R}^3)^3 \to \mathrm{L}^2_{\mathrm{T}}(\Gamma)$ and we denote by $\mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)$ the range of this extension.

When considering traces of fields defined in Ω_- or Ω_+ we use the notation γ_{T}^- and γ_{T}^+. We also define $\gamma_\times$, $\gamma_\times^-$, and $\gamma_\times^+$ by composing γ_{T}, γ_{T}^-, and γ_{T}^+ with the rotation operator $(\cdot \times n)$. Thus we have for smooth fields u:

$$\gamma_\times u = u \times n. \tag{6}$$

For $s > 1$ we denote by $\mathrm{H}^s(\Gamma)$ the subspace of $\mathrm{H}^1(\Gamma)$ consisting of traces of $\mathrm{H}^{s+1/2}(\mathbb{R}^3)$. The gradient operator is defined as a continuous map grad_Γ : $\mathrm{H}^1(\Gamma) \to \mathrm{L}^2_{\mathrm{T}}(\Gamma)$. It maps $\mathrm{H}^{3/2}(\Gamma)$ to $\mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)$, continuously. We denote by div_Γ the adjoint operator, which is thus continuous $\mathrm{L}^2_{\mathrm{T}}(\Gamma) \to \mathrm{H}^{-1}(\Gamma)$, and $\mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)' \to \mathrm{H}^{3/2}(\Gamma)'$.

We denote by X the space:

$$X = \{u \in \mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)' \ : \ \mathrm{div}_\Gamma u \in \mathrm{H}^{1/2}(\Gamma)'\}. \tag{7}$$

We denote by Δ_Γ the Laplace-Beltrami operator defined as the composition $\mathrm{div}_\Gamma \circ \mathrm{grad}_\Gamma \ : \ \mathrm{H}^1(\Gamma) \to \mathrm{H}^{-1}(\Gamma)$. Let $P : X \to \mathrm{L}^2_{\mathrm{T}}(\Gamma)$ be the map which to any $u \in X$ associates $\mathrm{grad}_\Gamma p$ where p is a solution of:

$$\Delta_\Gamma p = \mathrm{div}_\Gamma u. \tag{8}$$

Proposition 1.1 *The map P determines a continuous projector $X \to X$. In particular we have a direct sum:*

$$X = V \oplus W. \tag{9}$$

where W is kernel of P, which is also the kernel of the divergence operator, and V is the range of P.There is $C > 0$ such that:

$$\forall u \in V \quad \|u\|_X \leq C|\,\mathrm{div}\, u|_{\mathrm{H}^{-1/2}(\Gamma)} \tag{10}$$

Moreover the injection of V into $\mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)'$ is compact.

–Proof: The norm estimate follows from the fact that $\mathrm{div}_\Gamma : X \to \mathrm{H}^{-1/2}(\Gamma)$ has closed range, ans is injective on V hence determines an isomorphism from V onto its range. The compactness follows from the fact that the injection of $\mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)$ into $\mathrm{L}^2_{\mathrm{T}}(\Gamma)$ is compact. □

One shows that $\gamma^-_\times$ and $\gamma^+_\times$ have unique continuous extensions to maps:

$$\mathrm{H}_{\mathrm{curl}}(\Omega_-) \to X \quad \text{and} \quad \mathrm{H}_{\mathrm{curl}}(\Omega_+)_{loc} \to X \tag{11}$$

respectively, and that these extensions are surjective. Moreover γ^-_{T}, γ^+_{T} have unique continuous extensions to maps:

$$\mathrm{H}_{\mathrm{curl}}(\Omega_-) \to X' \quad \text{and} \quad \mathrm{H}_{\mathrm{curl}}(\Omega_+)_{loc} \to X' \tag{12}$$

Moreover the operator $(\cdot \times n)$ extends to isomorphisms $X \to X'$ and $X' \to X$. We define an operator C_k on tangent fields on X by:

$$C_k u = (1/2)(\gamma^-_{\mathrm{T}} + \gamma^+_{\mathrm{T}})\,\mathrm{curl}\,\Phi_k u. \tag{13}$$

Proposition 1.2 *The operator C_k is well-defined and continuous $X \to X'$. Moreover:*

$$\gamma^\pm_{\mathrm{T}}\,\mathrm{curl}\,\Phi_k u = \pm u \times n + C_k u, \tag{14}$$

$C_k - C_0$ is compact $X \to X'$ and C_0 is symmetric.

–Proof: We first remark that for any $u \in X$ we have in $\Omega_- \cup \Omega_+$:

$$\mathrm{curl}\,\mathrm{curl}\,\Phi_k u = (\mathrm{grad}\,\mathrm{div} - \Delta)\Phi_k u = \mathrm{grad}\,\Phi_k\,\mathrm{div}_\Gamma\, u - k^2 \Phi_k u. \tag{15}$$

The continuity of C_k and compactness of $C_k - C_0$ then follow straightforwardly from the mapping properties of the singlelayer potential. The trace relations follow from the jump formulas, see e.g. [13] p. 145.

