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Macroscopic modeling of magnetic hysteresis

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Seminar für Angewandte Mathematik
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Abstract

We formulate a time incremental macroscopical rate–independent model of magnetic hysteresis. This model stems out from a mesoscopical description recently given in [29]. We show uniqueness of a solution to the time–discretized problem and existence of discrete periodic solutions. As our macroscopical model has a convex structure we solve corresponding Euler–Lagrange equations at each time step. A numerical realization of those equations is given and computational examples are presented.

Keywords: micromagnetics, hysteresis, relaxation, Young measures finite elements, a priori error analysis

Subject Classification: 64M07, 65K10, 65N30, 73C50, 73S10, 65N15, 65N30

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1. Introduction – static model

The theory of rigid ferromagnetic bodies ([17, 18]) assumes that a *magnetization* $m : \omega \to \mathbb{R}^n$, describes the state of the body $\omega \subset \mathbb{R}^n$, $n = 2, 3$. Its value depends on a position $x \in \omega$ and has a given and temperature dependent magnitude

$$|\, m(x) \,| = w(\vartheta) \text{ for almost all } x \in \omega \ ,$$

where $w(\vartheta) = 0$ for temperatures $\vartheta \geq \vartheta_0$, the so-called Curie point. From now on we normalize $w(\vartheta)$ and assume that $|\, m \,| = 1$ almost everywhere in ω and that ω is a bounded Lipschitz domain. In the no-exchange formulation the energy of a large rigid ferromagnetic body $\omega \subset \mathbb{R}^n$ consists of three parts, and the variational principle governing equilibrium configurations can be stated as follows (see e.g. [2, 3, 4, 12]):

$$\text{minimize } E(m) = \int_\omega \varphi(m)\,\mathrm{d}x - \int_\omega H \cdot m\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\, \nabla u_m \,|^2\,\mathrm{d}x \ , \tag{1.1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous, $m : \omega \to \mathbb{R}^n$, $|\, m \,| = 1$ a.e. in ω, $H : \mathbb{R}^n \to \mathbb{R}^n$ is an applied external magnetic field and $u_m : \mathbb{R}^n \to \mathbb{R}$, a potential of an induced magnetic field. The first term is an anisotropy energy with a density φ which is an even nonnegative function depending on material properties and exhibiting crystallographic symmetry. Throughout the paper we suppose that $\varphi \geq 0$ on $\mathcal{S}^{n-1} := \{A \in \mathbb{R}^n;\ |\, A \,| = 1\}$, and it is zero only at the points $(0, \pm 1, 0)$ if $n = 3$, or $(0, \pm 1)$ if $n = 2$. Here and in the sequel $|\, A \,|^2 = \langle A, A \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean dot product in $\mathbb{R}^n$. This means that φ corresponds to a uniaxial material with the easy axis $e \in \mathbb{R}^n$ which coincides with the y-axis of our Cartesian coordinate system. The second term is the interaction energy between m and an external magnetic field H and the last term is a magnetostatic energy penalizing non–divergence free magnetization vectors and it is coupled with the magnetization field through the equation

$$\operatorname{div}(-\nabla u_m + m\chi_\omega) = 0 \ \text{ in } \mathbb{R}^n \ , \tag{1.2}$$

where $\chi_\omega : \mathbb{R}^n \to \{0, 1\}$ is the characteristic function of ω. This equation stems from the Maxwell equations (omitting constants)

$$\operatorname{div} B = 0 \ \ , \ \mathit{curl}\ \tilde{H} = 0 \ ,$$

where B is the magnetic induction and $\tilde{H}$ the intensity of the magnetic field. By definition, we have $B = \tilde{H} + m$ and $\tilde{H} = -\nabla u_m$.

We assume standard definitions of Lebesgue and Sobolev spaces as given e.g. in [6]. Let us define the set of admissible magnetizations

$$\mathcal{A} := \{\tilde{m} \in L^2(\omega; \mathbb{R}^n);\ |\, \tilde{m}(x) \,| = 1 \text{ for almost all } x \in \omega\} \ .$$

Eventually, we are concerned with

$$\inf_{m \in \mathcal{A}} E(m)\,, \quad \text{subject to (1.4).} \tag{1.3}$$

We have for any $v \in W^{1,2}(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} [-\nabla u_m + m\chi_\omega] \nabla v\,\mathrm{d}x = 0 \ . \tag{1.4}$$

Thus putting $v := u_m$, we obtain

$$\int_{\mathbb{R}^n} |\nabla u_m|^2 \, \mathrm{d}x = \int_\omega m \cdot \nabla u_m \, \mathrm{d}x \ .$$

Hölder's inequality gives the estimate $\| \nabla u_m \|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le \| m \|_{L^2(\omega;\mathbb{R}^n)}$.

It follows from the Lax-Milgram lemma that (1.4) has a unique solution $u_m \in W^{1,2}(\mathbb{R}^n)$ for any $m \in L^2(\omega;\mathbb{R}^n)$, and that the mapping $m \mapsto \nabla u_m$ is linear and weakly continuous. Hence the magnetostatic energy is sequentially weakly lower semicontinuous.

As $\mathcal{A}$ is not convex we cannot rely on direct methods (cf. [6]) in proving the existence of a solution. In fact, the solution to (1.3) need not exist in $\mathcal{A}$; cf. [12, 19] for the uniaxial case. Due to nonconvexity of $\mathcal{A}$ weak limits of minimizing sequences of E do not necessarily exist in $\mathcal{A}$.

Therefore, we have to look for a more general notion of solution and to solve the *relaxed problem*. Roughly speaking, we look for some functional, say $\mathcal{E}$, attaining its infimum and satisfying the following conditions

- $\inf E = \min \mathcal{E}$,
- minimizers of $\mathcal{E}$ are weak limits of minimizing (sub)sequences of E and
- minimizing (sub)sequences of E tend weakly to minimizers of $\mathcal{E}$.

2. Relaxation in terms of Young measures

We need to describe $\lim_{k\to\infty} E(m^k)$, where $\{m^k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ is a minimizing sequence of E. In what follows $C(S)$ stands for the linear space of continuous functions $S \to \mathbb{R}$. There is a result (see [1, 26, 31]) that for an open bounded set $\omega \subset \mathbb{R}^n$ and from any sequence $\{w_k\}_{k\in\mathbb{N}}$ of measurable functions $\omega \to S$, with $S \Subset \mathbb{R}^m$ a compact set, we can extract a subsequence (denoted by the same indices) such that there exists a family of probability measures $\nu = \{\nu_x\}_{x\in\omega}$, where $\operatorname{supp} \nu_x \subset S$ and

$$\lim_{k\to\infty} v \circ w_k = v \bullet \nu \quad \text{weak* in } L^\infty(\omega) \ , \tag{2.1}$$

for any continuous function $v : \mathbb{R}^m \to \mathbb{R}$. We use the shorthand notation $[v \bullet \nu](x) = \int_S v(A)\nu_x(\mathrm{d}A)$, for almost all $x \in \omega$. Conversely, having a family of probability measures $\{\nu_x\}_{x\in\omega}$ with ν_x supported on S, $x \in \omega$, and with $v \bullet \nu$ measurable for any $v \in C(S)$, i.e., $x \mapsto \nu_x$ is weakly measurable, then there exists a sequence of functions $\omega \to S$ such that (2.1) is fulfilled. A family of parameterized probability measures obtained by this way is called a Young measure.

