

Locking-Free DGFEM for Elasticity Problems in Polygons

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Abstract

The h -version of the discontinuous Galerkin finite element method (h -DGFEM) for nearly incompressible linear elasticity problems in polygons is analyzed. It is proved that the scheme is robust (locking-free) with respect to volume locking, even in the absence of H^2 -regularity of the solution. Furthermore, it is shown that an appropriate choice of the finite element meshes leads to robust and optimal algebraic convergence rates of the DGFEM even if the exact solutions are singular.

Keywords: DGFEM, locking, elasticity problems, singular solutions, graded meshes, discontinuous Galerkin methods

1 Introduction

In mechanical engineering, partial differential equations are often solved by low-order finite element methods (FEMs). In many applications, the convergence of these schemes may strongly depend on various problem parameters. Unfortunately, this can result in non-robustness of the convergence i.e. the asymptotic convergence regime of the method is reached only at such high numbers of degrees of freedom that the scheme is practically not feasible. In computational mechanics, this non-robustness of the FEM is termed *locking*.

There exist different kinds of locking: *Shear locking* typically appears if the corresponding domains are very thin and plate and shell theories, which include shear deformation, are used. In addition, in shell theories and their finite element models, there arises *membrane locking* which is caused by the interaction between bending and membrane energies. Finally, problems dealing with nearly incompressible materials are often accompanied by the so-called *volume locking*; this type of locking is very typical for elasticity problems in biology and will be explored in this paper.

In order to overcome locking, a wide variety of alternative approaches have been suggested. For example, low-order mixed FEMs, where an extra variable for the divergence term is introduced, yield adequate numerical results (cf. [8]). These methods are closely related to under-integration schemes. A further possibility is the use of non-conforming methods, where the global continuity of the numerical solutions is not anymore enforced (see [12], for example).

In 1983, M. Vogelius proved absence of volume locking for the p -version of the FEM on smooth domains [16]. Moreover, in 1992, I. Babuška, M Suri [5] showed that, on polygonal domains, the h -FEM is locking-free on regular triangular elements with $p \geq 4$. In addition, they proved that, for conforming methods, locking cannot be avoided on quadrilateral meshes for any $p \geq 1$. Recently, P. Hansbo and M. G. Larson [11] suggested the use of a discontinuous FEM (DGFEM). Assuming at least H^2 regularity, they showed that the h -version of the DGFEM does not lock for all $p \geq 1$.

Following the classical approach of M. F. Wheeler [17] and B. Rivière, M. F. Wheeler [14], this paper is devoted to the exploration of the DGFEM for linear elasticity problems (with mixed boundary conditions) in convex and non-convex polygons. Based on a recent regularity result by B. Q. Guo and C. Schwab [10] it will be proved here that, even if the exact solutions of the elasticity problems are singular (i.e. not H^2 anymore), the h -version of the DGFEM is locking-free. Additionally, the use of so-called 'graded meshes' lead the DGFEM to converge at an optimal algebraic rate (independently of the compressibility of the material).

The DGFEM above is closely related to non-conforming methods of Crouzeix-Raviart type. In 1992, S. C. Brenner, L. Sung [7] already showed that these schemes are locking-free even for $p = 1$. However, their results are based on the assumption that the displacements are H^2 regular, and therefore, the case of non-convex polygons is in general not covered by that work. Nevertheless, applying the regularity results and the mesh refinement strategies presented in this paper (Theorem 3.4, Theorem 5.10), it may be proved that the convergence statements in [7] are

extensible to the case where the exact solutions of the elasticity problems exhibit corner singularities.

The outline of the paper is as follows: In Section 2 and 3, the linear elasticity problem and its regularity on polygons are presented. In Section 4, the DGFEM is introduced. Section 5 contains the error analysis of the DGFEM and terminates with the proof of the main result (optimal, robust convergence of the DGFEM). In Section 6, the theoretical results are confirmed with some numerical examples.

2 Problem Formulation

Let Ω be a polygon in \mathbb{R}^2 . Its boundary $\Gamma := \partial\Omega$ is assumed to consist of a Dirichlet part Γ_D with $|\Gamma_D| > 0$ and a Neumann part Γ_N :

$$\bar{\Gamma} = \bar{\Gamma}_D \cup \bar{\Gamma}_N.$$

The linear elasticity problem reads as follows:

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma}(\underline{u}) = \underline{f} & \text{in } \Omega \\ \underline{u} = \underline{g}_D & \text{on } \Gamma_D \\ \boldsymbol{\sigma}(\underline{u}) \cdot \underline{n}_\Omega = \underline{g}_N & \text{on } \Gamma_N. \end{cases} \quad (1)$$

Here, $\underline{u} = (u_1, u_2)$ is the *displacement* and $\boldsymbol{\sigma} = \{\sigma_{ij}\}_{i,j=1}^2$ is the *stress tensor* for homogeneous isotropic material given by

$$\boldsymbol{\sigma}(\underline{u}) = 2\mu\boldsymbol{\epsilon}(\underline{u}) + \lambda\nabla \cdot \underline{u} \mathbf{1}_{2 \times 2},$$

where $\boldsymbol{\epsilon}(\underline{u}) = \{\epsilon_{ij}(\underline{u})\}_{i,j=1}^2$ with

$$\epsilon_{ij}(\underline{u}) = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i) \quad (2)$$

is the *symmetric gradient* of \underline{u} . Furthermore, μ and λ are the so-called *Lamé coefficients* satisfying

$$0 < \min\{\mu, \mu + \lambda\},$$

and \underline{n}_Ω is the unit outward vector of Ω on Γ .

3 Regularity

3.1 Weighted Sobolev Spaces

The regularity of (1) will be measured in terms of certain weighted Sobolev spaces. In order to do so, set

$$SP(\Omega, \Gamma_D, \Gamma_N) := \{A_i : i = 1, 2, \dots, M\},$$

where $A_i, i = 1, \dots, M$, denote the 'singular vertices', e.g. corners and points of changing boundary condition type of Ω . Moreover, introduce a weight vector $\underline{\beta} = (\beta_1, \dots, \beta_M)$ with $0 \leq \beta_i < 1$, and for any number $k \in \mathbb{R}$ set $\underline{\beta} \pm k := (\beta_1 \pm k, \dots, \beta_M \pm k)$. Then, let $\Phi_{\underline{\beta}}$ be a *weight function* on Ω given by

$$\Phi_{\underline{\beta}}(x) = \prod_{i=1}^M r_i^*(x)^{\beta_i}, \quad r_i^*(x) = |x - A_i|.$$

Furthermore, for any integers $m \geq l \geq 0$, denote by $H_{\underline{\beta}}^{m,l}(\Omega)^2$ the so-called *weighted Sobolev Spaces* on Ω (cf. [2], [3], [9]) which are understood to be the completions of $C^\infty(\bar{\Omega})^2$ with respect to the norms

$$\begin{aligned} \|\underline{u}\|_{H_{\underline{\beta}}^{m,l}(\Omega)}^2 &= \|\underline{u}\|_{H^{l-1}(\Omega)}^2 + \sum_{k=l}^m \| |D^k \underline{u}| \Phi_{\underline{\beta}+k-l} \|_{L^2(\Omega)}^2, \quad l \geq 1, \\ \|\underline{u}\|_{H_{\underline{\beta}}^m(\Omega)}^2 &= \sum_{k=0}^m \| |D^k \underline{u}| \Phi_{\underline{\beta}+k} \|_{L^2(\Omega)}^2, \quad l = 0. \end{aligned}$$

Convention 3.1 Since the weight function $\Phi_{\underline{\beta}}$ controls the local behaviour of the solution in the vicinity of a (singular) vertex, it is obvious to work locally with the weight function $\Phi_{\beta} = r^{\beta}$ with

$$\beta := \beta_i \quad \text{and} \quad r(x) := |x - A_i|,$$

where A_i denotes the corresponding vertex of the polygon.

Remark 3.2 In this paper, the spaces $H_{\underline{\beta}}^{2,2}(\Omega)^2$ will play a main role and it may be proved easily that for all $\varepsilon > 0$ and for each function $u \in H_{\underline{\beta}}^{2,2}(\Omega)^2$, there holds $\underline{u}|_{\tilde{\Omega}_\varepsilon} \in H^2(\tilde{\Omega}_\varepsilon)^2$, where

$$\tilde{\Omega}_\varepsilon := \Omega \setminus \bigcup_{i=1}^M \{x \in \mathbb{R}^2 : |x - A_i| < \varepsilon\}.$$

Moreover, $H_{\underline{0}}^{2,2}(\Omega) = H^2(\Omega)$.

Finally, the spaces $H_{\underline{\beta}}^{k-1/2, l-1/2}(\gamma)^2$, $l = 1, 2$, are defined as the trace spaces of $H_{\underline{\beta}}^{k,l}(\Omega)$ on $\gamma \subset \Gamma$ and

$$\|\underline{g}\|_{H_{\underline{\beta}}^{k-1/2, l-1/2}(\gamma)} := \inf_{\substack{\underline{G} \in H_{\underline{\beta}}^{k,l}(\Omega)^2 \\ \underline{G}|_\gamma = \underline{g}}} \|\underline{G}\|_{H_{\underline{\beta}}^{k,l}(\Omega)}.$$

3.2 Regularity of Generalized Stokes Problems

In order to obtain a regularity result for the elasticity problem (1), the following *generalized Stokes problem* in the polygon Ω is considered:

$$\left\{ \begin{array}{l} -\nabla \cdot \boldsymbol{\sigma}(\underline{u}, p) = \underline{f} \quad \text{in } \Omega \\ -\nabla \cdot \underline{u} = h \quad \text{in } \Omega \\ \underline{u} = \underline{g}_D \quad \text{on } \Gamma_D \\ \boldsymbol{\sigma}(\underline{u}, p) \cdot \underline{n}_\Omega = \underline{g}_N \quad \text{on } \Gamma_N. \end{array} \right. \quad (3)$$

Here, \underline{u} is the velocity field, p a Lagrange multiplier corresponding to the (hydrostatic) pressure in the incompressible limit and $\boldsymbol{\sigma}(\underline{u}, p)$ the hydrostatic stress tensor of \underline{u} defined by

$$\boldsymbol{\sigma}(\underline{u}, p) = -p \mathbf{1} + 2\nu \boldsymbol{\epsilon}(\underline{u}),$$

where $\boldsymbol{\epsilon}(\underline{u})$ is given as in (2) and $\nu > 0$ is the (kinematic) viscosity. If $\Gamma_N = \emptyset$, the following compatibility condition is supposed to be fulfilled:

$$\int_{\Omega} h \, dx + \int_{\partial\Omega} \underline{g}_D \cdot \underline{n}_\Omega \, ds = 0 \quad (4)$$

In [10] the following regularity result was proved:

Theorem 3.3 *Let $k \geq 0$ and $|\Gamma_D| > 0$. In addition, if $\Gamma_N = \emptyset$, let (4) be satisfied. Then there exists a weight vector $\underline{\beta} = (\beta_1, \dots, \beta_M)$ with $0 \leq \beta_i < 1$, $i = 1, \dots, M$, such that for $\underline{f} \in H_{\underline{\beta}}^{k,0}(\Omega)^2$, $h \in H_{\underline{\beta}}^{k+1,1}(\Omega)$, $\underline{g}_D \in H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)^2$ and $\underline{g}_N \in H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)^2$ the generalized Stokes problem (3) admits a unique solution $(\underline{u}, p) \in H_{\underline{\beta}}^{k+2,2}(\Omega)^2 \times H_{\underline{\beta}}^{k+1,1}(\Omega)$ and the a-priori estimate*

$$\begin{aligned} & \| \underline{u} \|_{H_{\underline{\beta}}^{k+2,2}(\Omega)} + \| p \|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} \\ & \leq C \left(\| \underline{f} \|_{H_{\underline{\beta}}^{k,0}(\Omega)} + \| h \|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} + \| \underline{g}_D \|_{H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)} + \| \underline{g}_N \|_{H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)} \right) \end{aligned} \quad (5)$$

holds true.

