Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Analytic regularity of Stokes flow in polygonal domains

B.Q. Guo* and C. Schwab

Research Report No. 2000-18
December 2000

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*Department of Applied Mathematics, University of Manitoba, Winnipeg, MB, Canada R3T 2N2

Analytic regularity of Stokes flow in polygonal domains

B.Q. Guo* and C. Schwab

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 2000-18 December 2000

Abstract

We investigate the analytic regularity of the Stokes problem in a polygonal domain $\Omega \subset \mathbb{R}^2$ with straight sides and piecewise analytic data. We establish a shift theorem in weighted Sobolev spaces of arbitrary order with explicit control of the order-dependence of the constants. The shift-theorem in countably normed weighted Sobolev spaces implies in particular interior analyticity and Gevrey-type analytic regularity near the corners.

*Department of Applied Mathematics, University of Manitoba, Winnipeg, MB, Canada R3T 2N2

1 Introduction

Modeling and simulation of viscous, incompressible flow is a basic problem in many engineering disciplines. Practically all models that are in use lead, upon linearization, to the Stokes problem

$$\begin{aligned} \underline{u}_t - \nu\Delta\underline{u} + \nabla p &= \underline{f} \quad \text{in} \quad \Omega\times(0,T)\,, \\ \nabla\cdot\underline{u} &= h \quad \text{in} \quad \Omega\times(0,T) \end{aligned} \tag{1.1}$$

plus suitable initial- and boundary values. This problem dictates, already in the stationary case $\underline{u}_t = \underline{0}$, the variational setting: in two dimensions, it is a system of order

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

which is elliptic in the sense of Agmon, Douglis and Nirenberg. Consequently, if the boundary $\partial\Omega$ is smooth, elliptic regularity holds for (1.1) with $\underline{u}_t = \underline{0}$, i.e. for any $k \geq 0$ holds a **shift-theorem**

$$(\underline{f}, h) \in H^k(\Omega)^2 \times H^{k+1}(\Omega) \Longrightarrow (\underline{u}, p) \in H^{k+2}(\Omega)^2 \times H^{k+1}(\Omega)\,, \tag{1.2}$$

and if the data are analytic in $\overline{\Omega}$, so are the solutions, even up to (analytic pieces of) $\partial\Omega$, [20]. We refer to [10] for more on the regularity (1.2).

If $\partial\Omega$ is not smooth, however, it is well known [8, 14, 15, 16, 18, 19] that $(\underline{u}, p)$ has singularities at corners, even if $(\underline{f}, h)$ are smooth.

Analytic regularity results for the Navier-Stokes equations have been obtained in [17] and in [9], the latter work in space and time, however for periodic boundary conditions. This analytic regularity explains, to some extent, the success of spectral methods for flow problems at low and moderately high Reynolds numbers, even though analyticity does not hold in the presence of corners.

To prove analytic regularity for Stokes flow in polygonal domains for piecewise analytic data is the purpose of the present paper. We establish a shift theorem for piecewise analytic data where regularity is measured in countably normed spaces which were introduced by Babuška and Guo in [2, 3] (see also [5, 6]). The results obtained here are analogous to those for the $2-d$, linearized elasticity in [13]. The analysis there can, however, **not** be transferred directly to (1.1) with $\underline{u}_t = \underline{0}$ since the structure of the problem changes considerably in the incompressible limit. In particular, [13] does not give regularity of the pressure p in (1.1): this, however, is essential for the numerical solution of the Stokes problem by mixed Finite Elements. Moreover, the results in [1], heavily used in [13], do not apply here directly.

The present paper deals only with the linearized problem (1.1). However, the full Navier-Stokes problem admits similar regularity, at least in $2-d$. The corresponding analysis shall be given in a future paper. We remark that the present regularity results imply the exponential convergence of suitably designed spectral discretizations, in particular of mixed hp-finite element methods, see [12, 22, 23]. Even in the case of finite regularity, our estimates appear to be new. For example, we prove that for $\underline{u}_t = \underline{0}$, $h = 0$, and $\underline{f} \in L^2(\Omega)^2$, the solution $(\underline{u}, p)$ of (1.1) belongs to $H^{2,2}_\beta(\Omega)^2 \times H^{1,1}_\beta(\Omega)$ (cf. Theorem 5.4) for any polygon Ω and any combination of Dirichlet- and Neumann boundary conditions. This result contains the $H^2(\Omega)^2 \times H^1(\Omega)$ regularity of $(\underline{u}, p)$ in convex polygons for the stationary Dirichlet problem of (1.1) as special case and gives rise to optimal convergence rates of low order FEM on graded meshes, cf. [21].

Acknowledgement: This research was initiated during a visit of B. Guo to the Research Institute for Mathematics (FIM) at ETH Zürich, whose support is gratefully acknowledged.

2 Preliminaries

By $\Omega \subset \mathbb{R}^2$, we denote a polygon with vertices A_i and open edges Γ_i connecting A_i and A_{i+1}, $1 \le i \le M$. Throughout, we understand the subscript i modulo M, i.e. $A_{M+1} = A_1$, $\Gamma_{M+1} = \Gamma_1$, see Figure 1. Let $\mathcal{D}$ and $\mathcal{N}$ be disjoint subsets of $\{1, 2, \dots, M\}$ and set $\Gamma^0 = \bigcup_{i\in\mathcal{D}} \overline{\Gamma}_i$, $\Gamma^1 := \bigcup_{i\in\mathcal{N}} \overline{\Gamma}_i$. Then $\Gamma = \partial\Omega = \Gamma^0 \cup \Gamma^1$. Let further $\beta = (\beta_1, \dots, \beta_M)$ be an M-tuple of real numbers satisfying $0 < \beta_i < 1$, $1 \le i \le M$, and define the weight function

$$\Phi_{\beta+k}(x) := \prod_{i=1}^{M} (r_i(x))^{\beta_i+k} ,$$

where $r_i(x) = \mathrm{dist}(x, A_i)$ and k is any integer.

By $H^k(\Omega)$ we denote the usual Sobolev spaces, and by $H^{k,\ell}_\beta(\Omega)$, $k \ge \ell \ge 0$, weighted Sobolev spaces equipped with norm

$$\|u\|^2_{H^{k,\ell}_\beta(\Omega)} := \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{|\alpha|\ge\ell}^{k} \|\Phi_{\beta+|\alpha|-\ell}\, D^\alpha u\|^2_{L^2(\Omega)} ,$$

where the $\|u\|_{H^{\ell-1}(\Omega)}$-term is to be dropped if $\ell = 0$. If $k = \ell = 0$, we write also $L_\beta(\Omega)$ for $H^{0,0}_\beta(\Omega)$. We adopted here the multi-index notation $D^\alpha u$, i.e.

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}} , \quad \alpha = (\alpha_1, \alpha_2), \;\; |\alpha| = \alpha_1 + \alpha_2$$

for higher derivatives in cartesian coordinates.

Let $Q = \{(r, \theta) : 0 < r < \delta,\ 0 < \theta < \omega\}$ be a finite ($\delta < \infty$) or an infinite ($\delta = \infty$) sector. Then we denote by $\mathcal{D}^\alpha u$ the α-th derivative with respect to polar-coordinates (r, θ), i.e.

$$\mathcal{D}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial r^{\alpha_1}\, \partial \theta^{\alpha_2}} , \quad \alpha = (\alpha_1, \alpha_2), \;\; |\alpha| = \alpha_1 + \alpha_2 ,$$

and denote by $\mathcal{H}^{k,\ell}_\beta(Q)$ the weighted Sobolev spaces in terms of polar-coordinates, equipped with the norm

$$\|u\|^2_{\mathcal{H}^{k,\ell}_\beta(Q)} = \|u\|^2_{H^{\ell-1}(Q)} + \sum_{|\alpha|\ge\ell}^{k} \|r^{\alpha_1-\ell+\beta}\, \mathcal{D}^\alpha u\|^2_{L^2(Q)} .$$

We shall further require the weighted spaces $W^k_\beta(Q)$ introduced by Kondratev ([14, 15, 16]) which are defined in terms of the norms

$$\|u\|^2_{W^k_\beta(Q)} = \sum_{|\alpha|\le k} \int_Q r^{2(\beta-k+\alpha_1)} |\mathcal{D}^\alpha u|^2\, r dr\, d\theta .$$

By $D = \{(\tau, \theta) :\ \tau \in \mathbb{R},\ 0 < \theta < \omega\}$ we denote the strip of width $\omega > 0$ and define, for an integer $k \ge 0$ and any $h > 0$ the spaces

$$H^k_h(D) = \{u \in L^2(D) :\ \|u\|_{H^k_h(D)} < \infty\} ,$$

where

$$\|u\|^2_{H^k_h(D)} := \sum_{|\alpha|\le k} \int_D e^{2h\tau}\, |D^\alpha u|^2 \, d\tau \, d\theta \, .$$

In the polygon Ω, we define the countably normed spaces $B^\ell_\beta(\Omega)$, $\ell = 0, 1, 2$ by

$$\begin{aligned} B^\ell_\beta(\Omega) = \;\; & \Big\{ u \in \bigcap_{k\ge \ell} H^{k,\ell}_\beta(\Omega) : \; \|\Phi_{\beta+k-\ell}\, D^\alpha u\|_{L^2(\Omega)} \le C d^{k-\ell}(k-\ell)! \\ & \text{for } |\alpha| = k = \ell,\ \ell+1, \ldots \text{ and some } C, d \ge 1 \text{ independent of k} \Big\} \, . \end{aligned}$$

The spaces $\mathcal{B}^\ell_\beta(S)$ on a finite sector S are defined analogously, however in terms of polar coordinates (r, θ). Clearly, B^ℓ_β and $\mathcal{B}^\ell_\beta$ depend on the constants C, d in their definitions and we shall write $B^\ell_\beta(\Omega, C, d)$, $\mathcal{B}^\ell_\beta(\Omega, C, d)$ if this dependence is considered. The spaces defined in cartesian and polar coordinates are equivalent for $0 \le \ell \le 2$. More precisely, there holds (see Theorem 2.1 of [2]).

Proposition 2.1. *Let $S = \{(r, \theta) : 0 < r < \delta,\ 0 < \theta < \omega\}$ be a finite sector and $\Phi_\beta(x) = r^\beta$, $0 < \beta < 1$. Then, for $\ell = 0, 1, 2,$ and for all k*

$$u \in H^{k,\ell}_\beta(S) \Longleftrightarrow u \in \mathcal{H}^{k,\ell}_\beta(S)\,, \quad u \in B^\ell_\beta(S) \Longleftrightarrow u \in \mathcal{B}^\ell_\beta(S)\, .$$

We define $H^{k-\frac{1}{2},\ell-\frac{1}{2}}_\beta(\Gamma_m)$ and $B^{\ell-\frac{1}{2}}_\beta(\Gamma_m)$ on Γ_m, $m \in \{\mathcal{D}, \mathcal{N}\}$, for $\ell = 1, 2$ as spaces of traces of $H^{k,\ell}_\beta(\Omega)$ and $B^\ell_\beta(\Omega)$, respectively, and equip them with the norms

$$\|\underline{g}\|_{H^{k-\frac{1}{2},\ell-\frac{1}{2}}_\beta(\Gamma_m)} = \inf_{\underline{G}|_{\Gamma_m} = \underline{g}} \|\underline{G}\|_{H^{k,\ell}_\beta(\Omega)} \, .$$

By $(u, v)_\Omega$ we denote the $L^2(\Omega)$ innerproduct, taken componentwise for vectors and tensors u, v.

3 Stokes Problem with data in weighted spaces

In the polygon Ω, we consider the Stokes problem

$$-\mathrm{div}(\boldsymbol{\sigma}\,[\underline{u}, p]) = \underline{f} \quad \text{in} \quad \Omega\,, \tag{3.1a}$$

$$-\mathrm{div}\,\underline{u} = h \quad \text{in} \quad \Omega\,, \tag{3.1b}$$

$$\underline{u}\big|_{\Gamma^0} = \underline{g}^0 \quad \text{on} \quad \Gamma^0\,, \tag{3.1c}$$

$$\boldsymbol{\sigma}\,[\underline{u}, p]\,\underline{n} = \underline{g}^1 \quad \text{on} \quad \Gamma^1\,. \tag{3.1d}$$

Here, $\underline{u}$ is the velocity field, p the (hydrostatic) pressure and $\underline{\sigma}\,[\underline{u}, p]$ the hydrostatic stress tensor of the fluid. For a Newtonian fluid, $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma}\,[\underline{u}, p] = -p\,\mathbf{1} + 2\nu\,\boldsymbol{\epsilon}\,[\underline{u}] \tag{3.2}$$

where $\boldsymbol{\epsilon}[\underline{u}]$ denotes the symmetric gradient of $\underline{u}$, i.e.

$$\boldsymbol{\epsilon}\,[\underline{u}] = \frac{1}{2}\,(\boldsymbol{\nabla}\underline{u} + (\boldsymbol{\nabla}\underline{u})^\top)\,,$$

and $\nu > 0$ denotes the (kinematic) viscosity of the fluid.

Remark 3.1. If $h = 0$ in (3.1b) we have $\operatorname{div} \boldsymbol{\epsilon}[\underline{u}] = \frac{1}{2}\Delta \underline{u}$, and (3.1a) becomes

$$-\nu \Delta \underline{u} + \nabla p = \underline{f}\,.$$

We will be interested in the variational formulation and in the solvability of (3.1) under the following assumptions on the data:

$$\underline{g}^\ell \in H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)^2, \quad \ell = 0, 1\,, \tag{3.3a}$$

$$\underline{f} \in L_\beta(\Omega)^2,\ h \in H_\beta^{1,1}(\Omega)\,. \tag{3.3b}$$

By the definition of the traces, there exists $\underline{G}^\ell \in H_\beta^{2-\ell,2-\ell}(\Omega)$ such that

$$\underline{g}^\ell = \underline{G}^\ell\big|_{\Gamma^\ell},\ \ \|\underline{g}^\ell\|_{H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)} \cong \|\underline{G}^\ell\|_{H_\beta^{2-\ell,2-\ell}(\Omega)},\ \ell = 0, 1\,. \tag{3.4}$$

The first step to a variational formulation is to transform (3.1) to homogeneous Dirichlet data. Writing $\underline{u}_1 = \underline{u} - \underline{G}^0$ with $\underline{G}^0$ in (3.4), we find that $\underline{u}_1 \in W_0$ where

$$W_0 := \{\underline{u} \in H^1(\Omega)^2 :\ \underline{u}\big|_{\Gamma^0} = \underline{0}\}\,. \tag{3.5}$$

Then (3.1b) becomes

$$-\operatorname{div} \widetilde{\underline{u}}_1 = \widetilde{h} := h + \operatorname{div} \underline{G}^0\,. \tag{3.6}$$

If $\partial\Omega = \Gamma^0$, i.e. for the Dirichlet problem, we have

$$\int_\Omega (h + \operatorname{div} \underline{G}^0)dx = \int_\Omega \operatorname{div} \underline{u}_1\, dx = \int_{\partial\Omega} \underline{u}_1 \cdot \underline{n}\, ds = 0$$

and since also $\int_\Omega \operatorname{div} \underline{G}^0 = \int_{\partial\Omega} \underline{g}^0 \cdot \underline{n}\, ds$, it follows that we must necessarily have the compatibility condition

$$\int_\Omega h\, dx + \int_{\partial\Omega} \underline{g}^0 \cdot \underline{n}\, ds = 0 \ \text{ if } \ \Gamma^0 = \partial\Omega\,. \tag{3.7}$$

To reduce (3.6) to homogeneous data, we require

$$J_0 := \{\underline{v} \in W_0 :\ \operatorname{div} \underline{v} = 0\} \ \text{ and } \ J_0^\perp := \{\underline{v} \in \underline{W}_0 :\ (\underline{v}, \underline{w})_\Omega = 0 \quad \forall \underline{w} \in J_0\}\,. \tag{3.8}$$

The following result is classical (see eg. [10, 11]).

Proposition 3.2. *Define the space L_0 by*

$$L_0 := \begin{cases} L^2(\Omega) \ \ \textit{if} \ |\Gamma^1| > 0\,, \\ L_0^2(\Omega) = \{q \in L^2(\Omega) :\ (q, 1)_\Omega = 0 \ \ \textit{if} \ |\Gamma^1| = 0\,. \end{cases} \tag{3.9}$$

Then the mapping

$$\operatorname{div} : J_0^\perp \longrightarrow L_0$$

is bijective. In particular, for every $h \in L_0$ there exists a unique $\underline{u}_h \in J_0^\perp$ such that $-\operatorname{div} \underline{u}_h = h$.

For the weak formulation of (3.1), we introduce the bilinear forms

$$a(\underline{u}, \underline{v}) := 2\nu(\boldsymbol{\epsilon}\,[\underline{u}],\ \boldsymbol{\epsilon}\,[\underline{u}])_\Omega,\ b(p, \underline{v}) := -(p, \operatorname{div} \underline{v})_\Omega\,. \tag{3.10}$$

Then the variational formulation of (3.1) reads:
Find $\underline{u}_1 \in W_0$ and $p \in L_0$ such that

$$\begin{aligned} a(\underline{u}_1, \underline{v}) + b(p, \underline{v}) &= (\underline{f}, \underline{v})_\Omega + \langle \underline{g}^1, \underline{v} \rangle_{\Gamma^1} - a(\underline{G}^0, \underline{v}) \quad \forall \underline{v} \in W_0\,, \\ b(q, \underline{u}_1) &= (h, q)_\Omega - b(q, \underline{G}^0) \quad \forall q \in L_0\,. \end{aligned} \tag{3.11}$$

The solvability of (3.11) is a consequence of the Babuška-Brezzi theorem.

