Eidgenössische Technische Hochschule Zürich
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Swiss Federal Institute of Technology Zurich

Mixed *hp*-DGFEM for incompressible flows

D. Schötzau[†], Ch. Schwab[*] and A. Toselli[*]

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

[†]Department of Mathematics, University of Basel, Rheinsprung 21, CH-4051 Basel, Switzerland, email: `schotzau@math.unibas.ch`

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Abstract

We consider several mixed discontinuous Galerkin approximations of the Stokes problem and propose an abstract framework for their analysis. Using this framework we derive a priori error estimates for hp-approximations on tensor product meshes. We also prove a new stability estimate for the discrete divergence bilinear form.

Keywords: hp-FEM, discontinuous Galerkin methods, Stokes problem

Subject Classification: 65N30, 65N35, 65N12, 65N15

[†]Department of Mathematics, University of Basel, Rheinsprung 21, CH-4051 Basel, Switzerland, email: `schotzau@math.unibas.ch`

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1 Introduction

Discontinuous Galerkin (DG) methods for incompressible flow problems allow one to use discrete velocity spaces consisting of piecewise polynomial functions with no interelement continuity. Well-posedness of the discrete formulations is then achieved by numerical fluxes, i.e., by introducing suitable bilinear forms defined on the interfaces between the elements of the mesh. This choice presents considerable advantages for certain types of problems, especially those modeling phenomena where transport is dominant; see the state-of-the-art surveys in [18], the monograph [15], the recent review [20], and the references therein. In addition, DG approximations allow for non-conforming meshes.

Even if transport may be the dominant effect of a problem, diffusive terms still need to be accounted for and correctly discretized in a DG framework. For the Oseen or the incompressible Navier-Stokes equations, for instance, if advective terms are properly treated by, e.g., suitable upwinding techniques, stability and convergence only depend on the diffusive part of the operator and can then be studied for the simpler Stokes problem; see, e.g., [39, 25, 10, 33, 9]. In particular, suitable velocity-pressure space pairs are required to ensure stability and convergence. This separation of advective and diffusive effects was employed in [5] for the first definition of DG methods for convection-diffusion problems, in [19, 16, 7] for the so-called local discontinuous Galerkin and Baumann-Oden methods, respectively, and also in [27] for the hp-DG approximation of scalar advection-diffusion problems.

The recent works in [32] and [2] have unified the formulation and analysis of DG approximations for purely diffusive problems, where virtually all the available DG methods can be analyzed in a unified framework. In particular, several assumptions on the discrete spaces and bilinear forms have been given and analyzed that can be used to ensure a priori error estimates for the methods.

While extensive work has been done for diffusion or advection-diffusion problems, there are considerably fewer works for DG discretizations of saddle-point problems describing, e.g., nearly incompressible solids or incompressible fluid flows. We mention [4, 28], where an interior penalty approximation with discontinuous, piecewise divergence-free velocities and continuous pressures is employed for the Stokes and incompressible Navier-Stokes equations, respectively. In [17], a local discontinuous Galerkin approximation for the Stokes problem is proposed. There, the introduction of certain pressure stabilization terms allows one to choose velocity and pressure spaces of the same polynomial order k. Optimal error estimates for h-approximations are proved. In [26], an h-approximation for incompressible and nearly incompressible elasticity based on an interior penalty DG method is introduced and studied. Triangular and tetrahedral meshes are employed, together with polynomial spaces of total degree k and $k-1$ for the velocity and pressure, respectively. Optimal error estimates in h are derived, which remain valid in the incompressible limit. A similar approach was considered in [40] for hp-approximations of the Stokes problem on tensor product meshes in two and three dimensions. Stability estimates for the discrete divergence bilinear form that are explicit in h and k are obtained.

Numerical results point out that these estimates are not sharp in the order k, at least for conforming two-dimensional meshes. In the present work we indeed prove sharper estimates for the same DG approximation.

The present work has two purposes. In the first part, we develop an abstract framework for mixed DG approximations of the Stokes problem. In particular, we give a set of assumptions on the approximation spaces and on the velocity and divergence bilinear forms which allows us to obtain a priori error estimates. All available mixed DG methods for the Stokes problem can be analyzed in the presented framework by introducing lifting operators similar to the ones used in [2] for the Laplace equation. However, unlike in the analysis of [2], our error estimates are derived by using a variant of Strang's lemma, combined with the techniques developed in [40] that give abstract estimates for the errors in the velocity and the pressure. With respect to the use of Strang's lemma, our approach is closely related to the setting proposed in [30, 31] for the analysis of local discontinuous Galerkin methods for purely elliptic problems.

Our second result is a new proof of the inf-sup condition of the discrete DG divergence bilinear form for tensor product meshes and $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ elements. In particular, we prove a sharper bound than that given in [40]. Our analysis is valid for shape-regular two- and three-dimensional tensor product meshes, possibly with hanging nodes. Even though our estimate does not appear to be sharp, at least in two dimensions (see the numerical results in [40]), we are able to ensure the same convergence rate for the velocity and the pressure as that of conforming $\mathbb{Q}_k - \mathbb{Q}_{k-2}$ elements in three dimensions, but with a gap in the polynomial degree of the velocity-pressure pair of just one.

Our framework and analysis can be adapted to the case of nearly incompressible elasticity in a straightforward way. We note that equal-order conforming discretizations are possible both in nearly incompressible elasticity and incompressible flows, but that the bilinear forms need to be suitably modified. These stabilization techniques typically rely on local terms that are added to the bilinear forms and are constructed with the residual of the differential equations on each element; see [22, 21, 24]. The calculation of these terms is not often a simple matter for higher-order hp-approximations. On the other hand, DG approximations allow to narrow or eliminate the polynomial degree gap between the velocity and pressure spaces by employing a discontinuous velocity space and suitable bilinear forms on the interfaces. This brings in an increase of the velocity degrees of freedom, that, in the case of p- and hp-approximations, is not however of the same order of magnitude as the number of degrees of freedom of the corresponding conforming discretization, as is the case of lower-order approximations.

The rest of this paper is organized as follows: We start by reviewing the Stokes problem in section 2, and then present our abstract framework in section 3. Suitable assumptions on the bilinear forms allow us to derive a priori error estimates in section 4. In section 5 we discuss some particular choices for the bilinear forms and the approximation spaces. Section 6 contains the proofs of the inf-sup condition of the discrete divergence bilinear form. In sections 7 and 8 we establish the remaining assumptions for our DG approximations. Finally, we derive hp-error estimates in section 9.

2 The Stokes problem

Let Ω be a bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, respectively, with $\mathbf{n}$ denoting the outward normal unit vector to its boundary $\partial\Omega$. Given a source term $\mathbf{f} \in L^2(\Omega)^d$ and a Dirichlet datum $\mathbf{g} \in H^{1/2}(\partial\Omega)^d$ satisfying the usual compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\, ds = 0$, the Stokes problem in incompressible fluid flow is to find a velocity field $\mathbf{u}$ and a pressure p such that

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega. \end{aligned} \tag{1}$$

If we define

$$\mathbf{V} := H^1(\Omega)^d, \qquad Q := L^2_0(\Omega) = \{\, q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0 \,\},$$

and

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu\nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x}, \qquad B(\mathbf{u}, p) = -\int_\Omega p\, \nabla \cdot \mathbf{u}\, d\mathbf{x},$$

then the corresponding variational problem consists in finding $(\mathbf{u}, p) \in \mathbf{V} \times Q$, with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$, such that

$$\left\{ \begin{array}{lclcc} A(\mathbf{u}, \mathbf{v}) & + & B(\mathbf{v}, p) & = & \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} \\ B(\mathbf{u}, q) & & & = & 0 \end{array} \right. \tag{2}$$

for all $\mathbf{v} \in H^1_0(\Omega)^d$ and $q \in Q$.

The well-posedness of (2) is ensured by the continuity of $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$, the coercivity of $A(\cdot, \cdot)$, and the following inf-sup condition

$$\inf_{0 \neq q \in L^2_0(\Omega)} \sup_{\mathbf{0} \neq \mathbf{v} \in H^1_0(\Omega)^d} \frac{-\int_\Omega q\, \nabla \cdot \mathbf{v}\, d\mathbf{x}}{|\mathbf{v}|_1 \|q\|_0} \geq \gamma > 0, \tag{3}$$

with an inf-sup constant γ only depending on Ω; see, e.g., [10, 25]. Here, we denote by $\|\cdot\|_{s,\mathcal{D}}$ and $|\cdot|_{s,\mathcal{D}}$ the norm and seminorm of $H^s(\mathcal{D})$ and $H^s(\mathcal{D})^d$, $s \geq 0$. In case $\mathcal{D} = \Omega$, we drop the subscript.

3 Mixed discretizations with nonconforming velocity spaces

Let $\mathbf{V}_h$ be a nonconforming finite element space approximating the velocities. We introduce the space

$$\mathbf{V}(h) := \mathbf{V} + \mathbf{V}_h,$$

and endow it with a suitable norm $\|\cdot\|_h$. Furthermore, let $Q_h \subset Q$ be a conforming finite element space for the pressure, equipped with the L^2-norm $\|\cdot\|_0$.

Given forms $A_h : \mathbf{V}(h)\times\mathbf{V}(h) \to \mathbb{R}$, $B_h : \mathbf{V}(h)\times Q \to \mathbb{R}$ and functionals $F_h : \mathbf{V}_h \to \mathbb{R}$, $G_h : Q_h \to \mathbb{R}$, chosen to discretize the Laplacian and the divergence constraint, we consider mixed methods of the form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\left\{ \begin{array}{lclcl} A_h(\mathbf{u}_h, \mathbf{v}) & + & B_h(\mathbf{v}, p_h) & = & F_h(\mathbf{v}) \\ B_h(\mathbf{u}_h, q) & & & = & G_h(q) \end{array} \right. \tag{4}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

Let us make precise our assumptions on the forms A_h and B_h. First, they are assumed to satisfy the following continuity properties

$$|A_h(\mathbf{v}, \mathbf{w})| \le \alpha_1 \|\mathbf{v}\|_h \|\mathbf{w}\|_h, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{V}(h), \tag{5}$$

$$|B_h(\mathbf{v}, q)| \le \alpha_2 \|\mathbf{v}\|_h \|q\|_0, \qquad (\mathbf{v}, q) \in \mathbf{V}(h) \times Q, \tag{6}$$

with constants $\alpha_1 > 0$ and $\alpha_2 > 0$. Further, let us define $\mathbf{Z}(G_h) \subset \mathbf{V}_h$ by

$$\mathbf{Z}(G_h) = \{\, \mathbf{v} \in \mathbf{V}_h : B_h(\mathbf{v}, q) = G_h(q)\ \forall q \in Q_h \,\}. \tag{7}$$

We require the form A_h to be coercive on the kernel of B_h, i.e.,

$$A_h(\mathbf{v}, \mathbf{v}) \ge \beta \|\mathbf{v}\|_h^2, \qquad \mathbf{v} \in \mathbf{Z}(0), \tag{8}$$

for a coercivity constant $\beta > 0$. The form B_h is assumed to satisfy the discrete inf-sup condition

$$\inf_{0 \ne q \in Q_h} \sup_{\mathbf{0} \ne \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \ge \gamma_h, \tag{9}$$

with a stability constant $\gamma_h > 0$. Finally, we assume the exact solution $\mathbf{u} \in \mathbf{V}$ to fulfill the consistency condition

$$B_h(\mathbf{u}, q) = G_h(q), \qquad \forall q \in Q_h. \tag{10}$$

We do not impose any consistency requirements on the form A_h; instead we will work with the residual

$$R_h(\mathbf{u}, p; \mathbf{v}) := A_h(\mathbf{u}, \mathbf{v}) + B_h(\mathbf{v}, p) - F_h(\mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h, \tag{11}$$

where $(\mathbf{u}, p) \in \mathbf{V} \times Q$ is again the exact solution. Our abstract error estimates will then be expressed in terms of $\mathcal{R}_h(\mathbf{u}, p)$ given by

$$\mathcal{R}_h(\mathbf{u}, p) := \sup_{\mathbf{0} \ne \mathbf{v} \in \mathbf{V}_h} \frac{|R_h(\mathbf{u}, p; \mathbf{v})|}{\|\mathbf{v}\|_h}. \tag{12}$$

For all the DG methods we introduce in section 5 the quantity $\mathcal{R}_h(\mathbf{u}, p)$ is optimally convergent.

We note that if F_h and G_h are continuous functionals on $\mathbf{V}_h$ and Q_h, respectively, the mixed problem (4) has a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$.