3.3 Regularity of Linear Elasticity Problems

A regularity result for linear elasticity problems in polygons was proved in [9], Theorem 5.2. However, referring to the previous Theorem 3.3, a more specific statement, which clarifies the regularity of the linear elasticity problem (1) in dependence on the Lamé coefficient λ , may be developed.

Theorem 3.4 *Let Ω be a polygon in \mathbb{R}^2 and $|\Gamma_D| > 0$. Then there exists a weight vector $\underline{\beta} = (\beta_1, \dots, \beta_M)$ with $0 \leq \beta_i < 1$, $i = 1, \dots, M$, such that for $\underline{f} \in H_{\underline{\beta}}^{k,0}(\Omega)^2$,*

$\underline{g}_D \in H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)^2$ and $\underline{g}_N \in H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)^2$ the linear elasticity problem (1) has a unique solution $\underline{u} \in H_{\underline{\beta}}^{k+2,2}(\Omega)^2$. In addition, there exists a constant $C > 0$ independent of λ such that the ensuing estimate holds true:

$$\begin{aligned} & \|\underline{u}\|_{H_{\underline{\beta}}^{k+2,2}(\Omega)} + |\lambda| \|\nabla \cdot \underline{u}\|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} \\ & \leq C \left(\|\underline{f}\|_{H_{\underline{\beta}}^{k,0}(\Omega)} + \|\underline{g}_D\|_{H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)} + \|\underline{g}_N\|_{H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)} \right). \end{aligned} \quad (6)$$

Proof: As already mentioned above, the unique solution $\underline{u}_{\text{elast}}$ of the linear elasticity problem (1) belongs to $H_{\underline{\beta}}^{k+2,2}(\Omega)$ ([9], Theorem 5.2). Therefore, the choice

$$h := -\nabla \cdot \underline{u}_{\text{elast}} \in H_{\underline{\beta}}^{k+1,1}(\Omega)$$

leads to the following solution (\underline{u}, p) of the generalized Stokes problem (3):

$$p = -\lambda \nabla \cdot \underline{u}_{\text{elast}}$$

and

$$\underline{u} = \underline{u}_{\text{elast}}.$$

Hence, using (5) implies that

$$\begin{aligned} \|\underline{u}\|_{H_{\underline{\beta}}^{k+2,2}(\Omega)} + |\lambda| \|\nabla \cdot \underline{u}\|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} & \leq C \left(\|\underline{f}\|_{H_{\underline{\beta}}^{k,0}(\Omega)} + \|\nabla \cdot \underline{u}\|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} \right. \\ & \left. + \|\underline{g}_D\|_{H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)} + \|\underline{g}_N\|_{H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)} \right). \end{aligned} \quad (7)$$

Thus, if $|\lambda| < 2C$, it follows that

$$\begin{aligned} & \|\underline{u}\|_{H_{\underline{\beta}}^{k+2,2}(\Omega)} + |\lambda| \|\nabla \cdot \underline{u}\|_{H_{\underline{\beta}}^{k+1,1}(\Omega)} \\ & \leq \tilde{C} \|\underline{u}\|_{H_{\underline{\beta}}^{k+2,2}(\Omega)} \\ & \leq \tilde{C} \left(\|\underline{f}\|_{H_{\underline{\beta}}^{k,0}(\Omega)} + \|\underline{g}_D\|_{H_{\underline{\beta}}^{k+3/2,3/2}(\Gamma_D)} + \|\underline{g}_N\|_{H_{\underline{\beta}}^{k+1/2,1/2}(\Gamma_N)} \right) \end{aligned}$$

for a constant \tilde{C} independent of $|\lambda| \in (0, 2C)$. In the last step, Theorem 5.2 in [9] was applied.

Alternatively, if $|\lambda| \geq 2C$, the term $C \|\nabla \cdot \underline{u}\|_{H_{\underline{\beta}}^{k+1,1}(\Omega)}$ in the right-hand side of (7) may obviously be absorbed into the left-hand side. \square

4 The DGFEM

4.1 Finite Element Meshes

Consider a *regular** partition (FE mesh) \mathcal{T} of Ω into open triangles K :

$$\mathcal{T} = \{K_i\}_i, \quad \bigcup_{K \in \mathcal{T}} \bar{K} = \bar{\Omega}.$$

*i.e. a FE mesh without any hanging nodes

The elements $K \in \mathcal{T}$ are images of the reference triangle

$$\hat{T} := \{(\hat{x}, \hat{y}) : 0 \leq \hat{y} \leq \sqrt{3}(1 - |\hat{x}|), \hat{x} \in (-1, 1)\} \quad (8)$$

under affine maps F_K , i.e. for each $K \in \mathcal{T}$ there exists a constant matrix $\mathbf{A}_K \in \mathbb{R}^{2 \times 2}$ and a constant vector $\underline{b}_K \in \mathbb{R}^2$ such that with

$$F_K(\underline{x}) = \mathbf{A}_K \underline{x} + \underline{b}_K \quad (9)$$

there holds

$$K = F_K(\hat{T}). \quad (10)$$

Moreover, for each $K \in \mathcal{T}$, introduce

$$h_K := \text{diam}(K)$$

and

$$\rho_K := \sup\{\text{diam}(B) : B \text{ is a ball contained in } K\}.$$

Finally, the so-called *mesh width* of \mathcal{T} is given by

$$h_{\mathcal{T}} := \sup_{K \in \mathcal{T}} h_K. \quad (11)$$

Henceforth, the FE meshes are assumed to be shape regular:

Definition 4.1 *Let $\mathcal{G} = \{\mathcal{T}_i\}_{i \in \mathbb{N}}$ be a family of FE meshes. Then \mathcal{G} is called shape regular if there exists a constant $\mu > 0$ independent of i such that*

$$\mu \leq \min_{K \in \mathcal{T}_i} \frac{h_K}{\rho_K} \leq \max_{K \in \mathcal{T}_i} \frac{h_K}{\rho_K} \leq \mu^{-1}, \quad \forall i \in \mathbb{N}. \quad (12)$$

4.2 FE Spaces

Let \mathcal{T} be a regular finite element mesh consisting of shape regular triangles $K \in \mathcal{T}$. The discontinuous finite element spaces that will be appropriate for the DGFEM are defined as follows:

$$\mathcal{S}^{1,0}(\Omega, \mathcal{T}) := \{\underline{u} \in L^2(\Omega)^2 : \underline{u}|_K \in \mathcal{P}_1(K)^2, K \in \mathcal{T}\}. \quad (13)$$

Here,

$$\mathcal{P}_1(K) := \{u(x, y) = ax + by + c : a, b, c \in \mathbb{R}\}$$

is the space of all linear functions on K .

4.3 Variational Formulation

First of all, assume that there exists an index set $\mathcal{I} \subset \mathbb{N}$ such that the elements in the subdivision \mathcal{T} are numbered in a certain way:

$$\mathcal{T} = \{K_i\}_{i \in \mathcal{I}}.$$

Furthermore, denote by \mathcal{E} the set of all element edges associated with the mesh \mathcal{T} . Additionally, let Γ_{int} be the union of all edges $e \in \mathcal{E}$ not lying on $\partial\Omega$:

$$\Gamma_{\text{int}} := \bigcup_{\substack{e \in \mathcal{E}: \\ e \cap \Omega = \emptyset}} e.$$

Moreover, define

$$\Gamma_{\text{int},D} := \Gamma_{\text{int}} \cup \{e \in \mathcal{E} : e \subset \Gamma_D\}.$$

Obviously, for each $e \in \Gamma_{\text{int}}$, there exist two indices i and j with $i > j$ such that K_i and K_j share the interface e :

$$e = \partial K_i \cap \partial K_j.$$

Thus, the following mapping is well-defined:

$$\begin{aligned} \varphi : \Gamma_{\text{int}} &\longrightarrow \mathbb{N}^2 \\ e &\longmapsto \begin{pmatrix} \varphi_1(e) := i \\ \varphi_2(e) := j \end{pmatrix}. \end{aligned}$$

If $e \in \mathcal{E} \setminus \Gamma_{\text{int}}$, i.e. if e is a boundary edge, there is a unique element $K_i \in \mathcal{T}$ such that

$$e = \partial K_i \cap \Gamma.$$

Hence, the above definition may be expanded as follows:

$$\begin{aligned} \varphi : \mathcal{E} \setminus \Gamma_{\text{int}} &\longrightarrow \mathbb{N} \\ e &\longmapsto \varphi(e) := i. \end{aligned}$$

As the DGFEM is based on functions in

$$H^1(\Omega, \mathcal{T})^2 := \{\underline{v} : \underline{v} \in H^1(K)^2, \forall K \in \mathcal{T}\} \not\subset \mathcal{C}^0(\Omega)^2,$$

the discontinuities over element boundaries have to be controlled in a certain way. In order to do so, consider $\underline{v} \in H^1(\Omega, \mathcal{T})^2$, $e \in \Gamma_{\text{int}}$ and $x \in e$, and introduce the so-called (numbering-dependent) *jump operator* of \underline{v} in x ,

$$[\underline{v}]_e(x) := \gamma_{K_{\varphi_1(e)}, e} \underline{v}(x) - \gamma_{K_{\varphi_2(e)}, e} \underline{v}(x),$$

and the so-called *average operator* of \underline{v} in x ,

$$\langle \underline{v} \rangle_e(x) := \frac{1}{2} (\gamma_{K_{\varphi_1(e)}, e} \underline{v}(x) + \gamma_{K_{\varphi_2(e)}, e} \underline{v}(x)).$$

Here, $\gamma_{K_{\varphi_i(e)}, e} \underline{v}$ denotes the trace of $\underline{v}|_{K_{\varphi_i(e)}}$ onto e , $i = 1, 2$. For $e \subset \Gamma$, define:

$$[\underline{v}]_e \equiv \langle \underline{v} \rangle_e \equiv \gamma_{K_{\varphi(e)}, e} \underline{v}.$$

Additionally, on $e \in \Gamma_{\text{int}}$, introduce a normal vector $\underline{\nu}_e$ pointing from $K_{\varphi_1(e)}$ to $K_{\varphi_2(e)}$ (for boundary edges $e \subset \Gamma$, set $\underline{\nu}_e := \underline{n}_K$, where \underline{n}_K is the unit outward vector of K on ∂K).