Coming back to our problem, we set

$$\bar{\mathcal{A}} = \left\{ \nu = \{\nu_x\}_{x\in\omega};\ \operatorname{supp} \nu_x \subset \mathcal{S}^{n-1},\ x \mapsto \nu_x \text{ weakly measurable} \right\} . \tag{2.2}$$

It is well-known (see e.g. [8, 23, 25]) that the role of the functional $\mathcal{E}$ from above can be played by the functional $\bar{E}$, where

$$\bar{E}(\nu, m) := \int_\omega \int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A) \, \mathrm{d}x - \int_\omega H \cdot m \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_m|^2 \, \mathrm{d}x \ .$$

This functional is minimized over the set $\bar{\mathcal{A}} \times L^2(\omega;\mathbb{R}^n)$, subject to (1.4) and $m(x) = \int_{\mathcal{S}^{n-1}} A\nu_x(\mathrm{d}A)$, for a.a. $x \in \omega$.

The infimum of $\bar{E}$ is then attained and it is equal to the infimum of E; cf. [23]. Finally, we can write

$$\bar{E}(\nu, m) = e(\nu, m) - \int_\omega H \cdot m \, \mathrm{d}x \ , \tag{2.3}$$

where

$$e(\nu, m) = \int_\omega \int_{\mathcal{S}^{n-1}} \varphi(A) \nu_x(\mathrm{d}A) \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} |\, \nabla u_m |^2 \, \mathrm{d}x \ .$$

Relaxation by means of Young measures provides us with averaged macroscopical quantities such as magnetization vectors (i.e., the first momenta of the Young measure) as well as with information about a minimizing sequence of E. Namely, the support of ν_x on the unit sphere tells us what magnetization vectors from $\mathcal{A}$ must be combined in a weakly converging sequence in order to achieve the observed macroscopical magnetization. This pattern of the minimizing sequence is called a microstructure and it is fully encoded in the Young measure; cf. [8] for details. This level of relaxation is usually called mesoscopical as it does not provide information about processes on an atomic scale but it records more than average macroscopic quantities.

3. An evolutionary Model

The static model introduced above can describe quasistatic evolution of soft magnetic materials with sufficient accuracy. As for many uniaxial materials a minimizer of $\bar{E}$ over $\bar{\mathcal{A}} \times \mathcal{A}^{**}$ is unique (see [15]), and varying the external magnetic field H just leads to a functional graph (or a loop with the zero thickness) in a m/H diagram. Therefore minimization of $\bar{E}$ cannot be used to study hysteresis behavior of hard ferromagnets where the thickness of the loop is significant.

Recently, a new model describing *rate-independent hysteresis losses* has been developed in [16, 29]. For the sake of simplicity we will suppose that we have a uniaxial magnet whose energy density has only two minima, the north pole and the south pole, laying on the line through the center of the sphere. In [29] the authors described the mechanism that dissipates a prescribed amount of energy needed to change magnetization from one pole into the other, no matter how fast this process is. In fact, there are many contributions to energetic losses if a ferromagnet is exposed to a switching external magnetic field. Besides hysteresis losses which are independent of an external field frequency and which we are going to model here there are also intrinsic damping, disaccommodation and eddy currents. Except for hysteresis losses all others are rate–dependent. We refer e.g. to [11] for more details. In this sense, hysteresis losses are considered as a limit for frequencies tending to zero. On the other hand, for ferromagnets, rate-independence holds with a good approximation in a fairly wide range of frequencies.

In order to define the evolution $t \mapsto q(t) \equiv (\, \nu(t), m(\cdot, t) \,)$, where for any time $t \in [0, T]$

$$q \in Q = \big\{ (\, \nu(t), m(\cdot, t) \,) \in \bar{\mathcal{A}} \times L^2(\omega; \mathbb{R}^n); \ \ m(\cdot, t) = \mathrm{id} \bullet \nu(t) \big\} \ ,$$

we must also postulate the generalized *impulse* $\dot{q} \equiv (\, \dot{\nu}, \dot{m} \,)$, with the dot indicating the time derivative. The convex geometry of Q is taken to be induced from the linear space $L^1\big(\omega; C(\mathcal{S}^{n-1})\big)^* \times L^2(\omega; \mathbb{R}^n)$. Borrowing from models used in (quasi)plasticity we take a

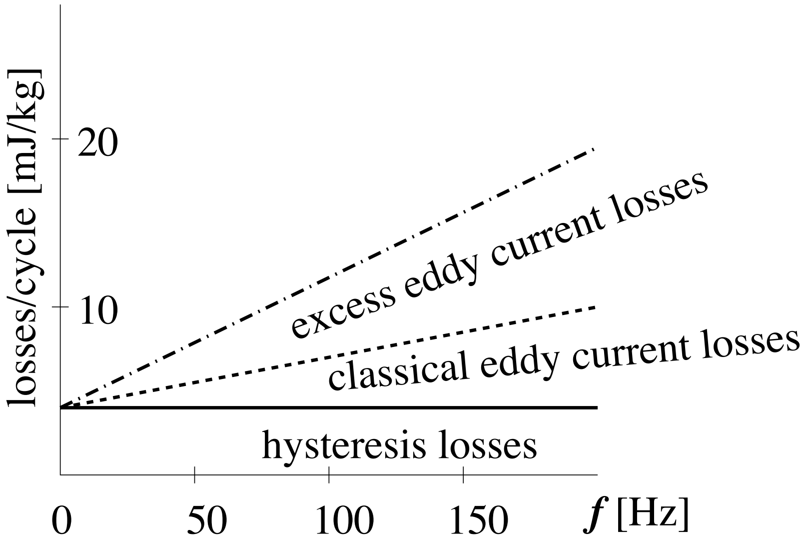

FIGURE 1. Frequency dependence of different kinds of energetic losses in a transformer steel (after [11]).

non-differentiable and degree-1 positively homogeneous dissipative function of the form

$$R(\dot q) \equiv R(\dot\nu, \dot m) := \tilde R(\dot\nu) := \int_\omega \left|\xi \bullet \dot\nu\right| \,\mathrm{d}x\,.$$

Time t is the process time, being zero at the beginning of our process. Recall that we have $[\xi \bullet \dot\nu](x) := \int_{\mathcal{S}^{n-1}} \xi(A)\dot\nu_x(\mathrm{d}A)$. On a mesoscopical level, the function $\xi : \mathcal{S}^{n-1} \to \mathbb{R}$ reflects the dissipation mechanism during pole transformation; the simplest form of ξ is a linear function. As in [29], we call ξ the *pole indicator*. Thus one can find that the energy $\rceil_{\text{north}-\text{south}}$ needed for pole transformation between the north and the south pole (per unit volume) equals

$$\rceil_{\text{north}-\text{south}} = |\,\xi(A_{\text{north}}) - \xi(A_{\text{south}})\,|\,.$$