Proposition 3.3. *Let X, M be real Hilbert spaces and let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be continuous bilinear forms*

$$a : X \times X \to \mathbb{R}, \quad b : M \times X \to \mathbb{R}\,.$$

Let further $F \in X'$, $\ell \in M'$ be continuous linear functionals and define $V_\ell := \{v \in X : b(q, \underline{v}) = \ell(q)\ \forall q \in M\}$. Assume that $a(\cdot,\cdot)$ is X-coercive, i.e. there is $\alpha > 0$ such that

$$a(u, u) \geq \alpha\, \|u\|_X^2 \quad \forall u \in X\,, \tag{3.12a}$$

and that the form $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\inf_{q \in M} \sup_{v \in X} \frac{b(q, v)}{\|v\|_X\, \|q\|_M} \geq \beta > 0\,. \tag{3.12b}$$

Then there exists a unique $(u, p) \times X \in M$ solution of the saddle point problem:
Find $(u, p) \in X \times M$ such that

$$\begin{aligned} a(u, v) + b(p, v) &= F(v) \quad \forall v \in X\,, \\ b(q, u) &= \ell(v) \quad \forall q \in M\,, \end{aligned} \tag{3.13}$$

and there holds the a-priori estimate

$$\begin{aligned} \|u\|_X &\leq \frac{1}{\alpha}\, \|F\|_{X'} + \frac{1}{\beta} \left(\frac{C}{\alpha}\right) \|\ell\|_{M'}\,, \\ \|p\|_M &\leq \frac{1}{\beta} \left(1 + \frac{C}{\alpha}\right) \|F\|_{X'} + \frac{C}{\beta^2} \left(1 + \frac{C}{\alpha}\right) \|\ell\|_{M'}\,. \end{aligned} \tag{3.14}$$

For a proof, see for example [7].

We apply Proposition 3.3 to the Stokes problem (3.11). With the bilinear forms as in (3.10), we select $X = W_0$, $M = L_0$ as in (3.5) and (3.9), respectively, and the functionals $F(\cdot)$, $\ell(\cdot)$ are

$$F(\underline{v}) := (\underline{f}, \underline{v})_\Omega - a(\underline{G}^0, \underline{v}) + \langle \underline{g}^1\, \underline{v} \rangle_{\Gamma^1}\,, \tag{3.15}$$

$$\ell(q) := (h, q)_\Omega - b(q, \underline{G}^0)\,. \tag{3.16}$$

We get

Theorem 3.4. *Under the assumption $|\Gamma^0| > 0$ and (3.3), (3.7), the problem (3.11) admits a unique variational solution $(\underline{u}_1, p) \in W_0 \times L_0$. The solution satisfies the a-priori estimates*

$$\|u_1\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \Big\{ \|f\|_{L_\beta(\Omega)} + \|h\|_{H^{1,1}_\beta(\Omega)} + \sum_{\ell=0}^{1} \|\underline{g}^\ell\|_{H_\beta^{\frac{3}{2}-\ell, \frac{3}{2}-\ell}(\Gamma^\ell)} \Big\}\,. \tag{3.17}$$

Proof: We apply Proposition 3.3 with $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ as in (3.11). We select $X = W_0$, $M = L_0$. Since $|\Gamma^0| > 0$, by the first Korn inequality $a(\underline{u},\underline{u}) \geq \alpha \|\underline{u}\|^2_{H^1(\Omega)}$ for every $\underline{u} \in W_0$. By Proposition 3.2, the form $b(\cdot,\cdot)$ satisfies (3.12) for some $\beta > 0$, since $M = L_0$. It remains to check that $F(\cdot)$ and $\ell(\cdot)$ in (3.15), (3.16) are in X', M', respectively:

$$\begin{aligned} |\ell(q)| &\leq \|h\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}\|\mathrm{div}\,\underline{G}^0\|_{L^2(\Omega)} \\ &\leq \left(\|h\|_{H^{1,1}_\beta(\Omega)} + \|\underline{G}^0\|_{H^1(\Omega)}\right)\|q\|_{L^2(\Omega)} \,. \end{aligned} \tag{3.18}$$

As in [13], one verifies that

$$\begin{aligned} |F(\underline{v})| &\leq C\,\|\underline{f}\|_{L_\beta(\Omega)}\|\underline{v}\|_{H^1(\Omega)} + C\,\|\underline{G}^0\|_{H^1(\Omega)}\|\underline{v}\|_{H^1(\Omega)} \\ &+ C\,\|\underline{v}\|_{H^1(\Omega)}\Big(\|\underline{g}^1\|_{H^{\frac{1}{2},\frac{1}{2}}_\beta(\Gamma^1)} + \|\boldsymbol{\sigma}\,[\underline{G}^0,0]\,\underline{n}\|_{H^{\frac{1}{2},\frac{1}{2}}_\beta(\Gamma^1)}\Big) \\ &\leq C\,\|\underline{v}\|_{H^1(\Omega)}\Big(\|\underline{f}\|_{L_\beta(\Omega)} + \|\underline{G}^0\|_{H^{2,2}_\beta(\Omega)} + \|\underline{g}^1\|_{H^{\frac{1}{2},\frac{1}{2}}_\beta(\Gamma^1)}\Big) \\ &\overset{(3.4)}{=} C\,\|\underline{v}\|_{H^1(\Omega)}\Big(\|\underline{f}\|_{L_\beta(\Omega)} + \|\underline{g}^0\|_{H^{\frac{3}{2},\frac{3}{2}}_\beta(\Gamma^0)} + \|\underline{g}^1\|_{H^{\frac{1}{2},\frac{1}{2}}_\beta(\Gamma^1)}\Big) \,. \end{aligned} \tag{3.19}$$

The terms in parentheses in (3.18), (3.19) are upper bounds for $\|\ell\|_{M'}$, and $\|F\|_{X'}$, respectively.

Finally, to verify that $\ell(\cdot) \in M'$ in the case $\Gamma^0 = \partial\Omega$, we have $M = L^2_0$ and hence we must have $\ell(1) = 0$. By (3.16), this is (3.7). Now Proposition 3.3 implies the existence and (3.14) implies with (3.18), (3.19) the estimate (3.17). □

Remark 3.5. In Theorem 3.4, we assumed $|\Gamma^0| > 0$. If $|\Gamma^0| = 0$, an existence result holds with modified spaces, since in the proof of Theorem 3.4 the first Korn inequality cannot be used to verify the coercivity (3.12). To cover this case, we introduce the "rigid body motions"

$$\mathcal{R} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}\right\}. \tag{3.20}$$

Evidently, $\boldsymbol{\epsilon}\,[\underline{r}] = \underline{0}$ for all $\underline{r} \in \mathcal{R}$. The variational formulation (3.11) involves now the space

$$W_0 = \{\underline{u} \in H^1(\Omega)^2 : (\underline{u},\underline{r})_\Omega = 0 \quad \forall \underline{r} \in \mathcal{R}\}\,. \tag{3.21}$$

The data $\underline{f}$ and $\underline{g}^1$ in (3.1) must now be equilibrated, i.e. they must satisfy

$$(\underline{f},\underline{r})_\Omega + \langle \underline{g}^1,\underline{r}\rangle_{\partial\Omega} = 0 \quad \forall \underline{r} \in \mathcal{R}\,. \tag{3.22}$$

The compatibility condition (3.7) is not necessary now and from (3.9) we have $L_0 = L^2(\Omega)$. It is also not necessary to remove $\underline{h}^0$ now and Proposition 3.3 can be applied directly, if we use the second Korn inequality: there exists $\alpha > 0$ with

$$(\boldsymbol{\epsilon}\,[\underline{u}],\, \boldsymbol{\epsilon}\,[\underline{u}])_\Omega \geq \alpha\,\|\underline{u}\|^2_{H^1(\Omega)} \quad \forall \underline{u} \in W_0\,, \tag{3.23}$$

to establish (3.12). Hence there exists a (unique up to rigid body motions) solution $(\underline{u}, p)$ of (3.13) and the a-priori estimate (3.14) holds, now, however with $M = L^2(\Omega)$ and with $X = H^1(\Omega)^2/\mathcal{R}$, the factor space with respect to the rigid body motions.

Remark 3.6. (On the Neumann boundary condition).

As we indicated in Remark 3.1, if $\nabla \cdot \underline{u} = 0$ in Ω the momentum conservation (3.1a) can be written equivalently as

$$-\nu \, \Delta \underline{u} + \underline{\nabla p} = \underline{f} \text{ in } \Omega .$$

If this equation is cast into the weak form (3.11), its bilinear form $a(\cdot,\cdot)$ reads

$$a(\underline{u},\underline{v}) = \nu \, (\underline{\underline{\operatorname{grad} u}}, \ \underline{\underline{\operatorname{grad} v}})_\Omega = \nu \sum_{i=1}^{2} (\nabla u_i, \nabla u_i)_\Omega \tag{3.24}$$

which is different from (3.10). *These variational formulations are not equivalent:* in the pure Neumann case considered in Remark 3.5, for example, the function space W_0 in (3.21) is replaced by

$$\widetilde{W}_0 = \{\underline{u} \in H^1(\Omega)^2 : \ (u_\alpha, 1)_\Omega = 0, \ \alpha = 1,2\} \tag{3.25}$$

since then (3.24) satisfies $a(\underline{u},\underline{u}) \geq \alpha \, \|\underline{u}\|_1^2$ for every $\underline{u} \in \widetilde{W}_0$ by the second Poincaré inequality. Note also that in this case the compatibility conditions (3.22) for the data change - no equilibrium of momentum is required. Finally, and most importantly for our analysis, both formulations imply different Neumann boundary conditions. In the formulation based on (3.10), the boundary condition (3.1d) is enforced whereas in (3.11) with (3.24), the boundary condition

$$-p\underline{n} + \nu \, \frac{\partial \underline{u}}{\partial \underline{n}} = \underline{g}^1 \text{ on } \Gamma^1 \tag{3.26}$$

is in effect imposed. This boundary condition is not equivalent to (3.1d), even if $\nabla \cdot \underline{u} = 0$, and leads, in fact, to different corner singularities, as we will show in Remark 4.1 in the next section. In what follows, we will therefore always consider the boundary condition (3.1d), but occasionally comment on (3.26) to emphasize the essential differences between the two.

Remark 3.7. In Theorem 3.4, we obtained the a-priori estimate for (u_1, p). It is easy to deduce estimates for (u,p), though, since, by the embedding $H^{2,2}_\beta(\Omega) \subset H^1(\Omega)$ and by (3.4),

$$\begin{aligned} \|\underline{u}\|_{H^1(\Omega)} &\leq \|\underline{u}_1\|_{H^1(\Omega)} + \|\underline{G}^0\|_{H^1(\Omega)} \\ &\leq \|\underline{u}_1\|_{H^1(\Omega)} + C \, \|\underline{G}^0\|_{H^{2,2}_\beta(\Omega)} \\ &= \|\underline{u}_1\|_{H^1(\Omega)} + C \, \|\underline{g}^0\|_{H^{\frac{3}{2},\frac{3}{2}}_\beta(\Gamma^0)} . \end{aligned} \tag{3.27}$$

Using now (3.17) gives the same estimate also for $(\underline{u}, p)$.

4 Stokes Problem in an infinite sector

Let Q denote the infinite sector described in polar coordinates (r,θ) by

$$Q = \{(r,\theta) : \ 0 < r < \infty, \ 0 < \theta < \omega\} .$$

In Q, we consider the Stokes equations (3.1), in components

$$\begin{aligned} -\nu\Big(2\,\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial}{\partial x_2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)\Big) + \frac{\partial p}{\partial x_1} &= f_1 , \\ -\nu\Big(\frac{\partial}{\partial x_1}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) + 2\,\frac{\partial^2 u_2}{\partial x_2^2}\Big) + \frac{\partial p}{\partial x_2} &= f_2 , \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \qquad\qquad &= h \end{aligned} \tag{4.1}$$

with any one of the following boundary conditions

$$u\big|_{\theta=0,\omega} \;=\; \underline{g}^0 = (g_1^0, g_2^0)^\top \quad \text{(Dirichlet)}\,, \tag{4.2a}$$

$$(-p\underline{n} + 2\nu\,\boldsymbol{\epsilon}(\underline{u})\underline{n})\big|_{\theta=0,\omega} \;=\; \underline{g}^1 = (g_1^1, g_2^1)^\top \quad \text{(Neumann)}\,, \tag{4.2b}$$

$$\underline{u}\big|_{\theta=0} \;=\; \underline{g}^0, \quad -p\underline{n} + 2\nu\,\boldsymbol{\epsilon}(\underline{u})\underline{n}\big|_{\theta=\omega} = \underline{g}^1 \quad \text{(Mixed)}\,, \tag{4.2c}$$

where

$$\boldsymbol{\epsilon}(\underline{u}) = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\Big(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\Big) & \dfrac{\partial u_2}{\partial x_2} \end{pmatrix}, \quad \underline{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

and

$$2\nu\boldsymbol{\epsilon}(\underline{u})\underline{n} = \nu \begin{pmatrix} 2n_1\,\dfrac{\partial u_1}{\partial x_1} + n_2\,\Big(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\Big) \\ n_1\,\Big(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\Big) + 2n_2\,\dfrac{\partial u_2}{\partial x_2} \end{pmatrix}.$$

4.1 A system with a complex parameter $\boldsymbol{\lambda}$

We study the regularity of solution $(\underline{u}, p)$ to (4.1). To this end, we rewrite (4.1) in polar coordinates (r, θ)

$$\begin{aligned} -\nu(\overline{\Delta}u_r - r^{-2}\,u_r - 2r^{-2}\,\partial_\theta u_\theta + \partial_r(\overline{\nabla}\cdot\overline{u})) + \partial_r\,p \;&=\; f_r\,, \\ -\nu(\overline{\Delta}u_\theta - r^{-2}\,u_\theta + 2r^{-2}\,\partial_\theta u_r + r^{-1}\,\partial_\theta(\overline{\nabla}\cdot\overline{u})) + r^{-1}\,\partial_\theta\,p \;&=\; f_\theta\,, \\ \partial_r u_r + r^{-1}(u_r + \partial_\theta u_\theta) \;&=\; h \end{aligned} \tag{4.3}$$

where $\overline{\Delta} = \overline{\Delta}_{r\theta} = (\partial_r^2 + r^{-1}\,\partial_r + r^{-2}\,\partial_\theta^2)$, $\overline{\nabla}\cdot\overline{u} = (\partial_r u_r + r^{-1}u_r + r^{-1}\,\partial_\theta u_\theta)$, and the components (u_r, u_θ) of $\overline{u}$ and (f_r, f_θ) of $\overline{\underline{f}}$ are given by

$$\overline{u} = \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \overline{\underline{f}} = \begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \boldsymbol{A}\,\underline{f}\,.$$

The boundary conditions (4.2) read in polar coordinates:

$$\overline{u}\big|_{\theta=0,\omega} \;=\; \overline{\underline{g}}^0 = (\overline{g}_r^0, \overline{g}_\theta^0)^\top \;\; \text{(Dirichlet)}, \tag{4.4a}$$

$$\overline{\sigma}_n\,(\overline{u}, p)\big|_{\theta=0,\omega} \;=\; (2\nu\,\overline{\epsilon}_{r\theta}, -p + 2\nu\,\overline{\epsilon}_{\theta\theta})\big|_{\theta=0,\omega} = \overline{\underline{g}}^1 := (\overline{g}_r^1,\ \overline{g}_\theta^1)^\top \;\; \text{(Neumann)}, \tag{4.4b}$$

where $\overline{\sigma}_n(\overline{u}, p) = -p\underline{n} + 2\nu\,\boldsymbol{\epsilon}(\overline{u})\underline{n}$, $\underline{n} = (0,1)^\top$ and

$$\overline{\boldsymbol{\epsilon}}(\overline{u}) = \begin{pmatrix} \overline{\epsilon}_{rr} & \overline{\epsilon}_{r\theta} \\ \overline{\epsilon}_{\theta r} & \overline{\epsilon}_{\theta\theta} \end{pmatrix}$$

with

$$\overline{\epsilon}_{rr} = \partial_r u_r,\; \overline{\epsilon}_{\theta\theta} = r^{-1}(\partial_\theta u_\theta + u_r),\; \overline{\epsilon}_{r\theta} = \frac{1}{2}\,(r^{-1}\,\partial_\theta u_r + \partial_r u_\theta - r^{-1}\,u_\theta)$$

and the boundary conditions of mixed type

$$\begin{aligned} \overline{u} &= \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}\Big|_{\theta=0} = \underline{g}^0_0 = \begin{pmatrix} \overline{g}^0_r \\ \overline{g}^0_\theta \end{pmatrix}, \\ \overline{\sigma}_n &= (\overline{u}, p)\Big|_{\theta=\omega} = \underline{g}^1_\omega = \begin{pmatrix} \overline{g}^1_r \\ \overline{g}^1_\theta \end{pmatrix}. \end{aligned} \tag{4.4c}$$

Remark 4.1. The Stokes equations

$$\begin{aligned} -\nu(\Delta \underline{u} + \underline{\operatorname{grad}}\,(\operatorname{div} \underline{u})) + \underline{\operatorname{grad}}\, p &= \underline{f}\,, \\ \operatorname{div} \underline{u} &= h\,, \end{aligned}$$

are occasionally also written in the form

$$-\nu\, \Delta \underline{u} + \underline{\operatorname{grad}}\, p = \underline{f} + \underline{\operatorname{grad}}\, h =: f^*,\ \operatorname{div} \underline{u} = h\,.$$

For this differential operator, the variational Neumann boundary condition differs from (3.1d). It reads

$$\Big(- p\underline{n} + \nu\, \frac{\partial \underline{u}}{\partial \underline{n}} \Big)\Big|_{\theta=0,\omega} = \underline{g}^1\,.$$

In polar coordinates, it differs from (4.2b) in the $\overline{g}^1_r$-component:

$$\overline{g}^1_r = r^{-1}(\partial_\theta u_r - r^{-1} u_\theta)\,.$$

This boundary condition excludes rigid swirl flows ($\underline{u} = (x_2, -x_1)^\top$) as solutions of the homogeneous Neumann problem. In the sequel, we analyze the regularity for the physical problem (4.2), (4.4) and comment, where appropriate, on the Neumann boundary conditions.