Finally, in order to define a variational formulation for the linear elasticity problem (1), the following product operator on $L^2(K)^{2 \times 2} \times L^2(K)^{2 \times 2}$, $K \in \mathcal{T}$ has to be introduced:

$$\boldsymbol{\alpha} : \boldsymbol{\beta} := \sum_{i,j=1}^2 \alpha_{ij} \beta_{ij}$$

with the induced norm

$$\|\boldsymbol{\alpha}\|_K := \sqrt{\int_K \boldsymbol{\alpha} : \boldsymbol{\alpha} \, dx}.$$

Definition 4.2 (DGFEM) Define a bilinear form B_{DG} by

$$\begin{aligned} B_{DG}(\underline{u}, \underline{v}) &:= \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\sigma}(\underline{u}) : \boldsymbol{\epsilon}(\underline{v}) \, dx \\ &\quad - \sum_{e \in \Gamma_{\text{int},D}} \int_e (\langle \boldsymbol{\sigma}(\underline{u}) \cdot \underline{\nu}_e \rangle_e \cdot [\underline{v}]_e - [\underline{u}]_e \cdot \langle \boldsymbol{\sigma}(\underline{v}) \cdot \underline{\nu}_e \rangle_e) \, ds \\ &\quad + \mu \sum_{e \in \Gamma_{\text{int},D}} \frac{1}{|e|} \int_e [\underline{u}]_e \cdot [\underline{v}]_e \, ds, \end{aligned}$$

and a corresponding linear functional L_{DG} by

$$\begin{aligned} L_{DG}(\underline{v}) &:= \sum_{K \in \mathcal{T}} \int_K \underline{f} \cdot \underline{v} \, dx + \int_{\Gamma_N} \underline{g}_N \cdot \underline{v} \, ds \\ &\quad + \int_{\Gamma_D} (\boldsymbol{\sigma}(\underline{v}) \cdot \underline{n}_\Omega) \cdot \underline{g}_D \, ds + \mu \sum_{\substack{e \in \mathcal{E}: \\ e \subset \Gamma_D}} \frac{1}{|e|} \int_e \underline{g}_D \cdot \underline{v} \, ds. \end{aligned}$$

Then, the DGFEM for the linear elasticity problem (1) reads as follows:

Find $\underline{u}_{DG} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T})$ such that

$$B_{DG}(\underline{u}_{DG}, \underline{v}) = L_{DG}(\underline{v}) \quad \forall \underline{v} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T}). \quad (14)$$

Proposition 4.3 (Consistency) If the exact solution \underline{u}_{ex} of the linear elasticity problem (1) belongs to $H_{\underline{\beta}}^{2,2}(\Omega)^2$ for any weight vector $\underline{\beta} = (\beta_1, \dots, \beta_M)$ with $\beta_i \in [0, 1)$, $i = 1, \dots, M$, then the DGFEM (14) is consistent:

$$B_{DG}(\underline{u}_{ex}, \underline{v}) = L(\underline{v}) \quad \forall \underline{v} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T}). \quad (15)$$

Proof: Cf. [18]. □

Finally, the DGFEM will be associated with the following norm:

$$\|\underline{u}\|_{DG}^2 := \sum_{K \in \mathcal{T}} \|\epsilon(\underline{u})\|_K^2 + \frac{\mu}{m_{\text{elast}}} \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \int_e |[\underline{u}]_e|^2 ds, \quad (16)$$

where

$$m_{\text{elast}} := 2 \min\{\mu, \mu + \lambda\}.$$

Remark 4.4 The norm in (16) is equivalent to the elementwise H^1 norm. A corresponding result may be found in [6], where a discrete Korn inequality was proved.

Proposition 4.5 (Coercivity) *The bilinear form B_{DG} is coercive on $\mathcal{S}^{1,0}(\Omega, \mathcal{T})$. More precisely,*

$$B_{DG}(\underline{u}, \underline{u}) \geq 2m_{\text{elast}} \|\underline{u}\|_{DG}^2$$

for all $\underline{u} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T})$.

Proof: Set

$$\epsilon_0(\underline{u}) := \epsilon(\underline{u}) - \frac{1}{2} \nabla \cdot \underline{u} \mathbf{1}_{2 \times 2}.$$

Then, for $K \in \mathcal{T}$, there holds that

$$\begin{aligned} \int_K \boldsymbol{\sigma}(\underline{u}) : \epsilon(\underline{u}) dx &= 2\mu \int_K \epsilon(\underline{u}) : \epsilon(\underline{u}) dx + \lambda \int_K |\nabla \cdot \underline{u}|^2 dx \\ &= 2\mu \int_K (\epsilon_0(\underline{u}) + \frac{1}{2} \nabla \cdot \underline{u} \mathbf{1}_{2 \times 2}) : (\epsilon_0(\underline{u}) + \frac{1}{2} \nabla \cdot \underline{u} \mathbf{1}_{2 \times 2}) dx \\ &\quad + \lambda \int_K |\nabla \cdot \underline{u}|^2 dx \\ &= 2\mu \int_K \{\epsilon_0(\underline{u}) : \epsilon_0(\underline{u}) + \frac{1}{2} |\nabla \cdot \underline{u}|^2\} dx + \lambda \int_K |\nabla \cdot \underline{u}|^2 dx \\ &= 2\mu \int_K \epsilon_0(\underline{u}) : \epsilon_0(\underline{u}) dx + (\mu + \lambda) \int_K |\nabla \cdot \underline{u}|^2 dx. \end{aligned}$$

Moreover, since

$$\begin{aligned} \int_K \epsilon(\underline{u}) : \epsilon(\underline{u}) dx &= \int_K (\epsilon_0(\underline{u}) + \frac{1}{2} \nabla \cdot \underline{u} \mathbf{1}_{2 \times 2}) : (\epsilon_0(\underline{u}) + \frac{1}{2} \nabla \cdot \underline{u} \mathbf{1}_{2 \times 2}) dx \\ &= \int_K \{\epsilon_0(\underline{u}) : \epsilon_0(\underline{u}) + \frac{1}{2} |\nabla \cdot \underline{u}|^2\} dx, \end{aligned}$$

it follows that

$$\int_K \boldsymbol{\sigma}(\underline{u}) : \epsilon(\underline{u}) dx \geq m_{\text{elast}} \int_K \epsilon(\underline{u}) : \epsilon(\underline{u}) dx.$$

Thus,

$$\begin{aligned} B_{DG}(\underline{u}, \underline{u}) &\geq m_{\text{elast}} \sum_{K \in \mathcal{T}} \int_K \epsilon(\underline{u}) : \epsilon(\underline{u}) dx + \mu \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \int_e |[\underline{u}]_e|^2 ds \\ &\geq m_{\text{elast}} \|\underline{u}\|_{DG}^2. \end{aligned}$$

□

5 Error Analysis of the DGFEM

5.1 Interpolants

Proposition 5.1 *Let $K \in \mathcal{T}$ be a triangle with vertices A_1, A_2, A_3 . Then, for each $\beta \in [0, 1)$ and for $\Phi_\beta(x) = r^\beta = |x - A_1|^\beta$, there exists an interpolant*

$$\pi_K : H_\beta^{2,2}(K)^2 \longrightarrow \mathcal{P}_1(K)^2$$

such that the following properties are satisfied:

- a) $\int_e \underline{u} - \pi_K \underline{u} \, ds = \underline{0}, \quad \forall e \in \mathcal{E}_K := \{e \in \mathcal{E} : e \subset \partial K\};$
- b) $\int_e (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_e \, ds = 0, \quad \forall e \in \mathcal{E}_K, (\underline{n}_e \text{ is the unit outward vector of } K \text{ on } e);$
- c) $\int_K \nabla \cdot (\underline{u} - \pi_K \underline{u}) \, dx = 0;$

Proof: For $\underline{u} \in H_\beta^{2,2}(K)^2$ the interpolant $\pi_K \underline{u} \in \mathcal{P}_1(K)^2$ is uniquely defined by

$$\pi_K \underline{u}(x_e^M) := \frac{1}{|e|} \int_e \underline{u} \, ds, \quad \forall e \in \mathcal{E}_K,$$

where x_e^M denotes the midpoint of $e \in \mathcal{E}_K$. Then, a) and b) follow directly from this definition. c) results from b) and from Green's formula:

$$\begin{aligned} \int_K \nabla \cdot (\underline{u} - \pi_K \underline{u}) \, dx &= \int_{\partial K} (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_{\partial K} \, ds \\ &= \sum_{e \in \mathcal{E}_K} \int_e (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_e \, ds \\ &= 0. \end{aligned}$$

□

Proposition 5.2 *For $\underline{u} \in H_\beta^{2,2}(K)^2$, $K \in \mathcal{T}$, the interpolant $\pi_K \underline{u}$ from Proposition 5.1 satisfies the following estimates:*

$$\|\underline{u} - \pi_K \underline{u}\|_{L^2(K)} + h_K |\underline{u} - \pi_K \underline{u}|_{H^1(K)} \leq Ch_K^{2-\beta} |\underline{u}|_{H_\beta^{2,2}(K)} \quad (17)$$

$$|\underline{u} - \pi_K \underline{u}|_{H_\beta^{2,2}(K)} \leq |\underline{u}|_{H_\beta^{2,2}(K)} \quad (18)$$

and

$$\|\nabla \cdot (\underline{u} - \pi_K \underline{u})\|_{L^2(K)} \leq Ch_K^{1-\beta} |\nabla \cdot \underline{u}|_{H_\beta^{1,1}(K)} \quad (19)$$

$$|\nabla \cdot (\underline{u} - \pi_K \underline{u})|_{H_\beta^{1,1}(K)} \leq |\nabla \cdot \underline{u}|_{H_\beta^{1,1}(K)}. \quad (20)$$

$C > 0$ is a constant independent of h_K and \underline{u} .

Proof: Set $\underline{U} := \underline{u} - \pi_K \underline{u}$. Then, since $\pi_K \underline{u} \in \mathcal{P}_1(K)^2$, there holds:

$$|\underline{U}|_{H_\beta^{2,2}(K)} = |\underline{u}|_{H_\beta^{2,2}(K)} \quad \text{and} \quad |\nabla \cdot \underline{U}|_{H_\beta^{1,1}(K)} = |\nabla \cdot \underline{u}|_{H_\beta^{1,1}(K)}.$$

Thus, applying Lemma A.2 to \underline{U} and Lemma A.3 to $\nabla \cdot \underline{U}$, terminates the proof. □

5.2 Stability

In a polygon Ω consider a FE mesh \mathcal{T} satisfying the conditions from Section 4.1. Moreover, let $\underline{\beta} = (\beta_1, \dots, \beta_M)$ be a weight vector and $\Phi_{\underline{\beta}}$ the corresponding weight function described in Section 3.1. Then, on $\mathcal{S}^{1,0}(\Omega, \mathcal{T})$, define an interpolant

$$\Pi_{\mathcal{T}} : H_{\underline{\beta}}^{2,2}(\Omega)^2 \longrightarrow \mathcal{S}^{1,0}(\Omega, \mathcal{T})$$

by

$$\Pi_{\mathcal{T}}|_K \underline{u} = \pi_K \underline{u}, \quad \forall K \in \mathcal{T},$$

where π_K , $K \in \mathcal{T}$ is the interpolant from Proposition 5.1.