We now turn to the symmetry of C_0. The proof is a variant of the proof of Theorem 3.9 in [11], whcih dealt with the case of positive frequencies.

Choose u and v in X. Put:

$$\gamma^-_{\mathrm{T}}\,\mathrm{curl}\,\Phi_0 u = a \quad , \quad \gamma^+_{\mathrm{T}}\,\mathrm{curl}\,\Phi_0 u = a', \tag{16}$$

$$\gamma^-_{\mathrm{T}}\,\mathrm{curl}\,\Phi_0 v = b \quad , \quad \gamma^+_{\mathrm{T}}\,\mathrm{curl}\,\Phi_0 v = b'. \tag{17}$$

One checks that for any bilinear form c on X one has:

$$c\,((a - a'), 1/2(b + b')) + c\,(1/2(a + a'), (b - b')) = c(a, b) - c(a', b') \tag{18}$$

We denote by $\langle \cdot, \cdot \rangle$ the dualities of Sobolev spaces on Γ. We have:

$$\begin{aligned} \langle C_0 u, v\rangle &= \langle 1/2(a + a'), (b - b') \times n\rangle && (19)\\ &= -\langle (a - a'), 1/2(b + b') \times n\rangle + \langle a, b \times n\rangle - \langle a', b' \times n\rangle && (20)\\ &= \langle C_0 v, u\rangle + \langle a, b \times n\rangle - \langle a', b' \times n\rangle. && (21) \end{aligned}$$

For any $R > 0$ let B_R be the ball with radius R and senter 0, let S_R be the corresponding sphere and let n_R be the outward normal on S_R. For any R such that $\Omega_- \subset B_R$ we can perform the following integrations by parts in $\Omega_+ \cap B_R$:

$$\begin{aligned}
&-\langle a', b' \times n\rangle && (22)\\
= &\int_{\Omega_+\cap B_R} \operatorname{curl}\operatorname{curl}\Phi_0 u \cdot \operatorname{curl}\Phi_0 v - \operatorname{curl}\Phi_0 u \cdot \operatorname{curl}\operatorname{curl}\Phi_0 v \\
&-\int_{S_R} \operatorname{curl}\Phi_0 u \cdot \operatorname{curl}\Phi_0 v \times n_R && (23)\\
= &\int_{\Omega_+\cap B_R} \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u \cdot \operatorname{curl}\Phi_0 v - \operatorname{curl}\Phi_0 u \cdot \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma v \\
&-\int_{S_R} \operatorname{curl}\Phi_0 u \cdot \operatorname{curl}\Phi_0 v \times n_R && (24)\\
= &\langle \gamma_{\mathrm{T}}^+ \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u, (\gamma_{\mathrm{T}}^+\Phi_0 v)\times n\rangle + \langle \gamma_{\mathrm{T}}^+\Phi_0 u, (\gamma_{\mathrm{T}}^+ \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma v)\times n\rangle \\
&-\int_{S_R} \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u, (\Phi_0 v)\times n_R + \Phi_0 u \cdot (\operatorname{grad}\Phi_0 \operatorname{div}_\Gamma v)\times n_R \\
&-\int_{S_R} \operatorname{curl}\Phi_0 u \cdot \operatorname{curl}\Phi_0 v \times n_R && (25)
\end{aligned}$$

The properties of the single layer potential yield:

$$|\Phi_0 u| = O(1/|x|) \quad (26)$$
$$|\operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u| = O(1/|x|^2) \quad (27)$$
$$|\operatorname{curl}\Phi_0 u| = O(1/|x|^2) \quad (28)$$

Thus we can consider the limit as $R \to \infty$:

$$\begin{aligned}
&-\langle a', b' \times n\rangle = && (29)\\
&\langle \gamma_{\mathrm{T}}^+ \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u, (\gamma_{\mathrm{T}}^+\Phi_0 v)\times n\rangle + \langle \gamma_{\mathrm{T}}^+\Phi_0 u, (\gamma_{\mathrm{T}}^+ \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma v)\times n\rangle
\end{aligned}$$

In the interior domain Ω_- we apply the same integration by parts formula. Since the orientation of n appears twice, the signs cancel and we obtain:

$$\begin{aligned}
&-\langle a, b \times n\rangle = && (30)\\
&\langle \gamma_{\mathrm{T}}^- \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma u, (\gamma_{\mathrm{T}}^-\Phi_0 v)\times n\rangle + \langle \gamma_{\mathrm{T}}^-\Phi_0 u, (\gamma_{\mathrm{T}}^- \operatorname{grad}\Phi_0 \operatorname{div}_\Gamma v)\times n\rangle
\end{aligned}$$

Noticing that the singlelayer does not jump across Γ, nor the tangential component of its gradient, we obtain:

$$\langle a', b' \times n\rangle = \langle a, b \times n\rangle. \quad (31)$$

This yields the symmetry of C_0. □

We will also use a normal trace operator defined on smooth vector fields by:

$$\gamma_{\mathrm{n}} \ : \ u \mapsto u|_\Gamma \cdot n. \quad (32)$$

It has unique continuous extensions to a linear maps $\mathrm{H}_{\mathrm{div}}(\Omega_\pm) \to \mathrm{H}^{-1/2}(\Gamma)$.