To analyze the problem (4.1), (4.2) in the infinite sector, we introduce in (4.3) the variable $t = \ell n(1/r)$, thereby converting (4.3) into the problem

$$\begin{aligned} -\nu\big(2(\partial^2_{tt}\, \widetilde{u}_t - \widetilde{u}_t) + \partial^2_{\theta\theta}\, \widetilde{u}_t - \partial^2_{t\theta}\, \widetilde{u}_\theta - 3\partial_\theta \widetilde{u}_\theta\big) - (\partial_t\, \widetilde{p} + \widetilde{p}) &= \widetilde{f}_t \\ -\nu\big(- \partial^2_{t\theta} + 3\partial_\theta\, \widetilde{u}_t + \partial^2_{tt}\, \widetilde{u}_\theta + 2\partial^2_{\theta\theta}\, \widetilde{u}_\theta - \widetilde{u}_\theta\big) + \partial_\theta\, \widetilde{p} &= \widetilde{f}_\theta \\ -\partial_t\, \widetilde{u}_t + \widetilde{u}_t + \partial_\theta\, \widetilde{u}_\theta &= \widetilde{h} \end{aligned} \tag{4.5}$$

in the infinite strip $D = \{(t,\theta) : -\infty < t < \infty,\ 0 < \theta < \omega\}$ and the boundary conditions (4.4) become

$$\widetilde{\underline{u}}\big|_{\theta=0,\omega} = (\widetilde{u}_t, \widetilde{u}_\theta)^\top\big|_{\theta=0,\omega} = \widetilde{\underline{g}}^0 \quad \text{(Dirichlet)}\,, \tag{4.6a}$$

$$\widetilde{\underline{\sigma}}(\widetilde{\underline{u}}, \widetilde{p})\big|_{\theta=0,\omega} := \pm(2\nu\, \widetilde{\epsilon}_{t\theta},\, -\widetilde{p} + 2\nu\, \widetilde{\epsilon}_{\theta\theta})^\top\big|_{\theta=0,\omega} = \widetilde{\underline{g}}^1 \quad \text{(Neumann)}, \tag{4.6b}$$

$$\widetilde{\underline{u}}\big|_{\theta=0} = \widetilde{\underline{g}}^0_0, \quad \widetilde{\underline{\sigma}}(\widetilde{\underline{u}}, \widetilde{p})\big|_{\theta=\omega} = \widetilde{\underline{g}}^1_\omega \ \text{(Mixed)}\,. \tag{4.6c}$$

Here, $\widetilde{\underline{u}}(t,\theta) := \overline{\underline{u}}(e^{-t},\theta)$, $\widetilde{p}(t,\theta) := e^{-t}\,\overline{p}(e^{-t},\theta)$, $\widetilde{\underline{f}}(t,\theta) := e^{-2t}\,\widetilde{\underline{f}}(e^{-t},\theta)$ and $\widetilde{h}(t,\theta) = e^{-t}\,h(e^{-t},\theta)$. The boundary data are $\widetilde{\underline{g}}^\ell(t,\theta) = e^{-\ell t}\,\overline{\underline{g}}(e^{-t},\theta)$, $\ell = 0, 1$, and $(\widetilde{\epsilon}_{t\theta}, \widetilde{\epsilon}_{\theta\theta}) := (\frac{1}{2}\,(\partial_\theta\, \widetilde{u}_t - \partial_t\, \widetilde{u}_\theta - \widetilde{u}_\theta),\, (\partial_\theta\, \widetilde{u}_\theta + \widetilde{u}_t))$.

Finally, we apply the Fourier-transform to (4.5), (4.6) with respect to t, i.e.

$$\widehat{\underline{u}} = \mathcal{F}(\widetilde{\underline{u}}) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda t}\, \widetilde{\underline{u}}(t,\theta)dt,\ \lambda \in \xi + i\eta,\ -\infty < \xi < \infty,\ \eta > 0\,,$$

and denote $\widehat{p} = \mathcal{F}(\widetilde{p})$, $\widehat{h} = \mathcal{F}(\widetilde{h})$, $\widehat{\underline{g}}^{\ell} = \mathcal{F}(\widetilde{\underline{g}}^{\ell})$, $\ell = 0, 1$, and $(\widehat{\epsilon}_{t\theta},\, \widehat{\epsilon}_{\theta\theta}) := (\frac{1}{2}\,(\partial_\theta\, \widehat{u}_t - (1+\lambda i)\widehat{u}_\theta),$ $(\partial_\theta\, \widehat{u}_\theta + \widehat{u}_t))$. This yields the two-point boundary value problem in the interval $I = (0,\omega)$ depending on the complex parameter λ:

$$\begin{aligned} -\nu(\partial^2_{\theta\theta}\, \widehat{u}_t - 2(1+\lambda^2)\, \widehat{u}_t - (3+\lambda i)\, \partial_\theta\, \widehat{u}_\theta) - (1+i\lambda)\, \widehat{p} \;&=\; \widehat{f}_t\,, \\ -\nu\big((3-\lambda i)\, \partial_\theta\, \widehat{u}_t + 2\partial^2_{\theta\theta}\, \widehat{u}_\theta - (1+\lambda^2)\, \widehat{u}_\theta\big) + \partial_\theta\, \widehat{p} \;&=\; \widehat{f}_\theta\,, \\ (1-i\lambda)\widehat{u}_t + \partial_\theta\, \widehat{u}_\theta \;&=\; \widehat{h} \end{aligned} \tag{4.7}$$

equipped with one of the following boundary conditions

$$\widehat{\underline{u}}\big|_{\theta=0,\omega} = (\widehat{u}_t, \widehat{u}_\theta)^\top\big|_{\theta=0,\omega} \;=\; \widehat{\underline{g}}^0 \quad \text{(Dirichlet)}\,, \tag{4.8a}$$

$$\widehat{\underline{\sigma}}(\widehat{\underline{u}}, \widehat{p}) = \pm(2\nu\, \widehat{\epsilon}_{t\theta},\, -\widehat{p} + 2\nu\, \widehat{\epsilon}_{\theta\theta})^\top\big|_{\theta=0,\omega} \;=\; \widehat{\underline{g}}^1 \quad \text{(Neumann)}, \tag{4.8b}$$

$$\widehat{\underline{u}}\big|_{\theta=0} = \widehat{\underline{g}}^0_0\,,\ \ \widehat{\underline{\sigma}}(\widehat{\underline{u}}, \widehat{p})\big|_{\theta=\omega} \;=\; \widehat{\underline{g}}^1_\omega \quad \text{(Mixed)}\,. \tag{4.8c}$$

Denoting $\partial_\theta = iD$, (4.7), (4.8) may be written in symbolic form as

$$\begin{aligned} \widehat{L}(D,\lambda)(\widehat{\underline{u}}, \widehat{p}) \;&=\; (\widehat{\underline{f}}, \widehat{h}) \quad \text{on} \quad I = (0,\omega)\,, \\ \widehat{B}(D,\lambda)(\widehat{\underline{u}}, \widehat{p}) \;&=\; (\widehat{\underline{g}}^0, \widehat{\underline{g}}^1) \quad \text{on} \quad \partial I = \{0,\omega\} \end{aligned} \tag{4.9}$$

with the differential operator pencil

$$\widehat{L}(D,\lambda) = \begin{pmatrix} \nu D^2 + 2\nu(1+\lambda^2) & \nu(3+\lambda i)Di & -(1+\lambda i) \\ -\nu(3-\lambda i)Di & 2\nu\, D^2 + \nu(1+\lambda^2) & iD \\ (1-i\lambda) & iD & 0 \end{pmatrix}$$

and the boundary operator pencils

$$\begin{aligned} \widehat{B}(D,\lambda)\big|_{\theta=0,\omega} \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ \text{(Dirichlet)}\,, \\ \widehat{B}(D,\lambda)\big|_{\theta=0,\omega} \;&=\; \pm \begin{pmatrix} \nu\, Di & -\nu(1+\lambda i) & 0 \\ 2\nu & 2\nu\, Di & -1 \end{pmatrix} \ \text{(Neumann)}\,, \\ \widehat{B}(D,\lambda)\big|_{\theta=0} \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \widehat{B}(D,\lambda)\big|_{\theta=\omega} = \begin{pmatrix} \nu\, Di & -\nu(1+\lambda i) & 0 \\ 2\nu & 2\nu\, Di & -1 \end{pmatrix} \ \text{(Mixed)}\,. \end{aligned}$$

The principal parts $\widehat{L}_0(D,\lambda)$, $\widehat{B}_0(D,\lambda)$ of these operators are

$$\widehat{L}_0(D,\lambda) = \begin{pmatrix} \nu(D^2 + 2\lambda^2) & -\nu\lambda D & -i\lambda \\ -\nu\lambda D & \nu(2D^2 + \lambda^2) & iD \\ -i\lambda & iD & 0 \end{pmatrix}$$

and

$$\widehat{B}_0(D,\lambda)\big|_{\theta=0,\omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ (Dirichlet)},$$

$$\widehat{B}_0(D,\lambda)\big|_{\theta=0,\omega} = \pm \begin{pmatrix} \nu\, Di & -\nu\lambda i & 0 \\ 0 & 2\nu\, Di & -1 \end{pmatrix} \text{ (Neumann)},$$

$$\widehat{B}_0(D,\lambda)\big|_{\theta=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \widehat{B}_0(D,\lambda)\big|_{\theta=\omega} = \begin{pmatrix} \nu\, Di & -\nu\lambda i & 0 \\ 0 & 2\nu\, Di & -1 \end{pmatrix} \text{ (Mixed)}.$$

We will prove certain a-priori estimates for solutions of (4.9) which are the basis for the regularity theory. To this end, we introduce norms in I which depend on λ: for integer $k \geq 0$, we set

$$|||\underline{\widehat{u}}|||^2_{H^k(I)} := \sum_{\ell=0}^{k} |\lambda|^{2\ell}\, \|\underline{\widehat{u}}\|^2_{H^{k-\ell}(I)}\,. \tag{4.10a}$$

Due to the interpolation of norms, there is an equivalent norm

$$\|\underline{\widehat{u}}\|^2_{H^k(I)} + |\lambda|^{2k}\|\underline{\widehat{u}}\|^2_{L^2(I)} \leq |||\underline{\widehat{u}}|||^2_{H^k(I)} \leq C\big(\|\underline{\widehat{u}}\|^2_{H^k(I)} + |\lambda|^{2k}\|\underline{\widehat{u}}\|^2_{L^2(I)}\big) \tag{4.10b}$$

where $C > 0$ is independent of λ.

The rest of this section is devoted to the proof of the a-priori estimate for the problem (4.9): for $\lambda = \xi + i\eta$ with $\eta > 0$ fixed, there holds for $-\infty < \xi < \infty$ and for any integer $k \geq 2$

$$|||\underline{\widehat{u}}|||^2_{H^k(I)} + |||\underline{\widehat{p}}|||^2_{H^{k-1}(I)} \leq C\Big(|||\underline{\widehat{f}}|||^2_{H^{k-2}(I)} + \sum_{\ell=0}^{1} |\lambda|^{3-2\ell}|\underline{\widehat{g}}^\ell|^2 + |||\underline{\widehat{h}}|||^2_{H^{k-1}(I)}\Big) \tag{4.11}$$

where the constant C is independent of ξ.

Remark 4.2. The operators $\widehat{L}(D,\lambda)$, $\widehat{B}(D,\lambda)$ in (4.9) are not of homogeneous degree in D and λ. Hence the a-priori estimates in [1] and [14] for a parameter-dependent system of homogeneous degree cannot be applied to (4.9). Therefore we shall prove next a-priori estimates (4.11) for solutions of (4.9). The proof will be self-contained and follows the basic steps in [1]: First (4.11) is established for the principal parts $\widehat{L}_0$, $\widehat{B}_0$ on the whole real line ($I = \mathbb{R}$) resp. on the half-line ($I = \mathbb{R}_+$), then on the bounded interval $I = (0,\omega)$ by a localization argument. Finally, (4.11) will be obtained for $\widehat{L}(D,\lambda)$, $\widehat{B}(D,\lambda)$ by a perturbation argument.

4.2 A-priori estimates on the entire line $\mathbb{R}^1$

Consider the principal part of the system (4.9):

$$\widehat{L}_0(D,\lambda)(\underline{\widehat{u}},\widehat{p}) = (\underline{\widehat{f}},\widehat{h}) \ \text{ on } \ I = \mathbb{R} = (-\infty,\infty)\,. \tag{4.12}$$

By a Fourier-transformation with respect to θ,

$$\widehat{\underline{\widehat{u}}}(\xi,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi\theta}\, \underline{\widehat{u}}(\theta,\lambda)d\theta = \widehat{\mathcal{F}}(\underline{\widehat{u}})\,,$$

(4.12) is reduced to an algebraic system

$$\widehat{L}_0(\xi,\lambda)(\widehat{\underline{\widehat{u}}},\widehat{\widehat{p}}) = (\widehat{\underline{\widehat{f}}},\widehat{\underline{\widehat{h}}})\,, \tag{4.13}$$

with

$$\det(\widehat{L}_0(\xi,\lambda)) = \nu(\xi^2+\lambda^2)^2\,,$$

Therefore, if $\xi^2+\lambda^2 \neq 0$, $\widehat{L}_0^{-1}(\xi,\lambda)$ exists and (4.13) admits the unique solution $(\widehat{\underline{\widehat{u}}},\widehat{\widehat{p}}) = \widehat{L}_0^{-1}(\xi,\lambda)(\widehat{\underline{\widehat{f}}},\widehat{\underline{\widehat{h}}})$ where

$$(\widehat{L}_0(\xi,\lambda))^{-1} = \frac{1}{\nu(\xi^2+\lambda^2)^2}\begin{pmatrix} \xi^2 & \lambda\xi & i\nu\lambda(\xi^2+\lambda^2) \\ \lambda\xi & \lambda^2 & -i\nu\xi(\xi^2+\lambda^2) \\ i\nu\lambda(\xi^2+\lambda^2) & -i\nu\xi(\xi^2+\lambda^2) & 2\nu^2(\xi^2+\lambda^2)^2 \end{pmatrix}. \tag{4.14}$$

In what follows, we denote for $\phi_0 \in (0,\pi/2)$ the sector $\sum_{\phi_0}$ by

$$\textstyle\sum_{\phi_0} = \{\lambda \in \mathbb{C} \,|\, |\arg\lambda| < \phi_0 \;\text{ or }\; |\pi - \arg\lambda| < \phi_0\}\,.$$

Theorem 4.3. *For $\lambda \in \sum_{\phi_0}$ with $|\lambda| \geq \lambda_0$, $\lambda_0 > 0$ arbitrary and for any $(\widehat{\underline{f}},\widehat{h}) \in H^{k-2}(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$, $k \geq 2$, the principal system* (4.12) *has a unique solution $(\widehat{\underline{u}},\widehat{p}) \in H^k(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$ satisfying*

$$|||\widehat{\underline{u}}|||^2_{H^k(\mathbb{R})} + |||\widehat{\underline{p}}|||^2_{H^{k-1}(\mathbb{R})} \leq C\big(|||\widehat{\underline{f}}|||^2_{H^{k-2}(\mathbb{R})} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R})}\big) \tag{4.15}$$

where the constant C depends only on λ_0, ϕ_0, k but is independent of $\widehat{\underline{f}}, \widehat{\underline{h}}$.

Proof: For real ξ and for $\lambda \in \sum_{\phi_0}$, $|\lambda| \geq \lambda_0 > 0$ arbitrary, $\det(\widehat{L}_0) \neq 0$. Thus, (4.13) is uniquely solvable and due to (4.14) we have, for any $k \geq 2$,

$$\begin{aligned}(1+|\lambda|+|\xi|)^{2k}\,|\widehat{\underline{\widehat{u}}}|^2 + (1+|\lambda|+|\xi|)^{2(k-1)}|\widehat{\widehat{p}}|^2 \leq\; & C\big\{(1+|\lambda|+|\xi|)^{2(k-2)}|\widehat{\underline{\widehat{f}}}|^2 \\ & + (1+|\lambda|+|\xi|)^{2(k-1)}|\widehat{\underline{\widehat{h}}}|^2\big\}\,.\end{aligned} \tag{4.16}$$

Further, $(\widehat{\underline{u}},\widehat{p}) = (\widehat{\mathcal{F}}^{-1}(\widehat{\underline{\widehat{u}}}),\, \widehat{\mathcal{F}}^{-1}(\widehat{\widehat{p}}))$ is the solution of (4.12), and (4.16) directly gives (4.15). □

To establish (4.15) for the full system (4.9), i.e.

$$\widehat{L}(D,\lambda)(\widehat{\underline{u}},\widehat{p}) = (\widehat{\underline{f}},\widehat{h}) \;\text{ on }\; I = \mathbb{R}\,,$$

we need a perturbation lemma.

Lemma 4.4. *For any integer $k \geq 2$ and for $|\lambda| \geq \lambda_0 > 0$ with arbitrary $\lambda_0 > 0$, it holds for $I = \mathbb{R}, \mathbb{R}_+, (0,\omega)$ that*

$$|||(\widehat{L}(D,\lambda)(\widehat{\underline{u}},\widehat{p}) - \widehat{L}_0(D,\lambda)(\widehat{\underline{u}},\widehat{p}))_{12}|||^2_{H^{k-2}(I)} \leq C\big\{|||\widehat{\underline{u}}|||^2_{H^{k-1}(I)} + |||\widehat{p}|||^2_{H^{k-2}(I)}\big\}\,,$$

$$|||(\widehat{L}(D,\lambda)(\widehat{\underline{u}},\widehat{p}) - \widehat{L}_0(D,\lambda)(\widehat{\underline{u}},\widehat{p}))_3||| \leq C\big\{|||\widehat{\underline{u}}|||^2_{H^{k-1}(I)} + |||\widehat{p}|||^2_{H^{k-2}(I)}\big\}\,.$$

with C independent of λ and of $(\widehat{\underline{u}},\widehat{p})$, but depending on λ_0.