Motivated by plasticity models in metals and shape-memory alloys (see also [21, 22, 27, 28]) we showed that the desired dissipation/hysteretic effects can be achieved by the evolution $t \mapsto q(t)$ governed by the following first-order evolution inclusion:

$$\partial R(\frac{\mathrm{d}q}{\mathrm{d}t}) + e'(q) + N_Q(q) \ni H \otimes \mathrm{id}\ , \qquad q(0) = q^0\ , \tag{3.1}$$

where ∂R denotes the subdifferential of R which is a set-valued monotone mapping and $N_Q(q)$ is the normal cone to Q at q. The operation "$\otimes$" is defined for any $h \in L^2(\omega;\mathbb{R}^n)$ and any S, a continuous mapping $\mathcal{S}^{n-1} \to \mathbb{R}^n$, by $[h \otimes S](x, A) = \sum_{i=1}^n h_i(x)S_i(A)$. Also, $q^0 \equiv (\,\nu^0, m^0\,)$ is the initial configuration; in fact, only the momenta of q^0 involved in R are to be set up which means that one must set up the *initial volume fraction* of both (i.e., one) poles.

The following regularization of e, e_ζ, $\zeta > 0$, has been considered in [29]

$$e_\zeta(\nu, m) = e(\nu, m) + \frac{\zeta}{2}\|\, D^\gamma(\xi \bullet \nu)\,\|^2_{L^2(\omega;\mathbb{R}^n)}\ , \tag{3.2}$$

where D^γ denotes the fractional derivative, $\gamma \in (0, 0.5)$.

For the sake of simplicity, we will put $\zeta = 0$ in the sequel, but at the end we comment on the extension of our results also to the case $\zeta > 0$.

4. Approximation of weak solutions

We will approximate a solution to (3.1) by the implicit Euler formula in time (with a time step τ) and for $1 \leq k \leq T/\tau \in \mathbb{N}$

$$\partial R\left(\frac{q_\tau^k - q_\tau^{k-1}}{\tau}\right) + e(q_\tau^k) + N_Q(q_\tau^k) \ni H_\tau^k \otimes id \; . \tag{4.1}$$

where

$$q_\tau^k = (\,\nu_\tau^k, m_\tau^k\,) \quad , \quad \nu_\tau^0 = \nu^0 \;, m_\tau^0 = \mathrm{id} \bullet \nu_\tau^0 \;, \quad m_\tau^k := \mathrm{id} \bullet \nu_\tau^k \;, \quad H_\tau^k(x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} H(t,x)\,\mathrm{d}t \; .$$

It is shown in [29] that an approximate solution to (4.1) can be obtained as a solution to the following minimization problem

$$\begin{cases} \text{minimize} \quad e(\nu, m) + \displaystyle\int_\omega \tau \left| \xi \bullet \frac{\nu - \nu_\tau^{k-1}}{\tau} \right| - \int_\omega H_\tau^k \cdot m \,\mathrm{d}x, \\ \text{subject to} \quad (1.4)\;, \quad \nu \in \bar{\mathcal{A}}\;, \quad m = \mathrm{id} \bullet \nu \in L^2(\omega; \mathbb{R}^n)\,, \end{cases} \tag{4.2}$$

whose solution, denoted by $(\,\nu_\tau^k, m_\tau^k\,)$, exists by a direct method argument. We refer to [13] for an effective numerical solution to (4.2).

In what follows we specialize ourselves to the case where $\xi : \mathcal{S}^{n-1} \to \mathbb{R}$ can be affinely extended to the ball $\{A \in \mathbb{R}^n; |A| \leq 1\}$. Then, if μ is a probability measure on $\mathcal{S}^{n-1}$ we have $\int_{\mathcal{S}^{n-1}} \xi(A)\mu(\mathrm{d}A) = \xi(\int_{\mathcal{S}^{n-1}} A\mu(\mathrm{d}A))$.

In particular, the dissipative term in (4.2) can be written as

$$\begin{aligned} \int_\omega \tau \left| \xi \bullet \frac{\nu - \nu_\tau^{k-1}}{\tau} \right| \,\mathrm{d}x &= \int_\omega \left| \int_{\mathcal{S}^{n-1}} \xi(A)\nu_x(\mathrm{d}A) - \int_{\mathcal{S}^{n-1}} \xi(A)\nu_{\tau,x}^{k-1}(\mathrm{d}A) \right| \,\mathrm{d}x \\ &= \int_\omega \left| \xi\left(\int_{\mathcal{S}^{n-1}} A\nu_x(\mathrm{d}A) \right) - \xi\left(\int_{\mathcal{S}^{n-1}} A\nu_{\tau,x}^{k-1}(\mathrm{d}A) \right) \right| \,\mathrm{d}x \\ &= \int_\omega \left| \xi(m) - \xi(m_\tau^{k-1}) \right| \,\mathrm{d}x \;, \end{aligned}$$

where we use the fact that $m(x) = \int_{\mathcal{S}^{n-1}} A\nu_x(\mathrm{d}A)$, for almost all $x \in \omega$ and, similarly, $m_\tau^{k-1}(x) = \int_{\mathcal{S}^{n-1}} A\nu_{\tau,x}^{k-1}(\mathrm{d}A)$, for almost all $x \in \omega$.

Therefore, (4.2) can now be written as

$$\begin{cases} \text{minimize} \quad e(\nu, m) + \displaystyle\int_\omega \left| \xi(m) - \xi(m_\tau^{k-1}) \right| \,\mathrm{d}x - \int_\omega H_\tau^k \cdot m \,\mathrm{d}x\,, \\ \text{subject to} \quad (1.4)\;, \quad \nu \in \bar{\mathcal{A}}\;, \quad m = \mathrm{id} \bullet \nu \in L^2(\omega; \mathbb{R}^n). \end{cases} \tag{4.3}$$

Assuming that $m = \mathrm{id} \bullet \nu$, we denote

$$\bar{E}_\tau^k(\nu) := e(\nu, m) + \int_\omega \left| \xi(m) - \xi(m_\tau^{k-1}) \right| \,\mathrm{d}x - \int_\omega H_\tau^k \cdot m \,\mathrm{d}x \quad , \tag{4.4}$$

and we get

$$
\begin{aligned}
(4.5) \qquad \bar{E}_\tau^k(\nu) \;=\;& \int_\omega \int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\,\nabla u_m\,|^2\,\mathrm{d}x - \int_\omega H_\tau^k \cdot m\,\mathrm{d}x \\
&+ \int_\omega |\,\xi(m) - \xi(m_\tau^{k-1})\,|\,\mathrm{d}x\ .
\end{aligned}
$$