2 E-based boundary integral equation for the eddy current problem

The eddy current problem is to find the electromagnetic field (E, H) solution of:

$$\operatorname{curl} E = -i\mu_r H \quad \text{in} \quad \mathbb{R}^3, \tag{33}$$

$$\operatorname{curl} H = \begin{cases} \tau^2 E & \text{in} \quad \Omega_-, \\ J_s & \text{in} \quad \Omega_+. \end{cases} \tag{34}$$

where $\mu_r = 1$ in Ω_+, $\tau > 0$ and, for simplicity, J_s is a divergence-free excitation with compact support in Ω_+. Moreover we impose the decay conditions:

$$E(x) = O(1/|x|) \quad \text{and} \quad H(x) = O(1/|x|). \tag{35}$$

Define E_s by:

$$E_s(x) = -i \int_{\mathbb{R}^3} \frac{J_s(y)}{4\pi|x-y|} \mathrm{d}y. \tag{36}$$

Then the eddy current problem can be reformulated as the following transmission problem for the electric field E:

$$\operatorname{curl}\operatorname{curl} E + i\tau^2\mu_r E = 0 \quad \text{in} \quad \Omega_-, \tag{37}$$

$$\operatorname{curl}\operatorname{curl} U = 0, \ \operatorname{div} U = 0 \quad \text{in} \quad \Omega_+, \tag{38}$$

$$\gamma_{\mathrm{T}}^- E - \gamma_{\mathrm{T}}^+ U = \gamma_{\mathrm{T}}^- E_s, \tag{39}$$

$$(1/\mu_r)\gamma_{\mathrm{T}}^- \operatorname{curl} E - \gamma_{\mathrm{T}}^+ \operatorname{curl} U = \gamma_{\mathrm{T}}^- \operatorname{curl} E_s, \tag{40}$$

where E is recovered in Ω_+ as $E = U + E_s$. This system of equations is supplied with the decay condition:

$$U/(1+|x|^2)^{1/2} \in \mathrm{L}^2(\Omega_+)^3 \quad \text{and} \quad \operatorname{curl} U \in \mathrm{L}^2(\Omega_+)^3. \tag{41}$$

Define κ by:

$$\kappa = (\sqrt{2}/2)(1+i)\tau\sqrt{\mu_r}, \tag{42}$$

so that we have:

$$i\tau^2\mu_r = \kappa^2. \tag{43}$$

In the interior domain Ω_- we have:

$$\operatorname{curl}\operatorname{curl} E + \kappa^2 E = 0. \tag{44}$$

from which one deduces the following representation formula:

$$E = (1 - 1/\kappa^2 \operatorname{grad}\operatorname{div})\Phi_\kappa \gamma_\times^- \operatorname{curl} E + \operatorname{curl} \Phi_\kappa \gamma_\times^- E. \tag{45}$$

It gives the following two identities:

$$\gamma_{\mathrm{T}}^- E = \gamma_{\mathrm{T}}^- (1 - 1/\kappa^2 \operatorname{grad}\operatorname{div})\Phi_\kappa \gamma_\times^- \operatorname{curl} E + 1/2\gamma_{\mathrm{T}}^- E + C_\kappa \gamma_\times^- E, \tag{46}$$

and:

$$\gamma_{\mathrm{T}}^- \operatorname{curl} E = 1/2\gamma_{\mathrm{T}}^- \operatorname{curl} E + C_\kappa \gamma_\times^- \operatorname{curl} E + \gamma_{\mathrm{T}}^- (-\kappa^2 + \operatorname{grad}\operatorname{div})\Phi_\kappa \gamma_\times^- E. \quad (47)$$

In the exterior domain Ω_+ we have:

$$\Delta U = 0 \quad \text{and} \quad \operatorname{div} U = 0, \quad (48)$$

which gives the representation formula:

$$U = -\Phi_0 \gamma_\times^+ \operatorname{curl} U - \operatorname{grad} \Phi_0 \gamma_{\mathrm{n}}^+ U - \operatorname{curl} \Phi_0 \gamma_\times^+ U. \quad (49)$$