Proof: This follows directly from the definition of $\widehat{L}_0(D,\lambda)$. □

Lemma 4.4 gives immediately

Theorem 4.5. *For $\lambda \in \sum_{\phi_0}$ and for $(\underline{\widehat{f}}, \widehat{h}) \in H^{k-2}(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$, there exists $\lambda_0 > 0$ such that the system*

$$\widehat{L}(D,\lambda)(\underline{\widehat{u}}, \widehat{p}) = (\underline{\widehat{f}}, \widehat{g}) \quad on \ \ I = \mathbb{R} \tag{4.17}$$

has a unique solution $(\underline{\widehat{u}}, \widehat{p}) \in H^k(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$ and the a-priori estimate (4.15) *holds.*

Proof: We have

$$\det(\widehat{L}(\xi,\lambda)) = \nu(\lambda^2 + (\xi+1)^2)(\lambda^2 + (\xi-1)^2)\,.$$

Therefore, for real ξ and $\lambda \in \sum_{\phi_0}$ with $|\lambda| \geq \lambda_0$ and arbitrary $\lambda_0 > 0$, $\det(\widehat{L}(\xi,\lambda)) \neq 0$.

For these (ξ,λ) we may argue as in the proof of Theorem 4.3. The solution $(\underline{\widehat{u}}, \widehat{p})$ of (4.17) hence exists in $H^k(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$ if $(\underline{\widehat{f}}, \widehat{h}) \in H^{k-2}(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$. Due to (4.15), there holds the a-priori estimate

$$\begin{aligned}|||\underline{\widehat{u}}|||^2_{H^k(\mathbb{R})} + |||\widehat{p}|||^2_{H^{k-1}(R)} \;\leq\; & C\{|||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R})} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R})} \\ & + |||\big(L(D,\lambda)(\underline{\widehat{u}}, \widehat{p}) - \widehat{L}_0(D,\lambda)(\underline{\widehat{u}}, \widehat{p})\big)_{12}|||^2_{H^{k-2}(\mathbb{R})} \\ & + |||\big(L(D,\lambda)(\underline{\widehat{u}}, \widehat{p}) - \widehat{L}_0(D,\lambda)(\underline{\widehat{u}}, \widehat{p})\big)_{3}|||^2_{H^{k-1}(\mathbb{R})}\}\end{aligned}$$

where $(\ )\ldots$ denote the components of the vectors and, by Lemma 4.4, we get

$$|||\underline{\widehat{u}}|||^2_{H^k(\mathbb{R})} + |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R})} \leq C\{|||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R})} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R})} + |||\underline{\widehat{u}}|||^2_{H^{k-1}(\mathbb{R})} + |||\widehat{p}|||^2_{H^{k-2}(\mathbb{R})}\}\,.$$

By the equivalence (4.10b), we have for $|\lambda| > \lambda_0$ with $\lambda_0 > 0$ sufficiently large

$$|||\underline{\widehat{u}}|||^2_{H^{k-1}(\mathbb{R})} + |||\widehat{p}|||^2_{H^{k-2}(\mathbb{R})} \leq \frac{1}{2C}\left(|||\underline{\widehat{u}}|||^2_{H^k(\mathbb{R})} + |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R})}\right)$$

which leads to the desired estimate (4.15). □

4.3 A-priori estimate on the half-line $\mathbb{R}_+$

We consider again the principal system

$$\widehat{L}_0(D,\lambda)(\underline{\widehat{u}}, \widehat{p}) \;=\; (\underline{\widehat{f}}, \widehat{h}) \;\text{ on }\; I = \mathbb{R}_+ = (0,\infty)\,, \tag{4.18a}$$

$$\widehat{B}_0(D,\lambda)(\underline{\widehat{u}}, \widehat{p}) \;=\; (\underline{\widehat{g}}^0, \underline{\widehat{g}}^1)\,. \tag{4.18b}$$

We may assume that $\underline{\widehat{f}} = \underline{0}$, $\widehat{h} = 0$. Then the solution of the homogeneous system of second order can be written as linear combination of fundamental solutions. Since $\widehat{L}_0(D,\lambda)$ has constant coefficients, the fundamental solutions have the form $e^{b\theta}\,\underline{E}$ where b satisfies

$$\det(\widehat{L}_0(-ib,\lambda)) = \nu(\lambda^2 - b^2)^2 = 0\,.$$

Hence, $b = \pm\lambda$ with multiplicity 2 if $\lambda \neq 0$ and $b = 0$ with multiplicity 4 if $\lambda = 0$. Note that $e^{\pm\lambda\theta}\,\underline{E}$ with $\lambda \neq 0$ may not be integrable on $I = (0,\infty)$ if $\lambda \neq 0$. There are only two solutions which tend to zero as $\theta \to \infty$. Since either $Re\,\lambda < 0$ or $Re(-\lambda) < 0$, we may assume that

$Re\,\lambda < 0$ for $\lambda \neq 0$. Then $\underline{\widehat{w}}_1 = e^{\lambda\theta}(1, i, 0)^\top$ and $\underline{\widehat{w}}_2 = e^{\lambda\theta}(1 + \lambda\theta, i\lambda\theta, 2i\nu\lambda)^\top$ are two stable fundamental solutions for $\lambda \neq 0$ and the solution of (4.18) can be written in the form

$$(\underline{\widehat{u}}, \widehat{p})^\top = c_1\, \underline{\widehat{w}}_1 + c_2\, \underline{\widehat{w}}_2\,.$$

The coefficients c_1, c_2 are determined from (4.18b).

For the **Dirichlet condition** $\underline{\widehat{u}}\big|_{\theta=0} = \underline{\widehat{g}}^0 = (\widehat{g}^0_t, \widehat{g}^0_\theta)^\top$, we get $c_1 = -i\widehat{g}^0_\theta$, $c_2 = \widehat{g}^0_t + i\widehat{g}^0_\theta$ which gives

$$\Big|\frac{d^\ell\, \underline{\widehat{u}}}{d\theta^\ell}\Big|^2 \leq C\, e^{2\theta Re\,\lambda}|\lambda|^{2\ell}(1 + \theta^2\, |\lambda|^2)\, |\underline{\widehat{g}}^0|^2, \;\; \ell \geq 0\,, \tag{4.19a}$$

and

$$\Big|\frac{d^\ell\, \widehat{p}}{d\theta^\ell}\Big|^2 \leq C\, e^{2\theta Re\,\lambda}|\lambda|^{2(\ell+1)}\, |\underline{\widehat{g}}^0|^2, \;\; \ell \geq 0\,. \tag{4.19b}$$

We remark that $\lambda \in \sum_{\phi_0}$, $Re\,\lambda < 0$ and $|\lambda| \geq \lambda_0 > 0$ implies for any $m \geq 1$

$$\int\limits_0^\infty \theta^m\, e^{2\theta\, Re\,\lambda} d\theta \leq C_m\, |\lambda|^{-(m+1)} \tag{4.20}$$

which implies together with (4.19) for $k \geq 2$

$$\int\limits_0^\infty \Big\{\sum_{\ell=0}^k \Big|\frac{d^\ell \underline{\widehat{u}}}{d\theta^\ell}\Big|^2\, |\lambda|^{2(k-\ell)} + \sum_{\ell=0}^{k-1} \Big|\frac{d^\ell \widehat{p}}{d\theta^\ell}\Big|^2\, |\lambda|^{2(k-1-\ell)}\Big\} d\theta \leq C|\lambda|^{2k-1}\, |\underline{\widehat{g}}^0|^2\,. \tag{4.21}$$

For the **Neumann Condition**, $\underline{\widehat{\sigma}}_n(\underline{\widehat{u}}, \widehat{p})|_{\theta=0} = \underline{\widehat{g}}^1 = (\widehat{g}^1_t, \widehat{g}^1_\theta)^\top$, we have $c_1 = -\frac{i\nu}{2\lambda}\, \widehat{g}^1_\theta$, $c_2 = \frac{\nu}{2\lambda}\, (\widehat{g}^1_t + i\widehat{g}^1_\theta)$. Hence we get

$$\Big|\frac{d^\ell\, \underline{\widehat{u}}}{d\theta^\ell}\Big|^2 \leq C\, e^{2\theta Re\,\lambda}|\lambda|^{2(\ell-1)}(1 + |\lambda|^2\theta^2)\, |\underline{\widehat{g}}^1|^2\,, \tag{4.22a}$$

$$\Big|\frac{d^\ell\, \widehat{p}}{d\theta^\ell}\Big|^2 \leq C\, e^{2\theta Re\,\lambda}|\lambda|^{2\ell}\, |\underline{\widehat{g}}^1|^2\,. \tag{4.22b}$$

The bounds (4.22) and (4.20) lead to

$$\int\limits_0^\infty \Big\{\sum_{\ell=0}^k \Big|\frac{d^\ell \underline{\widehat{u}}}{d\theta^\ell}\Big|^2 |\lambda|^{2(k-\ell)} + \sum_{\ell=0}^{k-1} \Big|\frac{d^\ell \widehat{p}}{d\theta^\ell}\Big|^2\, |\lambda|^{2(k-1-\ell)}\Big\} d\theta \leq C\, |\lambda|^{2k-3} |\underline{\widehat{g}}^1|^2\,. \tag{4.23}$$

Combining (4.21) and (4.23), we have shown

Theorem 4.6. *For $\lambda \in \sum_{\phi_0}$ with $|\lambda| \geq \lambda_0 > 0$ sufficiently large, the principal problem (4.18) admits, for $\underline{\widehat{f}} \in H^{k-2}(\mathbb{R}_+)^2$, $\widehat{h} \in H^{k-1}(\mathbb{R}_+)$, $k \geq 2$, and any initial data $\underline{\widehat{g}}^\ell \in \mathbb{C}^2$, $\ell = 0, 1$, a unique solution $\underline{\widehat{u}} \in H^k(\mathbb{R})^2$, $\widehat{p} \in H^{k-1}(\mathbb{R})$ which satisfies the a-priori estimate*

$$\begin{aligned} &|||\underline{\widehat{u}}|||^2_{H^k(\mathbb{R}_+)} + |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R}_+)} \leq \\ &C\, \big\{|||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R}_+)} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R}_+)} + |\lambda|^{2k-1-2\ell}\, |\underline{\widehat{g}}^\ell|^2\big\}\,. \end{aligned} \tag{4.24}$$

Proof: i) For $\underline{\widehat{f}} = \underline{0}$, $\widehat{h} = 0$, we constructed the explicit solution and estimates (4.21), (4.23) lead to (4.24).

ii) For $\underline{\widehat{f}} \neq \underline{0}$, $\widehat{h} \neq 0$, we extend $\underline{\widehat{f}}$, $\widehat{h}$ to all of $\mathbb{R}$ preserving their norms. Theorem 4.3 implies that there is a solution $(\underline{\widehat{u}}_0, \widehat{p}_0) \in H^k(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$ satisfying (4.17) and the a-priori estimate (4.15).

Set now $(\underline{\widehat{v}}, \widehat{q}) := (\underline{\widehat{u}} - \underline{\widehat{u}}_0, \widehat{p} - \widehat{p}_0)$. Then

$$\begin{aligned} \widehat{L}_0(D,\lambda)(\underline{\widehat{v}}, \widehat{q}) &= \underline{0} \text{ in } \mathbb{R}_+\,, \\ \widehat{B}_0(D,\lambda)(\underline{\widehat{v}}, \widehat{q}) &= \begin{cases} \underline{\widehat{g}}^0 - \widehat{B}_0(D,\lambda)\,(\underline{\widehat{u}}_0, \widehat{p}_0) & \text{(Dirichlet) or} \\ \underline{\widehat{g}}^1 - \widehat{B}_0(D,\lambda)\,(\underline{\widehat{u}}_0, \widehat{p}_0)\,. \end{cases} \end{aligned}$$

By part i) of the proof, we have (4.24) for $(\underline{\widehat{v}}, \widehat{q})$, i.e.

$$|||\underline{\widehat{v}}|||^2_{H^k(\mathbb{R})} + |||\widehat{q}|||^2_{H^{k-1}(\mathbb{R})} \leq C \begin{cases} |\lambda|^{2k-1}\, |\underline{\widehat{g}}^0 - \widehat{B}_0(D,\lambda)\,(\underline{\widehat{u}}_0, p_0)|^2 \\ |\lambda|^{2k-3}\, |\underline{\widehat{g}}^1 - \widehat{B}_0(D,\lambda)\,(\underline{\widehat{u}}_0, p_0)|^2\,. \end{cases} \tag{4.25}$$

By Lemma 4.7 ahead, we have

$$\begin{aligned} |\lambda|^{2k-1}\, |\underline{\widehat{u}}_0(0)|^2 &\leq C \left\{ |\lambda|^{2k} \|\underline{\widehat{u}}_0\|^2_{L^2(\mathbb{R})} + |\lambda|^{2(k-1)} \left\| \frac{d\underline{\widehat{u}}_0}{d\theta} \right\|^2_{L^2(\mathbb{R})} \right\} \\ &\leq C\, |||\underline{\widehat{u}}|||^2_{H^k(\mathbb{R})} \\ &\leq C \left\{ |||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R}_+)} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R}_+)} \right\} \end{aligned} \tag{4.26}$$

by (4.15). Similarly, we have

$$\begin{aligned} |\lambda|^{2k-3}\, |\underline{\widehat{u}}_0'(0)|^2 &\leq C \left\{ |\lambda|^{2(k-1)} \|\underline{\widehat{u}}_0'\|^2_{L^2(\mathbb{R})} + |\lambda|^{2(k-2)} \|\underline{\widehat{u}}_0''\|^2_{L^2(\mathbb{R})} \right\} \\ &\leq C\, |||\underline{\widehat{u}}_0|||^2_{H^k(\mathbb{R})} \\ &\leq C \left\{ |||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R}_+)} + \|\widehat{h}|||^2_{H^{k-1}(\mathbb{R}_+)} \right\}, \end{aligned} \tag{4.27a}$$

again by (4.15), and

$$\begin{aligned} |\lambda|^{2k-3}\, |\widehat{p}_0(0)|^2 &\leq C \left\{ |\lambda|^{2(k-1)} \|\widehat{p}_0\|^2_{L^2(\mathbb{R})} + |\lambda|^{2(k-2)} \|\widehat{p}_0'\|^2_{L^2(\mathbb{R})} \right\} \\ &\leq C\, |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R})} \\ &\leq C \left\{ |||\underline{\widehat{f}}|||^2_{H^{k-2}(\mathbb{R}_+)} + |||\widehat{h}|||^2_{H^{k-1}(\mathbb{R}_+)} \right\}. \end{aligned} \tag{4.27b}$$

Now (4.25) - (4.27) yield the estimate (4.24) in the general case. □

It remains to prove

Lemma 4.7. *Let complex-valued functions $v \in H^1(I)$ be given where $I = \mathbb{R}, \mathbb{R}_+$ or $(0,\omega)$. Then for every $\lambda_0 > 0$ such that for $|\lambda| > \lambda_0$ holds*

$$|\lambda|\; |v(0)|^2 \leq C \big\{ \|v\|^2_{H^1(I)} + |\lambda|^2\, \|v\|^2_{L^2(I)} \big\} = C\, |||v|||^2_{H^1(I)}\,.$$

Proof: By the embedding theorem, $v \in C(I)$, and

$$(v(0))^2 = (v(x))^2 + \int_0^x 2v'(t)\, v(t) dt\,.$$

This immediately gives for $|\lambda| > \lambda_0$

$$\begin{aligned} |\lambda|\, |v(0)|^2 \;&\le\; |\lambda| \int_I |v(x)|^2\, dx + \int_I (|v'(x)|^2 + |\lambda|^2\, |v(x)|^2) dx \\ &\le\; C \left\{ |\lambda|^2\, \|v\|^2_{L^2(I)} + \|v'\|^2_{L^2(I)} \right\}. \end{aligned}$$

□

We now consider the system

$$\widehat{L}(D,\lambda)(\widehat{\underline{u}},\widehat{p}) \;=\; (\widehat{\underline{f}},\widehat{h}) \;\text{ on }\; I = \mathbb{R}_+ = (0,\infty)\,, \tag{4.28a}$$

$$\widehat{B}(D,\lambda)(\widehat{\underline{u}},\widehat{p}) \;=\; (\widehat{\underline{g}}^0,\widehat{\underline{g}}^1)\,. \tag{4.28b}$$

Theorem 4.8. *For $\lambda \in \sum_{\phi_0}$ with $|\lambda| \ge \lambda_0 > 0$ for sufficiently large $\lambda_0 > 0$,* (4.28) *has a unique solution $(\widehat{\underline{u}},\widehat{p}) \in H^k(\mathbb{R}_+)^2 \times H^{k-1}(\mathbb{R}_+)$ for any $(\widehat{\underline{f}},\widehat{h}) \in H^{k-2}(\mathbb{R}_+)^2 \times H^{k-1}(\mathbb{R}_+)$, $k \ge 2$, and the a-priori estimate* (4.24) *holds.*

Proof: The solution for (4.28) can be constructed as the one for the principal system (4.18). To prove the estimate (4.24), we need to show that there is $\lambda_0 > 0$ such that for $\lambda \in \sum_{\phi_0}$ with $|\lambda| > \lambda_0$ holds

$$\begin{aligned} &|||((\widehat{L}(D,\lambda) - \widehat{L}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R}_+)} + \\ &|||((\widehat{L}(D,\lambda) - \widehat{L}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p}))_3|||^2_{H^{k-1}(\mathbb{R}_+)} + |\lambda|^{2k-3}|(\widehat{B}(D,\lambda) - \widehat{B}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p})|^2 \\ &\le \frac{1}{2C}\left(|||\widehat{\underline{u}}|||^2_{H^k(\mathbb{R}_+)} + |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R}_+)}\right). \end{aligned}$$

As shown in Lemma 4.4,

$$|||(\widehat{L}(D,\lambda) - \widehat{L}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p})|||^2_{H^{k-2}(\mathbb{R}_+)} \le C\left\{|||\widehat{\underline{u}}|||^2_{H^{k-1}(\mathbb{R}_+)} + |||\widehat{p}|||^2_{H^{k-2}(\mathbb{R}_+)}\right\}$$

and

$$\begin{aligned} |(\widehat{B}(D,\lambda) - \widehat{B}_0(D,\lambda))\,(\widehat{\underline{u}},\widehat{p})|^2 \;&=\; 0 && \text{for Dirichlet boundary conditions}\,, \\ |(\widehat{B}(D,\lambda) - \widehat{B}_0(D,\lambda))\,(\widehat{\underline{u}},\widehat{p})|^2 \;&=\; \nu|\widehat{\underline{u}}(0)|^2 && \text{for Neumann boundary conditions}\,. \end{aligned}$$

By Lemma 4.7,

$$\begin{aligned} |\lambda|^{2k-3}|(\widehat{B}(D,\lambda) - \widehat{B}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p})|^2 \;&\le\; C\,|\lambda|^{2(k-2)}\left(|\lambda|^2\, \|\widehat{\underline{u}}\|^2_{L^2(\mathbb{R}_+)} + \|\widehat{\underline{u}}\|^2_{H^1(\mathbb{R}_+)}\right) \\ &\le\; C\,|||\widehat{\underline{u}}|||^2_{H^{k-1}(\mathbb{R}_+)}\,. \end{aligned}$$

Therefore, for $|\lambda| \ge \lambda_0 > 0$ with λ_0 sufficiently large

$$\begin{aligned} &|||((\widehat{L}(D,\lambda) - \widehat{L}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R}_+)} + \\ &|||((\widehat{L}(D,\lambda) - \widehat{L}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p}))_3|||^2_{H^{k-1}(\mathbb{R}_+)} + |\lambda|^{2k-3}|(\widehat{B}(D,\lambda) - \widehat{B}_0(D,\lambda))(\widehat{\underline{u}},\widehat{p})|^2 \\ &\le C\left\{|||\widehat{\underline{u}}|||^2_{H^{k-1}(\mathbb{R}_+)} + |||\widehat{p}|||^2_{H^{k-2}(\mathbb{R}_+)}\right\} \le \frac{1}{2}\left\{|||\widehat{\underline{u}}|||^2_{H^k(\mathbb{R}_+)} + |||\widehat{p}|||^2_{H^{k-1}(\mathbb{R}_+)}\right\}. \end{aligned}$$

□

4.4 A-priori estimate on the interval $I = (0, \omega)$

We prove the a-priori estimate (4.24) by a localization argument and by using Theorems 4.5 and 4.8.