Defining still

$$
(4.6) \qquad \hat{\varphi}(m) = \begin{cases} \varphi(m) & \text{if } |\,m\,| = 1\ , \\ +\infty & \text{otherwise}\ , \end{cases}
$$

and the convex envelope of $\hat{\varphi}$

$$
\varphi^{**} = \sup\{f : \mathbb{R}^n \to \mathbb{R} \text{ convex};\ f \le \hat{\varphi}\}\ ,
$$

we can set the following functional

$$
\begin{aligned}
(4.7) \qquad \left(E_\tau^{**}\right)^k(m) \;=\;& \int_\omega \varphi^{**}(m)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\,\nabla u_m\,|^2\,\mathrm{d}x - \int_\omega H_\tau^k \cdot m\,\mathrm{d}x \\
&+ \int_\omega \left|\xi(m) - \xi(m_\tau^{k-1})\right|\,\mathrm{d}x\ \ ,
\end{aligned}
$$

and

$$
(4.8) \qquad \mathcal{A}^{**} = \left\{m \in L^2(\omega;\mathbb{R}^n);\ |\,m(x)\,| \le 1 \text{ for a.a. } x \in \omega\right\}.
$$

We have the following proposition.

Proposition 4.1. *There exists $\hat{m} \in \mathcal{A}^{**}$ such that $\left(E_\tau^{**}\right)^k(\hat{m}) = \min_{m\in\mathcal{A}^{**}}\left(E_\tau^{**}\right)^k(m)$.*

Proof. The functional $\left(E_\tau^{**}\right)^k$ is convex and continuous w.r.t. the strong topology of $L^2(\omega,\mathbb{R}^n)$ and therefore weakly lower semicontinuous. Moreover, $\mathcal{A}^{**}$ is weakly closed. The assertion therefore follows by the direct method. $\square$

Proposition 4.2. *If φ is a nonnegative continuous function on $\mathcal{S}^{n-1}$, we have $\min_{\nu\in\bar{\mathcal{A}}} \bar{E}_\tau^k(\nu) = \min_{m\in\mathcal{A}^{**}}\left(E_\tau^{**}\right)^k(m)$.*

Proof. Let $\nu \in \bar{\mathcal{A}}$ and take $m \in \mathcal{A}^{**}$ such that $m = \mathrm{id}\bullet\nu$. Then we have by Jensen inequality

$$
\int_\omega \int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x \ge \int_\omega \varphi^{**}(m)\,\mathrm{d}x\,,
$$

and the other terms of $\left(E_\tau^{**}\right)^k$ and $\bar{E}_\tau^k$ are the same. Thus $E_\tau^k(\nu) \ge \left(E_\tau^{**}\right)^k(m)$. Therefore for a given $\nu \in \bar{\mathcal{A}}$ we always find $m \in \mathcal{A}^{**}$ such that $E_\tau^k(\nu) \ge \left(E_\tau^{**}\right)^k(m)$. Consequently, $\min_{\nu\in\bar{\mathcal{A}}} \bar{E}_\tau^k(\nu) \ge \min_{m\in\mathcal{A}^{**}}\left(E_\tau^{**}\right)^k(m)$.

Conversely, if $m \in \mathcal{A}^{**}$ by the definition of φ^{**} we can always find $\nu \in \bar{\mathcal{A}}$ (see [23, p. 93] such that

$$
(4.9) \qquad \varphi^{**}\big(m(x)\big) = \int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A)
$$

and

$$
m(x) = \int_{\mathcal{S}^{n-1}} A\nu_x(\mathrm{d}A)\ .
$$

Therefore, for a given $m \in \mathcal{A}^{**}$ we always find $\nu \in \bar{\mathcal{A}}$ such that $\bar{E}^k_\tau(\nu) = \left(E^{**}_\tau\right)^k(m)$. Consequently, $\min_{\nu\in\bar{\mathcal{A}}} \bar{E}^k_\tau(\nu) \le \min_{m\in\mathcal{A}^{**}} \left(E^{**}_\tau\right)^k(m)$. The proposition is proved. $\square$

The problems of minimizing $\bar{E}^k_\tau$ and $\left(E^{**}_\tau\right)^k$ are equivalent in the following sense.

Corollary 4.1. *Let φ be continuous and nonnegative. If $\nu \in \bar{\mathcal{A}}$ is a minimizer of $\bar{E}^k_\tau$, its first moment, $m = \mathrm{id}\bullet\nu$, is a minimizer of $\left(E^{**}_\tau\right)^k$ and (4.9) holds for almost all $x \in \omega$. Conversely, if $m \in \mathcal{A}^{**}$ minimizes $\left(E^{**}_\tau\right)^k$ and $\nu \in \bar{\mathcal{A}}$ is such that (4.9) holds for almost all $x \in \omega$, then ν minimizes $\bar{E}^k_\tau$.*

Proof. Suppose that $\nu \in \bar{\mathcal{A}}$ is a minimizer of $\bar{E}^k_\tau$ with the first moment m and suppose that

$$\int_\omega \int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x > \int_\omega \varphi^{**}\big(m(x)\big)\,\mathrm{d}x\ .$$

Then $\min_{\mu\in\bar{\mathcal{A}}} \bar{E}^k_\tau(\mu) = E^k_\tau(\nu) > \left(E^{**}_\tau\right)^k(m)$, which contradicts Proposition 4.2. Hence,

$$\int_\omega \left(\int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A) - \varphi^{**}(m(x))\right)\mathrm{d}x \le 0\,. \tag{4.10}$$

But since $\int_{\mathcal{S}^{n-1}} \varphi(A)\nu_x(\mathrm{d}A) - \varphi^{**}\big(m(x)\big) \ge 0$ for a.a. $x \in \omega$, we have equality in (4.10), m is a minimizer of $E^{**\ k}_\tau$, and (4.9) holds for a.a. $x \in \omega$.

If $m \in \mathcal{A}^{**}$ minimizes $E^{**\ k}_\tau$, $m = \mathrm{id}\bullet\nu$ and (4.9) holds then for a.a. $x \in \omega$; we have $\left(E^{**}_\tau\right)^k(m) = \bar{E}^k_\tau(\nu)$ and therefore ν minimizes $\bar{E}^k_\tau$. $\square$

We set the following problem for $k = 1,\dots,T/\tau$ and an initial condition $m^0_\tau \in \mathcal{A}^{**}$:

$$\begin{cases} \text{minimize} & \left(E^{**}_\tau\right)^k(m),\ m \in \mathcal{A}^{**}, \\ \text{subject to} & (1.4). \end{cases} \tag{4.11}$$