Since $\gamma_{\mathrm{T}}^+ \circ \operatorname{grad} = \operatorname{grad}_\Gamma \circ \gamma^+$, it follows that:

$$\gamma_{\mathrm{T}}^+ U = -\gamma_{\mathrm{T}}^+ \Phi_0 \gamma_\times^+ \operatorname{curl} U - \operatorname{grad}_\Gamma \gamma^+ \Phi_0 \gamma_{\mathrm{n}}^+ U + 1/2\gamma_{\mathrm{T}}^+ U - C_0 \gamma_\times^+ U, \quad (50)$$

and since $\operatorname{curl}\operatorname{grad} = 0$ and $\Delta = \operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}$ we also have:

$$\gamma_{\mathrm{T}}^+ \operatorname{curl} U = 1/2\gamma_{\mathrm{T}}^+ \operatorname{curl} U - C_0 \gamma_\times^+ \operatorname{curl} U - \gamma_{\mathrm{T}}^+ \operatorname{grad}\operatorname{div} \Phi_0 \gamma_\times^+ U. \quad (51)$$

Substracting (50) from (46), with the transmission condition (39), and testing against divergence-free μ gives:

$$\begin{aligned} &\langle \gamma_{\mathrm{T}}^- \Phi_\kappa \gamma_\times^- \operatorname{curl} E + C_\kappa \gamma_\times^- E, \mu \rangle + \\ &\quad \langle \gamma_{\mathrm{T}}^+ \Phi_0 \gamma_\times^+ \operatorname{curl} U + C_0 \gamma_\times^+ U, \mu \rangle = \langle (1/2)\gamma_{\mathrm{T}} E_s, \mu \rangle. \end{aligned} \quad (52)$$

Substracting (51) from $1/\mu_r$(47) and using with the transmission condition (40) gives:

$$\begin{aligned} &1/\mu_r \left(C_\kappa \gamma_\times^- \operatorname{curl} E + \gamma_{\mathrm{T}}^- (-\kappa^2 + \operatorname{grad}\operatorname{div}) \Phi_\kappa \gamma_\times^- E \right) + \\ &\qquad C_0 \gamma_\times^+ \operatorname{curl} U + \gamma_{\mathrm{T}}^- \operatorname{grad}\operatorname{div} \Phi_0 \gamma_\times^+ U = (1/2)\gamma_{\mathrm{T}} \operatorname{curl} E_s. \end{aligned} \quad (53)$$

We now introduce the quantities:

$$u = \gamma_\times^- E, \quad (54)$$
$$\lambda = 1/\mu_r \gamma_\times^- \operatorname{curl} E. \quad (55)$$

We remark first that λ is divergence-free. Indeed by (40) we have:

$$\operatorname{div}_\Gamma \lambda = \gamma_{\mathrm{n}}^+ \operatorname{curl}\operatorname{curl} U + \gamma_{\mathrm{n}} \operatorname{curl}\operatorname{curl} E_s \quad (56)$$
$$= \gamma_{\mathrm{n}} (\operatorname{grad}\operatorname{div} - \Delta) E_s = 0. \quad (57)$$

In the above equations (52) and (53) we eliminate $\gamma_\times^+ U$ and $\gamma_\times^+ \operatorname{curl} U$ using the transmission conditions (39) and (40). Put into variational form these equations give rise to the system:

$$\left\{ \begin{array}{l} u \in X \\ \lambda \in W \end{array} \right. \quad \left\{ \begin{array}{lll} \forall v \in X & a(u,v) + c(\lambda, v) & = f(v) \\ \forall \mu \in W & c(\mu, u) + b(\lambda, \mu) & = g(\mu) \end{array} \right. \quad (58)$$

where the bilinear forms are given on smooth fields by:

$$a(u,v) = -\iint (1/\mu_r) G_\kappa(x,y)\left(\operatorname{div}_\Gamma u(x)\operatorname{div}_\Gamma v(y) + \kappa^2 u(x)\cdot v(y)\right)\mathrm{ds}_x\mathrm{ds}_y$$
$$-\iint G_0(x,y)\operatorname{div}_\Gamma u(x)\operatorname{div}_\Gamma v(y)\mathrm{ds}_x\mathrm{ds}_y, \quad (59)$$
$$b(\lambda,\mu) = +\iint \left(\mu_r G_\kappa(x,y) + G_0(x,y)\right)\lambda(x)\mu(y)\mathrm{ds}_x\mathrm{ds}_y, \quad (60)$$
$$c(\lambda,v) = \langle (C_\kappa + C_0)\lambda, v\rangle, \quad (61)$$

and extended to X and W by continuity.