Remark 3.1. *If $\mathbf{V}_h \subset \mathbf{V}$ is chosen to be a conforming finite element space, the setting of this section coincides with the standard mixed finite element setting; see [10].*

Remark 3.2. *For the DG forms in section 5 the constants α_1 and β depend on the viscosity ν whereas α_2 and γ_h are independent of ν. More precisely, we have that $\alpha_1 = \nu\bar{\alpha}_1$ and $\beta = \nu\bar{\beta}$ with $\bar{\alpha}_1$ and $\bar{\beta}$ independent of ν.*

4 Abstract error estimates

Abstract error bounds for the mixed method in (4) can be obtained by proceeding as in [40, Sect. 8]. We give the details of the proofs for the sake of completeness.

4.1 Error in the velocity

First, we prove an error estimate for the velocities following [40, Lemma 8.1].

Proposition 4.1. *Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the exact solution of the Stokes problem and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ the mixed finite element approximation. Under the assumptions of section 3, we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \leq (1+\frac{\alpha_1}{\beta})(1+\frac{\alpha_2}{\gamma_h}) \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u}-\mathbf{v}\|_h + \frac{\alpha_2}{\beta} \inf_{q\in Q_h} \|p-q\|_0 + \beta^{-1}\mathcal{R}_h(\mathbf{u}, p).$$

Proof. First, we fix $\mathbf{w} \in \mathbf{Z}(G_h)$ and $q \in Q_h$. Since $\mathbf{w} - \mathbf{u}_h \in \mathbf{Z}(0)$, (8) and the definition of the residual yield

$$\begin{aligned}\beta\|\mathbf{w}-\mathbf{u}_h\|_h^2 &\leq A_h(\mathbf{w}-\mathbf{u}_h, \mathbf{w}-\mathbf{u}_h) \\ &= A_h(\mathbf{w}-\mathbf{u}, \mathbf{w}-\mathbf{u}_h) - B_h(\mathbf{w}-\mathbf{u}_h, p-p_h) + R_h(\mathbf{u}, p; \mathbf{w}-\mathbf{u}_h).\end{aligned}$$

Since $\mathbf{w}-\mathbf{u}_h \in \mathbf{Z}(0)$, we can replace p_h by q in the form B_h. Using the continuity properties in (5), (6), and the triangle inequality, we obtain

$$\|\mathbf{u}-\mathbf{u}_h\|_h \leq (1+\frac{\alpha_1}{\beta})\|\mathbf{u}-\mathbf{w}\|_h + \frac{\alpha_2}{\beta}\|p-q\|_0 + \beta^{-1}\mathcal{R}_h(\mathbf{u}, p), \tag{13}$$

for any $\mathbf{w} \in \mathbf{Z}(G_h)$ and $q \in Q_h$.

Second, we fix $\mathbf{v} \in \mathbf{V}_h$ and consider the problem of finding $\mathbf{z}(\mathbf{v}) \in \mathbf{V}_h$ such that

$$B_h(\mathbf{z}(\mathbf{v}), q) = B_h(\mathbf{u}-\mathbf{v}, q), \qquad \forall q \in Q_h.$$

Thanks to the discrete inf-sup condition in (9), the continuity of B_h in (6) and [10, Proposition 1.2, p. 39], the solution $\mathbf{z}(\mathbf{v})$ is well defined. Furthermore,

$$\gamma_h\|\mathbf{z}(\mathbf{v})\|_h \leq \sup_{0\neq q\in Q_h} \frac{B_h(\mathbf{z}(\mathbf{v}), q)}{\|q\|_0} = \sup_{0\neq q\in Q_h} \frac{B_h(\mathbf{u}-\mathbf{v}, q)}{\|q\|_0} \leq \alpha_2\|\mathbf{u}-\mathbf{v}\|_h, \tag{14}$$

where we have used the continuity of B_h. By construction and assumption (10), we have $\mathbf{z}(\mathbf{v}) + \mathbf{v} \in \mathbf{Z}(G_h)$. Inserting $\mathbf{z}(\mathbf{v}) + \mathbf{v}$ in (13) yields

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq (1 + \frac{\alpha_1}{\beta})\|\mathbf{u} - \mathbf{v}\|_h + (1 + \frac{\alpha_1}{\beta})\|\mathbf{z}(\mathbf{v})\|_h + \frac{\alpha_2}{\beta}\|p - q\|_0 + \beta^{-1}\mathcal{R}_h(\mathbf{u}, p).$$

This, together with (14), proves the assertion. $\square$

Remark 4.1. *Assuming that α_1, α_2 and β are independent of the discretization parameter h, the bound in Proposition 4.1 can be expressed in a simpler fashion as*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C\big[\gamma_h^{-1} \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_h + \inf_{q\in Q_h} \|p - q\|_0 + \mathcal{R}_h(\mathbf{u}, p)\big].$$

4.2 Error in the pressure

Next, we prove an error estimate for the pressure following the arguments in [40, Lemma 8.2].

Proposition 4.2. *Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the exact solution of the Stokes problem and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ the mixed finite element approximation. Under the assumptions of section 3, we have*

$$\|p - p_h\|_0 \leq (1 + \frac{\alpha_2}{\gamma_h}) \inf_{q\in Q_h} \|p - q\|_0 + \frac{\alpha_1}{\gamma_h}\|\mathbf{u} - \mathbf{u}_h\|_h + \gamma_h^{-1}\mathcal{R}_h(\mathbf{u}, p).$$

Proof. Fix $q \in Q_h$. From the inf-sup condition in (9) we have

$$\gamma_h\|q - p_h\|_0 \leq \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h} \frac{B_h(\mathbf{v}, q - p_h)}{\|\mathbf{v}\|_h}.$$

Since $B_h(\mathbf{v}, q - p_h) = B_h(\mathbf{v}, q - p) - A_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + R_h(\mathbf{u}, p; \mathbf{v})$ for any $\mathbf{v} \in \mathbf{V}_h$, we obtain from the continuity properties in (5) and (6) and the definition of $\mathcal{R}_h$ in (12)

$$\gamma_h\|q - p_h\|_0 \leq \alpha_2\|p - q\|_0 + \alpha_1\|\mathbf{u} - \mathbf{u}_h\|_h + \mathcal{R}_h(\mathbf{u}, p).$$

The assertion follows then from the triangle inequality. $\square$

Remark 4.2. *Taking into account the estimate for $\|\mathbf{u}-\mathbf{u}_h\|_h$ in Proposition 4.1 and assuming again that α_1, α_2 and β are independent of the discretization parameter h, the bound in Proposition 4.2 reduces to*

$$\|p - p_h\|_0 \leq C\big[\gamma_h^{-1} \inf_{q\in Q_h} \|p - q\|_0 + \gamma_h^{-2} \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_h + \gamma_h^{-1}\mathcal{R}_h(\mathbf{u}, p)\big].$$

5 Discontinuous Galerkin discretizations

In this section, we give several examples of mixed discontinuous Galerkin methods that can be cast into the setting of section 3 by using lifting operators similar to the ones introduced in [2] for the Laplacian.

5.1 Triangulations and finite element spaces

Let $\mathcal{T}_h$ be a shape-regular affine quadrilateral or hexahedral mesh on Ω. We denote by h_K the diameter of the element $K \in \mathcal{T}_h$. Further, we assign to each element $K \in \mathcal{T}_h$ an approximation order $k_K \geq 1$. The local quantities h_K are k_K are stored in the vectors $\underline{h} = \{h_K\}_{K\in\mathcal{T}_h}$ and $\underline{k} = \{k_K\}_{K\in\mathcal{T}_h}$, respectively. We set $h = \max_{K\in\mathcal{T}_h} h_K$ and $|\underline{k}| = \max_{K\in\mathcal{T}_h} k_K$. Finally, $\mathbf{n}_K$ denotes the outward normal unit vector to the boundary ∂K.

An interior face of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary face of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K \cap \partial\Omega$ which consists of entire faces of ∂K. We denote by $\mathcal{E}_\mathcal{I}$ the union of all interior faces of $\mathcal{T}_h$, by $\mathcal{E}_\mathcal{D}$ the union of all boundary faces, and set $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{D}$. Here and in the following, we refer generically to a "face" even in the two-dimensional case.

We allow for irregular meshes, i.e., meshes with hanging nodes (see [37, Sect. 4.4.1]), in general, but suppose that the intersection between neighboring elements is either a common vertex, or a common edge, or a common face, or an entire face of one of the two elements. We also assume the local mesh-sizes and approximation degrees to be of bounded variation, that is, there is a constant $\kappa > 0$ such that

$$\kappa h_K \leq h_{K'} \leq \kappa^{-1} h_K, \qquad \kappa k_K \leq k_{K'} \leq \kappa^{-1} k_K, \tag{15}$$

whenever K and K' share a common face.

We wish to approximate the velocities and pressures in the discontinuous finite element spaces $\mathbf{V}_h$ and Q_h given by

$$\begin{aligned} \mathbf{V}_h &= \{\, \mathbf{v} \in L^2(\Omega)^d : \mathbf{v}|_K \in \mathbb{Q}_{k_K}(K)^d,\ K \in \mathcal{T}_h \,\}, \\ Q_h &= \{\, q \in L^2_0(\Omega) : q|_K \in \mathbb{Q}_{k_K-1}(K),\ K \in \mathcal{T}_h \,\}, \end{aligned} \tag{16}$$

respectively, where $\mathbb{Q}_k(K)$ is the space of polynomials of maximum degree k in each variable on K.

For the derivation and analysis of the methods we will make use of the auxiliary space $\underline{\Sigma}_h$ defined by

$$\underline{\Sigma}_h := \{\, \underline{\tau} \in L^2(\Omega)^{d\times d} : \underline{\tau} \in \mathbb{Q}_{k_K}(K)^{d\times d},\ K \in \mathcal{T}_h \,\}.$$

Note that $\nabla_h \mathbf{V}_h \subset \underline{\Sigma}_h$, where ∇_h is the discrete gradient, taken elementwise, and given by $[\nabla \mathbf{v}]_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}$ on $K \in \mathcal{T}_h$.

5.2 Trace operators

In this section, we define the trace operators needed in our discontinuous Galerkin discretizations. To this end, let $e \subset \mathcal{E}_\mathcal{I}$ be an interior face shared by K^+ and K^-. Let $(\mathbf{v}, q, \underline{\tau})$ be a function smooth inside each element $K^\pm$ and let us denote by $(\mathbf{v}^\pm, q^\pm, \underline{\tau}^\pm)$ the traces of $(\mathbf{v}, q, \underline{\tau})$ on e from the interior of $K^\pm$. Then, we define the mean values $\{\!\{\cdot\}\!\}$ and normal jumps $[\![\cdot]\!]$ at $\mathbf{x} \in e$ as

$$\begin{aligned}
\{\!\{\mathbf{v}\}\!\} &:= (\mathbf{v}^+ + \mathbf{v}^-)/2, & [\![\mathbf{v}]\!] &:= \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}, \\
\{\!\{q\}\!\} &:= (q^+ + q^-)/2, & [\![q]\!] &:= q^+ \, \mathbf{n}_{K^+} + q^- \, \mathbf{n}_{K^-}, \\
\{\!\{\underline{\tau}\}\!\} &:= (\underline{\tau}^+ + \underline{\tau}^-)/2, & [\![\underline{\tau}]\!] &:= \underline{\tau}^+ \, \mathbf{n}_{K^+} + \underline{\tau}^- \, \mathbf{n}_{K^-}.
\end{aligned}$$

Note that the jumps $[\![q]\!]$ and $[\![\underline{\tau}]\!]$ are both vectors whereas the jump $[\![\mathbf{v}]\!]$ is a scalar. We also need to define a jump of the velocity $\mathbf{v}$ which is a matrix, namely,

$$\underline{[\![\mathbf{v}]\!]} := \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-},$$

where, for two vectors $\mathbf{a}$ and $\mathbf{b}$, we set $[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j$.

On a boundary face $e \subset \mathcal{E}_{\mathcal{D}}$ given by $e = \partial K \cap \partial\Omega$, we set accordingly

$$\{\!\{\mathbf{v}\}\!\} := \mathbf{v}, \qquad \{\!\{q\}\!\} := q, \qquad \{\!\{\underline{\tau}\}\!\} := \underline{\tau},$$

as well as

$$[\![\mathbf{v}]\!] := \mathbf{v} \cdot \mathbf{n}, \qquad \underline{[\![\mathbf{v}]\!]} := \mathbf{v} \otimes \mathbf{n}, \qquad [\![q]\!] := q\mathbf{n}, \qquad [\![\underline{\tau}]\!] := \underline{\tau}\,\mathbf{n}.$$

We remark that, for the exact solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$, there holds $\underline{[\![\mathbf{u}]\!]} = \underline{0}$ and $[\![\nu\nabla\mathbf{u} - p\underline{I}]\!] = \mathbf{0}$ on $\mathcal{E}_{\mathcal{I}}$. The last property follows from the fact that $\nu\nabla\mathbf{u} - p\underline{I}$ belongs to $H(\mathrm{div};\Omega)$; see [40].