Defining for any $m \in \mathcal{A}^{**}$ and ξ affine the Helmholtz (or "stored") energy by

$$e^{**}(m) = \int_\omega \varphi^{**}(m)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\,\nabla u_m\,|^2\,\mathrm{d}x$$

and the dissipation

$$R^{**}(\dot{m}) := \int_\omega |\xi(\dot{m})|\mathrm{d}x,$$

where $\dot{m} = \partial m/\partial t$ we see that (3.1) can be rewritten as

$$\partial R^{**}(\frac{\partial m}{\partial t}) + (e^{**})'(m) + N_{\hat{\mathcal{A}}}(m) \ni H\ , \qquad m(0) = m^0\ , \tag{4.12}$$

where $N_{\hat{\mathcal{A}}}(m)$ denotes the normal cone to $\hat{\mathcal{A}} := \{m(\cdot,t) \in \mathcal{A}^{**}\}$ at m. The advantage of (4.12) over (3.1) is that the evolutionary problem is formulated in physical quantities, i.e., in the macroscopic magnetization rather than in Young measures which are just a mathematical tool to handle the problem. Of course, if one does not know the convex envelope φ^{**} explicitly, then the obvious program is to employ Young measures and to use (3.1) for the numerical treatment. Analogously as in [29] we get the formal energy balance per the process time $[0,T]$

$$\underbrace{\int_0^T \partial R^{**}(\dot m(t))\cdot \dot m(t)\,\mathrm{d}t}_{\text{dissipated energy}} + \underbrace{e^{**}(m(T)) - e^{**}(m(0))}_{\text{gain of Helmholtz's energy}} = \underbrace{\int_0^T H(t)\cdot \dot m(t)\,\mathrm{d}t}_{\text{work of external field}} .$$

Suppose now, that $n=2$ and that $\varphi(A_1,A_2) = c\,\langle A, e_\perp\rangle^2$, $c>0$. Then it is well-known (see [8]) that $\varphi^{**}(A_1,A_2) = c\,\langle A, e_\perp\rangle^2$, $|A|\le 1$.

Proposition 4.3. *Let $\varphi^{**}(A) = c\,\langle A, e_\perp\rangle^2$, $c>0$, $|A|\le 1$ and let $n=2$. Then the problem $\min_{m\in\mathcal{A}^{**}}\left(E_\tau^{**}\right)^k$ has a unique solution for any $k\ge 1$.*

Proof. We will proceed by induction. In what follows, we suppose that $m_\tau^0\in\mathcal{A}^{**}$ is given. Suppose that $m_\tau^{k-1}\in\mathcal{A}^{**}$, is given uniquely.

Let $m,\hat m\in\mathcal{A}^{**}$ be two different minimizers to $\left(E_\tau^{**}\right)^k$. Then $\nabla u_m = \nabla u_{\hat m}$ a.e. in $\mathbb{R}^2$. Indeed, if they were different, by convexity of $\left(E_\tau^{**}\right)^k$, strict convexity of the demagnetizing field energy, i.e. of $\|\cdot\|^2_{L^2(\mathbb{R}^2;\mathbb{R}^2)}$, and linearity of the map $\mathcal{A}^{**}\to L^2(\mathbb{R}^2,\mathbb{R}^2): m\mapsto \nabla u_m$, we could construct a magnetization $\theta m + (1-\theta)\hat m\in\mathcal{A}^{**}$, $0<\theta<1$ which gives a strictly lower energy than m and $\hat m$. Similarly, since φ^{**} is strictly convex in the first variable we get that $m_1=\hat m_1$, a.e. in ω. Put $\delta := m-\hat m$. Then $\delta\cdot(1,0)=0$ a.e. in ω, $\operatorname{div}\delta = 0$ a.e. in ω, and $u_\delta = 0$ a.e. in ω. So, we are exactly at the same situation as in the proof of [5, Th. 2.1]. We will follow its reasoning. We extend δ to the whole $\mathbb{R}^2$ by zero and as the normal components of δ at $\partial\omega$ are continuous we see (cf. [10, Th. 3.1, p. 37]) that $\delta\in\{v\in L^2(\omega;\mathbb{R}^2); \operatorname{div} v = 0\}$, and therefore $\delta = (\partial\eta/\partial x_2, -\partial\eta/\partial x_1)$ for some stream function $\eta\in W^{1,2}(\omega)$. Moreover, as $\delta=0$ outside ω, η is constant there. Further, as $\delta\cdot(1,0)=0$ we see that $\nabla\eta$ is parallel to $(1,0)$ and, hence, η depends only on x_1. Finally, we see that η must be constant on ω and thus $\delta=0$. The proposition is proved. □

Corollary 4.2. *If $n=2$, $\varphi(A) = c\,\langle A, e_\perp\rangle^2$, $c>0$, $|A|=1$ and ξ is affine on the unit ball in $\mathbb{R}^2$ the hysteresis loop coming from the solution to (4.11) for $k\ge 0$ is given uniquely. Consequently, the energy dissipated during the process is given uniquely in almost all material points $x\in\omega$.*

Proof. This is obvious, since the hysteresis loop is a plot of H_τ^k applied in some direction v vs. $|\,\omega\,|^{-1}\int_\omega m\,\mathrm{d}x\cdot v$, where m solves (4.2), $k\ge 0$. □

In order to show uniqueness of the Young measure solution to (4.2) we see from Corollary 4.1 that we must find a Young measure ν such that (4.9) holds. So, let $m\in\mathcal{A}^{**}$ minimize $\left(E_\tau^{**}\right)^k$. It has been shown in [8] that such a measure is unique and always at most two-atomic, i.e., $\nu_x = \theta(x)\delta_{A(x)} + (1-\theta(x))\delta_{B(x)}$, where $A,B\in\mathcal{S}^{n-1}$, and for almost all $x\in\omega$

$$\begin{aligned}
A(x) &= (1-m_1(x)^2)^{1/2}(0,1) + m_1(x)(1,0)\ ,\\
B(x) &= -(1-m_1(x)^2)^{1/2}(0,1) + m_1(x)(1,0)\ ,
\end{aligned}$$

where $\theta(x) = 1$ if $m(x) = (1, 0)$, or

$$\theta(x) = \frac{1}{2} + \frac{m_2(x)}{2(1 - m_1(x)^2)^{1/2}} \qquad \text{otherwise} .$$

Combining the uniqueness of the Young measure with Proposition 4.3 and Corollary 4.2 we have the following result.

Proposition 4.4. *If $n = 2$, $\varphi(A) = c\,\langle A, e_\perp \rangle^2$, $c > 0$, $|A| = 1$ and ξ is affine on the unit ball in $\mathbb{R}^2$, then (4.2) has a unique solution.*

Since we would like to derive Euler–Lagrange equations of $\left(E_\tau^{**}\right)^k$, we have to regularize the dissipative term. Thus instead of $|\xi(m) - \xi(m_\tau^{k-1})|$, we use

$$\sqrt{(\xi(m) - \xi(m_\tau^{k-1}))^2 + \varepsilon}, \quad \varepsilon > 0 .$$

Therefore we set

$$\begin{aligned} \left(E_{\tau,\varepsilon}^{**}\right)^k(m) \;&=\; \int_\omega \varphi^{**}(m)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,\mathrm{d}x - \int_\omega H_\tau^k \cdot m\,\mathrm{d}x \\ &\quad + \int_\omega \sqrt{(\xi(m) - \xi(m_\tau^{k-1}))^2 + \varepsilon}\,\mathrm{d}x , \end{aligned}$$

and consider for every $k \geq 1$ and a given $m_\tau^0 \in \mathcal{A}^{**}$,

$$\text{(4.13)} \qquad \begin{cases} \text{minimize} & \left(E_{\tau,\varepsilon}^{**}\right)^k(m), \;\; m \in \mathcal{A}^{**}, \\ \text{subject to} & (1.4). \end{cases}$$