The right hand sides are given by:

$$f(v) = \langle(-1/2(\cdot\times n) + C_0)\gamma_\times \operatorname{curl} E_s, v\rangle - \langle \gamma\Phi_0 \operatorname{div}_\Gamma \gamma_\times E_s, \operatorname{div}_\Gamma v\rangle, \quad (62)$$
$$g(\mu) = \langle(-1/2(\cdot\times n) + C_0)\gamma_\times E_s, \mu\rangle - \langle \gamma_{\mathrm{T}}\Phi_0\gamma_\times \operatorname{curl} E_s, \mu\rangle. \quad (63)$$

3 Fredholm property of the integral equations

Let d be the bilinear form on $X\times W$ defined by:

$$d\left((u,\lambda),(v,\mu)\right) = a(u,v) + c(\lambda,v) + c(\mu,u) + b(\lambda,\mu). \quad (64)$$

Let Θ be the isomorphism $(u,\lambda)\to(-Pu - 1/\kappa^2(I-P)u,\lambda)$. The analysis of the system (58) relies on:

Proposition 3.1 *There is a compact bilinear form k on $X\times W$ such that for a $C>0$ it holds:*

$$\mathfrak{Re}(d+k)((u,\lambda),\overline{\Theta(u,\lambda)}) \geq 1/C\|(u,\lambda)\|^2_{X\times W}. \quad (65)$$

–Proof: We denote by s the bilinear form associated with the single layer operator, both on scalar and on vector fields. Thus for scalar fields:

$$s(p,q) = \iint G_\kappa(x,y)p(x)q(y)\mathrm{ds}_x\mathrm{ds}_y. \quad (66)$$

For any $u\in X$ consider its decomposition $u = u^V + u^W$ with $u^V = Pu \in V$ and $u^W = (I-P)u\in W$. Associate with it $v = -\overline{u}^V - 1/\kappa^2\overline{u}^W$. Then we have:

$$a(u,v) = (1/\mu_r + 1)s(\operatorname{div}_\Gamma u^V, \operatorname{div}_\Gamma \overline{u}^V) + (1/\mu_r)s(u^W,\overline{u}^W) \quad (67)$$
$$+(\kappa^2/\mu_r)\left(s(u^V,\overline{u}^V) + s(u^V, 1/\kappa^2\overline{u}^W) + s(u^W,\overline{u}^V)\right). \quad (68)$$

It follows from the compactness of the injection $V\to \mathrm{H}^{1/2}_{\mathrm{T}}(\Gamma)'$ (Proposition 1.1), that all the terms on line (68) are compact. Moreover it follows from the coercivity on $\mathrm{H}^{-1/2}(\Gamma)$ of the single layer operator that, up to compact bilinear forms, the sum of the terms on line (67) is coercive on X.

Using also $\mu = \overline{\lambda}$ we have:

$$c(\lambda, v) + c(\mu, u) = c(\lambda, -\overline{u}^V) + c(\overline{\lambda}, u^V) + c(\lambda, -1/\kappa^2 \overline{u}^W) + c(\overline{\lambda}, u^W). \quad (69)$$

It follows from the symmetry of C_0 (Proposition 1.2 and the compactness of $C_\kappa - C_0$ that the sum of the first two terms is, up to a compact term, purely imaginary. Moreover the two last terms are compact. □

It follows that if d is left injective then it determines an isomorphism $X \times W \to (X \times W)^\star$ and we have the Inf-Sup estimate:

$$\inf_{(u,\lambda) \in X \times W} \sup_{(v,\mu) \in X \times W} \frac{|d((u,\lambda),(v,\mu))|}{\|(u,\lambda)\| \|(v,\mu)\|} > 0, \quad (70)$$

where $\|\cdot\|$ denotes the standard norm on $X \times W$.

Aiming at the discretization of this equation, we remark that it is in general a hard problem to construct explicit subspaces of W, when Γ is allowed to have arbitrary topology. In particular the algorithm proposed by Hiptmair [17] involves an N^2 algorithm to determine surface cycles of trianglulations with N vertices. This is far more than the $N(\log N)^a$ complexity of multipole matrix-vector products.

We therefore chose to enforce the condition that $\lambda \in W$ with Lagrange multipliers. Let $(\Gamma_i)_{i \in I}$ be the family of connected components of Γ. Let Y denote the space:

$$Y = \{q \in \mathrm{H}^{-1/2}(\Gamma) \ : \ \forall i \in I \quad \langle q, 1 \rangle_{\Gamma_i} = 0\}. \quad (71)$$

Notice first that $Y = \operatorname{div}_\Gamma X$ and that if $q \in Y$, if $u \in V$ is the solution of $\operatorname{div}_\Gamma u = q$ then by Proposition 1.1 we have an estimate of the form $\|u\|_X \leq C\|q\|_Y$.