Then, the DG-error $\underline{e} := \underline{u}_{\text{ex}} - \underline{u}_{DG}$, where $\underline{u}_{\text{ex}}$ is the exact solution of the linear elasticity problem (1) and \underline{u}_{DG} is the solution of the DGFEM (14), may be represented as follows:

$$\underline{e} = \underbrace{\underline{u}_{\text{ex}} - \Pi_{\mathcal{T}} \underline{u}_{\text{ex}}}_{=: \underline{\eta}} + \underbrace{\Pi_{\mathcal{T}} \underline{u}_{\text{ex}} - \underline{u}_{DG}}_{=: \underline{\xi}}. \quad (21)$$

Remark 5.3 Since $H_{\underline{\beta}}^{2,2}(\Omega)^2 \subset C^0(\bar{\Omega})^2$ (cf. [4]), $\underline{u}_{\text{ex}} \in H_{\underline{\beta}}^{2,2}(\Omega)$ implies that

$$\int_e [\underline{\eta}]_e ds = 0$$

for all edges $e \in \Gamma_{\text{int}}$.

Proposition 5.5 shows that $\|\underline{\xi}\|_{DG}$ is bounded by $\|\underline{\eta}\|_{DG}$. Therefore, the error $\underline{e} = \underline{u}_{\text{ex}} - \underline{u}_{DG}$ of the DGFEM may be controlled by $\underline{\eta}$ only.

In order to prove this, consider the following notations:

$$\mathcal{K}_0 := \{K \in \mathcal{T} : \partial K \cap SP(\Omega, \Gamma_D, \Gamma_N) \neq \emptyset\},$$

and

$$\partial \mathcal{K}_0 := \{e \in \mathcal{E} : \exists K \in \mathcal{K}_0 \text{ such that } e \subset \partial K\}.$$

Furthermore, consider the following Lemma, which will be useful for the error analysis of the DGFEM.

Lemma 5.4 *Let $\underline{u} \in H_{\underline{\beta}}^{2,2}(\Omega)$. Then,*

$$\begin{aligned} & \mu^2 \sum_{K \in \mathcal{T}} \|\epsilon(\underline{\eta})\|_K^2 + \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int}, D}}} \|\gamma_{K,e}(\sigma(\underline{\eta}) \cdot \underline{\nu}_e)\|_{L^1(e)}^2 + \mu^2 \sum_{e \in \Gamma_{\text{int}, D}} |e|^{-1} \|[\underline{\eta}]_e\|_{L^2(e)}^2 \\ & \leq C \left\{ \mu^2 \left[\sum_{K \in \mathcal{T}} (h_K^{-2} \|\underline{\eta}\|_{L^2(K)}^2 + \|\underline{\eta}\|_{H^1(K)}^2) + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 \|\underline{\eta}\|_{H^2(K)}^2 \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} \|\underline{\eta}\|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] + \lambda^2 \left[\sum_{K \in \mathcal{T}} \|\nabla \cdot \underline{\eta}\|_{L^2(K)}^2 \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 \|\nabla \cdot \underline{\eta}\|_{H^1(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} \|\nabla \cdot \underline{\eta}\|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right] \right\}, \end{aligned}$$

where $\underline{\eta} := \underline{u} - \Pi_{\mathcal{T}} \underline{u}$.

Proof: Obviously,

$$\sum_{K \in \mathcal{T}} \|\epsilon(\underline{\eta})\|_K^2 \leq C \sum_{K \in \mathcal{T}} |\underline{\eta}|_{H^1(K)}^2.$$

Furthermore, Lemma A.4 and Remark 3.2 imply that

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int}, D}}} \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}^2 \\ & \leq C \left[\mu^2 \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int}, D}}} \|\gamma_{K,e}(\underline{\epsilon}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}^2 + \lambda^2 \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int}, D}}} \|\gamma_{K,e}(\nabla \cdot \underline{\eta})\|_{L^1(e)}^2 \right] \\ & \leq C \mu^2 \left[\sum_{K \in \mathcal{T}} \|\nabla \underline{\eta}\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\underline{\eta}|_{H^2(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\underline{\eta}|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] \\ & \quad + C \lambda^2 \left[\sum_{K \in \mathcal{T}} \|\nabla \cdot \underline{\eta}\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\nabla \cdot \underline{\eta}|_{H^1(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\nabla \cdot \underline{\eta}|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right]. \end{aligned}$$

Additionally, by the Trace Theorem (cf. [15], Theorem A.11), there holds

$$\begin{aligned} \sum_{e \in \Gamma_{\text{int}, D}} |e|^{-1} \|\llbracket \underline{\eta} \rrbracket_e\|_{L^2(e)}^2 & \leq C \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int}, D}}} |e|^{-1} \|\gamma_{K,e} \underline{\eta}\|_{L^2(e)}^2 \\ & \leq C \sum_{e \in \Gamma_{\text{int}, D}} (|e|^{-2} \|\underline{\eta}\|_{L^2(K)}^2 + \|\nabla \underline{\eta}\|_{L^2(K)}^2) \\ & \leq C \sum_{e \in \Gamma_{\text{int}, D}} (h_K^{-2} \|\underline{\eta}\|_{L^2(K)}^2 + \|\nabla \underline{\eta}\|_{L^2(K)}^2). \end{aligned}$$

□

Proposition 5.5 (Stability) *Let the exact solution $\underline{u}_{\text{ex}}$ of the linear elasticity problem (1) be in $H_{\underline{\beta}}^{2,2}(\Omega)$, where Ω is a polygon in \mathbb{R}^2 . Then, there holds the following stability inequality for the DGFEM (14)*

$$\begin{aligned} \|\underline{\xi}\|_{DG}^2 & \leq C C_{\mu, \lambda} \left\{ \mu^2 \left[\sum_{K \in \mathcal{T}} (h_K^{-2} \|\underline{\eta}\|_{L^2(K)}^2 + |\underline{\eta}|_{H^1(K)}^2) + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\underline{\eta}|_{H^2(K)}^2 \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\underline{\eta}|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] + \lambda^2 \left[\sum_{K \in \mathcal{T}} \|\nabla \cdot \underline{\eta}\|_{L^2(K)}^2 \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 |\nabla \cdot \underline{\eta}|_{H_{\underline{\beta}}^{1,1}(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\nabla \cdot \underline{\eta}|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right] \right\}, \end{aligned}$$

where $\underline{\eta}$ and $\underline{\xi}$ are defined in (21) and where

$$C_{\mu, \lambda} := \max \left\{ 1, \sqrt{\frac{2 \min\{\mu, \mu + \lambda\}}{\mu}} \right\}$$

is bounded independently of λ and μ as $\lambda \rightarrow \infty$, and $C > 0$ is a constant independent of μ, λ and of $\{h_K : K \in \mathcal{T}\}$.

Proof: Due to the consistency of the DGFEM (cf. Proposition 4.3), it holds that

$$B_{DG}(\underline{\xi}, \underline{\xi}) = B_{DG}(\underline{e} - \underline{\eta}, \underline{\xi}) = -B_{DG}(\underline{\eta}, \underline{\xi}).$$

Therefore, by Proposition 4.5

$$2m_{\text{elast}} \|\underline{\xi}\|_{DG}^2 \leq -B_{DG}(\underline{\eta}, \underline{\xi}). \quad (22)$$

Furthermore,

$$\begin{aligned} B_{DG}(\underline{\eta}, \underline{\xi}) &= \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\sigma}(\underline{\eta}) : \boldsymbol{\epsilon}(\underline{\xi}) \, dx \\ &\quad - \sum_{e \in \Gamma_{\text{int}, D}} \int_e (\langle \boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e \rangle_e \cdot [\underline{\xi}]_e - [\underline{\eta}]_e \cdot \langle \boldsymbol{\sigma}(\underline{\xi}) \cdot \boldsymbol{\nu}_e \rangle_e) \, ds \\ &\quad + \mu \sum_{e \in \Gamma_{\text{int}, D}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, ds \\ &= 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\underline{\eta}) : \boldsymbol{\epsilon}(\underline{\xi}) \, dx + \lambda \sum_{K \in \mathcal{T}} \nabla \cdot \underline{\xi} \int_K \nabla \cdot \underline{\eta} \, dx \\ &\quad - \sum_{e \in \Gamma_{\text{int}, D}} \left(\int_e \langle \boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e \rangle_e \cdot [\underline{\xi}]_e \, ds - \langle \boldsymbol{\sigma}(\underline{\xi}) \cdot \boldsymbol{\nu}_e \rangle_e \cdot \int_e [\underline{\eta}]_e \, ds \right) \\ &\quad + \mu \sum_{e \in \Gamma_{\text{int}, D}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, ds. \end{aligned}$$

Applying Proposition 5.1 and Remark 5.3 results in

$$\begin{aligned} B_{DG}(\underline{\eta}, \underline{\xi}) &= 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\underline{\eta}) : \boldsymbol{\epsilon}(\underline{\xi}) \, dx - \sum_{e \in \Gamma_{\text{int}, D}} \int_e \langle \boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e \rangle_e \cdot [\underline{\xi}]_e \, ds \\ &\quad + \mu \sum_{e \in \Gamma_{\text{int}, D}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, ds \\ &= I - II + III. \end{aligned}$$

By Hölder's inequality, there holds that

$$\begin{aligned} |I| &= \left| 2\mu \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\epsilon}(\underline{\eta}) : \boldsymbol{\epsilon}(\underline{\xi}) \, dx \right| \\ &\leq \left[4\mu^2 \sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\underline{\eta})\|_K^2 \right]^{1/2} \left[\sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\underline{\xi})\|_K^2 \right]^{1/2}. \end{aligned}$$

A bound for II is obtained as follows:

$$\begin{aligned}
|II| &\leq \sum_{e \in \Gamma_{\text{int},D}} \int_e |\langle \boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e \rangle_e| \|\underline{\xi}\|_e ds \\
&\leq \sum_{e \in \Gamma_{\text{int},D}} \|\underline{\xi}\|_e \|L^\infty(e)\| \|\langle \boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e \rangle_e\|_{L^1(e)} \\
&\leq \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \Gamma_{\text{int}} \\ e \in \partial K}} \|\underline{\xi}\|_e \|L^\infty(e)\| \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)} \\
&\quad + \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_D}} \|\underline{\xi}\|_e \|L^\infty(e)\| \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)} \\
&\leq C \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} \|\underline{\xi}\|_e \|L^\infty(e)\| \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}.
\end{aligned}$$

Furthermore, Lemma A.1 implies that

$$\begin{aligned}
|II| &\leq C \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} |e|^{-1/2} \|\underline{\xi}\|_e \|L^2(e)\| \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)} \\
&\leq C \left[\sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} |e|^{-1} \|\underline{\xi}\|_e^2 \|L^2(e)\| \right]^{1/2} \left[\sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}^2 \right]^{1/2} \\
&= C \sqrt{\frac{m_{\text{elast}}}{\mu}} \left[\frac{\mu}{m_{\text{elast}}} \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \|\underline{\xi}\|_e^2 \|L^2(e)\| \right]^{1/2} \\
&\quad \cdot \left[\sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}^2 \right]^{1/2}.
\end{aligned}$$

Finally,

$$|III| \leq \sqrt{\frac{m_{\text{elast}}}{\mu}} \left[\mu^2 \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \|\underline{\eta}\|_e^2 \|L^2(e)\| \right]^{1/2} \left[\frac{\mu}{m_{\text{elast}}} \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \|\underline{\xi}\|_e^2 \|L^2(e)\| \right]^{1/2}.$$