We take $\xi(m) = H_c(m \cdot e)$, where H_c is the so-called *coercive force*. In general it is a rate-dependent material parameter and its value is related to the thickness of the hysteresis loop. In our setting, H_c denotes the limit of coercive forces along frequencies going to zero. Our choice of ξ is in accordance with the experimental observation that the thickness of the hysteresis loop vanishes if we magnetize the specimen in a direction perpendicular to the easy axis e. Existence and uniqueness of a solution to (4.13) can be proven in the identical way as it was done for $\varepsilon = 0$. Note that $\left(E_{\tau,\varepsilon}^{**}\right)^k$ is even strictly convex and therefore the uniqueness of a solution easily follows. We see that for given $k \geq 0$, (4.13) is equivalent to

$$\text{(4.14)} \qquad \begin{cases} \nabla u + D\varphi^{**}(m) + \dfrac{\xi(m) - \xi(m_\tau^{k-1})}{\sqrt{(\xi(m) - \xi(m_\tau^{k-1}))^2 + \varepsilon}} D\xi(m) + \lambda m = H_\tau^k, \text{ a.e. in } \omega , \\ \text{subject to } (1.4) , m \in \mathcal{A}^{**} , \lambda(1 - |m|)_+ = 0 \text{ a.e. in } \omega , \end{cases}$$

almost everywhere in ω. Above, "D" denotes the gradient operator with respect to m. Model (4.14) is very similar to the one introduced by Visintin in [30].

5. Numerical Realization and Scientific Computation

In order to solve numerically (4.14), we follow the approach by [5, 24] and limit far field effects of the demagnetizing field by assuming that $u = 0$ outside a domain $\mathbb{R}^2 \supset \Omega \Supset \omega$, cf. Figure 2.

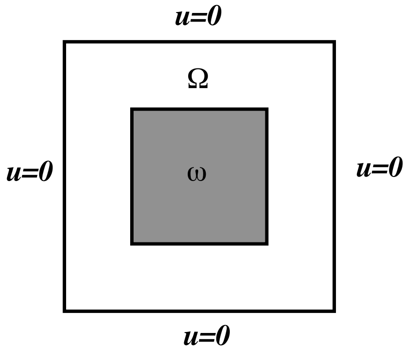

FIGURE 2. Geometry of the domains Ω and ω and the boundary conditions.

Given $m^0 \in \mathcal{A}^{**}$, we need to solve[1] for every $k \geq 1$

$$(5.1)\quad \begin{cases} \nabla u + D\varphi^{**}(m) + \dfrac{\xi(m) - \xi(m^{k-1})}{\sqrt{(\xi(m) - \xi(m^{k-1}))^2 + \varepsilon}} D\xi(m^k) + \lambda m = H^k, \text{ a.e. in } \omega\ , \\ \text{subject to} \int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x = \int_\omega m \cdot \nabla w \,\mathrm{d}x \quad \forall\, w \in W_0^{1,2}(\Omega), \\ \qquad m \in \mathcal{A}^{**}\ , \lambda(1 - |\, m\,|)_+ = 0 \text{ a.e. in } \omega\ , \end{cases}$$

for almost every $x \in \omega$.

The domain Ω is discretized by means of a regular triangular mesh $\mathcal{T}_h$ with $h > 0$ being the diameter of triangular elements $K \in \mathcal{T}_h$.

We also denote $H_h^k\big|_K := |\, K\,|^{-1} \int_K H^k \,\mathrm{d}x$, for every $K \in \mathcal{T}_h$. Moreover, we use the following notation for the formulation of the problem:

$$\begin{aligned} \mathcal{L}^0(\mathcal{T}_h\big|_\omega) &= \left\{ v_h \in L^\infty(\omega) :\ \forall\, K \in \mathcal{T}_h\big|_\omega,\ v_h\big|_K \text{ constant} \right\}, \\ \mathcal{S}^{1,NC}(\mathcal{T}_h) &= \left\{ v_h \in L^\infty(\Omega) :\ v_h\big|_K \text{ affine},\ v_h(z) \text{ continuous } \forall\, z \in \mathcal{M}_h \right\}, \\ \mathcal{S}_0^{1,NC}(\mathcal{T}_h) &= \left\{ v_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h) :\ v_h(z) = 0\ \forall\, z \in \mathcal{M}_h \cap \partial\omega \right\}. \end{aligned}$$

The latter two are assembled from Crouzeix-Raviart type finite elements, which are continuous along midpoints of edges $\Gamma \subset \partial K$, for all $K \in \mathcal{T}_h$. The set of all these midpoints is denoted by $\mathcal{M}_h$.

We denote

$$\mathcal{A}_h^{**} = \mathcal{A}^{**} \cap \mathcal{L}^0(\mathcal{T}_h\big|_\omega)\ .$$

For every $k \geq 1$, and given $m_h^0 \in \mathcal{L}^0(\mathcal{T}\big|_\omega)^2$, a penalized finite element version then reads for a given triangulation $\mathcal{T}_h$ of Ω: Find $(\, u_h, m_h, \lambda_h\,) \in \mathcal{S}_0^{1,NC}(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h\big|_\omega)^2 \times \mathcal{L}^0(\mathcal{T}_h\big|_\Omega)$ that satisfies

$$(5.2)\quad \begin{cases} \nabla_h u_h + D\varphi^{**}(m_h) + \dfrac{\xi(m_h) - \xi(m_h^{k-1})}{\sqrt{(\xi(m_h) - \xi(m_h^{k-1}))^2 + \varepsilon}} D\xi(m_h) + \lambda_h m_h = H_h^k\ , \text{ a.e. in } \omega\ , \\ \text{subject to } \int_\Omega \nabla_h u_h \cdot \nabla w_h \,\mathrm{d}x = \int_\omega m_h \cdot \nabla w_h \,\mathrm{d}x \qquad \forall\, w_h \in \mathcal{S}_0^{1,NC}(\mathcal{T}_h)\ , \\ \qquad \lambda_h = \frac{1}{\delta} \frac{(|\, m_h\,| - 1)_+}{|\, m_h\,|}\ , \quad \delta > 0 \text{ a.e. in } \omega. \end{cases}$$

[1]In the sequel, we drop the index τ to ease notation.

Definition 5.1. *A solution* $\{m_h^k\}_{k=1}^{T/\tau}$ *to* $\min_{\mathcal{A}_h^{**}}\big(E_\tau^{**}\big)^k$, $k=1,\dots,T/\tau$ *with an initial condition* m_h^0 *is called periodic if* $m_h^{T/\tau}=m_h^0$.

Proposition 5.1. *There exists a periodic solution to* $\min_{\mathcal{A}_h^{**}}\big(E_\tau^{**}\big)^k$, $k=1,\dots,T/\tau$.