Theorem 4.9. *There exists $\lambda_0 > 0$ such that for $\lambda \in \sum_{\phi_0}$, $|\lambda| > \lambda_0$, and any $k \geq 1$, and for $(\widehat{\underline{f}}, \widehat{h}) \in H^{k-1}(I)^2 \times H^{k-1}(I)$ the system*

$$\widehat{L}(D,\lambda)(\widehat{\underline{u}}, \widehat{p}) = (\widehat{\underline{f}}, \widehat{h}) \quad on \quad I = (0,\omega)\,, \tag{4.29a}$$

$$\widehat{B}(D,\lambda)(\widehat{\underline{u}}, \widehat{p}) = (\widehat{\underline{g}}^0, \widehat{\underline{g}}^1) \tag{4.29b}$$

has a unique solution $(\widehat{\underline{u}}, \widehat{p}) \in H^k(I)^2 \times H^{k-1}(I)$ and the a-priori estimate (4.24) holds with $I = (0,\omega)$ in place of $\mathbb{R}_+$.

Proof: We deal with the problem with mixed type boundary condition. The Neumann or the Dirichlet problem are similar. Let $\{I_i\}_{i=1}^n$ be a covering $\overline{I} = [0,\omega]$ and let $\{\varphi_i\}_{i=1}^n$ be a subordinate partition of unity, $\sum_{i=1}^n \varphi_i(x) = 1 \ \forall x \in \overline{I}$. Let further $(\widehat{\underline{u}}_i, \widehat{p}_i) := (\varphi_i\, \widehat{\underline{u}},\ \varphi_i\, \widehat{p})$ with support $\overline{I}_i$ and $I_i \subset I$, $2 \leq i \leq n-1$. Then $(\widehat{\underline{u}}_i, \widehat{p}_i)$ satisfies, for $i = 1, \dots, n$,

$$\widehat{L}(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i) + \widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i) = (\varphi_i\, \widehat{\underline{f}}, \varphi_i\, \widehat{h}) \quad \text{on} \quad I\,, \tag{4.30a}$$

$$\widehat{B}(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i) + \widehat{B}_i(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i) = (\widehat{\underline{g}}_i^0, \widehat{\underline{g}}_i^1)\,, \tag{4.30b}$$

where $\widehat{L}_i$ are matrix differential operators of one degree lower than $\widehat{L}$, and

$$|||\widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i)|||^2_{H^{k-2}(I)} \leq C\left\{|||\widehat{\underline{u}}_i|||^2_{H^{k-1}(I)} + |||\widehat{p}_i|||^2_{H^{k-2}(I)}\right\}, \tag{4.31}$$

where $\widehat{B}_i(D,\lambda) \equiv 0$ at $\theta = 0$, resp. $\widehat{B}_i(D,\lambda)$ is a boundary operator of one order lower than $\widehat{B}(D,\lambda)$ for Neumann conditions, and by Lemma 4.7

$$|\lambda|\,|\widehat{B}_1(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p})|^2 \leq C\,|\lambda|\,|\widehat{\underline{u}}_i(\omega)|^2 \leq C\big\{\|\widehat{\underline{u}}_i\|^2_{H^1(I)} + |\lambda|^2\, \|\widehat{\underline{u}}_i\|^2_{L^2(I)}\big\} \tag{4.32}$$

with $\widehat{\underline{g}}_i^\ell = 0$, $2 \leq i \leq n-1$ and $\widehat{\underline{g}}_i^\ell = \widehat{\underline{g}}^\ell$, $i = 1, n$, $\ell = 0, 1$.

For $i = 2, \dots, n-1$, the system (4.30) can be extended to the whole line $\mathbb{R}$, and for $i = 1, n$ to the half-line $\mathbb{R}_+$. According to Theorem 4.5 and 4.8, $(\widehat{\underline{u}}_i, \widehat{p}_i)$ are the unique solutions of these extended problems and there exists $\lambda_0 > 0$ such that for $\lambda \in \sum_{\phi_0}$, $|\lambda| > \lambda_0$ and any $k \geq 2$ the a-priori estimate

$$\begin{aligned} |||\widehat{\underline{u}}_i|||^2_{H^k(J)} + |||\widehat{p}_i|||^2_{H^{k-1}(J)} \leq\ & C\,\{|||\widehat{\underline{f}}_i|||^2_{H^{k-2}(J)} + |||\widehat{h}_i|||^2_{H^{k-1}(J)} + |||(\widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, p_i))_{12}|||^2_{H^{k-2}(I)} \\ & + |||(\widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i))_3|||^2_{H^{k-1}(J)} \\ & + |\lambda|^{2k-1}\, |\widehat{\underline{g}}_i^0|^2 + |\lambda|^{2k-3}(|\widehat{\underline{g}}_i^1|^2 + |\widehat{B}_{1i}(D,\lambda)(\widehat{\underline{u}}_i, \widehat{p}_i)|^2)\big\} \end{aligned}$$

holds with $J = \mathbb{R}$ if $2 \leq i \leq n-1$, $J = \mathbb{R}_+$ if $i = 1, n$.

By (4.31), (4.32), we may estimate for any $k \geq 1$

$$\begin{aligned} |||\widehat{\underline{u}}_i|||^2_{H^k(J)} + |||\widehat{p}_i|||^2_{H^{k-1}(J)} \leq\ & C\,\{|||\widehat{\underline{f}}|||^2_{H^{k-2}(J)} + |||\widehat{h}|||^2_{H^{k-1}(J)} + |||(\widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, p_i))_{12}|||^2_{H^{k-2}(I)} \\ & + |||(\widehat{L}_i(D,\lambda)(\widehat{\underline{u}}_i, p_i))_3|||^2_{H^{k-1}(I)} + |\lambda|^{2k-1}\, |\widehat{\underline{g}}^0|^2 + |\lambda|^{2k-3}\, |\widehat{\underline{g}}^1|^2\}\,. \end{aligned}$$

Selecting $\lambda_0 > 0$ so large that $C/\lambda_0 < \frac{1}{2}$, we get

$$|||\widehat{\underline{u}}_i|||^2_{H^k(J)} + |||p_i|||^2_{H^{k-1}(J)} \le C\,\{|||\widehat{\underline{f}}|||^2_{H^{k-2}(J)} + |||\widehat{h}|||^2_{H^{k-1}(J)} + |\lambda|^{2k-1}\,|\widehat{\underline{g}}^0|^2 + |\lambda|^{2k-3}\,|\widehat{\underline{g}}^1|^2\}\,.$$

Summation over i gives

$$\begin{aligned} |||\widehat{\underline{u}}|||^2_{H^k(I)} + |||\widehat{p}|||^2_{H^{k-1}(I)} &\le C\sum_{i=1}^{n}\left(|||\widehat{\underline{u}}_i|||^2_{H^k(J)} + |||\widehat{p}_i|||^2_{H^{k-1}(J)}\right) \\ &\le C\,|||\widehat{\underline{f}}|||^2_{H^{k-2}(I)} + |||\widehat{h}|||^2_{H^{k-1}(I)} + \sum_{\ell=0}^{1}|\lambda|^{2(k-\ell)-1}\,|\widehat{\underline{g}}^\ell|^2\Big)\,. \end{aligned}$$

□

For Neumann or respectively Dirichlet boundary conditions on ∂I, the above estimate holds with the term $\widehat{\underline{g}}^0$ or $\widehat{\underline{g}}^1$ omitted in the upper bound.

This establishes the a-priori estimate (4.24) on $I = (0,\omega)$ for $\lambda \in \sum_{\phi_0}$, $|\lambda| > \lambda_0$, λ_0 sufficiently large. To obtain it also in the bounded set $|\lambda| \le \lambda_0$, we must investigate the poles of the resolvent.

4.5 Poles of the resolvent $R(\lambda)$ near the real line

Let $\mathcal{U}(\lambda) := [\widehat{L}(D,\lambda),\, \widehat{B}(D,\lambda)]$ denote the operator pencil in (4.9) which depends polynomially on the complex parameter λ. Arguing as in [1], $\mathcal{U}(\lambda) : H^k(I)^2 \times H^{k-1}(I) \to H^{k-2}(I)^2 \times H^{k-1}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$ realizes an isomorphism for all $\lambda \in \mathbb{C}$ except at certain isolated points. Consequently, the resolvent $\mathcal{R}(\lambda) = \mathcal{U}(\lambda)^{-1}$ is an operator-valued, meromorphic function of λ with poles of finite multiplicity. The set of all poles of $R(\lambda)$ (resp. of eigenvalues of $\mathcal{U}(\lambda)$) shall be denoted by $\Lambda \subset \mathbb{C}$.

The eigenvalues λ are such that the homogeneous equation

$$\mathcal{U}(\lambda)(\widehat{\underline{v}}, \widehat{q}) = \underline{0}$$

admits nontrivial solutions $(\widehat{\underline{v}}, \widehat{q})$, the corresponding eigenfunctions.

Let $\lambda = iz$. Then, according to the theory in [21], [13], λ is an eigenvalue of $\mathcal{U}(\lambda)$ if and only if z is a root of the transcendental equations

$$\sin^2(z\omega) \;=\; z^2\sin^2(\omega) \quad \text{(Dirichlet or Neumann)}\,, \tag{4.33a}$$

$$\cos^2(z\omega) \;=\; z^2\sin^2(\omega) \quad \text{(Mixed type)}\,. \tag{4.33b}$$

In [13], it has been shown that $z = 0$ is not an eigenvalue and that $\mathcal{R}(\lambda)$ has no pole on the real line for the problem with Dirichlet and mixed boundary conditions, and that $\lambda = 0$ is the only pole on the real line with multiplicity 2 for the homogeneous Neumann problem, with corresponding eigenfunctions

$$\underline{e}_1 = (\cos\theta, -\sin\theta)^\top,\;\; \underline{e}_2 = (\sin\theta, \cos\theta)^\top\,.$$

By (4.33), for any eigenvalue $\lambda \in \mathbb{C}$, also $\overline{\lambda}$ is an eigenvalue. We denote by κ_1 the smallest positive imaginary part of the nonzero eigenvalues with positive imaginary part, and by $\mathcal{J}_h$ the strip $\{\lambda \in \mathbb{C}\,|\, -h < Im\,\lambda < h\}$ with $0 < h < \kappa_1$. Then, $\mathcal{R}(\lambda)$ has no poles in $\mathcal{J}_h$ for the Dirichlet and the mixed boundary conditions and $\lambda = 0$ is the only pole of $\mathcal{R}(\lambda)$ in $\mathcal{J}_h$ for the Neumann problem.

Remark 4.10. If the Neumann boundary conditions (3.26) are adopted, (4.33a) is replaced by

$$\sin^2(z\omega) = 9z^2 \sin^2 \omega \,. \tag{4.33c}$$

We shall next establish (4.13) for $\lambda = \xi + ih$ with $-\infty < \xi < \infty$ and $h \in (0, \kappa_1)$.

Evidently, $z = \pm 1$ is a root of (4.33a) in the Neumann case with corresponding eigenfunction $\underline{u}^* = (x_2, -x_1)^\top$, the velocity field of a swirl flow. This (physical) eigensolution is absent in (4.33c), since $\underline{u}^*$ does not satisfy (3.26) with $\underline{g}^1 = \underline{0}$.

Theorem 4.11. *Let $\mathcal{L}_h = \{\lambda \in \mathbb{C} : Im\,\lambda = h\}$. If $\mathcal{R}(\lambda)$ has no poles on the line $\mathcal{L}_h$, the system (4.6), (4.7) admits a unique solution $(\widehat{\underline{u}}, \widehat{p}) \in H^k(I) \times H^{k-1}(I)$ provided $(\widehat{\underline{f}}, \widehat{h}, \widehat{\underline{g}}^0, \widehat{\underline{g}}^1) \in H^{k-2}(I) \times H^{k-1}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$, and it holds for all $\lambda \in \mathcal{L}_h$:*

$$|||\widehat{\underline{u}}|||^2_{H^k(I)} + |||\widehat{p}|||^2_{H^{k-1}(I)} \le C \left\{ |||\widehat{\underline{f}}|||^2_{H^{k-2}(I)} + |||\widehat{h}|||^2_{H^{k-1}(I)} + \sum_{\ell=0,1} |\lambda|^{2k-2\ell-1} \, |\widehat{\underline{g}}^\ell|^2 \right\} \tag{4.34}$$

with C independent of ξ.

Proof: Since the line $\mathcal{L}_h$ is free of poles of the resolvent $\mathcal{R}(\lambda)$, $\lambda = \xi + ih$, the solution $(\widehat{\underline{u}}, \widehat{p}) \in H^k(I) \times H^{k-1}(I)$ exists if $\widehat{\underline{f}}, \widehat{h}$ and $\widehat{\underline{g}}^\ell$ are in $H^{k-2}(I) \times H^{k-1}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$. By Theorem 4.9, there exists $\lambda_0 > 0$ such that for $\lambda \in \mathcal{L}_h$, $|\lambda| \ge \lambda_0$, (4.34) holds. For $\lambda \in \mathcal{L}_h$, $|\lambda| \le \lambda_0$, by assumption $\mathcal{R}(\lambda)$ is bounded, i.e. for any $k \ge 2$ the a-priori estimates

$$\|\widehat{\underline{u}}\|^2_{H^k(I)} + \|\widehat{p}\|^2_{H^{k-1}(I)} \le C \left\{ \|\widehat{\underline{f}}\|^2_{H^{k-2}(I)} + \|\widehat{h}\|^2_{H^{k-1}(I)} + \|\widehat{h}\|^2_{H^{k-1}(I)} + \sum_{\ell=0,1} |\lambda|^{2(k-\ell)-1} \, |\widehat{\underline{g}}^\ell|^2 \right\}$$

and, for $\lambda \in \mathcal{L}_h$, $|\lambda| \le \lambda_0$:

$$\begin{aligned} &|\lambda|^{2k} \|\widehat{u}\|^2_{L^2(I)} + |\lambda|^{2(k-1)} \|\widehat{p}\|^2_{L^2(I)} \\ &\le \widetilde{C} \left\{ |\lambda|^{2(k-2)} \|\widehat{\underline{f}}\|^2_{L^2(I)} + |\lambda|^{2(k-1)} \|\widehat{h}\|^2_{L^2(I)} + \|\widehat{h}\|^2_{H^{k-1}(I)} + \sum_{\ell=0,1} |\lambda|^{2(k-\ell)-1} |\widehat{g}^\ell|^2 \right\} \end{aligned}$$

hold with $\widetilde{C}$ depending λ_0 but not on $|\lambda|$. Combining these two inequalities leads to (4.34) and completes the proof. □

4.6 Regularity of the Stokes problem in the infinite sector

We now prove the regularity for the Stokes problem (4.1), (4.2) in the infinite sector Q. We will employ weighted spaces $W^k_\beta(Q)$ of Kondratev-type on Q, and also weighted spaces $H^k_h(D)$ over the infinite strip D. These spaces are equipped with norms given by

$$\|v\|^2_{W^k_\beta(Q)} = \sum_{|\alpha| \le k} \|r^{\beta+\alpha_1-k} \mathcal{D}^\alpha v\|^2_{L^2(Q)}, \quad k \ge 0 \,,$$

where $\mathcal{D}^\alpha v$ is as in Section 2, and

$$\|\widetilde{v}\|^2_{H^k_h(D)} := \sum_{|\alpha| \le k} \|e^{ht} \, \mathcal{D}^\alpha \widetilde{v}\|^2_{L^2(D)}, \quad k \ge 0 \,,$$

where $\mathcal{D}^\alpha \widetilde{v} := \dfrac{\partial^{|\alpha|} \widetilde{v}}{\partial t^{\alpha_1} \, \partial \theta^{\alpha_2}}$. We need the following lemmas from [2]:

Lemma 4.12. *If $v(r,\theta) \in W^k_\beta(Q)$, $k \geq 0$, then $\overline{v}(t,\theta) := v(e^{-t},\theta) \in H^k_h(D)$ with $h = k-1-\beta$, and*

$$C_1 \, \|\widetilde{v}\|_{H^k_h(D)} \leq \|v\|_{W^k_\beta(Q)} \leq C_2 \, \|\widetilde{v}\|_{H^k_h(D)} \,. \tag{4.35}$$

Moreover, for $0 \leq \ell \leq 1$, $\widetilde{v}_\ell(t,\theta) = e^{(\ell-2)t}\, \widetilde{v}(e^{-t},\theta) \in H^k_h(D)$, with $h = k+1-\ell-\beta$, and

$$C_1 \, \|\widetilde{v}_\ell\|_{H^k_h(D)} \leq \|v_\ell\|_{W^k_\beta(Q)} \leq C_2 \, \|\widetilde{v}_\ell\|_{H^k_h(D)} \,. \tag{4.36}$$

Here C_1 and C_2 are independent of $v, \widetilde{v}$.