Summing up and using (22) yields

$$\begin{aligned}
\|\underline{\xi}\|_{DG}^2 &\leq \frac{1}{2m_{\text{elast}}} |B_{DG}(\underline{\eta}, \underline{\xi})| \\
&\leq \frac{1}{2m_{\text{elast}}} (|I| + |II| + |III|) \\
&\leq C \max \left\{ 1, \sqrt{\frac{m_{\text{elast}}}{\mu}} \right\} \|\underline{\xi}\|_{DG} \cdot \left[\mu^2 \sum_{K \in \mathcal{T}} \|\boldsymbol{\epsilon}(\underline{\eta})\|_K^2 \right. \\
&\quad \left. + \sum_{K \in \mathcal{T}} \sum_{\substack{e \in \mathcal{E}_K \\ e \in \Gamma_{\text{int},D}}} \|\gamma_{K,e}(\boldsymbol{\sigma}(\underline{\eta}) \cdot \boldsymbol{\nu}_e)\|_{L^1(e)}^2 + \mu^2 \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \|\underline{\eta}\|_e^2 \|L^2(e)\| \right]^{1/2}.
\end{aligned}$$

Applying Lemma 5.4 completes the proof immediately. \square

A direct sequence of the statement above is the ensuing

Corollary 5.6 *Let the assumptions of Proposition 5.5 be satisfied. Then, the following a priori error estimate holds true*

$$\begin{aligned} \|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG}^2 &\leq C\tilde{C}_{\mu,\lambda} \left\{ \mu^2 \left[\sum_{K \in \mathcal{T}} (h_K^{-2} \|\underline{\eta}\|_{L^2(K)}^2 + \|\underline{\eta}\|_{H^1(K)}^2) + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 \|\underline{\eta}\|_{H^2(K)}^2 \right. \right. \\ &\quad \left. \left. + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} \|\underline{\eta}\|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] + \lambda^2 \left[\sum_{K \in \mathcal{T}} \|\nabla \cdot \underline{\eta}\|_{L^2(K)}^2 \right. \right. \\ &\quad \left. \left. + \sum_{K \in \mathcal{T} \setminus \mathcal{K}_0} h_K^2 \|\nabla \cdot \underline{\eta}\|_{H_{\underline{\beta}}^{1,1}(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} \|\nabla \cdot \underline{\eta}\|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right] \right\}. \end{aligned}$$

Here, $\underline{u}_{\text{ex}}$ is the exact solution of (1), \underline{u}_{DG} is the solution of the DGFEM (14) and

$$\tilde{C}_{\mu,\lambda} = \max\{\mu^{-2}, \mu^{-1}m_{\text{elast}}^{-1}, C_{\mu,\lambda}\},$$

where $C_{\mu,\lambda}$ is the constant from Proposition 5.5.

Remark 5.7 Obviously, the constant $\tilde{C}_{\mu,\lambda}$ from the above Proposition loses its dependence on λ if λ is sufficiently large, i.e.:

$$\exists \lambda_0(\mu) : \tilde{C}_{\mu,\lambda} \leq \tilde{C}_{\mu} \quad \forall \lambda > \lambda_0,$$

where \tilde{C}_{μ} is a constant independent of λ .

Proof: From the error splitting (21) it follows that

$$\begin{aligned} \|\underline{e}\|_{DG}^2 &\leq C(\|\underline{\eta}\|_{DG}^2 + \|\underline{\xi}\|_{DG}^2) \\ &\leq C \left[\sum_{K \in \mathcal{T}} \|\underline{\epsilon}(\underline{\eta})\|_{DG}^2 + \frac{\mu}{m_{\text{elast}}} \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \int_e |[\underline{\eta}]|^2 ds + \|\underline{\xi}\|_{DG}^2 \right] \\ &\leq C \max\{\mu^{-2}, \mu^{-1}m_{\text{elast}}^{-1}\} \left[\mu^2 \sum_{K \in \mathcal{T}} \|\underline{\epsilon}(\underline{\eta})\|_{DG}^2 + \mu^2 \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \int_e |[\underline{\eta}]|^2 ds \right] \\ &\quad + C\|\underline{\xi}\|_{DG}^2. \end{aligned}$$

Thus, using Lemma 5.4 and inserting the stability bound from Proposition 5.5 completes the proof. \square

5.3 Convergence Rates of the DGFEM

It is a well-known fact that, if $\underline{u}_{\text{ex}} \in H^2(\Omega)^2$, where $\underline{u}_{\text{ex}}$ denotes the exact solution of (1), the standard (continuous) finite element method (and also the DGFEM) converges at an optimal algebraic rate, i.e.

$$\|\underline{u}_{\text{ex}} - \underline{u}_{FE}\| \leq CN^{-1/2},$$

where $N = \dim(\mathcal{S}^{1,0}(\Omega, \mathcal{T}))$ is the number of degrees of freedom and \mathcal{T} is a uniform mesh on Ω . Unfortunately, this result is typically not anymore true if the assumption $\underline{u}_{\text{ex}} \in H^2(\Omega)^2$ is weakened, i.e. $\underline{u}_{\text{ex}} \in H_{\underline{\beta}}^{2,2}(\Omega)^2$ with $\underline{\beta} \succ \underline{0}$. Moreover, C depends on λ , $C \sim \sqrt{\lambda}$ as $\lambda \rightarrow \infty$.

Although, the convergence rate remains algebraic in this case, the optimal order $\mathcal{O}(N^{-1/2})$ is usually reduced to $\mathcal{O}(N^{-\alpha/2})$ with $\alpha \ll 1$. This effect is even more pronounced at higher orders of approximation.

The aim of this section is to prove that the optimal convergence rate may be preserved even if the exact solution is singular, i.e. $\underline{u}_{\text{ex}} \notin H^2(\Omega)$. The main idea is to replace the uniform meshes by so-called 'graded meshes' which are able to resolve the singularities without the need of additional degrees of freedom.

5.3.1 Graded Meshes

There are several possibilities to introduce graded meshes. The following approach may be found in [4].

Definition 5.8 *Let $\underline{\gamma}$ be a weight vector as defined in Section 3.1 and $\Phi_{\underline{\gamma}}$ the corresponding weight function on Ω . Then, a mesh $\mathcal{T}_{\underline{\gamma}}$ on Ω is called a **graded mesh with grading vector $\underline{\gamma}$** if there exists a constant $L > 0$ such that the following properties are satisfied:*

i) if $K \in \mathcal{T}_{\underline{\gamma}} \setminus \mathcal{K}_0$ then

$$L^{-1}h_{\mathcal{T}_{\underline{\gamma}}}\Phi_{\underline{\gamma}}(x) \leq h_K \leq Lh_{\mathcal{T}_{\underline{\gamma}}}\Phi_{\underline{\gamma}}(x) \quad \forall x \in K;$$

ii) if $K \in \mathcal{K}_0$ then

$$L^{-1}h_{\mathcal{T}_{\underline{\gamma}}}\sup_{x \in K}\Phi_{\underline{\gamma}}(x) \leq h_K \leq Lh_{\mathcal{T}_{\underline{\gamma}}}\sup_{x \in K}\Phi_{\underline{\gamma}}(x).$$

Here, $h_{\mathcal{T}_{\underline{\gamma}}}$ is the mesh width of $\mathcal{T}_{\underline{\gamma}}$ (cf. (11)).

Graded meshes have asymptotically the same number of degrees of freedom as uniform meshes:

Lemma 5.9 *Let $\mathcal{T}_{\underline{\gamma}}$ be a graded mesh as in Definition 5.8. Then,*

$$N := \dim(\mathcal{S}^{1,0}(\Omega, \mathcal{T}_{\underline{\gamma}})) \leq Ch_{\mathcal{T}_{\underline{\gamma}}}^{-2},$$

where $C > 0$ is a constant independent of $\{h_K : K \in \mathcal{T}_{\underline{\gamma}}\}$.

Proof: See [4], Lemma 4.1. □

5.3.2 Main Result

Theorem 5.10 (Robust Optimal Convergence) *Let the assumptions of Theorem 3.4 be satisfied. Moreover, let \mathcal{T}_γ with $(1, 1, \dots, 1) \succ \underline{\gamma} \succeq \underline{\beta}$ be a graded mesh as introduced in Definition 5.8. Then, for the h -DGFEM (14) it holds the following optimal error estimate:*

$$\|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG} \leq C\tilde{C}_{\mu,\lambda}N^{-1/2}.$$

Here, $\underline{u}_{\text{ex}} \in H_{\underline{\beta}}^{2,2}(\Omega)^2$ is the exact solution of the linear elasticity problem (1), \underline{u}_{DG} is the solution of the DGFEM (14), $N = \dim(\mathcal{S}^{1,0}(\mathcal{T}_\gamma, \Omega))$, $\tilde{C}_{\mu,\lambda}$ is the constant from Corollary 5.6 (independent of λ as $\lambda \rightarrow \infty$) and $C > 0$ is a constant independent of N and the Lamé coefficients μ and λ .

Proof: Let $\Pi_{\mathcal{T}_\gamma}$ be the global interpolant from Section 5.2, i.e.

$$\Pi_{\mathcal{T}_\gamma}|_K = \pi_K, \quad K \in \mathcal{T}_\gamma,$$

where π_K is the interpolant from Proposition 5.1. Referring to Corollary 5.6 the following error bound for the DGFEM may be obtained:

$$\begin{aligned} & \|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG}^2 \\ & \leq C\tilde{C}_{\mu,\lambda} \left\{ \mu^2 \left[\sum_{K \in \mathcal{T}_\gamma} (h_K^{-2} \|\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}\|_{L^2(K)}^2 + |\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|_{H^1(K)}^2) \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} h_K^2 |\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|_{H^2(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] \right. \\ & \quad \left. + \lambda^2 \left[\sum_{K \in \mathcal{T}_\gamma} \|\nabla \cdot (\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} h_K^2 |\nabla \cdot (\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})|_{H^1(K)}^2 \right. \right. \\ & \quad \left. \left. + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\nabla \cdot (\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right] \right\}. \end{aligned}$$

Moreover, inserting the estimates from Proposition 5.2 yields

$$\begin{aligned} & \|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG}^2 \\ & \leq C\tilde{C}_{\mu,\lambda} \left\{ \mu^2 \left[\sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} h_K^2 |\underline{u}_{\text{ex}}|_{H^2(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\underline{u}_{\text{ex}}|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right] \right. \\ & \quad \left. + \lambda^2 \left[\sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} h_K^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H^1(K)}^2 + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} |\nabla \cdot \underline{u}_{\text{ex}}|_{H_{\underline{\beta}}^{1,1}(K)}^2 \right] \right\} \\ & = C\tilde{C}_{\mu,\lambda} \left\{ \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} h_K^2 (\mu^2 |\underline{u}_{\text{ex}}|_{H^2(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H^1(K)}^2) \right. \\ & \quad \left. + \sum_{K \in \mathcal{K}_0} h_K^{2-2\beta} (\mu^2 |\underline{u}_{\text{ex}}|_{H_{\underline{\beta}}^{2,2}(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_{\underline{\beta}}^{1,1}(K)}^2) \right\}. \end{aligned} \tag{23}$$