Next let s_0 be the bilinear form obtained by substituting 0 for κ in s. We define a form e by:

$$e((u,\lambda), q) = s_0(\operatorname{div}_\Gamma \lambda, q). \quad (72)$$

By the coercivity of s_0 on $\mathrm{H}^{-1/2}(\Gamma)$ it follows that we have the Babuska-Brezzi compatibility estimate:

$$\inf_{q \in Y} \sup_{(u,\lambda) \in X \times X} \frac{|e((u,\lambda), q)|}{\|(u,\lambda)\| \ \|q\|} > 0. \quad (73)$$

We extend d from a bilinear form on $X \times W$ to one on $X \times X$ by keeping the expressions (59), (60) and (61) in the definition (64) of d. The original problem (58) is equivalent to one of the form: find $(u,\lambda) \in X \times X$ and $q \in Y$, such that:

$$\left\{ \begin{array}{lllll} \forall (v,\mu) \in X \times X & d((u,\lambda),(v,\mu)) + e((v,\mu), q) & = & l((v,\mu)) \\ \forall p \in Y & e((u,\lambda), p) & = & 0 \end{array} \right. \quad (74)$$

Here $l((v,\mu)) = f(v) + \tilde{g}(\mu)$ where $\tilde{g}$ is any continuous extension of g from W to X. By the above Inf-Sup conditions it follows from Nicolaides' [22] theorem that this problem is uniquely solvable when d is left injective.

Notice that for our purposes, in (72), s_0 can be replaced by any bilinear form which is coercive on $\mathrm{H}^{-1/2}(\Gamma)$.

4 Discrete Inf-Sup condition

Suppose we have Galerkin spaces $X_h \subset X$ and $Y_h \subset Y$. Then we consider the problem of finding $(u, \lambda) \in X_h \times X_h$ and $q \in Y_h$, such that:

$$\left\{ \begin{array}{llcl} \forall (v,\mu) \in X_h \times X_h & d((u,\lambda),(v,\mu)) + e((v,\mu),q) & = & l((v,\mu)) \\ \forall p \in Y_h & e((u,\lambda),p) & = & 0 \end{array} \right. \tag{75}$$

In general this problem does not satisfy discrete uniform Inf-Sup conditions. Therefore we shall make some additional assumptions on X_h and Y_h. First we suppose that:

$$Y_h = \operatorname{div}_\Gamma X_h = \{\operatorname{div}_\Gamma u \ : \ u \in X_h\}. \tag{76}$$

Furthermore we define:

$$W_h = \{u \in X_h \ : \ \operatorname{div}_\Gamma u = 0\} \tag{77}$$

and:

$$V_h = \{u \in X_h \ : \ \forall w \in W_h \quad \int u \cdot w = 0\}. \tag{78}$$

Then we have:

$$X_h = V_h \oplus W_h, \tag{79}$$

but in general V_h is not a subspace of V.

Recall that for any two closed subspaces X_0 and X_1 of X the gap $\delta(X_0, X_1)$ is defined by:

$$\delta(X_0, X_1) = \sup_{u_0 \in X_0} \inf_{u_1 \in X_1} \|u_0 - u_1\| / \|u_0\|. \tag{80}$$

We say that the family (X_h) of spaces is approximating if:

$$\forall u \in X \quad \lim_{h \to 0} \inf_{u_h \in X_h} \|u - u_h\| = 0. \tag{81}$$

Theorem 4.1 *If d is left injective, (X_h) is approximating and $\delta(V_h, V) \to 0$ then the system (75) satisfies uniform discrete Inf-Sup conditions for small enough h.*

–Proof: (i) Compatibility condition: Let P be the projector onto V parallel to W. Then we have:

$$\begin{aligned} \forall u_h \in V_h \quad \|u_h\| &\leq \|u_h - Pu_h\|_X + \|Pu_h\| && (82) \\ &\leq \|I - P\| \delta(V_h, V) \|u_h\| + \|\operatorname{div}_\Gamma u_h\|_Y. && (83) \end{aligned}$$

Therefore we have an estimate of the form there is $h_0 > 0$ and $C > 0$ such that for all $h < h_0$:

$$\forall u_h \in V_h \quad \|u_h\|_X \leq C \|\operatorname{div}_\Gamma u_h\|_Y. \tag{84}$$

Then, given $q \in Y$, if $u_h \in V_h$ is the solution of $\operatorname{div}_\Gamma u_h = q$ we have:

$$e((0, \overline{u}_h), q) / \|u_h\|_X = s_0(\operatorname{div}_\Gamma \overline{u}_h, q) \geq 1/C \|q\|. \tag{85}$$