5.3 Lifting operators

We introduce the following lifting operators. First, for a face $e \subset \mathcal{E}$ we define $\underline{\mathcal{L}}_e : \mathbf{V}(h) \to \underline{\Sigma}_h$ by

$$\int_\Omega \underline{\mathcal{L}}_e(\mathbf{v}) : \underline{\tau}\, d\mathbf{x} = \int_e \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds, \qquad \forall \underline{\tau} \in \underline{\Sigma}_h.$$

Note that the support of $\underline{\mathcal{L}}_e(\mathbf{v})$ is contained in the elements that share the face e. For a boundary face $e \subset \mathcal{E}_{\mathcal{D}}$, we introduce the lifting $\underline{\mathcal{G}}_e \in \underline{\Sigma}_h$ of the Dirichlet datum $\mathbf{g}$ given by

$$\int_\Omega \underline{\mathcal{G}}_e : \underline{\tau}\, d\mathbf{x} = \int_e (\mathbf{g} \otimes \mathbf{n}) : \underline{\tau}\, ds, \qquad \forall \underline{\tau} \in \underline{\Sigma}_h.$$

For the exact solution $\mathbf{u} \in \mathbf{V}$, we have

$$\underline{\mathcal{L}}_e(\mathbf{u}) = \underline{0}, \qquad \forall e \subset \mathcal{E}_{\mathcal{I}}, \qquad\qquad \underline{\mathcal{L}}_e(\mathbf{u}) = \underline{\mathcal{G}}_e, \qquad \forall e \subset \mathcal{E}_{\mathcal{D}}. \tag{17}$$

Globally, we define $\underline{\mathcal{L}} : \mathbf{V}(h) \to \underline{\Sigma}_h$ and $\underline{\mathcal{G}} \in \underline{\Sigma}_h$ by

$$\underline{\mathcal{L}} := \sum_{e \subset \mathcal{E}} \underline{\mathcal{L}}_e, \qquad \underline{\mathcal{G}} := \sum_{e \subset \mathcal{E}_{\mathcal{D}}} \underline{\mathcal{G}}_e.$$

These operators can be characterized by

$$\int_\Omega \underline{\mathcal{L}}(\mathbf{v}) : \underline{\tau}\, d\mathbf{x} = \int_\mathcal{E} \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds, \qquad \forall \underline{\tau} \in \underline{\Sigma}_h,$$
$$\int_\Omega \underline{\mathcal{G}} : \underline{\tau}\, d\mathbf{x} = \int_{\mathcal{E}_\mathcal{D}} (\mathbf{g} \otimes \mathbf{n}) : \underline{\tau}\, ds, \qquad \forall \underline{\tau} \in \underline{\Sigma}_h.$$

Finally, we need the lifting operator $\mathcal{M} : \mathbf{V}(h) \to Q_h$ defined by

$$\int_\Omega \mathcal{M}(\mathbf{v})\varphi\, d\mathbf{x} = \int_\mathcal{E} [\![\mathbf{v}]\!]\, \{\!\{\varphi\}\!\}\, ds, \qquad \forall \varphi \in Q_h.$$

For the exact solution $\mathbf{u} \in \mathbf{V}$, there holds

$$\int_\Omega \mathcal{M}(\mathbf{u})\, \varphi\, d\mathbf{x} = \int_{\mathcal{E}_\mathcal{D}} \varphi\, \mathbf{g} \cdot \mathbf{n}\, ds, \qquad \forall \varphi \in Q_h. \tag{18}$$

5.4 Mixed discontinuous Galerkin problems

We introduce mixed discontinuous Galerkin methods of the form (4) for the mixed-order spaces in (16):
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{cases} A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}) \\ B_h(\mathbf{u}_h, q) = G_h(q) \end{cases} \tag{19}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

The form $B_h : \mathbf{V}(h) \times Q \to \mathbb{R}$ and the functional $G_h : Q_h \to \mathbb{R}$ will always be chosen as:

$$B_h(\mathbf{v}, q) = -\int_\Omega q\, [\nabla_h \cdot \mathbf{v} - \mathcal{M}(\mathbf{v})]\, d\mathbf{x}, \qquad \mathbf{v} \in \mathbf{V}(h),\ q \in Q,$$
$$G_h(q) = \int_{\mathcal{E}_\mathcal{D}} q\, \mathbf{g} \cdot \mathbf{n}\, ds, \qquad q \in Q_h,$$

Restricted to discrete functions $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, we have

$$B_h(\mathbf{v}, q) = -\int_\Omega q\, \nabla_h \cdot \mathbf{v}\, d\mathbf{x} + \int_\mathcal{E} \{\!\{q\}\!\} [\![\mathbf{v}]\!]\, ds. \tag{20}$$

Thus, we obtain exactly the form B_h and the functional G_h considered in the mixed DG approaches in [17, 26, 40]. We remark that (10) is satisfied thanks to (18).

For discrete functions, we have equivalently,

$$B_h(\mathbf{v}, q) = \int_\Omega \nabla_h q \cdot \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}_\mathcal{I}} [\![q]\!] \cdot \{\!\{\mathbf{v}\}\!\}\, ds, \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h. \tag{21}$$

This follows by integration by parts and elementary manipulations, see equation (4.7) in [40].

The space $\mathbf{V}(h) = \mathbf{V} + \mathbf{V}_h$ is endowed with the broken norm

$$\|\mathbf{v}\|_h^2 = \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 + \int_{\mathcal{E}} \sigma |\underline{[\![\mathbf{v}]\!]}|^2 \, ds, \qquad \mathbf{v} \in \mathbf{V}(h), \tag{22}$$

where $\sigma \in L^\infty(\mathcal{E})$ is the so-called discontinuity stabilization function that we choose in terms of the local mesh-sizes and the polynomial degrees as follows. Define the functions $\mathtt{h} \in L^\infty(\mathcal{E})$ and $\mathtt{k} \in L^\infty(\mathcal{E})$ by

$$\mathtt{h}(\mathbf{x}) := \begin{cases} \min\{h_K, h_{K'}\}, & \mathbf{x} \text{ in the interior of } \partial K \cap \partial K', \\ h_K, & \mathbf{x} \text{ in the interior of } \partial K \cap \partial \Omega, \end{cases}$$

$$\mathtt{k}(\mathbf{x}) := \begin{cases} \max\{k_K, k_{K'}\}, & \mathbf{x} \text{ in the interior of } \partial K \cap \partial K', \\ k_K, & \mathbf{x} \text{ in the interior of } \partial K \cap \partial \Omega. \end{cases}$$

Then we set

$$\sigma = \sigma_0 \mathtt{h}^{-1} \mathtt{k}^2, \tag{23}$$

with a parameter $\sigma_0 > 0$ that is independent of $\mathtt{h}$ and $\mathtt{k}$.

For the form A_h related to the Laplacian several choices are possible. Let us discuss the stable and consistent forms in the sense of [2].

The interior penalty forms $\mathbf{A_h}$

The symmetric interior penalty (IP) form has been used in the mixed DG method introduced in [26]. It is obtained by first defining the stabilization form I_h^σ as

$$I_h^\sigma(\mathbf{u}, \mathbf{v}) := \nu \int_{\mathcal{E}} \sigma \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]} \, ds, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{V}(h), \tag{24}$$

where σ is the discontinuity stabilization function in (23), and then by taking, for $\mathbf{u}, \mathbf{v} \in \mathbf{V}(h)$,

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu \big[\nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{u}) : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v}) : \nabla_h \mathbf{u} \big] \, d\mathbf{x} + I_h^\sigma(\mathbf{u}, \mathbf{v}), \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \nu \int_\Omega \underline{\mathcal{G}} : \nabla_h \mathbf{v} \, d\mathbf{x} + \nu \int_{\mathcal{E}_\mathcal{D}} \sigma \mathbf{g} \cdot \mathbf{v} \, ds. \end{aligned} \tag{25}$$

Restricted to discrete functions $\mathbf{u}, \mathbf{v} \in \mathbf{V}_h$, we have

$$A_h(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{v} \, d\mathbf{x} - \int_{\mathcal{E}} \big(\{\!\{\nu \nabla_h \mathbf{v}\}\!\} : \underline{[\![\mathbf{u}]\!]} + \{\!\{\nu \nabla_h \mathbf{u}\}\!\} : \underline{[\![\mathbf{v}]\!]} \big) \, ds + I_h^\sigma(\mathbf{u}, \mathbf{v}).$$

The non-symmetric variant of the IP form has been studied in the mixed DG approach in [40] (see also [35, 27] for scalar convection-diffusion problems). It is obtained by choosing, for $\mathbf{u}, \mathbf{v} \in \mathbf{V}(h)$,

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu \big[\nabla_h \mathbf{u} : \nabla_h \mathbf{v} + \underline{\mathcal{L}}(\mathbf{u}) : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v}) : \nabla_h \mathbf{u} \big] \, d\mathbf{x} + I_h^\sigma(\mathbf{u}, \mathbf{v}), \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \nu \int_\Omega \underline{\mathcal{G}} : \nabla_h \mathbf{v} \, d\mathbf{x} + \nu \int_{\mathcal{E}_\mathcal{D}} \sigma \mathbf{g} \cdot \mathbf{v} \, ds. \end{aligned} \tag{26}$$

Remark 5.1. *For $\sigma \equiv 0$ the form A_h in (26) coincides with the form given by the so-called Baumann-Oden method [7, 29]. Further, realizations of the methods of Baker, Jureidini and Karakashian [4, 28] are obtained with the IP form A_h in (25), if we choose the spaces $\widetilde{\mathbf{V}}_h = \{\mathbf{v} \in \mathbf{V}_h : \mathbf{v}|_K$ is divergence free on each $K \in \mathcal{T}_h\}$ and $\widetilde{Q}_h = Q_h \cap C^0(\overline{\Omega})$, respectively.*

The LDG form $\mathbf{A_h}$

The local discontinuous Galerkin (LDG) form is closely related to the IP forms since it is also expressed in terms of the stabilization form I_h^σ in (24). In the context of the Stokes problem, it has been studied in [17] (see also [19, 13, 31]). In the primal variables, the LDG form is given by taking, for $\mathbf{u}, \mathbf{v} \in \mathbf{V}(h)$,

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu \left[\nabla_h \mathbf{u} - \underline{\mathcal{L}}(\mathbf{u})\right] : \left[\nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v})\right] d\mathbf{x} + I_h^\sigma(\mathbf{u}, \mathbf{v}), \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \nu \int_\Omega \underline{\mathcal{G}} : \left(\nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v})\right) d\mathbf{x} + \nu \int_{\mathcal{E}_\mathcal{D}} \sigma \mathbf{g} \cdot \mathbf{v} \, ds. \end{aligned} \tag{27}$$

The Bassi-Rebay forms $\mathbf{A_h}$

These forms were inspired by the original Bassi-Rebay (BR) method in [5], which, in fact, is unstable. They are defined by introducing a different stabilization form I_h^η given by

$$I_h^\eta(\mathbf{u}, \mathbf{v}) = \nu \sum_{e \subset \mathcal{E}} \int_\Omega \eta \, \underline{\mathcal{L}}_e(\mathbf{u}) : \underline{\mathcal{L}}_e(\mathbf{v}) \, d\mathbf{x}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{V}(h), \tag{28}$$

for a parameter $\eta > 0$. The first form we present here was introduced in [6] and is obtained by choosing, for $\mathbf{u}, \mathbf{v} \in \mathbf{V}(h)$,

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu \big[\nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{u}) : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v}) : \nabla_h \mathbf{u}\big] \, d\mathbf{x} + I_h^\eta(\mathbf{u}, \mathbf{v}), \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \nu \int_\Omega \underline{\mathcal{G}} : \nabla_h \mathbf{v} \, d\mathbf{x} + \nu \sum_{e \subset \mathcal{E}_\mathcal{D}} \int_\Omega \eta \, \underline{\mathcal{G}}_e : \underline{\mathcal{L}}_e(\mathbf{v}) \, d\mathbf{x}. \end{aligned} \tag{29}$$

In [11], the following variant of the BR form has been proposed:

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu \big[\nabla_h \mathbf{u} - \underline{\mathcal{L}}(\mathbf{u})\big] : \big[\nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v})\big] \, d\mathbf{x} + I_h^\eta(\mathbf{u}, \mathbf{v}), \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \nu \int_\Omega \underline{\mathcal{G}} : \nabla_h \mathbf{v} \, d\mathbf{x} + \nu \sum_{e \subset \mathcal{E}_\mathcal{D}} \int_\Omega \eta \, \underline{\mathcal{G}}_e : \underline{\mathcal{L}}_e(\mathbf{v}) \, d\mathbf{x}. \end{aligned} \tag{30}$$

6 Divergence stability

In this section, we establish an inf-sup condition for the form $B_h(\cdot, \cdot)$ with respect to the norm $\|\cdot\|_h$ in (22)-(23) and for the $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ spaces in (16). We recall that the divergence bilinear form is the same for all the methods that we consider.