Furthermore, from the definition of the graded meshes (Definition 5.8) it follows that

$$\begin{aligned} & \|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG}^2 \\ & \leq C\tilde{C}_{\mu,\lambda} \left\{ h_{\mathcal{T}_\gamma}^2 \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} \int_K r^{2\gamma} (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2) dx \right. \\ & \quad \left. + \sum_{K \in \mathcal{K}_0} h_{\mathcal{T}_\gamma}^{2-2\beta} (\sup_{x \in K} r^\gamma)^{2-2\beta} (\mu^2 |\underline{u}_{\text{ex}}|_{H_\beta^{2,2}(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_\beta^{1,1}(K)}^2) \right\}. \end{aligned}$$

For all $K \in \mathcal{K}_0$ there holds $r \leq h_K$. Hence,

$$h_K \leq Ch_{\mathcal{T}_\gamma} \sup_{x \in K} r^\gamma \leq Ch_{\mathcal{T}_\gamma} h_K^\gamma,$$

and therefore

$$h_K \leq Ch_{\mathcal{T}_\gamma}^{\frac{1}{1-\gamma}}.$$

This implies that

$$\sup_{x \in K} r^\gamma \leq Ch_K^\gamma \leq Ch_{\mathcal{T}_\gamma}^{\frac{\gamma}{1-\gamma}} \leq Ch_{\mathcal{T}_\gamma}^{\frac{\beta}{1-\beta}}.$$

Thus,

$$\begin{aligned} & \|\underline{u}_{\text{ex}} - \underline{u}_{DG}\|_{DG}^2 \\ & \leq C\tilde{C}_{\mu,\lambda} h_{\mathcal{T}_\gamma}^2 \left\{ \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} \int_K r^{2\gamma} (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2) dx \right. \\ & \quad \left. + \sum_{K \in \mathcal{K}_0} (\mu^2 |\underline{u}_{\text{ex}}|_{H_\beta^{2,2}(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_\beta^{1,1}(K)}^2) \right\} \\ & \leq C\tilde{C}_{\mu,\lambda} h_{\mathcal{T}_\gamma}^2 \left\{ \sum_{K \in \mathcal{T}_\gamma \setminus \mathcal{K}_0} \int_K \Phi_\beta^2 (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2) dx \right. \\ & \quad \left. + \sum_{K \in \mathcal{K}_0} (\mu^2 |\underline{u}_{\text{ex}}|_{H_\beta^{2,2}(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_\beta^{1,1}(K)}^2) \right\} \\ & \leq C\tilde{C}_{\mu,\lambda} h_{\mathcal{T}_\gamma}^2 \left\{ \int_\Omega \Phi_\beta^2 (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2) dx \right. \\ & \quad \left. + \sum_{K \in \mathcal{K}_0} (\mu^2 |\underline{u}_{\text{ex}}|_{H_\beta^{2,2}(K)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_\beta^{1,1}(K)}^2) \right\} \\ & \leq C\tilde{C}_{\mu,\lambda} h_{\mathcal{T}_\gamma}^2 (\mu^2 |\underline{u}_{\text{ex}}|_{H_\beta^{2,2}(\Omega)}^2 + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|_{H_\beta^{1,1}(\Omega)}^2). \end{aligned}$$

Finally, by Lemma 5.9, i.e.

$$h_{\mathcal{T}_\gamma} \leq CN^{-1/2},$$

and with the aid of Theorem 3.4, the proof is complete. \square

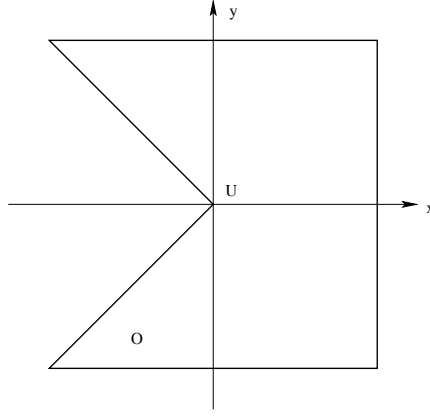


Figure 1: Polygonal domain Ω

Remark 5.11 On uniform meshes $\mathcal{T}_{\underline{\gamma}}$ it holds:

$$h_{\mathcal{T}_{\underline{\gamma}}} \sim h_K \sim \frac{1}{\sqrt{N}} \quad \forall K \in \mathcal{T}_{\underline{\gamma}}.$$

Therefore, (23) directly implies that, even if $\underline{\gamma} = \underline{0}$, the DGFEM still converges independently of μ and λ . However, due to the appearance of the term $h_K^{2-2\beta}$, the rate of convergence is not anymore optimal for $\underline{\beta} \succ \underline{0}$.

6 Numerical Results

The aim of this section is to confirm the previous theoretical results with some practical examples. More precisely, it will be shown that, even if the exact solutions of the corresponding problems are singular, the convergence rate of the DGFEM remains of order $\mathcal{O}(N^{-1/2})$, as expected. Moreover, the robustness of the method against volume locking will be illustrated.

6.1 L-shaped Domain

6.1.1 Model Problem

Let Ω be the polygonal domain with vertices

$$A_1 = (0, 0), A_2 = (-1, -1), A_3 = (1, -1), A_4 = (1, 1), A_5 = (-1, 1).$$

Note, that the origin $\mathcal{O} = (0, 0)$ is a reentrant corner of Ω (cf. Figure 6.1.1).

Then, consider the following model problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\underline{u}) &= \underline{0} & \text{in } \Omega \\ \underline{u} &= \underline{g}_D & \text{on } \Gamma_D = \partial\Omega \end{aligned} \quad (24)$$

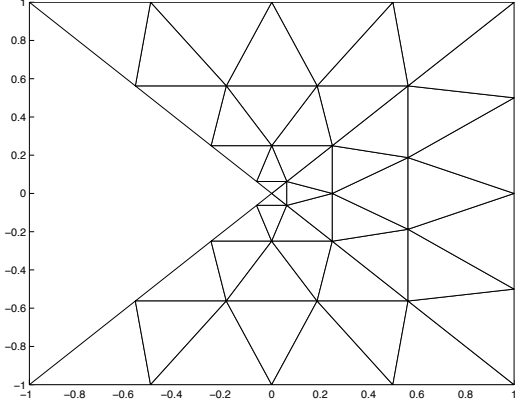


Figure 2: Graded mesh with refinement towards the origin ($\underline{\gamma} = (1/2, 0, 0, 0, 0)$)

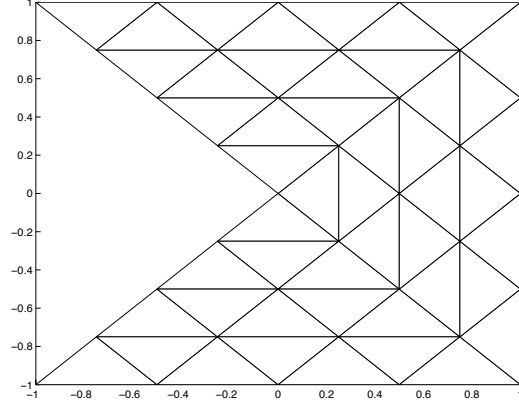


Figure 3: Uniform mesh (i.e. graded mesh with $\underline{\gamma} = (0, 0, 0, 0, 0)$)

Here, $\underline{g}_D := \underline{u}_{\text{ex}}|_{\Gamma_D}$, where $\underline{u}_{\text{ex}}$ is the exact solution of (24) given by its polar coordinates

$$\begin{aligned} u_r(r, \theta) &= \frac{1}{2\mu} r^\alpha (-(\alpha + 1) \cos((\alpha + 1)\theta) + (C_2 - (\alpha + 1))C_1 \cos((\alpha - 1)\theta)) \\ u_\theta(r, \theta) &= \frac{1}{2\mu} r^\alpha ((\alpha + 1) \sin((\alpha + 1)\theta) + (C_2 + \alpha - 1)C_1 \sin((\alpha - 1)\theta)), \end{aligned}$$

where $\alpha \approx 0.544484$ is the solution of the equation

$$\alpha \sin(2\omega) + \sin(2\omega\alpha) = 0$$

with $\omega = 3\pi/4$, and

$$C_1 = -\frac{\cos((\alpha + 1)\omega)}{\cos((\alpha - 1)\omega)}, \quad C_2 = \frac{2(\lambda + 2\mu)}{\lambda + \mu}.$$

6.1.2 Robust Optimal Convergence Rates on Graded Meshes

A few calculations show that the exact solution $\underline{u}_{\text{ex}}$ of the model problem (24) is in $H_{\underline{\beta}}^{2,2}(\Omega)^2$ with $\underline{\beta} = (\beta_1, 0, 0, 0, 0)$ for all $1 > \beta_1 > 1 - \alpha \approx 0.455516$. Thus, in order to obtain the optimal convergence rate, a graded mesh with refinement towards the origin must be used for the numerical simulations.

Figure 4 shows the errors of the DGFEM for $\lambda \in \{1, 100, 500, 1000, 5000\}$ ($\mu = 1$) in the energy norm

$$\|\underline{u}\|_{DG}^2 = \sum_{K \in \mathcal{T}} \|\epsilon(\underline{u})\|_K^2 + \frac{1}{m_{\text{elast}}} \sum_{e \in \Gamma_{\text{int},D}} |e|^{-1} \int_e |[\underline{u}]_e|^2 ds$$

on a graded mesh with weight vector $\underline{\gamma} = (1/2, 0, 0, 0, 0)$ (cf. Figure 2). Obviously, the convergence rate of the DGFEM is already almost optimal for approximately 5000 degrees of freedom (~ 800 elements). Moreover, the expected robustness of the DGFEM with respect to the Lamé coefficient λ is clearly visible.

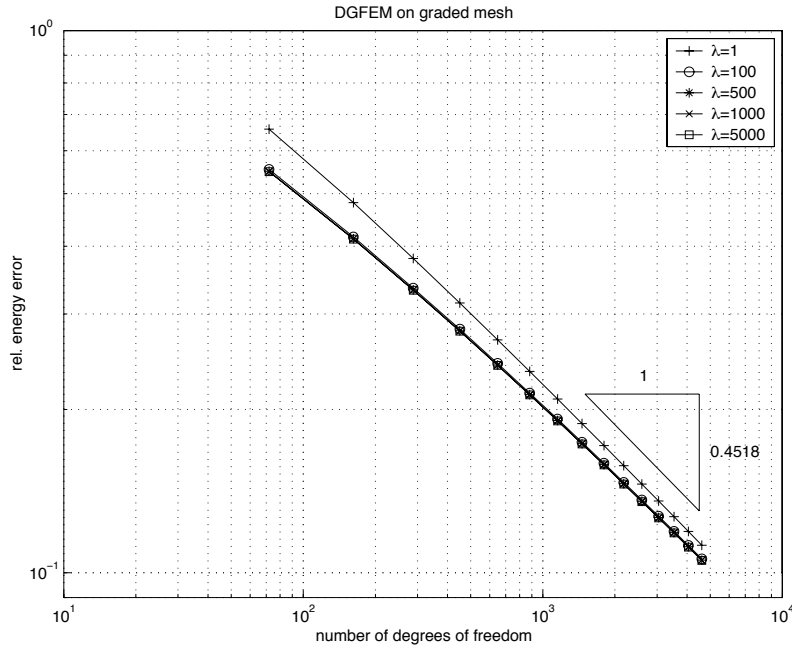


Figure 4: Performance of the DGFEM on the L-shaped domain with $\underline{\beta} = (1/2, 0, 0, 0, 0)$ (graded mesh)

In Figure 5 the energy error of the DGFEM on a uniform mesh (i.e. $\underline{\gamma} = (0, 0, 0, 0, 0)$) is presented. Although the DGFEM still converges robustly, the optimal convergence rate is not anymore achieved (cf. Remark 5.11) and the use of graded meshes is found to be justified.