(ii) Inf-Sup condition on the kernel. Remark that for all $(u, \lambda) \in X_h \times X_h$ if:

$$\forall q \in Y_h \quad e((u, \lambda), q) = 0, \tag{86}$$

then $\operatorname{div}_\Gamma \lambda = 0$, hence:

$$\forall q \in Y \quad e((u, \lambda), q) = 0. \tag{87}$$

In other words the left kernel of e on $(X_h \times X_h) \times Y_h$ is $X_h \times W_h$ which is a subspace of $X \times W$. Unfortunately this subspace is not stable under Θ. Let Θ_h denote the map:

$$\Theta_h(u_h, \lambda_h) \mapsto (-u_h^{V_h} - 1/\kappa^2 u_h^{W_h}, \lambda_h), \tag{88}$$

where u_h is decomposed as $u_h = u_h^{V_h} + u_h^{W_h}$ with $(u_h^{V_h}, u_h^{W_h}) \in V_h \times W_h$. By Lemma 6.1 part (ii), putting $\delta_h = \delta(V_h, V)$ we have an estimate of the form:

$$\|\Theta_h(u_h, \lambda_h) - \Theta(u_h, \lambda_h)\| \leq C\delta_h \|(u_h, \lambda_h)\|. \tag{89}$$

Using equation (65) it follows that for small enough h we have an estimate of the form:

$$\mathfrak{Re}(d + k)((u_h, \lambda_h), \overline{\Theta_h(u_h, \lambda_h)}) \geq 1/C \|(u_h, \lambda_h)\|^2_{X \times X}, \tag{90}$$

from which we can deduce the discrete Inf-Sup condition: There is $h_0 > 0$ and $C > 0$ such that for all $h < h_0$:

$$\inf_{(u_h, \lambda_h) \in X_h \times W_h} \sup_{(v_h, \mu_h) \in X_h \times W_h} \frac{|(d + k)((u_h, \lambda_h), (v_h, \mu_h))|}{\|(u_h, \lambda_h)\| \, \|(v_h, \mu_h)\|} \geq 1/C. \tag{91}$$

By virtue of the general theorems on injective compact pertubations of bilinear forms, a similar estimate holds with $(d + k)$ replaced by d when (X_h) is approximating, which is our claimed result. $\square$

Examples of Galerkin spaces for which the condition $\delta(V_h, V) \to 0$ holds include the case when we have triangulations $\mathcal{T}_h$ on Γ upon which we consider Raviart-Thomas finite elements for X_h and piecewise polynomials of degree one less and with zero integral on each Γ_i (this condition can be enforced with one Lagrange multiplier per connected component of Γ) for Y_h. This was proved in Hiptmair-Schwab [18]. A detailed proof can also be found in Christiansen [13]. Thus for lowest order finite elements we have two degrees of freedom per edge and one per face of the triangulation.

As is well-known the Inf-Sup condition implies quasi-optimal convergence of the Galerkin solution (u_h, λ_h) towards the continuous solution (u, λ), i.e:

$$\|(u, \lambda) - (u_h, \lambda_h)\|_{X \times X} \leq C \inf_{(u', \lambda') \in X_h \times X_h} \|(u, \lambda) - (u', \lambda')\|_{X \times X}. \tag{92}$$

5 Extension to curved boundaries

In many applications the scatterer is not piecewise flat, but only piecewise smooth. In this case parametric Finite Elements should be used, and it is our goal to define them here.

Suppose we have a triangulation $\mathcal{T}_h$ of Γ. We denote by Γ_h^1 the affine polyhedron it determines. We equip each triangle $T \in \mathcal{T}_h$ with the set Σ_T^p of points with barycentric coordinates which are multiples of $1/p$. Put $\Sigma_h^p = \cup_{T\in\mathcal{T}_h}\Sigma_T^p$. We suppose we are given a map $F_h^p : \Sigma^p \to \Gamma$. We extend F_h^p to a function $\overline{F}_h^p : \Gamma_h^1 \to \mathbb{R}^3$ by enforcing that on each $T \in \mathcal{T}_h$ its restriction is the P^p function coinciding with F_h^p on Σ_T^p. Then the P^p-approximation of Γ is defined as the range of $\overline{F}_h^p$ and denoted Γ_h^p. In the following $\overline{F}_h^p$ is denoted F_h^p.

By way of F_h^p standard finite element spaces on Γ_h^1 can be transported to Γ_h^p. The problem is now to define a transport function G_h^p from Γ_h^p to Γ. When Γ is globally smooth the orthogonal projection was used in Nédélec [20]. We extend the technique to piecewise smooth boundaries by a method used in Lenoir [19] to define parametric volume finite elements in smooth domains.

We suppose Γ can be decomposed as:

$$\Gamma = \cup_{i\in I_2}\gamma_i^2 \ \cup_{i\in I_1}\gamma_i^1 \ \cup_{i\in I_0}\gamma_i^0. \tag{93}$$

such that the sets γ_i^j are pairwise disjoint and such that for each i, j, γ_i^j is a smooth j-dimensional manifold.

We suppose the triangulation $\mathcal{T}_h$ is maid up of triangulations for the patches γ_i^2 which coincide along the edges γ_j^1 and cointain the vertices γ_k^0. A more precise definition would require the theory of simplicial complexes.