Proof. Let $m_h^0\in\mathcal{A}_h^{**}$ be an initial condition and let $\lim_{j\to\infty}m_{h,j}^0=m_h^0$ in $L^2(\omega;\mathbb{R}^2)$ for some $\{m_{h,j}^0\}_{j\in\mathbb{N}}\subset\mathcal{A}_h^{**}$. Denote further

$$
\begin{aligned}
(5.3)\qquad \mathcal{F}(m_h^0;m) \;:=\; & \int_\omega \varphi^{**}(m)\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2\,\mathrm{d}x-\int_\omega H_h^k\cdot m\,\mathrm{d}x\\
&+\int_\omega \big|\xi(m)-\xi(m_h^0)\big|\,\mathrm{d}x\ .
\end{aligned}
$$

Then $\lim_{j\to\infty}\mathcal{F}(m_{h,j}^0;\cdot)=\mathcal{F}(m_h^0;\cdot)$ uniformly. By Proposition 4.3, $\mathcal{F}(m_\tau^0;\cdot)$ has a unique minimizer; we get by [7, Cor. 7.24] that minimizers of $\mathcal{F}(m_{h,j}^0;\cdot)$ converge to a minimizer of $\mathcal{F}(m_h^0;\cdot)$ in $L^2(\omega;\mathbb{R}^2)$ as $j\to\infty$. This shows that a mapping $M^1:\mathcal{A}_h^{**}\to\mathcal{A}_h^{**}$ such that $M^1(m_h^0):=m_h^1$ is continuous. Repeating the argument we finally see that there exists a continuous mapping $M^k:\mathcal{A}_h^{**}\to\mathcal{A}_h^{**}$ such that $M^k(m_h^0):=m_h^k$. As $\mathcal{A}_h^{**}\neq\emptyset$ is convex and compact in $\mathbb{R}^N$ for $N=2*$(number of elements) we have by Brouwer's fixed point theorem that M^k has a fixed point in $\mathcal{A}_h^{**}$ and the proposition follows . □

Remark 5.1. The existence of periodic solutions to (4.13) can be proved exactly the same way.

We now turn to (5.2), where the constraint $|\,m_h^k(x)\,|\le 1$ is enforced by penalization; this problem has been studied numerically in [5] for the case where the term leaded by $\operatorname{sgn}_\varepsilon\big(\xi(m_h)-\xi(m_h^{k-1})\big):=\frac{\xi(m_h)-\xi(m_h^{k-1})}{\sqrt{(\xi(m_h^k)-\xi(m_h^{k-1}))^2+\varepsilon}}$ is absent; it has been shown that $\delta=\mathcal{O}(h)$ gives an optimal scaling in this problem for a quasiuniform mesh $\mathcal{T}_h$. Besides uniqueness of related solutions $(\,u_h,m_h,\lambda_h\,)$, convergence is shown

$$\|\,\nabla_h(u-u_h)\,\|_{L^2(\Omega,\mathbb{R}^2)}+\|\,D\varphi^{**}(m)-D\varphi^{**}(m_h)\,\|_{L^2(\Omega,\mathbb{R}^2)}+\|\,\lambda m-\lambda_h m_h\,\|_{L^2(\omega,\mathbb{R}^2)}\le c\,h\,,$$

provided that $(\,u,m,\lambda\,)\in W^{2,2}(\Omega)\times W^{1,2}(\omega,\mathbb{R}^2)\times W^{1,2}(\omega)$. Note that we are not provided with any convergence result in direction of the easy axis $e\in\mathbb{R}^2$ due to degeneracy of the problem, which prevents any information regarding convergence $m_h\to m$ in $L^2(\omega,\mathbb{R}^2)$; however, computational experiments in [24] indicate strong convergence of computed magnetizations at optimal rate even in direction of $e\in\mathbb{R}^2$.

Uniqueness of solutions to (5.2) can again be verified as in Proposition 4.3 by an inductive-type argument, exploiting discrete Helmholtz decomposition for the used finite element pairing as in [5] or by using the strict convexity argument. A convergence analysis for (5.2) requires $m_h^k\to m^k$ in $L^2(\omega,\mathbb{R}^2)$ to control contributions $\xi(m_h^k)\to\xi(m^k)$ in $L^2(\omega)$.

On the other hand, our computational experiments indicate stability and convergence of discretization (5.2) also in the present case of nonstationary relaxed micromagnetics for cases where $\varepsilon>0$ is sufficiently large with respect to $(\,h,k\,)$; cf. Figure 4.

Remark 5.2. Convergence $m_h \to m$ in $L^2(\omega, \mathbb{R}^2)$ in the stationary case can be verified for a finite element approach that employs additional stabilizing terms; cf. [24]. For the present situation, a corresponding stabilizing finite element formulation leads to the following modifications of (5.2): firstly, nonconforming functions from $\mathcal{S}_0^{1,NC}(\mathcal{T}_h)$ are replaced by $(W_0^{1,2}(\Omega)$-)conforming functions $w_h \in \mathcal{S}_0^1(\mathcal{T}_h)$, where $\mathcal{S}^1(\mathcal{T}_h\big|_\omega) = \mathcal{S}^{1,NC}(\mathcal{T}_h\big|_\omega) \cap C(\omega)$ (resp. $\mathcal{S}_0^1(\mathcal{T}_h) = \mathcal{S}^{1,NC}(\mathcal{T}_h) \cap C_0(\Omega)$). Secondly, the first line in (5.2) is replaced by the following one for $m_h \in \mathcal{S}^1(\mathcal{T}_h\big|_\omega)$, and every $\chi_h \in \mathcal{S}^1(\mathcal{T}_h\big|_\omega)$,

$$
\begin{aligned}
&\delta_0 \sum_\Gamma \int_\Gamma \big\langle [\partial_{\mathrm{n}} m_h], [\partial_{\mathrm{n}} \chi_h] \big\rangle \,\mathrm{d}x + \delta_1 \left(\nabla m_h, \nabla \chi_h\right) + \left(\nabla u_h, \chi_h\right) + \big(D\varphi^{**}(m_h), \chi_h\big) \\
&\quad + \big(\mathrm{sgn}_\varepsilon\big(\xi(m_h) - \xi(m_h^{k-1})\big) D\xi(m_h), \chi_h\big) + \left(\lambda_h m_h, \chi_h\right) = \left(H_h^k, \chi_h\right), \\
&\lambda_h = \frac{1}{\delta_2} \frac{(\,|\, m_h \,|\, - 1)_+}{|\, m_h \,|}, \qquad \partial_n m_h\big|_{\partial\omega} = 0\,.
\end{aligned}
\tag{5.4}
$$

Throughout this numerical realization, unphysical homogeneous Neumann boundary conditions for the computed magnetization are required. An optimal scaling $\delta_i = \mathcal{O}(h^{1+i/2})$, $i = 0, 1, 2$ was found in [24].