6.1 The discrete inf-sup condition

Let us begin by stating our main stability result.

Proposition 6.1. *Let $k_K \geq 2$ for all $K \in \mathcal{T}_h$. Then there are constants $c_1 > 0$ and $c_2 > 0$, independent of $\underline{h}$ and $\underline{k}$, such that for each $q \in Q_h$ there exists a discrete velocity field $\mathbf{v} \in \mathbf{V}_h$ such that*

$$B_h(\mathbf{v}, q) \geq c_1 \|q\|_0^2, \qquad \|\mathbf{v}\|_h \leq c_2 |\underline{k}|\, \|q\|_0.$$

From the above result, we immediately find the following stability result.

Theorem 6.1. *There exists a constant $c > 0$, independent of $\underline{h}$ and $\underline{k}$, such that, for $k_K \geq 2$,*

$$\inf_{0 \neq q \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \geq \gamma_h \geq c\, |\underline{k}|^{-1}. \tag{31}$$

Remark 6.1. *Theorem 6.1 establishes an hp-version divergence stability result for the $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ element family where the difference in the approximation orders for the velocity and the pressure is exactly one. It is well known that these elements are unstable in a conforming setting, although they are optimal in terms of the approximation properties of the finite element spaces. The use of discontinuous velocities overcomes these usual stability problems in a natural way. In addition, the bound (31) holds in two and three dimensions, and it is identical to the bound established in [38] for conforming mixed hp-FEM in three dimensions, although there $\mathbb{Q}_k - \mathbb{Q}_{k-2}$ spaces have been used.*

Remark 6.2. *The technique we use to prove this result is a combination of the h-version approach in [26] that makes use of $H(\mathrm{div})$-conforming projectors and of the work [40] that allows us to deal with hanging nodes. Indeed, we also decompose the pressure into piecewise constants and polynomials whose mean values vanish elementwise as in [40] (see also the analysis for conforming hp-methods in [38]) and use the low-order stability results in two and three dimensions of [36, 41] for $\mathbb{Q}_2 - \mathbb{Q}_0$ elements on irregular meshes. That is the reason why we assume $k_K \geq 2$ in Proposition 6.1. We remark that for $\mathbb{Q}_1 - \mathbb{Q}_0$ elements and conforming meshes, divergence stability can be obtained by establishing directly a Fortin property. We report on this case in more detail in section 6.5.*

Remark 6.3. *The numerical tests in [40] show that in two dimensions a stability constant independent of h* and *k is expected, indicating that the dependence on k in (31) is not likely to be sharp.*

Remark 6.4. *As can be inferred from its proof, the result of Theorem 6.1 holds in fact for the strictly smaller velocity space $\widetilde{\mathbf{V}}_h \subset \mathbf{V}_h$ given by*

$$\widetilde{\mathbf{V}}_h = \{\mathbf{v} \in \mathbf{V}_h \cap H_0(\mathrm{div}; \Omega) : \mathbf{v}|_K \in RT_{k_K - 1}(K),\ K \in \mathcal{T}_h\},$$

for the Raviart-Thomas space $RT_{k_K-1}(K)$ of degree k_K-1 introduced in the next section. Here, $H_0(\mathrm{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^d : \ \nabla\cdot\mathbf{v} \in L^2(\Omega), \ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$. Whether or not the dependence of the inf-sup constant γ_h on the polynomial degree in (31) is sharp for this space is an open issue.

The remaining part of this section is devoted to the proof of Proposition 6.1. We will carry out the proof for the three-dimensional case and note that the result in two dimensions is obtained completely analogously. We start in section 6.2 by defining Raviart-Thomas interpolation operators that we shall use as Fortin operators. In section 6.3, we establish new stability results for these operators. The proof of Proposition 6.1 is then given in section 6.4. In section 6.5, we report on some extensions of our stability result to uniform approximation orders and conforming meshes, also including $\mathbb{Q}_1 - \mathbb{Q}_0$ elements.

6.2 Raviart-Thomas spaces and interpolants

Given the reference cube $\hat{K} = (-1,1)^3$ and an integer $k \geq 0$, we consider the space

$$RT_k(\hat{K}) = \mathbb{Q}_{k+1,k,k}(\hat{K}) \times \mathbb{Q}_{k,k+1,k}(\hat{K}) \times \mathbb{Q}_{k,k,k+1}(\hat{K}),$$

where $\mathbb{Q}_{k_1,k_2,k_3}(\hat{K})$ is the space of polynomials of degree at most k_i in the ith variable. For an affinely mapped element $K \in \mathcal{T}_h$ the space $RT_k(K)$ is defined by suitably mapping functions in $RT_k(\hat{K})$ using a Piola transformation; see [10, Sect. 3.3] or [1, Sect. 3.3] for further details.

We denote the faces of $\hat{K}$ by γ_m, $m = 1,\ldots,6$. In particular, we set

$$\begin{array}{ll} \gamma_1 = \{x=-1\}, & \gamma_2 = \{x=1\}, \\ \gamma_3 = \{y=-1\}, & \gamma_4 = \{y=1\}, \\ \gamma_5 = \{z=-1\}, & \gamma_6 = \{z=1\}. \end{array}$$

We use the same notation for an affinely mapped element K, where the faces are obtained by mapping the corresponding ones of $\hat{K}$. Moreover, we denote by $\mathbb{Q}_{k,k}(\gamma_m)$ the space of polynomials of degree at most k in each variable on the face γ_m.

On the reference cube, there is a unique interpolation operator $\Pi_{\hat{K}} : H^1(\hat{K})^3 \to RT_k(\hat{K})$, such that

$$\begin{aligned} &\int_{\hat{K}} \big(\Pi_{\hat{K}}\mathbf{w} - \mathbf{w}\big)\cdot\mathbf{r}\,d\mathbf{x} = 0, \quad \mathbf{r} \in \mathbb{Q}_{k-1,k,k}(\hat{K}) \times \mathbb{Q}_{k,k-1,k}(\hat{K}) \times \mathbb{Q}_{k,k,k-1}(\hat{K}), \\ &\int_{\gamma_m} \big(\Pi_{\hat{K}}\mathbf{w} - \mathbf{w}\big)\cdot\mathbf{n}\,\varphi\,ds = 0, \quad \varphi \in \mathbb{Q}_{k,k}(\gamma_m), \quad m = 1,\ldots,6; \end{aligned} \tag{32}$$

see [10] or [1]. For $k = 0$, the first condition in (32) is void. For an element $K \in \mathcal{T}_h$, the interpolant $\Pi_K : H^1(K)^3 \to RT_k(K)$ can be defined by using a Piola transform in such a way that the orthogonality conditions in (32) also hold for Π_K; see, e.g., [1, Sect. 3.5].

6.3 Stability of the Raviart-Thomas interpolant

In order to prove our stability results for the operator Π_K, we need to introduce a representation formula, originally proposed in [1] for the two-dimensional case. We start by defining some additional operators for the reference cube $\hat{K}$. Given integers k_1, k_2 and k_3, we define

$$\hat{Q}_{k_1,k_2,k_3} = \pi^z_{k_3} \otimes \pi^y_{k_2} \otimes \pi^x_{k_1} : L^2(\hat{K}) \to \mathbb{Q}_{k_1,k_2,k_3}(\hat{K})$$

as the L^2-orthogonal projection onto $\mathbb{Q}_{k_1,k_2,k_3}(\hat{K})$. We note that $\hat{Q}_{k_1,k_2,k_3}$ is the tensor product of one-dimensional L^2-projections π_{k_i} on the reference interval $I = (-1, 1)$.

We next introduce extension operators from the faces γ_m. To that end, we denote by L_k, $k \geq 0$, the Legendre polynomial of degree k in I; see [9, Sect. 3]. For the face γ_1, we define $\mathcal{E}^{\gamma_1}_k : \mathbb{Q}_{k,k}(\gamma_1) \to \mathbb{Q}_{k+1,k,k}(\hat{K})$ as

$$(\mathcal{E}^{\gamma_1}_k \varphi)(x, y, z) := M^{\gamma_1}_k(x)\varphi(y, z), \qquad M^{\gamma_1}_k(x) := \frac{(-1)^{k+1}}{2}(L_{k+1}(x) - L_k(x)).$$

We note that

$$(\mathcal{E}^{\gamma_1}_k \varphi)|_{\gamma_1} = \varphi, \qquad (\mathcal{E}^{\gamma_1}_k \varphi)|_{\gamma_2} = 0,$$

and that $(\mathcal{E}^{\gamma_1}_k \varphi)|_{\gamma_m}$, $m = 3, \dots, 6$, does not vanish in general. Analogous definitions hold for the other faces γ_m, $m = 2, \dots, 6$.

Similar to [1, Lemma 3], for $\mathbf{w} = (w_x, w_y, w_z) \in H^1(\hat{K})^3$, the interpolant $\mathbf{v} = \Pi_{\hat{K}}\mathbf{w}$ can be written as $\mathbf{v} = (v_x, v_y, v_z)$ with

$$\begin{aligned}
v_x &= \hat{Q}_{k-1,k,k} w_x + \sum_{m=1}^{2} \mathcal{E}^{\gamma_m}_k (\pi^y_k \circ \pi^z_k) \left(w_x - \hat{Q}_{k-1,k,k} w_x \right), \\
v_y &= \hat{Q}_{k,k-1,k} w_y + \sum_{m=3}^{4} \mathcal{E}^{\gamma_m}_k (\pi^x_k \circ \pi^z_k) \left(w_y - \hat{Q}_{k,k-1,k} w_y \right), \\
v_z &= \hat{Q}_{k,k,k-1} w_z + \sum_{m=5}^{6} \mathcal{E}^{\gamma_m}_k (\pi^x_k \circ \pi^y_k) \left(w_z - \hat{Q}_{k,k,k-1} w_z \right),
\end{aligned} \tag{33}$$

where, e.g., $(\pi^y_k \circ \pi^z_k)(w_x - \hat{Q}_{k-1,k,k} w_x)$ is understood as $\pi^y_k \circ \pi^z_k$ applied to the restriction of $(w_x - \hat{Q}_{k-1,k,k} w_x)$ to γ_m, $m = 1, 2$.

Before proving our stability results, we need some technical lemmas. The results in the following lemma can be found using similar techniques as in Theorem 2.2 in [12], Lemma 3.9 in [27], and Theorems 3.91 and 3.92 in [37].

Lemma 6.1. *We have the following estimates.*

1. *Let $w \in H^1(\hat{K})$. Then there exists a constant $C > 0$ independent of k such that,*

$$|\hat{Q}_{k-1,k,k} w|^2_{1,\hat{K}} \leq C\, k\, |w|^2_{1,\hat{K}}, \tag{34}$$

$$\|w - \hat{Q}_{k-1,k,k} w\|^2_{0,\gamma_m} \leq C\, k^{-1}\, |w|^2_{1,\hat{K}}, \qquad m = 1, \dots, 6. \tag{35}$$

2. Let $I = (-1,1)$ and $w \in \mathbb{Q}_k(I)$. Then there exists a constant $C > 0$ such that

$$|w|_{1,I} \leq Ck^2 \|w\|_{0,I}, \tag{36}$$
$$\|w\|_{\infty,I} \leq Ck \|w\|_{0,I}. \tag{37}$$

The following lemma can be proved by using the properties of the Legendre polynomials given, e.g., in Theorem 3.2 and Remark 3.2 in [9], and Theorem 3.96 in [37].

Lemma 6.2. *Let*

$$M_k^{\gamma_1}(x) = \frac{(-1)^{k+1}}{2}(L_{k+1}(x) - L_k(x)).$$

Then

$$\|M_k^{\gamma_1}\|_{0,I}^2 \leq Ck^{-1}, \qquad |M_k^{\gamma_1}|_{1,I}^2 \leq Ck^3.$$

Similar estimates hold for the other faces γ_m.

We have the following stability result.