We now construct G_h^p for a patch γ_i^2. Let T be a triangle of $\mathcal{T}_h$. If T contains at most one point of $\partial\gamma_i^2$ then G_h^p is the orthogonal projection, which is well defined. If T contains two points on $\partial\gamma_i^2$, say P_0 and $P1$ belonging to a set $\overline{\gamma_j^1}$ we let the third point be P_3 and denote $\lambda_0, \lambda_1, \lambda_2$ the barycentric coordinates in T. The orthogonal projection onto $\overline{\gamma_j^1}$ induces a bicontinuous bijection P_j^1 from the edge $[P_0, P_1]$ to its range in $\overline{\gamma_j^1}$. We let $G_h^p(P)$ be the orthogonal projection onto $\overline{\gamma_i^2}$ of the point:

$$(\lambda_0 + \lambda_1)P_j^1\big(1/(\lambda_0 + \lambda_1)(\lambda_0 P_0 + \lambda_1 P_1)\big) + \lambda_2 P_2. \tag{94}$$

The other possibilities (in particular three points on the boundary) we rule out. On two neighouring triangles the definitions coincide, leading to a globally defined map $G_h^p : \Gamma_h^p \to \Gamma$. The triangulations are constructed such that this map is a bicontinuous bijection.

Let Ξ_h^p be the inverse of $G_h^p \circ F_h^p$. The Raviart-Thomas vector fields on Γ_h^1 are the transported to Γ by $G_h^p \circ F_h^p$ using Piola's transform:

$$u \mapsto \operatorname{Jac}\Xi_h^p \mathrm{D}^{-1}\Xi_h^p u \circ \Xi_h^p. \tag{95}$$

Piece-wise polynomials on Γ_h^1 are transported to Γ according to:

$$u \mapsto \operatorname{Jac}\Xi_h^p u \circ \Xi_h^p. \tag{96}$$

The transport formulas were chosen in order to satisfy the property:

$$\operatorname{div}_\Gamma \operatorname{Jac} \Xi_h^p \mathrm{D}^{-1} \Xi_h^p u \circ \Xi_h^p = \operatorname{Jac} \Xi_h^p \operatorname{div}_{\Gamma_h^1} u \circ \Xi_h^p. \tag{97}$$

With this property the analysis of the preceding sections carry over to the case of curved boundaries.

6 Appendix

Lemma 6.1 *Let X be a Banach space and $X = V \oplus W$ be a splitting of X into a direct sum of closed subspaces. Suppose (X_h) is a family of closed subspaces that can be split into direct sums of closed subspaces $X_h = V_h \oplus W_h$ such that $\delta_h = \max\{\delta(V_h, V), \delta(W_h, W)\} \to 0$.*

(i) Then there is $h_0 > 0$ and $C > 0$ such that for all $h < h_0$ and $(v_h, w_h) \in V_h \times W_h$:

$$\|v_h\| + \|w_h\| \leq C\|v_h + w_h\|. \tag{98}$$

(ii) Moreover there is $h_0 > 0$ and $C > 0$ such that for all $h < h_0$ and $u_h \in X_h$, if (v_h, w_h) is its decomposition in $V_h \times W_h$ and (v, w) is the one in $V \times W$ then:

$$\|v - v_h\| + \|w - w_h\| \leq C\delta_h\|u_h\|. \tag{99}$$

–Proof: (i) Let P denote the projection with range V and kernel W. For any $(v_h, w_h) \in V_h \times W_h$, with $u_h = v_h + w_h$ we have:

$$\begin{aligned} \|v_h\| &\leq \|P(v_h + w_h)\| + \|Pw_h\| + \|Pv_h - v_h\| && (100)\\ &\leq \|P\|\,\|u_h\| + \|P\|\delta_h\|w_h\| + \|I - P\|\delta_h\|v_h\|. && (101) \end{aligned}$$

Similarly:

$$\|w_h\| \leq \|I - P\|\,\|u_h\| + \|I - P\|\delta_h\|v_h\| + \|P\|\delta_h\|w_h\| \tag{102}$$

Putting $M = \max\{\|P\|, \|I - P\|\}$, adding and rearranging we have:

$$\|v_h\| + \|w_h\| \leq 2M/(1 - \delta_h 2M)^{-1}\|u\|_h. \tag{103}$$

This proves the first part of the lemma.

(ii) With the above notations we have:

$$\begin{aligned} \|Pu_h - v_h\| &\leq \|Pv_h - v_h\| + \|Pw_h\| && (104)\\ &\leq M\delta_h\|v_h\| + M\delta_h\|w_h\|. && (105) \end{aligned}$$

Similarly we have:

$$\|(I - P)u_h - w_h\| \leq M\delta_h\|w_h\| + M\delta_h\|v_h\|. \tag{106}$$

Together with the first estimate this gives the second part of the lemma. □

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