Computational hysteresis studies were performed for $\omega = (0.25, 0.75)^2$, $\Omega = (0,1)^2$, with $e = (0,1)$, and $H^k = 100 * \sin(5 \cdot 10^{-3}\pi k)$, for $1 \leq k \leq 600$, $H_c = 1$ and $T = 1$; the results are reported in Figures 3 and 4. Figures 4 evidence the effect of 'asymptotic' penalization of the side constraint $|\, m^k \,| = 1$ (in (a)), and instability for small choices of $\varepsilon > 0$ (which leads up to failure of convergence for too small choices; cf. (b)).

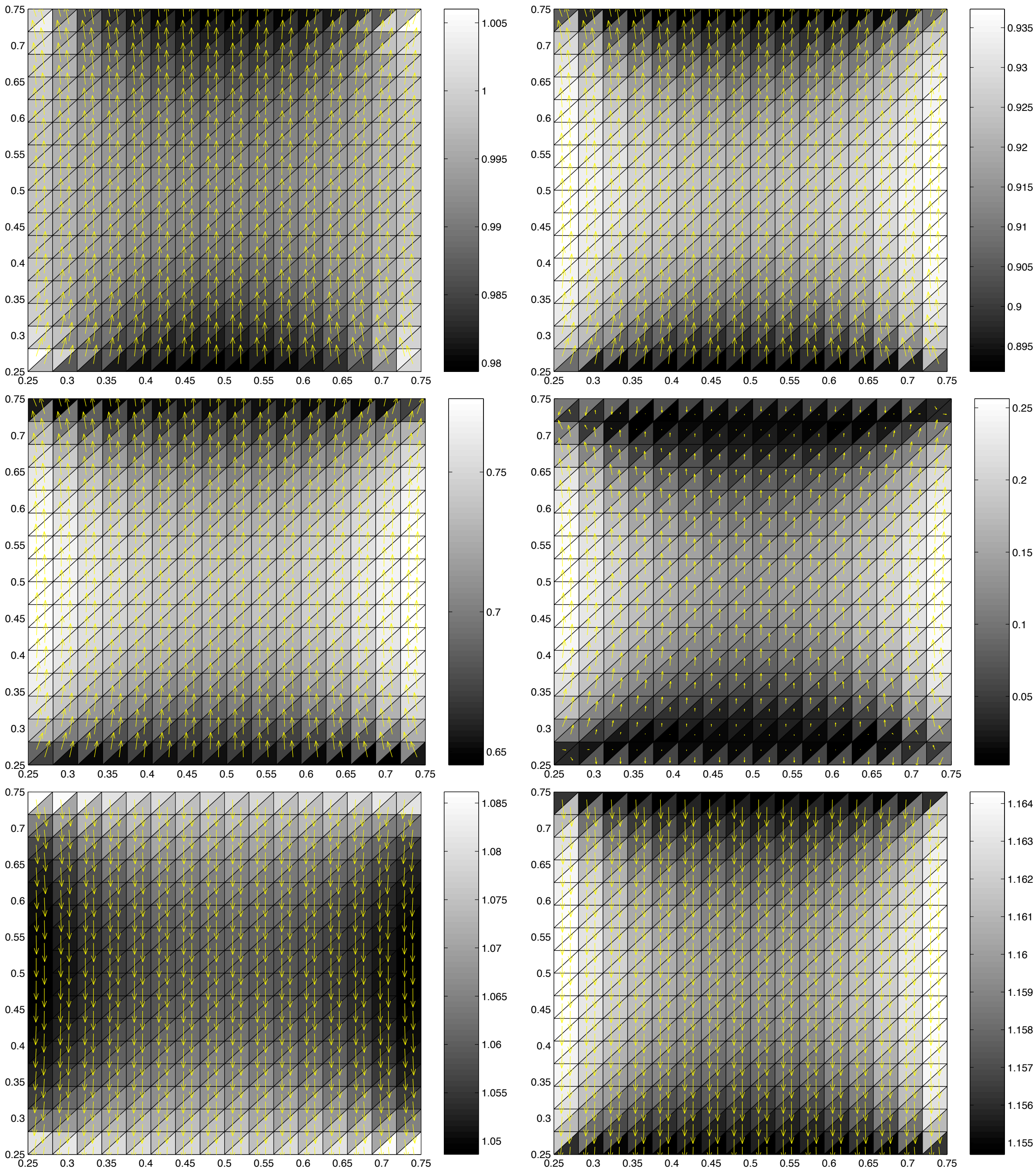

FIGURE 3. Plot of magnetization m_h^k close to first switching time at six subsequent time steps ($h = \frac{1}{32}$), ($\varepsilon = 10^{-3}$; color bars in the last row are scaled by 0.1)

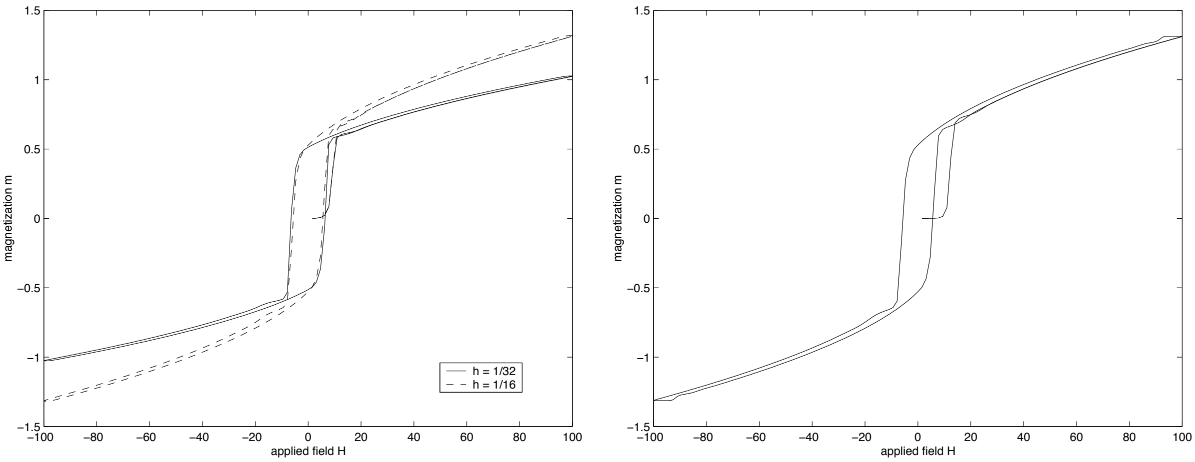

FIGURE 4. Hysteresis loops: applied field H_h^k vs. (averaged) magnetization $|\,\omega\,|^{-1}\int_\omega m_h^k\!\cdot e\,\mathrm{d}x$. (a) Usage of different $h=\frac{1}{16},\frac{1}{32}$ ($\varepsilon=10^{-3}$). (b) Instabilities visible as bubbles at saturation state for too small $\varepsilon=10^{-4}$ ($h=\frac{1}{16}$).

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01-03	C. Schwab, A.-M. Matache	Generalized FEM for Homogenization Problems