Lemma 6.3. *There exists a constant $C > 0$, independent of h_K and k, such that, for $\mathbf{w} \in H^1(K)^3$,*

$$|\Pi_K \mathbf{w}|_{1,K}^2 \leq Ck^2 |\mathbf{w}|_{1,K}^2.$$

Proof. Let $\mathbf{w} = (w_x, w_y, w_z)$. We set $\mathbf{v} = \Pi_K \mathbf{w}$ and $\mathbf{v} = (v_x, v_y, v_z)$. We only find a bound for the first component v_x. Bounds for v_y and v_z can be obtained similarly. In addition, we only consider the reference cube $\hat{K} = (-1,1)^3$ since a bound for an affinely mapped K can be easily deduced using a scaling argument. We consider the two terms of v_x in (33). Thanks to (34), we have

$$|\hat{Q}_{k-1,k,k} w_x|_{1,\hat{K}} \leq Ck^{\frac{1}{2}} |w_x|_{1,\hat{K}}. \tag{38}$$

We now consider the face γ_1. Using Lemma 6.2 and the stability of the L^2-projection, we can write

$$\begin{aligned}
\|\partial_x(\mathcal{E}_k^{\gamma_1}(\pi_k^y \circ \pi_k^z)(w_x - \hat{Q}_{k-1,k} w_x))\|_{0,\hat{K}}^2 &= |M_k^{\gamma_1}|_{1,I}^2 \, \|(\pi_k^y \circ \pi_k^z)(w_x - \hat{Q}_{k-1,k,k} w_x)\|_{0,\gamma_1}^2 \\
&\leq Ck^3 \|w_x - \hat{Q}_{k-1,k,k} w_x\|_{0,\gamma_1}^2,
\end{aligned}$$

and thanks to the estimate (35),

$$\|\partial_x(\mathcal{E}_k^{\gamma_1}(\pi_k^y \circ \pi_k^z)(w_x - \hat{Q}_{k-1,k,k} w_x))\|_{0,\hat{K}}^2 \leq Ck^2 |w_x|_{1,\hat{K}}^2. \tag{39}$$

Using Lemma 6.2 and the inverse estimate (36) we find

$$\begin{aligned}
&\|\partial_y(\mathcal{E}_k^{\gamma_1}(\pi_k^y \circ \pi_k^z)(w_x - \hat{Q}_{k-1,k,k} w_x))\|_{0,\hat{K}}^2 \\
&\quad = \|M_k^{\gamma_1}\|_{0,I}^2 \, \|\partial_y((\pi_k^y \circ \pi_k^z)(w_x - \hat{Q}_{k-1,k} w_x))\|_{0,\gamma_1}^2 \\
&\quad \leq Ck^{-1} k^4 \|w_x - \hat{Q}_{k-1,k,k} w_x\|_{0,\gamma_1}^2,
\end{aligned}$$

and due to the estimate (35),

$$\|\partial_y(\mathcal{E}_k^{\gamma_1}(\pi_k^y\circ\pi_k^z)(w_x-\hat{Q}_{k-1,k,k}w_x))\|_{0,\hat{K}}^2 \le Ck^2|w_x|_{1,\hat{K}}^2. \tag{40}$$

Analogously,

$$\|\partial_z(\mathcal{E}_k^{\gamma_1}(\pi_k^y\circ\pi_k^z)(w_x-\hat{Q}_{k-1,k,k}w_x))\|_{0,\hat{K}}^2 \le Ck^2|w_x|_{1,\hat{K}}^2. \tag{41}$$

Similar estimates can be found for the face γ_2. The proof is completed by combining (33), (38), (39), (40), and (41) with a scaling argument. □

On the boundary ∂K of an element K, we have the following bound.

Lemma 6.4. *There exists a constant $C > 0$, independent of h_K and k, such that, for $\mathbf{w} \in H^1(K)^3$,*

$$\|\mathbf{w}-\Pi_K\mathbf{w}\|_{0,\partial K}^2 \le Ch_K|\mathbf{w}|_{1,K}^2.$$

Proof. Let $\mathbf{v} = \Pi_K\mathbf{w}$. First, we find a bound for the first component v_x of $\mathbf{v}$ on the reference cube $\hat{K} = (-1,1)^3$.

On the face γ_1, we have

$$w_x - v_x = w_x - \hat{Q}_{k-1,k,k}w_x - (\pi_k^y\circ\pi_k^z)(w_x-\hat{Q}_{k-1,k,k}w_x).$$

Hence, by the triangle inequality, (35) and by the stability of the L^2-projection, we obtain

$$\|w_x-v_x\|_{0,\gamma_1}^2 \le Ck^{-1}|w_x|_{1,\hat{K}}^2.$$

An analogous estimate holds on γ_2.

Consider now the face γ_3. We have

$$\begin{aligned}\|w_x-v_x\|_{0,\gamma_3}^2 \le{}& C\|w_x-\hat{Q}_{k-1,k,k}w_x\|_{0,\gamma_3}^2\\ &+C\sum_{m=1}^{2}\|\mathcal{E}_k^{\gamma_m}(\pi_k^y\circ\pi_k^z)\Big(w_x-\hat{Q}_{k-1,k,k}w_x\Big)\|_{0,\gamma_3}^2.\end{aligned}$$

The first term above can be bounded again by using (35). Further, using Lemma 6.2, the inverse estimate (37), and the estimate (35), we find for $m = 1,2$,

$$\begin{aligned}&\int_{\gamma_3}\Big(\mathcal{E}_k^{\gamma_m}(\pi_k^y\circ\pi_k^z)\Big(w_x-\hat{Q}_{k-1,k,k}w_x\Big)\Big)^2\,dx\,dz\\ &\quad=\|M_k^{\gamma_m}\|_{0,I}^2\int_{-1}^{1}\Big((\pi_k^y\circ\pi_k^z)\Big(w_x-\hat{Q}_{k-1,k,k}w_x\Big)\Big)^2_{|_{y=-1}}\,dz\\ &\quad\le C\,k^{-1}\,k^2\int_{\gamma_m}\Big((\pi_k^y\circ\pi_k^z)\Big(w_x-\hat{Q}_{k-1,k,k}w_x\Big)\Big)^2\,dy\,dz\\ &\quad\le C\,|w_x|_{1,\hat{K}}^2.\end{aligned}$$

Hence, we obtain

$$\|w_x - v_x\|_{0,\gamma_3}^2 \le C\,|w_x|_{1,\hat{K}}^2.$$

The analogous bounds are obtained on γ_4, γ_5 and γ_6. This gives the desired result for the first component of $\mathbf{w} - \Pi_K \mathbf{w}$.

The proof is completed by observing that the same techniques give analogous bounds for the other components of $\mathbf{w} - \Pi_K \mathbf{w}$, and by a scaling argument. $\square$

6.4 Proof of Proposition 6.1

Fix $q \in Q_h$. We first proceed as in [40, Lemma 6.3] (see also [38] for conforming mixed hp-FEM), and decompose q into

$$q = q_0 + \bar{q} \tag{42}$$

where q_0 is the L^2-projection of q into the subspace of $L^2_0(\Omega)$ consisting of piecewise constant pressures.

Owing to the results in [25, 38] for conforming meshes and the results in [36, 41] (valid for two- and three-dimensional domains) for meshes with hanging nodes (see also [40]), there exists a piecewise quadratic velocity field $\mathbf{v}_0 \in \mathbf{V}_h \cap H^1_0(\Omega)^3$, such that

$$B_h(\mathbf{v}_0, q_0) = -\int_\Omega q_0 \nabla\cdot \mathbf{v}_0 \, d\mathbf{x} \ge \|q_0\|_0^2, \qquad \|\mathbf{v}_0\|_h = |\mathbf{v}_0|_1 \le C_0 \|q_0\|_0. \tag{43}$$

Further, for $K \in \mathcal{T}_h$, we set $\bar{q}_K = \bar{q}|_K$ and have, by construction, $\int_K \bar{q}_K \, d\mathbf{x} = 0$. Due to the continuous inf-sup condition [10, 25], there is a velocity field $\bar{\mathbf{w}}_K \in H^1_0(K)^3$ such that

$$-\int_K \bar{q}_K \nabla\cdot \bar{\mathbf{w}}_K \, d\mathbf{x} \ge \|\bar{q}_K\|_{0,K}^2, \qquad |\bar{\mathbf{w}}_K|_{1,K} \le C \|\bar{q}_K\|_{0,K}, \tag{44}$$

with a constant $C > 0$ solely depending on the shape-regularity of the mesh. Define $\bar{\mathbf{w}} \in H^1_0(\Omega)^3$ by $\bar{\mathbf{w}}|_K = \bar{\mathbf{w}}_K$ for all $K \in \mathcal{T}_h$, and let $\bar{\mathbf{v}} \in \mathbf{V}_h$ be given by

$$\bar{\mathbf{v}}|_K = \bar{\mathbf{v}}_K := \Pi_K \bar{\mathbf{w}}_K \in RT_{k_K-1}(K), \qquad K \in \mathcal{T}_h,$$

for the Raviart-Raviart projector Π_K of degree $k_K - 1$ on K. Since $\bar{\mathbf{w}}_K \in H^1_0(K)^3$, we have

$$\bar{\mathbf{v}}_K \cdot \mathbf{n}_K = 0 \qquad \text{on } \partial K, \tag{45}$$

due to the second conditions in (32) (valid for an affinely mapped element), and hence $[\![\bar{\mathbf{v}}]\!] = 0$ on $\mathcal{E}$. From the definition of B_h in (20), we thus have

$$B_h(\bar{\mathbf{v}}, \bar{q}) = -\int_\Omega \bar{q}\, \nabla_h \cdot \bar{\mathbf{v}} \, d\mathbf{x} = \sum_{K\in\mathcal{T}_h} \int_K \nabla \bar{q}_K \cdot \bar{\mathbf{v}}_K \, d\mathbf{x}.$$

Mapping $\bar{q}_K$ to $\hat{\bar{q}}_{\hat{K}}$ via the usual pullback operator and $\bar{\mathbf{v}}_K$ to $\hat{\bar{\mathbf{v}}}_{\hat{K}}$ via the Piola transformation, we obtain from [10, Sect. 3.1]

$$\int_K \nabla \bar{q}_K \cdot \bar{\mathbf{v}}_K \, d\mathbf{x} = \int_{\hat{K}} \hat{\nabla} \hat{\bar{q}}_{\hat{K}} \cdot \hat{\bar{\mathbf{v}}}_{\hat{K}} \, d\hat{\mathbf{x}}$$

We then note that, since $\hat{\bar{q}}_{\hat{K}} \in \mathbb{Q}_{k_K-1}(\hat{K})$, we have

$$\hat{\nabla}\hat{\bar{q}}_{\hat{K}} \in \mathbb{Q}_{k_K-2,k_K-1,k_K-1}(\hat{K}) \times \mathbb{Q}_{k_K-1,k_K-2,k_K-1}(\hat{K}) \times \mathbb{Q}_{k_K-1,k_K-1,k_K-2}(\hat{K}).$$

Using the orthogonality conditions in (32), we obtain

$$\int_{\hat{K}} \hat{\nabla}\hat{\bar{q}}_{\hat{K}} \cdot \hat{\bar{\mathbf{v}}}_{\hat{K}}\, d\hat{\mathbf{x}} = \int_{\hat{K}} \hat{\nabla}\hat{\bar{q}}_{\hat{K}} \cdot \hat{\bar{\mathbf{u}}}_{\hat{K}}\, d\hat{\mathbf{x}} = \int_K \nabla\bar{q}_K \cdot \bar{\mathbf{w}}_K\, d\mathbf{x}$$

and, therefore, from (44),

$$B_h(\bar{\mathbf{v}}, \bar{q}) = \sum_{K\in\mathcal{T}_h} \int_K \nabla\bar{q}_K \cdot \bar{\mathbf{w}}_K\, d\mathbf{x} = -\sum_{K\in\mathcal{T}_h} \int_K \bar{q}_K \nabla\cdot\bar{\mathbf{w}}_K\, d\mathbf{x} \geq \|\bar{q}\|_0^2. \quad (46)$$

Further, from the stability result in Lemma 6.3 and (44), we obtain

$$\sum_{K\in\mathcal{T}_h} |\bar{\mathbf{v}}_K|_{1,K}^2 \leq C \sum_{K\in\mathcal{T}_h} k_K^2 |\bar{\mathbf{w}}_K|_{1,K}^2 \leq C|\underline{k}|^2 \|\bar{q}\|_0^2. \quad (47)$$