In addition, the L^2 errors for the computations above are shown in Figures 7 and 6. Again, the performance of the DGFEM on a uniform mesh is notably worse. However, the convergence rate of the L^2 error seems to be twice as high as of the energy error. A similar super convergence was already discovered in [1].

6.1.3 Volume Locking

Figures 8 and 9 show that the standard (i.e. continuous) finite element method does not convergence independently of λ . Although the asymptotic rate of convergence is optimal on graded meshes, the onset of the errors' decay is remarkably retarded for $\lambda \rightarrow \infty$. This effect is widely known as 'volume locking' which, in contrast to the DGFEM, seems to be unavoidable for low-order standard h -FEMs in the primal variables.

6.2 An Example on the Unit Square

Consider the following problem on $\Omega = (0, 1)^2$:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\underline{u}) &= \underline{0} & \text{in } \Omega \\ \underline{u} &= \begin{pmatrix} g_D^{(1)} \\ 0 \end{pmatrix} & \text{on } \Gamma_D = \partial\Omega \end{aligned} \quad (25)$$

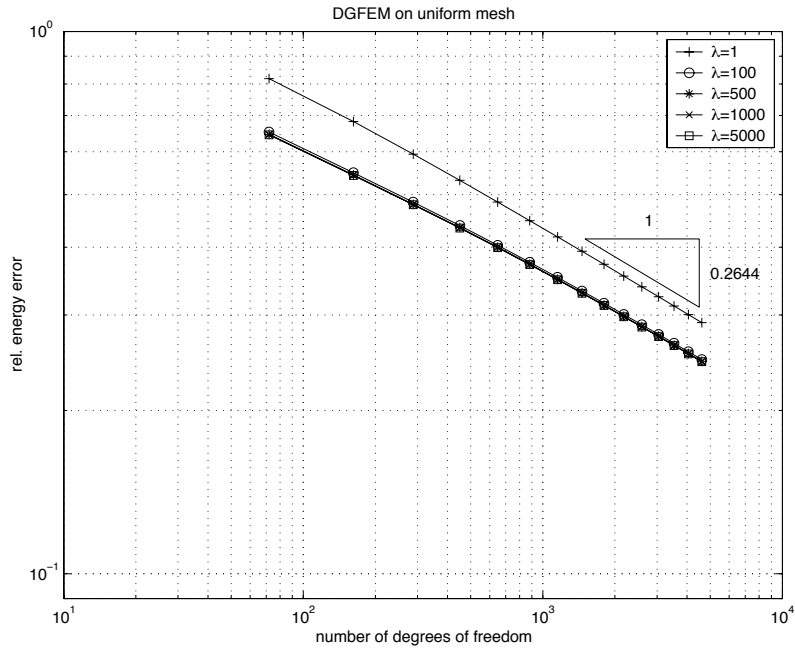


Figure 5: Performance of the DGFEM on the L-shaped domain with $\underline{\beta} = \underline{0}$ (uniform mesh)

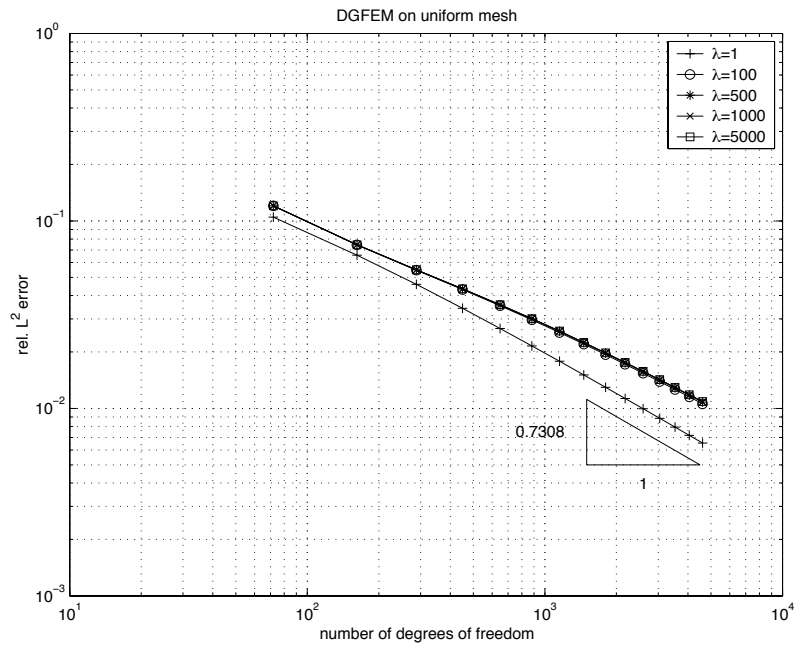


Figure 6: Performance of the DGFEM on the L-shaped domain with $\underline{\beta} = \underline{0}$ (uniform mesh)

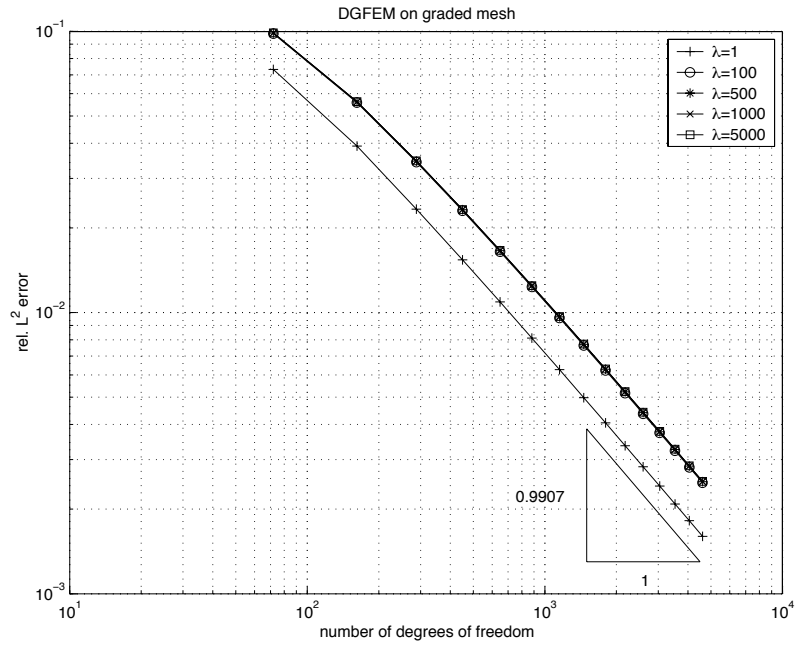


Figure 7: Performance of the DGFEM on the L-shaped domain with $\underline{\beta} = (1/2, 0, 0, 0, 0)$ (graded mesh)

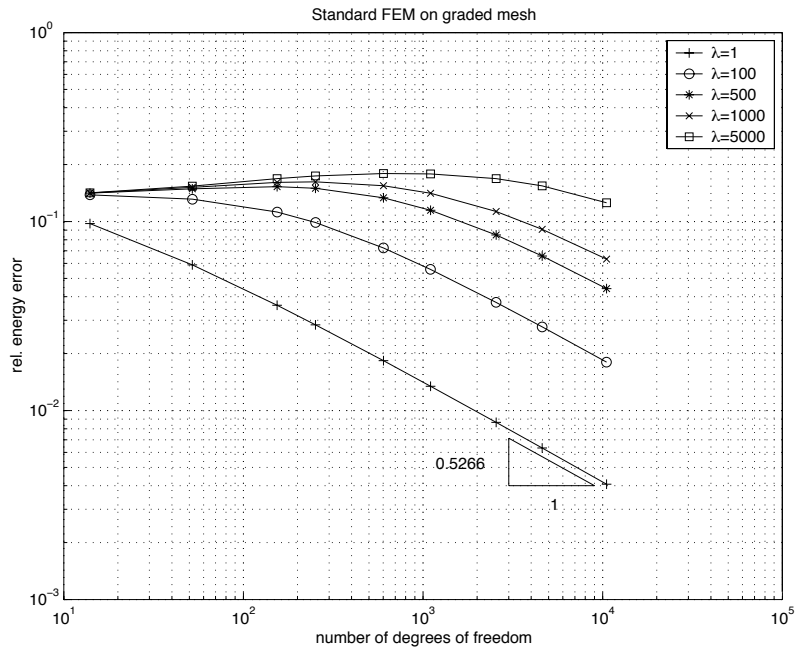


Figure 8: Performance of the Standard FEM on the L-shaped domain with $\underline{\beta} = \underline{0}$ (uniform mesh)

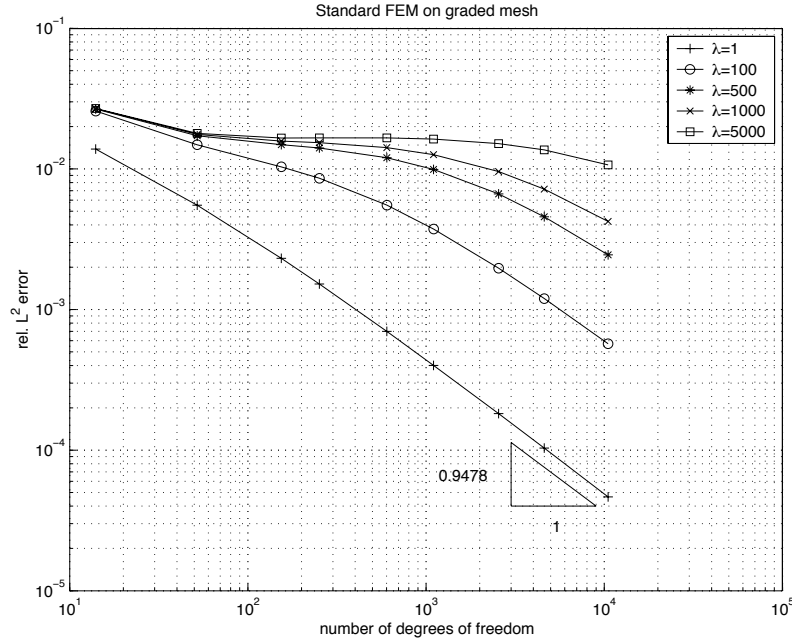


Figure 9: Performance of the Standard FEM on the L-shaped domain with $\underline{\beta} = (1/2, 0, 0, 0, 0)$ (graded mesh)

with

$$g_D^{(1)}(x, y) = \begin{cases} 1 - 4(x - 1/2)^2 & \text{if } (x, y) \in (0, 1) \times \{1\} \\ 0 & \text{else} \end{cases}$$

Due to Theorem 3.4, the exact solution of this problem belongs to $H^2(\Omega)^2$. Therefore, referring to the numerical analysis above, no mesh refinement is required for the DGFEM to converge optimally. The computational (uniform) mesh is shown in Figure 10. Additionally, the results for different choices of λ are presented (Figures 11–14). In contrast to the DGFEM, the standard FEM shows clear evidence of locking.