Lemma 4.13. *Define $D = \mathbb{R} \times I = (-\infty,\infty) \times (0,\omega)$, and let $\widetilde{v} \in H^k_h(D)$, $k \geq 0$. Then $\widehat{v}(\lambda,\cdot) = \mathcal{F}(\widetilde{v}) \in H^k(I)$, and*

$$C_1 \, \|\widetilde{v}\|_{H^k_h(D)} \leq \int\limits_{-\infty+ih}^{\infty+ih} |||\widehat{v}|||_{H^k(I)} \, d\lambda \leq C_2 \, \|\widetilde{v}\|_{H^k_h(D)} \,, \tag{4.37}$$

where the positive constants C_1 and C_2 are independent of $\widetilde{v}$.

For the proof of Lemmas 4.12 and 4.13, we refer to Lemmas 2.3 and 2.6 of [2].

Lemma 4.14. *Let $\overline{\underline{G}}^\ell(r,\theta) \in W^{k-\ell}_\beta(Q)^2$ with $\overline{\underline{G}}^\ell|_{\Gamma^\ell} = \overline{\underline{g}}^\ell$, $\ell = 0,1$, and let $\widehat{\underline{G}}^\ell = \mathcal{F}(\widetilde{\underline{G}}^\ell)$, with $\widetilde{\underline{G}}^\ell = e^{-\ell t}\, \overline{\underline{G}}(e^{-t},\theta)$. Then we have the a-priori estimate*

$$|\lambda|^{2(k-\ell-\frac{1}{2})} |\widehat{\underline{g}}^\ell|^2 \leq C \, |||\widehat{\underline{G}}^\ell|||_{H^{k-\ell}(I)} \,, \quad \ell = 0,1, \; k \geq 2 \,. \tag{4.38}$$

Proof: By Lemmas 4.12, 4.13, $\widehat{\underline{G}}^\ell \in H^{2-\ell}(I) \subset C^0(\overline{I})$. We have, by Lemma 4.7,

$$|\lambda|^{2k-3} \, |\widehat{\underline{g}}^1|^2 \leq C \, |\lambda|^{2(k-2)} |||\widehat{\underline{G}}^1|||^2_{H^1(I)} \leq C \, |||\widehat{\underline{G}}^1|||^2_{H^{k-1}(I)}$$

and

$$|\lambda|^{2k-1} \, |\widehat{\underline{g}}^0|^2 \leq C \, |\lambda|^{2(k-1)} |||\widehat{\underline{G}}^0|||^2_{H^1(I)} \leq C \, |||\widehat{\underline{G}}^0|||^2_{H^k(I)} \,,$$

which leads to (4.38). □

Theorem 4.15. *Let $\overline{\underline{f}} \in W^{k-2}_\beta(Q)^2$, $\overline{h} \in W^{k-1}_\beta(Q)$ and let $\overline{\underline{g}}^\ell \in W^{k-\ell-\frac{1}{2}}_\beta(\Gamma^\ell)^2$, $\ell = 0,1$, with $k \geq 2$. If $\mathcal{R}(\lambda)$ has no pole on the line $\mathcal{L}_h = \{\lambda : Im\,\lambda = h\}$ with $h = k-1-\beta$, then the Stokes problem (4.3), (4.4) has a unique solution $(\overline{\underline{u}}, \overline{p}) \in W^k_\beta(Q)^2 \times W^{k-1}_\beta(Q)$ and*

$$\|\overline{\underline{u}}\|_{W^k_\beta(Q)} + \|\overline{p}\|_{W^{k-1}_\beta(Q)} \leq C \left\{ \|\overline{\underline{f}}\|_{W^{k-2}_\beta(Q)} + \|\overline{h}\|_{W^{k-1}_\beta(Q)} + \sum_{\ell=0,1} \|\overline{\underline{g}}^\ell\|_{W^{k-\ell-\frac{1}{2}}(\Gamma^\ell)} \right\}. \tag{4.39}$$

Proof: By the definition of $W^{k-\ell-1/2}_\beta(\Gamma^\ell)$, there exists $\overline{\underline{G}}^\ell \in W^{k-\ell}_\beta(Q)^2$, $\ell = 0,1$, such that $\overline{\underline{G}}^\ell|_{\Gamma^\ell} = \overline{\underline{g}}^\ell$, and, for $\ell = 0,1$,

$$\frac{1}{2} \, \|\overline{\underline{G}}^\ell\|_{W^{k-\ell}_\beta(Q)} \leq \|\overline{\underline{g}}^\ell\|_{W^{k-\ell-\frac{1}{2}}_\beta(Q)} \leq \|\overline{\underline{G}}^\ell\|_{W^{k-\ell}_\beta(Q)} \,. \tag{4.40}$$

Due to Lemmas 4.12 and 4.13, the partial Fourier transforms $\widehat{\underline{f}} \in H^{k-2}(I)^2$, $\widehat{h} \in H^{k-1}(I)$ and $\widehat{\underline{G}}^{\ell} \in H^{k-\ell}(I)^2$, and (4.36), (4.37) hold. By Theorem 4.11, the system (4.29) has a unique solution $(\widehat{\underline{u}}, \widehat{p}) \in H^k(I)^2 \times H^{k-1}(I)$ for $k \geq 2$ and

$$|||\widehat{\underline{u}}|||^2_{H^k(I)} + |||\widehat{p}|||^2_{H^{k-1}(I)} \leq C \left\{ |||\widehat{\underline{f}}|||^2_{H^{k-2}(I)} + |||\widehat{h}|||^2_{H^{k-1}(I)} + \sum_{\ell=0,1} |\lambda|^{2(k-\ell)-1} |\widehat{\underline{g}}^{\ell}|^2 \right\},$$

where $\widehat{\underline{g}}^{\ell} = \widehat{G}^{\ell}|_{\theta=0}$ or $\widehat{\underline{G}}^{\ell}|_{\theta=\omega}$, and by Lemma 4.14 we have

$$|||\widehat{\underline{u}}|||^2_{H^k(I)} + |||\widehat{p}|||^2_{H^{k-1}(I)} \leq C \left\{ |||\widehat{\underline{f}}|||^2_{H^{k-2}(I)} + |||\widehat{h}|||^2_{H^{k-1}(I)} + \sum_{\ell=0,1} |||\widehat{\underline{G}}|||^2_{H^{k-\ell}(I)} \right\}.$$

Since $\mathcal{R}(\lambda)$ has no pole on the line $\mathcal{L}_h = \{\lambda : Im\,\lambda = h\}$ with $h = k - 1 - \beta$, the solution

$$(\widetilde{\underline{u}}, \widetilde{p}) := \mathcal{F}^{-1}(\widehat{\underline{u}}, \widehat{p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ih}^{\infty+ih} e^{i\lambda t} (\widehat{\underline{u}}, \widehat{p}) d\lambda$$

of the system (4.5), (4.6) is in $H^k_h(D)$ and by Lemma 4.12, 4.13 there holds

$$\|\widetilde{\underline{u}}\|^2_{H^k_h(D)} + \|\widetilde{p}\|^2_{H^{k-1}_h(D)} \leq C \left\{ \|\widetilde{\underline{f}}\|^2_{H^{k-2}_h(D)} + \|\widetilde{h}\|^2_{H^{k-1}_h(D)} + \sum_{\ell=0,1} \|\widetilde{G}^{\ell}\|_{H^{k-\ell}_h(D)} \right\}.$$

Consequently, $(\overline{\underline{u}}, \overline{p}) \in W^k_\beta(Q) \times W^{k-1}_\beta(Q)$ is the unique solution of the Stokes problem (4.3), (4.4) and it follows from (4.35), (4.36) that for any $k \geq 2$

$$\|\overline{\underline{u}}\|^2_{W^k_\beta(Q)} + \|\overline{p}\|^2_{W^{k-1}_\beta(Q)} \leq C(k) \left\{ \|\overline{\underline{f}}\|^2_{W^{k-2}_\beta(Q)} + \|\overline{h}\|^2_{W^{k-1}_\beta(Q)} + \sum_{\ell=0,1} \|\overline{\underline{G}}^{\ell}\|^2_{W^{k-\ell}_\beta(Q)} \right\} \tag{4.41}$$

which, together with (4.40), gives (4.39). □

Remark 4.16. The shift Theorem 4.15 is valid for $k \geq 2$, but we shall use it in the following mainly for $k = 2$. For $k > 2$, Theorem 4.15 is not applicable in practice since the regularity of the data in the right hand side of (4.41) implies, for $k > 2$, unrealistic compatibility conditions near the origin of Q. Therefore, we describe regularity in $W^2_\beta(Q)$ and in $H^k_\beta(Q)$ for $k > 2$ and for $\beta > 1 - \kappa_1$.

For the Stokes problem (4.1), (4.2) in Cartesian coordinates we have the following theorem.

Theorem 4.17. *The Stokes problem* (4.1), (4.2) *in the sector has a unique solution* $(\underline{u}, p) \in W^2_\beta(Q)^2 \times W^1_\beta(Q)$ *if* $\underline{f} \in W^0_\beta(Q)^2$, $h \in W^1_\beta(Q)$ *and* $\underline{g}^{\ell} \in W_\beta^{\frac{3}{2}-\ell}(\Gamma^{\ell})$, $\ell = 0, 1$, *provided that* $\beta > 1 - \kappa_1$, *and there holds the a-priori estimate*

$$\|\underline{u}\|^2_{W^2_\beta(Q)} + \|p\|^2_{W^1_\beta(Q)} \leq C \left\{ \|\underline{f}\|^2_{W^0_\beta(Q)} + \|h\|^2_{W^1_\beta(Q)} + \sum_{\ell=0,1} \|\underline{g}^{\ell}\|^2_{W_\beta^{\frac{3}{2}-\ell}(\Gamma^{\ell})} \right\}. \tag{4.42}$$

Proof: We start from (4.35) with $k = 2$ and pass from polar to Cartesian coordinates via $\underline{u} = \boldsymbol{A}^{-1}\,\overline{\underline{u}}$. Under such transformation, (4.42) follows from (4.41) as in [13], Corollary 4.2. □

5 Regularity of the Stokes problem on the polygon Ω

We discuss now the regularity of the weak solution $(\underline{u},p)$ of (3.1) over the polygonal domain Ω with data $\underline{f}\in L_\beta(\Omega)^2$, $h\in H^1_\beta(\Omega)$, $\underline{G}^\ell\in H^{2-\ell}_\beta(\Omega)$, $\ell=0,1$, and in particular the relation between the weak solution and the solution of the problem (4.1), (4.2) over the infinite sector Q with data $\underline{f},h,\underline{G}^\ell$ of bounded support in Q. This is then used to prove analytic regularity results for the Stokes problem (3.1), (3.2) in the scale of countably normed spaces. The analysis parallels [13] in the case of elasticity problems.

Assume therefore now that A_1 coincides with the origin and that Γ_1 coincides with the positive x_1-axis. Let (r,θ) denote polar coordinates centered at A_1 and let $S_\delta=\{(r,\theta):0<r<\delta,0,\ 0<\theta<\omega_1\}\subset\Omega$. By $\phi_\delta(r)$ we denote a cut-off function in $C^\infty(\mathbb{R})$ such that $\phi_\delta\equiv 1$ for $0<r<\delta/2$ and $\phi_\delta\equiv 0$ for $r>\delta$. Define $(\widetilde{\underline{u}},\widetilde{p})=\phi_\delta(\underline{u}-\underline{u}(A_1),p)$ where $(\underline{u},p)$ is the weak solution of (3.1), (3.2) with $\underline{f}\in L_\beta(\Omega)^2$, $h\in H^{1,1}_\beta(\Omega)$, $\underline{g}^\ell\in H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)$, $\ell=0,1$. By zero extension outside of S_δ, $(\widetilde{\underline{u}},\widetilde{p})$ is defined in the infinite sector $Q=\{(r,\theta)|0<r<\infty,$ $0<\theta<\omega_1\}$, and satisfies

$$L(\widetilde{\underline{u}},\widetilde{p})=\phi_\delta(\underline{f},h)+L_1(\underline{u},p,\phi_\delta)=:(\widetilde{\underline{f}},\widetilde{h}) \tag{5.1}$$

where L is the Stokes operator and

$$\widetilde{\underline{u}}\big|_{\theta=0,\omega}=\phi_\delta\,\underline{g}^0=:\widetilde{\underline{g}}^0\ \text{ if }\ \Gamma_1\cup\Gamma_M\subset\Gamma^0 \tag{5.2a}$$

$$\underline{\sigma}_n(\widetilde{\underline{u}},\widetilde{p})\big|_{\theta=0,\omega}=\phi_\delta\,\underline{g}^1+\ell_1(\underline{u},p,\phi_\delta)=\widetilde{\underline{g}}^1\ \text{ if }\ \Gamma_1\cup\Gamma_M\subset\Gamma^1\,, \tag{5.2b}$$

$$\widetilde{\underline{u}}\big|_{\theta=0}=\phi_\delta\,\underline{g}^0=:\widetilde{\underline{g}}^0,\ \underline{\sigma}_n(\widetilde{\underline{u}},\widetilde{p})\big|_{\theta=\omega}=\phi_\delta\,\widetilde{g}^1_\omega+\ell_1(\underline{u},p,\phi_\delta)=:\widetilde{g}^1_\omega\ \text{ if }\ \Gamma_1\subset\Gamma^0,\ \Gamma_M\subset\Gamma^1\,; \tag{5.2c}$$

here L_1 and ℓ_1 are lower order differential operators.

Consider next the solution $(\underline{v},q)$ of the Stokes problem (5.1), (5.2) in an infinite sector Q in the weighted spaces $W^2_\beta(Q)^2\times W^1_\beta(Q)$. We shall analyze the relation between $(\underline{v},q)$ and the weak solution $(\underline{u},p)\in H^1(\Omega)^2\times L^2(\Omega)$.

5.1 Relation between weak solutions in the polygon and in the infinite sector

We begin by observing that $\widetilde{\underline{f}}$ and $\widetilde{\underline{g}}^\ell$ have bounded support in Q and that there holds

$$\begin{aligned}
\|\widetilde{\underline{f}}\|_{L^2_\beta(Q)} &\le C\left\{\|\underline{f}\|_{L_\beta(S_\delta)}+\|(L_1(\underline{u},p,\phi_\delta))_{12}\|_{L_\beta(Q)}\right\}\\
&\le C\left\{\|\underline{f}\|_{L_\beta(S_\delta)}+\|\underline{u}\|_{H^1(S_\delta\backslash S_{\delta/2})}+\|p\|_{L^2(S_\delta\backslash S_{\delta/2})}\right\}\\
\|\widehat{h}\|_{W^1_\beta(Q)} &\le C\left\{\|h\|_{W^1_\beta(S_\delta)}+\|(L_1(\underline{u},p,\phi_\delta))_3\|_{W^1_\beta(Q)}\right\},
\end{aligned} \tag{5.3}$$

and

$$\|\widetilde{\underline{g}}^0\|_{W_\beta^{\frac{3}{2}}(\widetilde{\Gamma}^0)}\le C\,\|\underline{g}^0\|_{W_\beta^{\frac{3}{2}}(\Gamma_\delta)}\le C\,\|\underline{g}^0\|_{H_\beta^{\frac{3}{2},\frac{3}{2}}(\Gamma^0_\delta)} \tag{5.4a}$$

$$\begin{aligned}
\|\widetilde{\underline{g}}^1\|_{W_\beta^{\frac{1}{2}}(\widetilde{\Gamma}^1)} &\le C\left\{\|\underline{g}^1\|_{H_\beta^{\frac{1}{2},\frac{1}{2}}(\Gamma^1_\delta)}+\|\ell_1(\underline{u},p,\phi_\delta)\|_{H_\beta^{\frac{1}{2},\frac{1}{2}}(\Gamma^1_\delta)}\right\}\\
&\le C\left\{\|\underline{g}^1\|_{H_\beta^{\frac{1}{2},\frac{1}{2}}(\Gamma^1_\delta)}+\|\underline{u}\|_{H^1(S_\delta\backslash S_{\delta/2})}\right\}
\end{aligned} \tag{5.4b}$$

where $\widetilde{\Gamma}^\ell$, $\ell = 0, 1$ denotes the extension of $\Gamma^\ell \cap \partial S_\delta$ to the infinite sector Q with interior opening angle $\omega_1 < 2\pi$.