Then, since $\underline{[\![\bar{\mathbf{w}}]\!]} = \underline{0}$ on $\mathcal{E}$, we have with Lemma 6.4 and (15)

$$\begin{aligned}
\int_{\mathcal{E}} \sigma \underline{[\![\bar{\mathbf{v}}]\!]}^2 ds &= \int_{\mathcal{E}} \sigma \underline{[\![\bar{\mathbf{w}} - \bar{\mathbf{v}}]\!]}^2 ds \\
&\leq C \sum_{K\in\mathcal{T}_h} \frac{k_K^2}{h_K} \|\bar{\mathbf{w}}_K - \bar{\mathbf{v}}_K\|_{0,\partial K}^2 \leq C|\underline{k}|^2 \sum_{K\in\mathcal{T}_h} |\bar{\mathbf{w}}_K|_{1,K}^2 \leq C|\underline{k}|^2 \|\bar{q}\|_0^2.
\end{aligned}$$

Combining this estimate with (46) and (47) yields

$$B_h(\bar{\mathbf{v}}, \bar{q}) \geq \|\bar{q}\|_0, \qquad \|\bar{\mathbf{v}}\|_h^2 \leq \bar{C}|\underline{k}|^2 \|\bar{q}\|_0. \quad (48)$$

Next, we define

$$\mathbf{v} = \mathbf{v}_0 + \delta\bar{\mathbf{v}}$$

for a parameter $\delta > 0$ still at our disposal. First, we note that from (20) and (45),

$$B_h(\bar{\mathbf{v}}, q_0) = -\sum_{K\in\mathcal{T}_h} q_0|_K \int_K \nabla\cdot\bar{\mathbf{v}}_K\, d\mathbf{x} = -\sum_{K\in\mathcal{T}_h} q_0|_K \int_{\partial K} \bar{\mathbf{v}}_K \cdot \mathbf{n}_K\, ds = 0$$

since q_0 is piecewise constant. Further, $\mathbf{v}_0 \in \mathbf{V}_h \cap H_0^1(\Omega)^3$ and, therefore, we obtain from (43) and the arithmetic-geometric mean inequality

$$|B_h(\mathbf{v}_0, \bar{q})| = |\int_\Omega \bar{q}\,\nabla\cdot\mathbf{v}_0\, d\mathbf{x}| \leq C\|q_0\|_0 \|\bar{q}\|_0 \leq \frac{C_1}{\varepsilon}\|q_0\|_0^2 + \varepsilon C_2 \|\bar{q}\|_0^2,$$

with another parameter $\varepsilon > 0$ to be properly chosen. Combining the above results with (43) and (48), gives

$$\begin{aligned}
B_h(\mathbf{v}, q) &= B_h(\mathbf{v}_0, q_0) + B_h(\mathbf{v}_0, \bar{q}) + \delta B_h(\bar{\mathbf{v}}, \bar{q}) \\
&\geq (1 - \frac{C_1}{\varepsilon})\|q_0\|_0^2 + (\delta - \varepsilon C_2)\|\bar{q}\|_0^2.
\end{aligned}$$

It is then clear that we can choose δ and ε in such a way that

$$B_h(\mathbf{v}, q) \geq c_1 \|q\|_0^2, \tag{49}$$

with a constant c_1 independent of $\underline{h}$ and $\underline{k}$. Furthermore, from (43) and (48),

$$\|\mathbf{v}\|_h \leq |\mathbf{v}_0|_1 + \delta \|\bar{\mathbf{v}}\|_h \leq c_2 |\underline{k}| \|q\|_0, \tag{50}$$

with c_2 independent of $\underline{h}$ and $\underline{k}$. The assertion of Proposition 6.1 follows from (49) and (50).

6.5 Uniform approximation degrees and conforming meshes

For uniform approximation degrees $k_K = k$, $K \in \mathcal{T}_h$, and conforming meshes, the decomposition (42) is not necessary and we can establish the inf-sup condition directly via a Fortin property. In particular, this allows us to cover the case of $\mathbb{Q}_1 - \mathbb{Q}_0$ elements as well.

To do this, define the global interpolation operator Π by

$$\Pi \mathbf{w}_{|K} = \Pi_K \mathbf{w}, \quad K \in \mathcal{T}_h,$$

where Π_K is the Raviart-Thomas projector of degree $k-1$ on K. We note that $\Pi \mathbf{w}$ belongs to $\mathbf{V}_h$ and, in case $\mathbf{w} \in H_0^1(\Omega)^3$, the normal component of $\Pi \mathbf{w}$ is continuous across the interelement boundaries and vanishes on $\partial\Omega$, i.e., $[\![\Pi \mathbf{w}]\!] = 0$ on $\mathcal{E}$. This last property is no longer true if the mesh has hanging nodes.

We have the following Fortin property.

Lemma 6.5. *Assume that $\mathcal{T}_h$ is conforming and $k_K = k$, $K \in \mathcal{T}_h$. We have, for $\mathbf{w} \in H_0^1(\Omega)^3$ and $k \geq 1$,*

$$B_h(\Pi \mathbf{w}, q) = -\int_\Omega q \, \nabla \cdot \mathbf{w} \, d\mathbf{x}, \qquad q \in Q_h, \tag{51}$$

$$\|\Pi \mathbf{w}\|_h \leq Ck |\mathbf{w}|_1, \tag{52}$$

where $C > 0$ is independent of h and k.

Proof. We first note that, from (21), we have

$$\begin{aligned} B_h(\Pi \mathbf{w}, q) &= \sum_{K \in \mathcal{T}_h} \int_K \Pi \mathbf{w} \cdot \nabla q d\mathbf{x} - \int_{\mathcal{E}_\mathcal{I}} (q^+ - q^-) \frac{(\Pi \mathbf{w})^+ \cdot \mathbf{n}_{K^+} + (\Pi \mathbf{w})^- \cdot \mathbf{n}_{K^-}}{2} ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \Pi \mathbf{w} \cdot \nabla q d\mathbf{x} - \int_{\mathcal{E}_\mathcal{I}} (q^+ - q^-) \, \Pi \mathbf{w} \cdot \mathbf{n}_{K^+} \, ds, \end{aligned}$$

where we have used obvious notation to express the jumps and mean values. Again using the orthogonality conditions in (32), valid for an affinely mapped element K, we find

$$\begin{aligned} B_h(\Pi\mathbf{w}, q) &= \sum_{K\in\mathcal{T}_h} \int_K \mathbf{w}\cdot\nabla q d\mathbf{x} - \int_{\mathcal{E}_\mathcal{I}} (q^+ - q^-)\,\mathbf{w}\cdot\mathbf{n}_{K^+}\,ds \\ &= -\int_\Omega q\,\nabla\cdot\mathbf{w}\,d\mathbf{x}. \end{aligned}$$

The stability estimate in (52) follows from Lemma 6.3 and Lemma 6.4 as in the proof of Proposition 6.1. □

We note that the previous lemma is not true for irregular meshes. Combining Lemma 6.5 and the inf-sup condition (3) of the continuous problem, we find the following stability result.

Theorem 6.2. *Assume that $\mathcal{T}_h$ is conforming and $k_K = k$, $K \in \mathcal{T}_h$. There exists a constant $c > 0$, independent of h and k, such that for $k \geq 1$*

$$\inf_{0\neq q\in Q_h} \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h\|q\|_0} \geq \gamma_h \geq ck^{-1}. \tag{53}$$

We emphasize that, in particular, this result holds true for $k = 1$, thus covering $\mathbb{Q}_1 - \mathbb{Q}_0$ elements. We also remark that a similar non-conforming Stokes element, the so-called $\widetilde{\mathbb{Q}}_1 - \mathbb{Q}_0$ element, has been proposed and studied in [34, 8]. However, this element can be viewed as a natural quadrilateral analog of the well known Crouzeix-Raviart element whereas the $\mathbb{Q}_1 - \mathbb{Q}_0$ element here is based on completely discontinuous finite element spaces.

7 Continuity and coercivity

In this section, we establish the continuity and coercivity of the forms $A_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ with respect to the norm $\|\cdot\|_h$ in (22)-(23).

7.1 Stability of the lifting operators

We start by investigating the stability properties of the lifting operators. To this end, we need the following lemma concerning traces of polynomials, where we denote by $\mathbb{Q}_k(\gamma_m)$ the polynomials of degree at most k in each variable on the face γ_m.

Lemma 7.1. *Let $K \in \mathcal{T}_h$ and γ_m a face of ∂K. Then we have*

$$\|\varphi\|_{0,\gamma_m} \leq Ch_K^{-\frac{1}{2}} k \|\varphi\|_{0,K}, \qquad \forall\varphi\in\mathbb{Q}_k(K), \tag{54}$$

with a constant $C > 0$ just depending on the shape-regularity of the mesh. Conversely, for $\varphi \in \mathbb{Q}_k(\gamma_m)$ there is a polynomial extension $E(\varphi) \in \mathbb{Q}_k(K)$ with $E(\varphi)|_{\gamma_m} = \varphi$ and

$$\|E(\varphi)\|_{0,K} \leq Ch_K^{\frac{1}{2}} k^{-1} \|\varphi\|_{0,\gamma_m}, \tag{55}$$

with a constant $C > 0$ just depending on the shape-regularity of the mesh.

Proof. The first assertion follows from standard inverse inequalities, see, e.g., [37, Theorem 4.76].

We prove the second assertion only in three dimensions (the two-dimensional case is completely analogous). To this end, we consider first the reference cube $\hat{K} = (-1,1)^3$ and may assume that the face γ_m is given by $x = 1$. Fix $\varphi \in \mathbb{Q}_k(\gamma_m)$. Moreover, we consider the case where k is even and set

$$E(\varphi)(x,y,z) = \left(\frac{2}{k} \sum_{j=\frac{k}{2}+1}^{k} L_j(x) \right) \varphi(y,z),$$

where L_j denotes the Legendre polynomial of degree j on $(-1,1)$. Since $L_j(1) = 1$, we have

$$E(\varphi)|_{\gamma_m} = E(\varphi)(1,y,z) = \frac{2}{k}\,\frac{k}{2}\,\varphi(y,z) = \varphi(y,z).$$

Further,

$$\|E(\varphi)\|^2_{0,\hat{K}} = \|\varphi\|^2_{0,\gamma_m} \frac{4}{k^2} \sum_{j=\frac{k}{2}+1}^{k} \frac{2}{2j+1}.$$

We have

$$\begin{aligned} \sum_{j=\frac{k}{2}+1}^{k} \frac{2}{2j+1} &= \sum_{j=\frac{k}{2}+1}^{k} \frac{1}{(j+1)-\frac{1}{2}} \le \int_{\frac{k}{2}+1}^{k+1} \frac{1}{t-\frac{1}{2}}\,dt \\ &= \log(k+\frac{1}{2}) - \log(\frac{k}{2}+\frac{1}{2}) = \log(\frac{2k+1}{k+1}). \end{aligned}$$

The bound $\log(\frac{2k+1}{k+1}) \le C$, independent of k, proves the assertion for k even. If k is odd, the extension $E(\cdot)$ can be constructed similarly. This proves the assertion on the reference cube, the general case follows from a standard scaling argument. $\square$

We are now ready to prove the following stability result for the lifting $\underline{\mathcal{L}}_e$.

Lemma 7.2. *For a face $e \subset \mathcal{E}$, we have*

$$\begin{aligned} \|\underline{\mathcal{L}}_e(\mathbf{v})\|_0^2 &\ge C_1 \int_e \mathtt{k}^2\mathtt{h}^{-1} |\underline{[\![\mathbf{v}]\!]}|^2\,ds, \qquad \forall \mathbf{v} \in \mathbf{V}_h, \\ \|\underline{\mathcal{L}}_e(\mathbf{v})\|_0^2 &\le C_2 \int_e \mathtt{k}^2\mathtt{h}^{-1} |\underline{[\![\mathbf{v}]\!]}|^2\,ds, \qquad \forall \mathbf{v} \in \mathbf{V}(h), \end{aligned}$$

with constants $C_1 > 0$ and $C_2 > 0$ depending on the shape-regularity of the mesh. If e contains a hanging node, C_1 also depends on κ in (15).