A Appendix

Lemma A.1 *Let $I = [a, b]$, $a < b$ be an interval in \mathbb{R} and $h_I = b - a$. Then, for every $u \in \mathcal{P}_1(I)$ it holds that*

$$\|u\|_{L^\infty(I)} \leq 4\sqrt{2}h_I^{-1/2}\|u\|_{L^2(I)}.$$

Proof: See [13]. □

The proofs of the following lemmas may be found in [18].

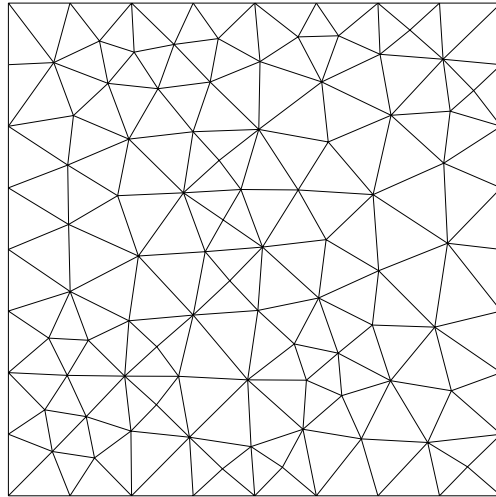


Figure 10: Computational mesh

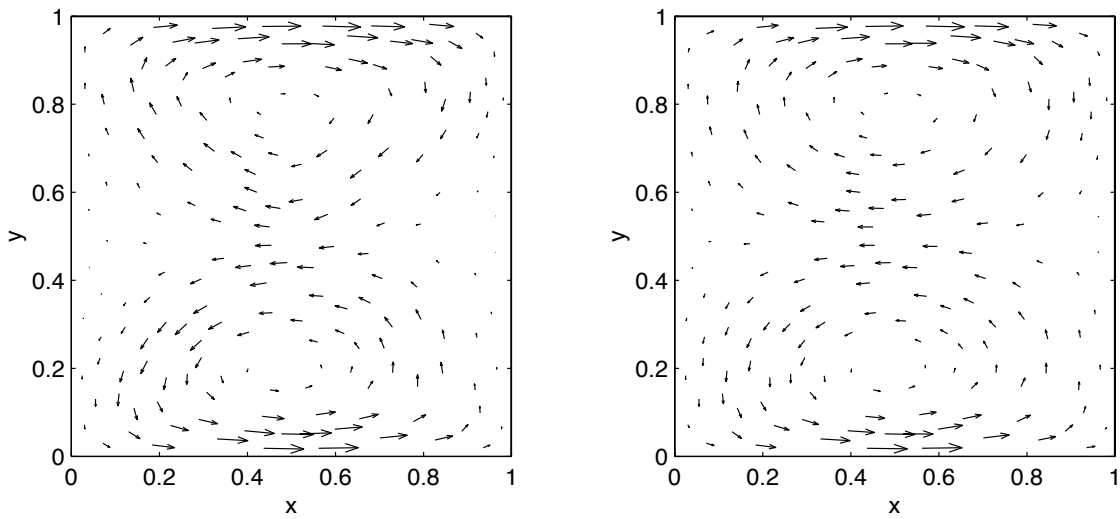


Figure 11: Standard FEM / DGFEM for $\lambda = 100$

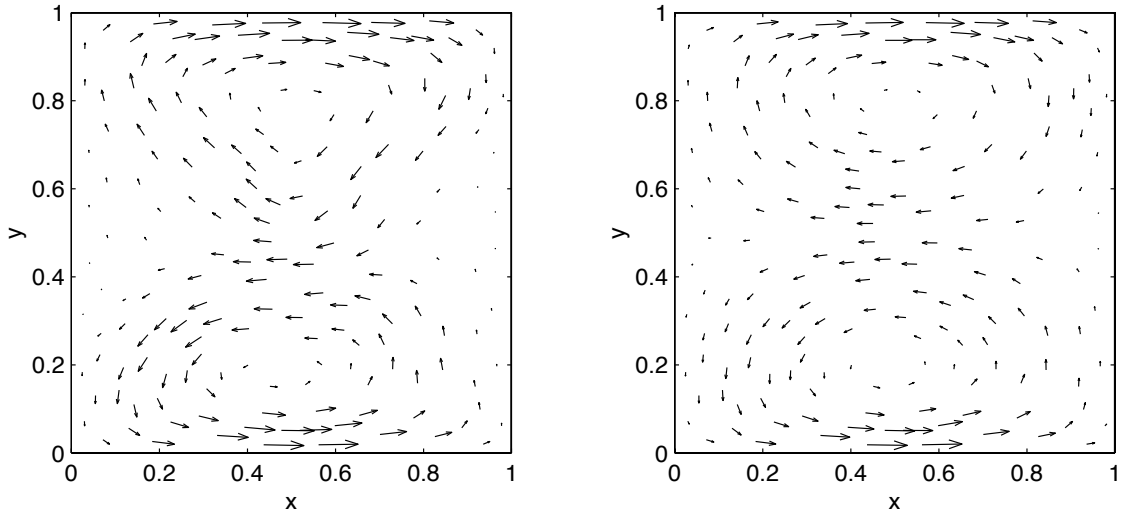


Figure 12: Standard FEM / DGFEM for $\lambda = 500$

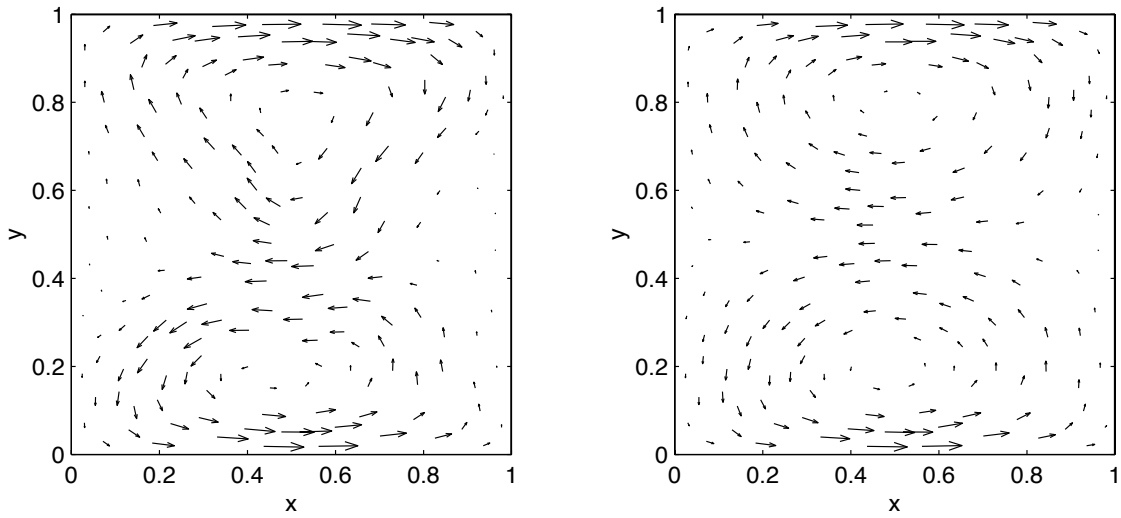


Figure 13: Standard FEM / DGFEM for $\lambda = 1000$

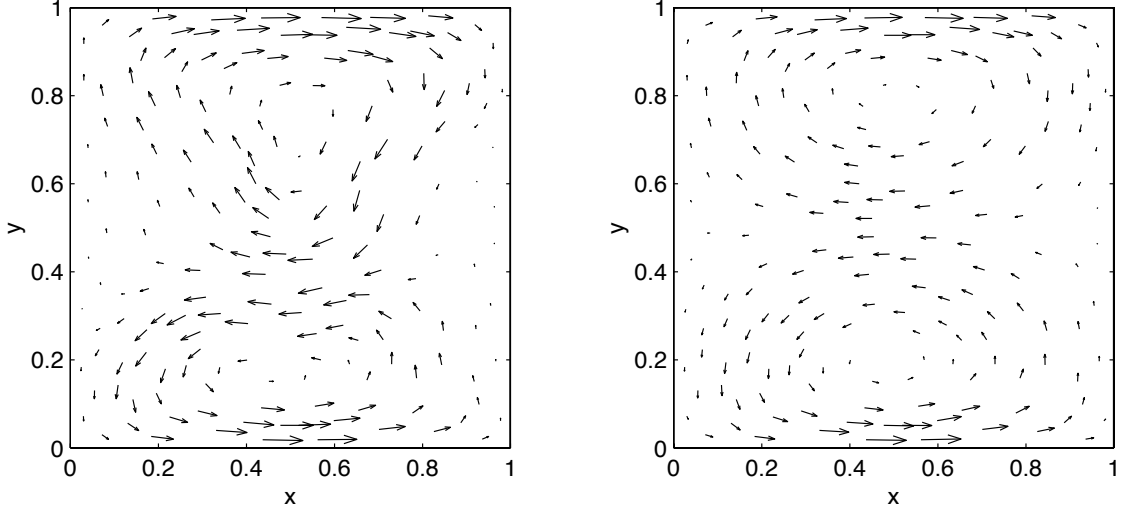


Figure 14: Standard FEM / DGFEM for $\lambda = 5000$

Lemma A.2 *Let $K \subset \mathbb{R}^2$ be a triangle with vertices A_1, A_2, A_3 . Then, for each $\underline{u} \in H_\beta^{2,2}(K)^2$, where $\beta \in [0, 1)$ and $\Phi_\beta(x) = r^\beta = |x - A_1|^\beta$, there holds:*

$$\|\underline{u}\|_{H_\beta^{2,2}(K)}^2 \leq C \left(\|\underline{u}\|_{H_\beta^{2,2}(K)}^2 + \sum_{e \in \mathcal{E}_K} \left| \int_e \underline{u} \, ds \right|^2 \right).$$

Here, $C > 0$ is a constant (independent of \underline{u}) and $\mathcal{E}_K = \{e_1, e_2, e_3\}$ is the set of all edges of K .

Lemma A.3 *Let the assumptions of Lemma A.2 be satisfied. In addition, let*

$$\int_K u \, dx = 0.$$

Then, there holds

$$\|u\|_{L^2(K)} \leq C \|u\|_{H_\beta^{1,1}(K)},$$

where $C > 0$ is a constant independent of u .

Lemma A.4 *Let the assumptions of Lemma A.2 be satisfied. Then, the following inequalities hold true:*

$$a) \|\underline{u}\|_{L^1(\partial K)} \leq C (\|\underline{u}\|_{L^2(K)} + h_K^{1-\beta} \|\underline{u}\|_{H_\beta^{1,1}(K)});$$

$$b) \|\nabla \underline{u}\|_{L^1(\partial K)} \leq C (\|\underline{u}\|_{H^1(K)} + h_K^{1-\beta} \|\underline{u}\|_{H_\beta^{2,2}(K)}).$$

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