By Theorem 4.17, the solution $(\underline{v}, q)$ of (5.1), (5.2) in the infinite sector Q exists and is unique in the space $W^2_\beta(Q)^2 \times W^1_\beta(Q)$. Since the data $\underline{\widetilde{f}}$ and $\underline{\widetilde{g}}^\ell$ have bounded support in Q, the solution $(\underline{v}, q)$ has additional properties which will be essential to establish the relation between $(\underline{\widetilde{u}}, \widetilde{p})$ and $(\underline{v}, q)$. To this end, let

$$|D^1 w|^2 = \sum_{|\alpha|=1} |D^\alpha w|^2 = \Big|\frac{\partial w}{\partial x_1}\Big|^2 + \Big|\frac{\partial w}{\partial x_2}\Big|^2 ,$$

and for vector-functions $\underline{w}$, $|D^1 \underline{w}|$ is defined by summing over the components. Then we have the following lemma:

Lemma 5.1. *For the solution $(\underline{v}, q)$ of problem (5.1) in the sector Q with either Dirichlet boundary conditions (5.2a) or mixed boundary conditions (5.2c) and $\underline{G}^0(A_1) = \underline{0}$ there holds*

$$\|D^1 \underline{v}\|^2_{L^2(Q)} + \|r^{-1}\underline{v}\|^2_{L^2(Q)} + \|q\|^2_{L^2(Q)} < \infty . \tag{5.5a}$$

For the solution $(\underline{v}, q)$ of (5.1) in Q with Neumann boundary conditions (5.2b), we have

$$\|D^1 \underline{v}\|^2_{L^2(Q)} + \|q\|^2_{L^2(Q)} < \infty . \tag{5.5b}$$

The proof parallels that of Theorem 4.4 and Corollary 4.3 of [13] and will therefore be omitted here. We can now prove

Theorem 5.2. *For the problem (5.1) in Q with either Dirichlet or mixed boundary conditions (5.2a), (5.2c), it holds that $(\underline{\widetilde{u}}, \widetilde{p}) = (\underline{v}, q)$ and there exists $C(\beta, \delta)$ such that*

$$\begin{aligned} \|\underline{\widetilde{u}}\|_{W^2_\beta(Q)} + \|\widetilde{p}\|_{W^1_\beta(Q)} \;\le\; & C \Big\{ \|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H^{1,1}_\beta(S_\delta)} + \sum_{\ell=0,1} \|\underline{g}^\ell\|_{H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell_\delta)} \\ & + \|\underline{u}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})} \Big\} , \end{aligned} \tag{5.6a}$$

where the term with $\ell = 1$ is omitted for pure Dirichlet problem.

For the solution of the Neumann problem (5.1), (5.2b) in Q, $(\underline{\widetilde{u}} - \underline{\widetilde{u}}(A_1), \widetilde{p}) = (\underline{v}, q)$ and

$$\begin{aligned} \|\underline{\widetilde{u}} - \underline{\widetilde{u}}(A_1)\|_{W^2_\beta(Q)} + \|\widetilde{p}\|_{W^1_\beta(Q)} \;\le\; & C \Big\{ \|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H^{1,1}_\beta(S_\delta)} + \|\underline{g}^1\|_{H_\beta^{\frac{1}{2},\frac{1}{2}}(\Gamma^1_\delta)} \\ & + \|\underline{u}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})} \Big\} . \end{aligned} \tag{5.6b}$$

Proof: We establish (5.6) for the Dirichlet boundary conditions (5.2a), the argument for (5.2c) being identical. Assume first that $\underline{g}^0(\underline{0}) = \underline{0}$; then $(\underline{\widetilde{u}}, \widetilde{p})$ satisfies

$$L(\underline{\widetilde{u}}, \widetilde{p}) = (\underline{\widetilde{f}}, \widetilde{h}) \;\text{ in }\; Q\,, \;\; \underline{\widetilde{u}}\big|_{\theta=0,\omega} = \underline{\widetilde{g}}^0 .$$

For any $(\underline{w}, \sigma)^2 \in \widetilde{H}^1_0(Q)^2 \times L^2_0(Q)$ where

$$\widetilde{H}^1_0(Q) = \{u \,|\, \|D^1 u\|_{L^2(Q)} < \infty,\; u\big|_{\partial Q} = 0\} ,$$

we have

$$\begin{aligned} a(\underline{\widetilde{u}}, \underline{w})_Q + b(\widetilde{p}, \underline{w})_Q &= \widetilde{F}(\underline{w})_Q , \\ b(\sigma, \underline{\widetilde{u}})_Q &= \widetilde{\ell}(\sigma)_Q . \end{aligned}$$

where $a(\cdot,\cdot)_Q$, $b(\cdot,\cdot)_Q$ are the bilinear forms in (3.10) with domain of integration taken over Q and

$$\widetilde{F}(\underline{w})_Q = \int\limits_Q \underline{\widetilde{f}} \cdot \underline{w}\, dx, \;\; \widetilde{\ell}(\sigma)_Q = \int\limits_Q \widetilde{h}\, \sigma\, dx\,.$$

On the other hand, for any $\underline{w} \in \widetilde{\widetilde{H}}{}^1_0(Q)^2 := \{\underline{w} \in \widetilde{H}^1_0(Q)^2 | \underline{w}$ has bounded support in $Q\}$ and any $\sigma \in L^2_0(Q)$ we have by integration by parts

$$\begin{aligned} a(\underline{v},\underline{w})_Q + b(q,\underline{w})_Q &= \widetilde{F}(\underline{w})_Q\,, \\ b(\sigma,\underline{v})_Q &= \widetilde{\ell}(\sigma)_Q \end{aligned}$$

which implies that for $(\underline{w},\sigma) \in \widetilde{\widetilde{H}}{}^1_0(Q)^2 \times L^2_0(Q)$

$$\begin{aligned} a(\underline{v}-\underline{\widetilde{u}},\underline{w})_Q + b(q-\widetilde{p},\underline{w})_Q &= 0\,, \\ b(\sigma,\underline{v}-\underline{\widetilde{u}})_Q &= 0\,. \end{aligned}$$

Since $\widetilde{\widetilde{H}}{}^1_0(Q)^2$ is dense in $\widetilde{H}^1_0(Q)^2$ and (5.5) holds, we have

$$a(\underline{v}-\underline{\widetilde{u}},\, \underline{v}-\underline{\widetilde{u}}) = 0$$

and

$$b(q-\widetilde{p},\underline{w}) = 0 \quad \forall \underline{w} \in \widetilde{H}^1_0(Q)^2\,.$$

Hence, $\underline{v}-\underline{\widetilde{u}} = \sum_{j=1}^3 \widetilde{c}_j\, \underline{e}_j$ is a rigid body motion with

$$\underline{e}_1 = (1,0)^\top,\; \underline{e}_2 = (0,1)^\top \;\text{ and }\; \underline{e}_3 = (-x_2,x_1)^\top\,. \tag{5.7}$$

Since $(\underline{v}-\underline{\widetilde{u}})|_{\partial Q} = 0$, it follows that $\underline{v} \equiv \underline{\widetilde{u}}$ in Q.

Due to Corollary 2.4 of [11], there exists $\underline{w} \in \widetilde{H}^1_0(Q_R)^2$ such that $\operatorname{div}\underline{w} = q-\widetilde{p}$ in Q_R and $\|\operatorname{div}\underline{w}\|_{L^2(Q_R)} \le c\|q-\widetilde{p}\|_{L^2(Q_R)} \le c\|q-\widetilde{p}\|_{L^2(Q)}$ where $Q_R = \{\underline{x} \in Q \big| \, |\underline{x}| < R\}$ denotes the truncated sector. We extend $\underline{w}$ to all of Q such that the norms are bounded and denote the extension still by $\underline{w}$. Then we let $\underline{\widetilde{w}} = \varphi_\delta \underline{w}$ with the cut-off function $\varphi_\delta \equiv 1$ in Q_R and $\varphi_\delta \equiv 0$ in $Q \backslash Q_{R+\delta}$. Then we have

$$0 = \int\limits_Q (q-\widetilde{p}) \operatorname{div}\underline{\widetilde{w}}\, dx = \int\limits_{Q_R} (q-\widetilde{p})^2\, dx + \int\limits_{Q_{R+\delta}\backslash Q_R} (q-\widetilde{p}) \operatorname{div}\underline{\widetilde{w}}\, dx\,.$$

We have, by the properties of the extension $\underline{w}$ to all of Q,

$$\Big| \int\limits_{Q_{R+\delta}\backslash Q_R} (q-\widetilde{p}) \operatorname{div}\underline{\widetilde{w}}\, dx \Big| \le C\, \|q-\widetilde{p}\|_{L^2(Q_{R+\delta}\backslash Q_R)} \|\underline{w}\|_{H^1(Q_R)}$$

and also, since $\|q-\widetilde{p}\|_{L^2(Q)} < \infty$, that $\|q-\widetilde{p}\|_{L^2(Q_{R+\delta}\backslash Q_R)} \to 0$ as $\delta \to 0$. This implies that $\|q-\widetilde{p}\|_{L^2(Q_R)} = 0$. Since R was finite, but otherwise arbitrary, we have $q = \widetilde{p}$ in Q.

We therefore conclude that $(\underline{\widetilde{u}},\widetilde{p}) = (\underline{v},q) \in W^2_\beta(Q)^2 \times W^1_\beta(Q)$, and that (5.6a) holds, since

$$\begin{aligned} \|\underline{\widetilde{u}}\|_{W^2_\beta(Q)} + \|\widetilde{p}\|_{W^1_\beta(Q)} &\le C\left\{\|\underline{\widetilde{f}}\|_{L_\beta(Q)} + \|\widetilde{h}\|_{W^1_\beta(Q)} + \|\underline{g}^0\|_{H^{\frac{3}{2},\frac{3}{2}}_\beta(\Gamma^0_\delta)}\right\} \\ &\le C\left\{\|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H^{1,1}_\beta(S_\delta)} + \|\underline{g}^0\|_{H^{\frac{3}{2},\frac{3}{2}}_\beta(\Gamma^0_\delta)} \right. \\ &\qquad \left. + \|\underline{u}\|_{H^1(S_\delta\backslash S_{\delta/2})} + \|p\|_{L^2(S_\delta\backslash S_{\delta/2})}\right\}. \end{aligned}$$

If $\underline{g}^0(0) = \underline{G}^0(A_1) \neq \underline{0}$, $\widetilde{\underline{u}} = \phi_\delta(\underline{u} - \underline{g}^0(0))$ satisfies (5.1), (5.2a) with data $\widetilde{\underline{f}}$, $\widetilde{\underline{g}}^0$ which, in turn, satisfies the estimates

$$\begin{aligned}\|\widetilde{\underline{f}}\|_{L_\beta(Q)} &\leq C\left\{\|\underline{f}\|_{L_\beta(S_\delta)} + \|\underline{u}\|_{H^1(S_\delta\setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta\setminus S_{\delta/2})} + |\underline{g}^0(0)|\right\} \\ &\leq C\left\{\|\underline{f}\|_{L_\beta(S_\delta)} + \|\underline{g}^0\|_{H_\beta^{\frac{3}{2},\frac{3}{2}}(\Gamma_\delta^0)} + \|\underline{u}\|_{H^1(S_\delta\setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta\setminus S_{\delta/2})}\right\}\end{aligned}$$

and

$$\|\widetilde{\underline{g}}^0\|_{W_\beta^{\frac{3}{2}}(\Gamma_\delta^0)} \leq C\left\{\|\underline{g}^0\|_{H_\beta^{\frac{3}{2},\frac{3}{2}}(\Gamma_\delta^0)} + |\underline{g}^0(0)|\right\} \leq C\left\{\|\underline{g}^0\|_{H_\beta^{\frac{3}{2},\frac{3}{2}}(\Gamma_\delta^0)}\right\}.$$

Applying the a-priori estimate for the homogeneous case to $\widetilde{\underline{u}} := \phi_\delta(\underline{u} - \underline{g}^0(0))$ and to $\widetilde{p} := \phi_\delta\, p$, we have (5.6a) for the general case.

We now show (5.6b) for the Neumann boundary conditions $\widetilde{\underline{u}} = \phi_\delta\,\underline{u}$, $\widetilde{p} = \phi_\delta\, p$ satisfying (5.1), (5.2b). For any $\underline{w} \in \widetilde{H}^1(Q)^2 := \{\underline{w} \mid \|D^1\underline{w}\|_{L^2(Q)} < \infty\}$, and any $\sigma \in L^2(Q)$ we have by integration by parts that

$$\begin{aligned} a(\widetilde{\underline{u}}, \underline{w})_Q + b(\widetilde{p}, w)_Q &= \widetilde{F}_Q(\underline{w}), \\ b(\sigma, \widetilde{\underline{u}})_Q &= \widetilde{\ell}_Q(\sigma)\end{aligned}$$

where

$$\widetilde{\ell}_Q(\sigma) = \int_Q \widetilde{h}\sigma\, dx, \quad \widetilde{F}_Q(\underline{w}) = \int_Q \widetilde{\underline{f}} \cdot \underline{w}\, dx + \int_{\widetilde{\Gamma}^1} \widetilde{\underline{g}}^1\, \underline{w}\, ds.$$

On the other hand, for any $\underline{w} \in \widetilde{\widetilde{H}}^1(Q) := \{\underline{w} \in \widetilde{H}^1(Q)\colon \underline{w}$ has bounded support in $Q\}$ and for $\sigma \in L^2(Q)$ we have

$$\begin{aligned} a(\underline{v}, \underline{w})_Q + b(q, \underline{w})_Q &= \widetilde{F}_Q(\underline{w}), \\ b(\sigma, \underline{v})_Q &= \widetilde{\ell}_Q(\sigma).\end{aligned}$$

which yields, for any $\underline{w} \in \widetilde{\widetilde{H}}^1(Q)$ and for any $\sigma \in L^2(Q)$,

$$\begin{aligned} a(\underline{v} - \widetilde{\underline{u}}, \underline{w}) + b(q - \widetilde{p}, \underline{w})_Q &= 0, \\ b(\sigma, \underline{v} - \widetilde{\underline{u}})_Q &= 0.\end{aligned}$$

Arguing as in the previous case, we have

$$a(\underline{v} - \widetilde{\underline{u}},\ \underline{v} - \widetilde{\underline{u}}) = 0, \quad \|q - \widetilde{p}\|_{L^2(Q)} = 0.$$

This implies that $\underline{v} - \widetilde{\underline{u}} = \sum_{j=1}^3 c_j\, \underline{e}_j$ with certain constants c_j where $\underline{e}_j$ are the rigid body motions (5.7). By Lemma 5.1, it holds $\|D^1(\underline{v} - \widetilde{\underline{u}})\|_{L^2(Q)} < \infty$ which implies $c_3 = 0$ and

$$\underline{v} - \widetilde{\underline{u}} = \sum_{j=1}^{2} c_j\, \underline{e}_j, \quad q = \widetilde{p}$$

and the a-priori estimate

$$\begin{aligned}\|\widetilde{\underline{u}} - \sum_{j=1}^{2} c_j\, \underline{e}_j\|_{W_\beta^2(Q)} + \|\widetilde{p}\|_{W_\beta^1(Q)} &\leq C\left\{\|\widetilde{\underline{f}}\|_{L_\beta(Q)} + \|\widetilde{h}\|_{W_\beta^1(Q)} + \|\widetilde{g}^1\|_{W_\beta^{\frac{1}{2}}(\widetilde{\Gamma}^1)}\right\} \\ &\leq C\left\{\|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H_\beta^{1,1}(S_\delta)} + \|g^1\|_{H_\beta^{\frac{1}{2},\frac{1}{2}}(\Gamma_\delta^1)} \right. \\ &\quad \left. + \|\underline{u}\|_{H^1(S_\delta\setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta\setminus S_{\delta/2})}\right\}.\end{aligned}$$

Observing that $W^2_\beta(Q) \subset H^{2,2}_\beta(S_\varepsilon) \subset C^0(\overline{S}_\varepsilon)$ for any $\varepsilon > 0$ (see [2]), we find $\underline{\widetilde{u}} \in C^0(\overline{Q})^2$ and thus $\underline{\widetilde{u}}(A_1)$ exists. Note finally that $r^{\beta-2}(\underline{\widetilde{u}} - \sum_{j=1}^2 c_j\, \underline{e}_j) \in L^2(Q)^2$ which means that $\underline{\widetilde{u}}(A_1) = \sum_{j=1}^2 c_j\, \underline{e}_j$ and that (5.6) holds also in the Neumann case (5.2b). □

5.2 Regularity of the second derivatives

We will now obtain one main result of our paper, namely the $H^{2,2}_\beta$-regularity for the velocities u of the Stokes problem in a polygon. This result will allow, for example, to prove the optimal convergence of low order FEM on properly refined meshes along the lines of [21], where explicit decompositions of (u, p) in regular and singular parts were used, however. In addition, the main result, Theorem 5.4, shall be the essential ingredient for the analytic regularity theory in the next section.

We start the analysis with a result at each vertex.

Theorem 5.3. *Let $(\underline{u}, p)$ be the weak solution of (4.1), (4.2). Then $(\underline{u}, p) \in H^{2,2}_\beta(S_{\delta/2})^2 \times H^{1,1}_\beta(S_{\delta/2})$ for any $\delta > 0$ and there holds the a-priori estimate*

$$\begin{aligned}\|\underline{u}\|_{H^{2,2}_\beta(S_{\delta/2})} + \|p\|_{H^{1,1}_\beta(S_{\delta/2})} \le C(\delta)\Big\{&\|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H^{1,1}_\beta(S_\delta)} + \sum_{\ell=0,1} \|\underline{g}^\ell\|_{H^{\frac32-\ell,\frac32-\ell}_\beta(\Gamma^\ell_\delta)} \\ &+ \|\underline{u}\|_{H^1(S_\delta\setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta\setminus S_{\delta/2})}\Big\}.\end{aligned} \tag{5.8}$$

Proof: Note that $(\underline{\widetilde{u}}, p) = (\underline{u} - \underline{u}(A_1), p)$ in $S_{\delta/2}$ and that $\|w\|_{H^{\ell,\ell}_\beta(S_{\delta/2})} \le \|w\|_{W^\ell_\beta(S_{\delta/2})}$ for any $\ell \ge 0$. It follows from Theorem 5.3 that

$$\begin{aligned}\|\underline{u} - \underline{u}(A_1)\|_{H^{2,2}_\beta(S_{\delta/2})} + \|p\|_{H^{1,1}_\beta(S_{\delta/2})} \le C\Big\{&\|\underline{f}\|_{L_\beta(S_\delta)} + \|h\|_{H^{1,1}_\beta(S_\delta)} + \sum_{\ell=0,1} \|g^\ell\|_{H^{\frac32-\ell,\frac32-\ell}_\beta(\Gamma^\ell_\delta)} \\ &+ \|\underline{u}\|_{H^1(S_\delta\setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta\setminus S_{\delta/2})}\Big\}.\end{aligned}$$

Since $H^{2,2}_\beta(S_{\delta/2}) \subset C^0(\overline{S}_{\delta/2})$, $\underline{u}$ is continuous on $\overline{S}_{\delta/2}$. If $|\Gamma^0_\delta| > 0$, $|\underline{u}(A_1)| = |\underline{g}^0(0)| \le C\,\|\underline{g}^0\|_{H^{\frac32,\frac32}_\beta(\Gamma^0_\delta)}$ which implies (5.8) immediately.