Proof. To prove the first estimate, fix $\mathbf{v} \in \mathbf{V}_h$ and let K be the element such that e is an entire face of ∂K. By Lemma 7.1, we can find a polynomial $\underline{\tau} \in \mathbb{Q}_{k_K}(K)^{d\times d}$ such that $\underline{\tau}|_e = \underline{[\![\mathbf{v}]\!]}$ and such that

$$\|\underline{\tau}\|_{0,K} \le C h_K^{\frac{1}{2}} k_K^{-1} \|\underline{[\![\mathbf{v}]\!]}\|_{0,e}.$$

Extending $\underline{\tau}$ by zero, we obtain a function also denoted by $\underline{\tau}$ in the finite element space $\underline{\Sigma}_h$. By definition of $\underline{\mathcal{L}}_e$ and construction of $\underline{\tau}$, we have

$$\frac{1}{2}\|\underline{[\![\mathbf{v}]\!]}\|_{0,e}^2 = \int_e \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds \leq \int_K |\underline{\mathcal{L}}_e(\mathbf{v}) : \underline{\tau}|\, d\mathbf{x} \leq C h_K^{\frac{1}{2}} k_K^{-1} \|\underline{\mathcal{L}}_e(\mathbf{v})\|_0 \|\underline{[\![\mathbf{v}]\!]}\|_{0,e}.$$

If e is also an entire face of a possible neighboring element K', we combine the above bound with the one for K' and obtain the desired result. If e is not an entire face of a neighboring element, we invoke (15) and obtain the bound.

Conversely, for $\mathbf{v} \in \mathbf{V}(h)$, we have

$$\begin{aligned}
\|\underline{\mathcal{L}}_e(\mathbf{v})\|_0 &= \sup_{\underline{\tau}\in\underline{\Sigma}_h} \frac{\int_\Omega \underline{\mathcal{L}}_e(\mathbf{v}) : \underline{\tau}\, d\mathbf{x}}{\|\underline{\tau}\|_0} = \sup_{\underline{\tau}\in\underline{\Sigma}_h} \frac{\int_e \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds}{\|\underline{\tau}\|_0} \\
&\leq \sup_{\underline{\tau}\in\underline{\Sigma}_h} \frac{\big(\int_e \mathtt{k}^2\mathtt{h}^{-1}|\underline{[\![\mathbf{v}]\!]}|^2\, ds\big)^{\frac{1}{2}} \big(C\sum_{K\in\mathcal{T}_h} k_K^{-2} h_K \|\underline{\tau}\|_{0,\partial K}^2\big)^{\frac{1}{2}}}{\|\underline{\tau}\|_0} \\
&\leq \sup_{\underline{\tau}\in\underline{\Sigma}_h} \frac{\big(\int_e \mathtt{k}^2\mathtt{h}^{-1}|\underline{[\![\mathbf{v}]\!]}|^2\, ds\big)^{\frac{1}{2}} \big(C\sum_{K\in\mathcal{T}_h} \|\underline{\tau}\|_{0,K}^2\big)^{\frac{1}{2}}}{\|\underline{\tau}\|_0} \\
&\leq C\Big(\int_e \mathtt{k}^2\mathtt{h}^{-1}|\underline{[\![\mathbf{v}]\!]}|^2\, ds\Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the definition of $\underline{\mathcal{L}}_e$, the Cauchy-Schwarz inequality and the trace estimate (54) from Lemma 7.1. $\square$

Remark 7.1. *Due to (17), we also have*

$$\|\underline{\mathcal{G}}_e\|_0^2 \leq C \int_e \mathtt{k}^2\mathtt{h}^{-1}|\mathbf{g}|^2\, ds$$

for any boundary face $e \subset \mathcal{E}_\mathcal{D}$.

In the same manner, we obtain the following stability estimates for $\underline{\mathcal{L}}$, $\mathcal{M}$ and $\underline{\mathcal{G}}$.

Lemma 7.3. *We have the stability estimates*

$$\begin{aligned}
\|\mathcal{M}(\mathbf{v})\|_0^2 &\leq C \int_\mathcal{E} \mathtt{k}^2\mathtt{h}^{-1}|\underline{[\![\mathbf{v}]\!]}|^2\, ds, \qquad \mathbf{v} \in \mathbf{V}(h), \\
\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 &\leq C \int_\mathcal{E} \mathtt{k}^2\mathtt{h}^{-1}|\underline{[\![\mathbf{v}]\!]}|^2\, ds, \qquad \mathbf{v} \in \mathbf{V}(h),
\end{aligned}$$

as well as

$$\|\underline{\mathcal{G}}\|_0^2 \leq C \int_{\mathcal{E}_\mathcal{D}} \mathtt{k}^2\mathtt{h}^{-1}|\mathbf{g}|^2\, ds,$$

with constants $C > 0$ solely depending on the shape-regularity of the mesh.

7.2 Continuity

The continuity conditions of $A_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ with respect to the the discrete norm $\|\cdot\|_h$ in (22) are established in the following lemma.

Lemma 7.4. *Let σ be given as in (23) with $\sigma_0 > 0$. Then:*

1. *All the forms A_h considered in section 5.4 are continuous,*

$$|A_h(\mathbf{v},\mathbf{w})| \le \nu\bar{\alpha}_1\|\mathbf{v}\|_h\|\mathbf{w}\|_h, \qquad \mathbf{v},\mathbf{w}\in\mathbf{V}(h),$$

   with a constant $\bar{\alpha}_1 > 0$ independent of $\underline{h}$ and $\underline{k}$. Hence, condition (5) is satisfied with $\alpha_1 = \nu\bar{\alpha}_1$.

2. *The form B_h is continuous,*

$$|B_h(\mathbf{v},q)| \le \alpha_2\|\mathbf{v}\|_h\|q\|_0, \qquad (\mathbf{v},q)\in\mathbf{V}(h)\times Q,$$

   with a constant $\alpha_2 > 0$ independent of $\underline{h}$ and $\underline{k}$.

Proof. This follows immediately from Lemma 7.2, Lemma 7.3 and Cauchy-Schwarz inequalities. ☐

7.3 Coercivity of $\mathbf{A}_h$

The coercivity condition in (8) of the different forms A_h is established in the following lemma.

Lemma 7.5. *Let σ be given as in (23) with $\sigma_0 > 0$. Then:*

1. *There is a constant $\sigma_{min} > 0$ (independent of $\underline{h}$ and $\underline{k}$) such that for $\sigma_0 \ge \sigma_{min}$ the symmetric interior penalty form A_h in (25) is coercive,*

$$A_h(\mathbf{v},\mathbf{v}) \ge \nu\bar{\beta}\|\mathbf{v}\|_h^2, \qquad \mathbf{v}\in\mathbf{V}_h,$$

   with a constant $\bar{\beta} > 0$ independent of $\underline{h}$ and $\underline{k}$. Hence, condition (8) is satisfied with $\beta = \nu\bar{\beta}$.

2. *The non-symmetric interior penalty form A_h in (26) is coercive on $\mathbf{V}(h)$ for any $\sigma_0 > 0$, with coercivity constant $\beta = \nu$.*

3. *The LDG form A_h in (27) is coercive on $\mathbf{V}_h$ for any $\sigma_0 > 0$, with a coercivity constant $\beta = \nu\bar{\beta}$ where $\bar{\beta} > 0$ is independent of $\underline{h}$ and $\underline{k}$.*

4. *There is a constant $\eta_{min} > 0$ (independent of $\underline{h}$ and $\underline{k}$) such that for $\eta \ge \eta_{min}$ the Bassi-Rebay form A_h in (29) is coercive on $\mathbf{V}_h$, with a coercivity constant $\beta = \nu\bar{\beta}$ where $\bar{\beta} > 0$ is independent of $\underline{h}$ and $\underline{k}$.*

5. *The Bassi-Rebay form A_h in (30) is coercive on $\mathbf{V}_h$ for any $\eta > 0$, with a coercivity constant $\beta = \nu\bar{\beta}$ where $\bar{\beta} > 0$ is independent of $\underline{h}$ and $\underline{k}$.*

Proof. These coercivity properties are obtained from Lemma 7.2, Lemma 7.3 and the arithmetic-geometric mean inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, for all $\varepsilon > 0$, see [2]. $\square$

Remark 7.2. *We chose to express the continuity and coercivity properties of the Bassi-Rebay methods in terms of the discrete norm* $\| \cdot \|_h$ *in (22)-(23) since this norm is explicit in the mesh-sizes and the approximation degrees. Instead, it is also possible to work with*

$$\|\mathbf{v}\|_h^2 = \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 + \sum_{e \subset \mathcal{E}} \int_\Omega \eta |\underline{\mathcal{L}}_e(\mathbf{v})|^2 \, d\mathbf{x}.$$

8 The residual

In this section, we study the residual $R_h(\mathbf{u}, p; \mathbf{v})$ in (11) for our DG methods and show that it is optimally convergent.

Proposition 8.1. *Let the exact solution* $(\mathbf{u}, p)$ *of the Stokes system (1) be in* $H^{s_K+1}(K)^d \times H^{s_K}(K)$ *for all* $K \in \mathcal{T}_h$ *and* $s_K \geq 1$*. Let* $\underline{Q}$ *and* Q *be the* L^2*-projections onto* $\underline{\Sigma}_h$ *and* Q_h*, respectively. Then the residual in* $R_h(\mathbf{u}, p; \mathbf{v})$ *in (11) is given by*

$$R_h(\mathbf{u}, p; \mathbf{v}) = \nu \int_{\mathcal{E}} \{\!\{\nabla \mathbf{u} - \underline{Q}(\nabla \mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]} \, ds - \int_{\mathcal{E}} \{\!\{p - Qp\}\!\} [\![\mathbf{v}]\!] \, ds, \qquad \forall \mathbf{v} \in \mathbf{V}_h,$$

for all forms discussed in section 5.4.
Furthermore, we have that $\mathcal{R}_h(\mathbf{u}, p)$ *in (12) can be estimated by*

$$\mathcal{R}_h(\mathbf{u}, p)^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h^{2\min(s_K, k_K)}}{k_K^{2s_K+1}} \left[\nu \|\mathbf{u}\|_{s_K+1,K}^2 + \nu^{-1} \|p\|_{s_K,K}^2\right],$$

with a constant $C > 0$ *independent of* $\underline{h}$, $\underline{k}$ *and* ν.

Proof. By (17), we have $\underline{\mathcal{L}}(\mathbf{u}) = \underline{\mathcal{G}}$ and obtain for all forms

$$R_h(\mathbf{u}, p; \mathbf{v}) = \nu \int_\Omega [\nabla \mathbf{u} : \nabla_h \mathbf{v} - \nabla \mathbf{u} : \underline{\mathcal{L}}(\mathbf{v})] \, d\mathbf{x} - \int_\Omega p [\nabla \cdot \mathbf{v} - \mathcal{M}(\mathbf{v})] \, d\mathbf{x} - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

Note that

$$\int_\Omega \nabla \mathbf{u} : \underline{\mathcal{L}}(\mathbf{v}) \, d\mathbf{x} = \int_\Omega \underline{Q}(\nabla \mathbf{u}) : \underline{\mathcal{L}}(\mathbf{v}) \, d\mathbf{x} = \int_{\mathcal{E}} \{\!\{\underline{Q}(\nabla \mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]} \, ds$$

and

$$\int_\Omega p \mathcal{M}(\mathbf{v}) \, d\mathbf{x} = \int_\Omega Qp \mathcal{M}(\mathbf{v}) \, d\mathbf{x} = \int_\Omega \{\!\{Qp\}\!\} [\![\mathbf{v}]\!] \, ds.$$

If now the exact solution belongs to $H^2(K)^d \times H^1(K)$, for all $K \in \mathcal{T}_h$, we obtain by integration by parts and elementary manipulations

$$\begin{aligned} R_h(\mathbf{u}, p; \mathbf{v}) &= \int_\Omega [-\nu\Delta\mathbf{u} + \nabla p - \mathbf{f}] \cdot \mathbf{v}\, d\mathbf{x} \\ &\quad + \nu \int_\mathcal{E} \{\!\{\nabla\mathbf{u} - \underline{Q}(\nabla\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]}\, ds - \int_\mathcal{E} \{\!\{p - Qp\}\!\} [\![\mathbf{v}]\!]\, ds. \end{aligned}$$

Here, we also used that $[\![\nu\nabla\mathbf{u} - p\underline{I}]\!] = \mathbf{0}$ on $\mathcal{E}_\mathcal{I}$. From the Stokes equations in (1) we obtain the first assertion.
From (15) and since $|[\![\mathbf{v}]\!]|^2 \le C|\underline{[\![\mathbf{v}]\!]}|^2$, the Cauchy-Schwarz equation yields

$$R_h(\mathbf{u}, p; \mathbf{v}) \le C\|\mathbf{v}\|_h \big(\nu \sum_{K\in\mathcal{T}_h} \frac{h_K}{k_K^2} \|\nabla\mathbf{u} - \underline{Q}(\nabla\mathbf{u})\|_{0,\partial K}^2 + \nu^{-1} \sum_{K\in\mathcal{T}_h} \frac{h_K}{k_K^2} \|p - Q(p)\|_{0,\partial K}^2\big)^{\frac{1}{2}},$$

from where the error estimate follows with the hp-approximation properties of the L^2-projection in [27]. $\square$

9 Error estimates

In this section, we make the abstract error estimates in section 4 explicit for our DG methods.

9.1 The main result

First, we consider general meshes with hanging nodes. We have the following result.

Theorem 9.1. *Let the exact solution* $(\mathbf{u}, p)$ *of the Stokes system (1) be in* $H^{s_K+1}(K)^d \times H^{s_K}(K)$ *for all* $K \in \mathcal{T}_h$ *and* $s_K \ge 1$. *Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_h^2 \le C \sum_{K\in\mathcal{T}_h} \Big[\gamma_h^{-2} \frac{h_K^{2\min(s_K,k_K)}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2 + \frac{h_K^{2\min(s_K,k_K)}}{k_K^{2s_K}} \|p\|_{s_K,K}^2\Big],$$

$$\|p - p_h\|_0^2 \le C \sum_{K\in\mathcal{T}_h} \Big[\gamma_h^{-4} \frac{h_K^{2\min(s_K,k_K)}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2 + \gamma_h^{-2} \frac{h_K^{2\min(s_K,k_K)}}{k_K^{2s_K}} \|p\|_{s_K,K}^2\Big],$$

with $C > 0$ *independent of* $\underline{h}$ *and* $\underline{k}$.

Proof. This follows from the choice of the stabilization parameter σ in (23), Proposition 4.1, Proposition 4.2, Proposition 8.1 and standard approximation properties of the finite element spaces, see, e.g., [3, Lemma 4.5] or [37]. In particular, we choose $\mathbf{v}$ in Proposition 4.1 and q in Proposition 4.2 as the locally-constructed interpolants of $\mathbf{u}$ and p, respectively, given in [3, Lemma 4.5]. $\square$

Remark 9.1. *The above hp-version estimates are optimal in the mesh-size* $\underline{h}$*, and slightly suboptimal in* $\underline{k}$ *(half a power is lost), up to the inf-sup constant* γ_h *(which depends on the polynomial degree* $\underline{k}$*). In the mesh-size* $\underline{h}$*, the same optimal bounds have been obtained in [26] for the IP method on simplicial and conforming meshes, and for* $\mathcal{P}_k - \mathcal{P}_{k-1}$ *elements, with* $\mathcal{P}_k$ *denoting polynomials of total degree at most* k*. We further note that, in the hp-version context, the same result was recently obtained in [40] for the NIP method, with different techniques.*

Remark 9.2. *The loss of half a power of* k *is typical of DG methods for second order problems. Indeed, in the case of elliptic diffusion problems in two– or three–dimensional domains, no better p–bounds can be found in the DG literature on general unstructured grids (see, e.g., the hp-version analyzes in [27, 32, 35, 31]). Improved p–bounds have been obtained in [14] for one–dimensional convection–diffusion problems, and recently in [23] for two–dimensional reaction–diffusion problems on affine quadrilateral grids containing hanging nodes and for solutions that belong to augmented Sobolev spaces. The latter results can be carried over immediately to the Stokes setting considered here.*

Remark 9.3. *Combining the above bound with the inf-sup constant* γ_h *in Theorem 6.1, results in a loss of* $k^{3/2}$ *in the approximation of the velocity, and in a loss of* $k^{5/2}$ *for the approximation of the pressure.*

9.2 Uniform approximation degrees and conforming meshes

In this section, we specialize the result of Theorem 9.1 to the case of uniform approximation orders, $k_K = k$, and conforming meshes with no hanging nodes. We also assume that the Dirichlet boundary datum $\mathbf{g}$ is piecewise polynomial, more precisely, we assume that there is a finite element function $\mathbf{G}_h \in \mathbf{V}_h$ such that $\mathbf{G}_h|_{\partial\Omega} = \mathbf{g}$.

In this particular situation, as in the analysis of [31] for the LDG method for pure diffusion problems, we can choose $\mathbf{v}$ in Proposition 4.1 as an optimal *hp*-approximant for the velocity which is continuous in the whole domain Ω according to [3, Theorem 4.6]. The discrete pressure q in Proposition 4.2 can be chosen as before. Since the residual $\mathcal{R}_h(\mathbf{u}, p)$ is optimally convergent, we obtain the following result.

Theorem 9.2. *Let the exact solution* $(\mathbf{u}, p)$ *of the Stokes system (1) be in* $H^{s+1}(\Omega)^d \times H^s(\Omega)$ *for* $s \geq 1$*. Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C \frac{h^{\min(s,k)}}{k^s} \Big[\gamma_h^{-1} \|\mathbf{u}\|_{s+1} + \|p\|_s \Big],$$
$$\|p - p_h\|_0 \leq C \frac{h^{\min(s,k)}}{k^s} \Big[\gamma_h^{-2} \|\mathbf{u}\|_{s+1} + \gamma_h^{-1} \|p\|_s \Big],$$

with $C > 0$ *independent of* $\underline{h}$ *and* $\underline{p}$*.*

This estimate is *optimal* in h and k, up to the inf-sup constant (which is independent of $\underline{h}$). With Theorem 6.1, we obtain exactly the same result as Stenberg and Suri in [38] for conforming mixed hp-FEM in three dimensions, but with an optimal gap of one order in the finite element spaces for the velocity and the pressure.

Remark 9.4. *The estimate in Theorem 9.2 also holds on meshes with certain kinds of hanging nodes provided that a conforming and optimal hp-approximant can be constructed. In two dimensions, results in this direction can be found in, e.g., [37].*

References

[1] M. Ainsworth and K. Pinchedez, *hp-approximation theory for BDFM/RT finite elements and applications*, SIAM J. Numer. Anal., to appear.

[2] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2001), 1749–1779.

[3] I. Babuška and M. Suri, *The hp-version of the finite element method with quasiuniform meshes*, RAIRO Anal. Numér. **21** (1987), 199–238.

[4] G.A. Baker, W.N. Jureidini, and O.A. Karakashian, *Piecewise solenoidal vector fields and the Stokes problem*, SIAM J. Numer. Anal. **27** (1990), 1466–1485.

[5] F. Bassi and S. Rebay, *A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations*, J. Comput. Phys. **131** (1997), 267–279.

[6] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini, *A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows*, Proceedings of the Second European Conference on Turbomachinery Fluid Dynamics and Thermodynamics, held in Antwerpen, Belgium (R. Decuypere and G. Dibelius, eds.), March 5-7, 1997, pp. 99–108.

[7] C.E. Baumann and J.T. Oden, *A discontinuous hp-finite element method for convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg. **175** (1999), 311–341.

[8] R. Becker and R. Rannacher, *Finite element solution of the incompressible Navier-Stokes equations on anisotropically refined meshes*, Proceedings of the 10th GAMM Seminar, Notes on Numer. Fluid Dynamics, vol. 49, Vieweg Publisher, 1995, pp. 52–62.

[9] C. Bernardi and Y. Maday, *Spectral methods*, Handbook of Numerical Analysis, vol. 5, North-Holland, Amsterdam, 1997, pp. 209–485.

[10] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Series in Computational Mathematics, vol. 15, Springer–Verlag, New York, 1991.

[11] F. Brezzi, M. Manzini, D. Marini, P. Pietra, and A. Russo, *Discontinuous Galerkin approximations for elliptic problems*, Num. Meth. Part. Diff. Eqs. **16** (2000), 365–378.

[12] C. Canuto and A. Quarteroni, *Approximation results for orthogonal polynomials in Sobolev spaces*, Math. Comp. **38** (1982), 67–86.

[13] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, *An a priori error analysis of the local discontinuous Galerkin method for elliptic problems*, SIAM J. Numer. Anal. **38** (2000), 1676–1706.

[14] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab, *Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection–diffusion problems*, Math. Comp. **71** (2002), 455–478.

[15] B. Cockburn, *Discontinuous Galerkin methods for convection-dominated problems*, High-Order Methods for Computational Physics (T. Barth and H. Deconink, eds.), vol. 9, Springer–Verlag, 1999, pp. 69–224.

[16] B. Cockburn and C. Dawson, *Some extensions of the local discontinuous Galerkin method for convection-diffusion equations in multidimensions*, The Proceedings of the 10th Conference on the Mathematics of Finite Elements and Applications: MAFELAP X (J. Whitemann, ed.), Elsevier, 2000, pp. 225–238.

[17] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab, *Local discontinuous Galerkin methods for the Stokes system*, SIAM J. Numer. Anal. **40** (2002), 319–343.

[18] B. Cockburn, G.E. Karniadakis, and C.-W. Shu (eds.), *Discontinuous Galerkin methods. Theory, computation and applications*, Lect. Notes Comput. Sci. Eng., vol. 11, Springer–Verlag, 2000.

[19] B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time–dependent convection–diffusion systems*, SIAM J. Numer. Anal. **35** (1998), 2440–2463.

[20] ______, *Runge–Kutta discontinuous Galerkin methods for convection–dominated problems*, J. Sci. Comp. **16** (2001), 173–261.

[21] L. Franca and S. L. Frey, *Stabilized finite element methods, II. The incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg. **99** (1992), 209–233.

[22] L. Franca and R. Stenberg, *Error analysis of some Galerkin-least-squares methods for the elasticity equations*, SIAM J. Numer. Anal. **28** (1991), 1680–1697.

[23] E.H. Georgoulis and E. Süli, *hp-DGFEM on shape–irregular meshes: reaction–diffusion*, Tech. Report NA 01–09, Oxford University Computing Laboratory, 2001.

[24] P. Gervasio and F. Saleri, *Stabilized spectral element approximation for the Navier-Stokes equations*, Num. Meth. Part. Diff. Eqs. **14** (1998), 115–141.

[25] V. Girault and P.A. Raviart, *Finite element methods for Navier–Stokes equations*, Springer–Verlag, New York, 1986.

[26] P. Hansbo and M.G. Larson, *Discontinuous finite element methods for incompressible and nearly incompressible elasticity by use of Nitsche's method*, Comput. Methods Appl. Mech. Engrg. **191** (2002), 1895–1908.

[27] P. Houston, C. Schwab, and E. Süli, *Discontinuous hp-finite element methods for advection–diffusion–reaction problems*, SIAM J. Numer. Anal. **39** (2002), 2133–2163.

[28] O.A. Karakashian and W.N. Jureidini, *A nonconforming finite element method for the stationary Navier-Stokes equations*, SIAM J. Numer. Anal. **35** (1998), 93–120.

[29] J.T. Oden, I. Babuška, and C.E. Baumann, *A discontinuous hp-finite element method for diffusion problems*, J. Comput. Phys. **146** (1998), 491–519.

[30] I. Perugia and D. Schötzau, *The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell's equations*, Tech. Report 1774, IMA, University of Minnesota, 2001, in press in Math. Comp.

[31] I. Perugia and D. Schötzau, *An hp-analysis of the local discontinuous Galerkin method for diffusion problems*, J. Sci. Comp. **17** (2002), 561–571.

[32] S. Prudhomme, F. Pascal, J.T. Oden, and A. Romkes, *Review of a priori error estimation for discontinuous Galerkin methods*, Tech. Report 2000-27, TICAM, University of Texas at Austin, 2000.

[33] A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer–Verlag, Berlin, 1994.

[34] R. Rannacher and S. Turek, *Simple nonconforming quadrilateral Stokes element*, Num. Meth. Part. Diff. Eqs. **8** (1992), 97–111.

[35] B. Rivière, M.F. Wheeler, and V. Girault, *Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems, Part I*, Computational Geosciences **3** (1999), 337–360.

[36] D. Schötzau, C. Schwab, and R. Stenberg, *Mixed hp-FEM on anisotropic meshes, II. Hanging nodes and tensor products of boundary layer meshes*, Numer. Math. **83** (1999), 667–697.

[37] C. Schwab, *p- and hp-FEM – Theory and application to solid and fluid mechanics*, Oxford University Press, Oxford, 1998.

[38] R. Stenberg and M. Suri, *Mixed hp-finite element methods for problems in elasticity and Stokes flow*, Numer. Math. **72** (1996), 367–389.

[39] R. Témam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland, Amsterdam, 1979.

[40] A. Toselli, *hp-discontinuous Galerkin approximations for the Stokes problem*, Tech. Report 02-02, Seminar for Applied Mathematics, ETH Zürich, 2002, in press in Math. Models Method. Appl. Sci.

[41] A. Toselli and C. Schwab, *Mixed hp-finite element approximations on geometric edge and boundary layer meshes in three dimensions*, Tech. Report 01-02, Seminar for Applied Mathematics, ETH Zürich, 2001, in press in Numer. Math.

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