If $|\Gamma^0_\delta| = 0$, we have

$$|\underline{u}(A_1)| = |\underline{\widetilde{u}}(r,0) - \underline{\widetilde{u}}(A_1)| \le \|\underline{\widetilde{u}}(\underline{x}) - \underline{\widetilde{u}}(A_1)\|_{C^0(S_\delta)} \le C\,\|\underline{\widetilde{u}} - \underline{\widetilde{u}}(A_1)\|_{W^2_\beta(Q)} \tag{5.9}$$

and

$$\|\underline{u}\|_{H^{2,2}_\beta(S_{\delta/2})} \le C\left\{|\underline{u}(A_1)| + \|\underline{\widetilde{u}} - \underline{\widetilde{u}}(A_1)\|_{W^2_\beta(Q)}\right\}. \tag{5.10}$$

Now (5.8) follows from (5.9), (5.10) and from (5.6). □

We combine the a-priori estimate Theorem 5.3 for each vertex to get

Theorem 5.4. *Let $\underline{f} \in L_\beta(\Omega)^2$, $h \in H^{1,1}_\beta(\Omega)$ and $\underline{g}^\ell \in H^{\frac32-\ell,\frac32-\ell}_\beta(\Omega)^2$. Assume further that $|\Gamma^0| > 0$ and that (3.7) holds if $\Gamma^0 = \partial\Omega$. Then (3.1), (3.2) has a unique solution $(\underline{u}, p) \in H^{2,2}_\beta(\Omega)^2 \times H^{1,1}_\beta(\Omega)$, for $\beta = (\beta_1, \dots, \beta_M)$ with $\beta_i > 1 - \kappa^i_1$, $0 < \kappa^i_1 \le 1$, and*

$$\|\underline{u}\|_{H^{2,2}_\beta(\Omega)} + \|p\|_{H^{1,1}_\beta(\Omega)} \le C\Big\{\|\underline{f}\|_{L_\beta(\Omega)} + \|h\|_{H^{1,1}_\beta(\Omega)} + \sum_{\ell=0,1} \|\underline{g}^\ell\|_{H^{\frac32-\ell,\frac32-\ell}_\beta(\Gamma^\ell)}\Big\}. \tag{5.11}$$

Proof: Theorem 3.4 implies existence and uniqueness of $(\underline{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ and (3.17) holds. Let $\Omega_{\delta/2} := \Omega \backslash \bigcup_{i=1}^M S_\delta^i$ for $\delta > 0$ sufficiently small so that $S_\delta^i \subset \Omega$. By the usual difference quotient argument, $\underline{u} \in H^2(\Omega_{\delta/2})^2$, $p \in H^1(\Omega_{\delta/2})$, and the a-priori estimate

$$\|\underline{u}\|_{H^2(\Omega_{\delta/2})} + \|p\|_{H^1(\Omega_{\delta/2})} \leq C \left\{ \|\underline{f}\|_{L_\beta(\Omega)} + \|h\|_{H_\beta^{1,1}(\Omega)} + \sum_{\ell=0,1} \|\underline{g}^\ell\|_{H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)} \right\} \tag{5.12}$$

holds.

In the vicinity S_δ^i of each vertex A_i we use the results of Section 4. More precisely, after localization of $(\underline{u}, p)$ near A_i, we are in the setting (5.1), (5.2) and have the a-priori estimates (5.3), (5.4) for the localized data. Theorem 5.3 shows that, in each neighborhood S_δ^i of A_i, $\delta > 0$, we have

$$(\underline{u}, p) \in H_{\beta_i}^{2,2}(S_\delta^i)^2 \times H_{\beta_i}^{1,1}(S_\delta^i), \quad 0 < \beta_i < 1$$

for $\beta_i > 1 - \kappa_1^i$ with $\kappa_1^i > 0$ denoting the smallest positive imaginary part of the roots λ of the transcendental equations (4.33), with ω replaced by ω_i. Moreover, the a-priori estimate

$$\|\underline{u}\|_{H_{\beta_i}^{2,2}(S_\delta^i)} + \|p\|_{H_{\beta_i}^{1,1}(S_\delta^i)} \leq C \left\{ \|\underline{f}\|_{L_\beta(\Omega)} + \|h\|_{H_\beta^{1,1}(\Omega)} + \sum_{\ell=0}^{1} \|\underline{g}^\ell\|_{H_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)} \right\} \tag{5.13}$$

holds.

Combining (5.12), (5.13) completes the proof of (5.11). □

5.3 Analytic Regularity

We can now prove the main result of this paper.

Theorem 5.5. *Let $k \geq 0$ and assume that $\underline{f} \in H_\beta^{k,0}(\Omega)^2$, $h \in H_\beta^{k+1,1}(\Omega)$ and $\underline{g}^\ell \in H^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma)$, $\ell = 0, 1$ with $\beta = (\beta_1, \ldots, \beta_M)$, $\beta_i > 1 - \kappa_1^i$ (where κ_1^i is defined in Section 4.5 for each vertex A_i), $1 \leq i \leq M$.*

Assume further $|\Gamma^0| > 0$ and that (3.7) holds if $\Gamma^0 = \partial\Omega$. Then the Stokes Problem (3.1), (3.2) admits a unique solution $(\underline{u}, p) \in H_\beta^{k+2,2}(\Omega)^2 \times H_\beta^{k+1,k}(\Omega)$ and the a-priori estimate

$$\begin{aligned} &\|\underline{u}\|_{H_\beta^{k+2,2}(\Omega)} + \|p\|_{H_\beta^{k+1,1}(\Omega)} \\ &\leq C \left\{ \|\underline{f}\|_{H_\beta^{k,0}(\Omega)} + \|h\|_{H_\beta^{k+1,1}(\Omega)} + \sum_{\ell=0,1} \|\underline{g}^\ell\|_{H^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)} \right\} \end{aligned} \tag{5.14}$$

holds for all $k \geq 0$.

Moreover, if $\underline{f} \in \mathcal{B}_\beta^0(\Omega)^2$, $\underline{g}^\ell \in \mathcal{B}_\beta^{\frac{3}{2}-\ell}(\Gamma^\ell)^2$, $\ell = 0, 1$, and if $h \in \mathcal{B}_\beta^1(\Omega)$, then $(\underline{u}, p) \in \mathcal{B}_\beta^2(\Omega)^2 \times \mathcal{B}_\beta^1(\Omega)$.

Remark 5.6. If $|\Gamma^0| = 0$ and if (3.22) holds, the above result also holds for the Neumann boundary conditions on all of $\partial\Omega$.

Proof: The case $k = 0$ is just Theorem 5.4.

Let first $\delta > 0$ be sufficiently small and define

$$S_\delta^i = \{(r_i, \theta_i) : 0 < r_i < \delta,\ 0 < \theta_i < \omega_i\} \subset \Omega\,,$$

the truncated sector at vertex A_i of opening angle $\omega_i < 2\pi$, and $\delta > 0$ selected so small that $S^i_\delta \cap S^j_\delta = \emptyset$ if $i \neq j$. Define further $\Omega_\delta := \Omega \backslash \bigcup_{i=1}^M S^i_\delta$. Then, for any $k \geq 0$, by standard elliptic regularity we have

$$\|\underline{u}\|_{H^{k+2}(\Omega_{\delta/2})} + \|p\|_{H^{k+1}(\Omega_{\delta/2})} \leq C \left\{ \|\underline{f}\|_{H^k(\Omega_{\delta/4})} + \|h\|_{H^{k+1}(\Omega_{\delta/4})} + \sum_{\ell=0}^{1} \|\underline{G}^\ell\|_{H^{k+2-\ell}(\Omega_{\delta/4})} \right\} \tag{5.15}$$

where $\underline{G}^\ell\big|_{\Gamma^\ell} = \underline{g}^\ell$, $\ell = 0, 1$ and

$$\|\underline{g}^\ell\|_{H^{k+\frac{3}{2}-\ell, \frac{3}{2}-\ell}(\Gamma^\ell)} \leq \|\underline{G}^\ell\|_{H^{k+2-\ell,2-\ell}_\beta(\Omega)} \leq 2\|\underline{g}^\ell\|_{H_\beta^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)}\,.$$

If $\underline{f} \in \mathcal{B}^0_\beta(\Omega)^2$, $\underline{g}^\ell \in \mathcal{B}_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)$ and $h \in \mathcal{B}^{1,1}_\beta(\Omega)$, these data are analytic in $\Omega_{\frac{\delta}{4}}$ and, by an argument of Morrey [20], $(\underline{u}, p)$ are analytic in $\Omega_{\delta/2}$, i.e. $(\underline{u}, p) \in \mathcal{B}^2_\beta(\Omega_{\delta/2})^2 \times \mathcal{B}^1_\beta(\Omega_{\delta/2})$.

It remains to establish regularity in the truncated sectors $S^i_{\delta/2}$. We shall prove that if

$$\underline{f} \in H^{k,0}_\beta(\Omega)^2,\ h \in H^{k+1,1}_\beta(\Omega) \ \text{ and } \ \underline{g}^\ell \in H_\beta^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)^2\,,$$

then

$$\begin{aligned}\|\underline{\overline{u}}\|_{H^{k+2,2}_{\beta_i}(S^i_{\delta/2})} + \|\overline{p}\|_{H^{k+1,1}_{\beta_i}(S^i_{\delta/2})} \ \leq\ C \Big\{ &\|\underline{\overline{f}}\|_{H^{k,0}_{\beta_i}(S^i_\delta)} + \|h\|_{H^{k+1,1}_{\beta_i}(S^i_\delta)} + \sum_{\ell=0}^{1} \|\underline{\overline{g}}^\ell\|_{H_{\beta_i}^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)} \\ &+ \|\underline{\overline{u}}\|_{H^{k+1,1}_{\beta_i}(S^i_\delta \backslash S^i_{\delta/2})} + \|p\|_{H^k_{\beta_i(S^i_\delta \backslash S^i_{\delta/2})}} \Big\}\,,\end{aligned} \tag{5.16}$$

and if $\underline{f} \in \mathcal{B}^0_\beta(\Omega)^2$, $h \in \mathcal{B}^1_\beta(\Omega)$ and $\underline{g}^\ell \in \mathcal{B}_\beta^{\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell)^2$, then for $|\alpha| = k+2$

$$\|r^{\beta_i+\alpha_1-2}\,\mathcal{D}^\alpha\,\underline{\overline{u}}\|_{L^2(S^i_{\delta/2})} + \|r^{\beta_i+\alpha_1-1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta/2})} \leq C L_i\, D_i^k\, E_i^{\,\overline{\alpha_2-2}}\, k! \tag{5.17}$$

for certain constants L_i, D_i, E_i that are independent of k. Here, $|\alpha| = k+2$, $k \geq 0$, L_i, P_i, D_i are sufficiently large constants, but independent of k, $\overline{\alpha_2 - 2} = \alpha_2 - 2$ if $\alpha_2 \geq 2$ and $\overline{\alpha_2 - 2} = 0$ if $\alpha_2 < 2$. This implies by Theorem 2.1 and Lemma 5.1 of [13] that

$$\begin{aligned}&\|\underline{u}\|_{H^{k+2,2}_{\beta_i}(S^i_{\delta/2})} + \|p\|_{H^{k+1,1}_{\beta_i}(S^i_{\delta/2})} \\ &\leq C \Big\{ \|\underline{f}\|_{H^{k,0}_{\beta_i}(S^i_\delta)} + \|h\|_{H^{k+1,1}_{\beta_i}(S^i_\delta)} + \sum_{\ell=0}^{1} \|\underline{g}^\ell\|_{H_{\beta_i}^{k+\frac{3}{2}-\ell,\frac{3}{2}-\ell}(\Gamma^\ell_\delta)} \Big\}\end{aligned} \tag{5.18}$$

and, for $|\alpha| = k+2$,

$$\|r^{\beta_i+\alpha_1-2}\, D^\alpha \underline{u}\|_{L^2(S^i_{\delta/2})} + \|r^{\beta_i+\alpha_1-1}\, D^\alpha p\|_{L^2(S^i_{\delta/2})} \leq C L_i\, D_i^k\, E_i^{\,\overline{\alpha_2-2}}\, k\,! \tag{5.19}$$

(5.18) and (5.19) imply the assertion.

It remains to show (5.16), (5.17). Without loss of generality we assume to this end that $\underline{g}^0 = \underline{0}$. Due to Theorem 5.4, (5.16) and (5.17) are true up to order $k+1$, $k \geq 1$ arbitrary but fixed in the following. To show (5.16) and (5.17) for $k+2$, we introduce

$$\underline{\overline{v}} := r^k\, \frac{\partial^k\, \underline{\overline{u}}}{\partial r^k}\,, \quad q := r^k\, \frac{\partial^k p}{\partial r^k}\,.$$

Then it is easy to verify that $(\underline{v}, q)$ solve in S^i_δ

$$L(\underline{v}, q) = r^{k-2}\,\frac{\partial^k}{\partial r^k}\,(r^2\,\underline{f}) - kr^{k-2}\Big(r\,\frac{\partial^k p}{\partial r^k} + (k-1)\,\frac{\partial^{k-1}p}{\partial r^{k-1}}\,,\;\frac{\partial^k p}{\partial r^{k-1}\partial\theta}\Big)^\top \tag{5.20}$$

together with one of the following boundary conditions: for $i = 0, 1, \dots$

$$\underline{v}\big|_{\theta=0,\theta=\omega_i} = \underline{0} \qquad \text{(Dirichlet)} \tag{5.21a}$$

$$\underline{\sigma}_n(\underline{v}, q)\big|_{\theta=0,\omega_i} = r^{k-1}\,\frac{\partial^k(r\underline{g}^1)}{\partial r^k} + \begin{pmatrix} 0 \\ kr^{k-1}\,\partial_r^{k-1}\,p \end{pmatrix}\Big|_{\theta=0,\omega_i} \qquad \text{(Neumann)} \tag{5.21b}$$

$$\underline{v}\big|_{\theta=0} = \underline{0},\; \underline{\sigma}_n(\underline{v}, q)\big|_{\theta=\omega_i} = r^{k-1}\,\frac{\partial^k(r\underline{g}^1_{\omega_i})}{\partial r^k} + \begin{pmatrix} 0 \\ kr^{k-1}\,\partial_r^{k-1}\,p \end{pmatrix}\Big|_{0=\omega_i}. \qquad \text{(Mixed)} \tag{5.21c}$$

Note that

$$q_0 := kr^{k-2}\Big(r\,\frac{\partial^k p}{\partial r^k} + (k-1)\,\frac{\partial^{k-1}p}{\partial r^{k-1}}\,,\;\frac{\partial^k p}{\partial r^{k-1}\partial\theta}\Big)^\top$$

in the right hand side of (5.20) is a lower-order term, and by the induction assumption in $S^i_{\delta/2}$ and by the analyticity of $(\underline{u}, p)$ in $S^i_{\delta/2}$ we have

$$\begin{aligned}\|q_0\|_{L^2_{\beta_i}(S^i_\delta)} \;\le\; C\Big\{&\|\underline{f}\|_{H^{k-1,0}_{\beta_i}(S^i_\delta)} + \|h\|_{H^{k,1}_{\beta_i}(S^i_\delta)} + \sum_{\ell=0}^{1}\|\underline{g}^\ell\|_{H^{k+\frac12-\ell,\frac32-\ell}_{\beta_i}(\Gamma^\ell_\delta)} \\ &+ \|\underline{u}\|_{H^k(S^i_\delta\setminus S^i_{\delta/2})} + \|p\|_{H^{k-1}(S^i_\delta\setminus S^i_{\delta/2})}\Big\}\end{aligned}$$

and

$$\|q_0\|_{L^2_{\beta_i}(S^i_\delta)} \le CL_i\,D_i^{k-1}(k-1)!$$

respectively.

Applying now Theorem 5.3 to $(\underline{v}, q)$ gives (5.18), (5.19) with $|\alpha| = k+1$ and $|\alpha_2| \le 2$.

Arguing exactly as in the proof of Theorem 5.2 of [13], we get then (5.18), (5.19) for $\mathcal{D}^\alpha\underline{u}$ for all α with $|\alpha| = k+2$ and for $\mathcal{D}^\alpha p$ with $|\alpha| = k+1$. By Proposition 2.1, we have (5.18), (5.19) for $D^\alpha u$ with $|\alpha| = k+2$, and for $D^\alpha p$ with $|\alpha| = k+1$.

Hence $(\overline{\underline{u}}, \overline{p}) \in H^{k+2,2}_{\beta_i}(S^i_{\delta/2})^2 \times H^{k+1,1}_{\beta_i}(S^i_{\delta/2})$ (resp. $(\overline{\underline{u}}, \overline{p}) \in B^2_{\beta_i}(S^i_{\delta/2})^2 \times B^1_{\beta_i}(S^i_{\delta/2})$). This implies that $(\underline{u}, p) \in H^{k+2,2}_{\beta_i}(S^i_{\delta/2})^2 \times H^{k+1,1}_{\beta_i}(S^i_{\delta/2})$ (resp. in $B^2_{\beta_i}(S^i_{\delta/2})^2 \times B^1_{\beta_i}(S^i_{\delta/2})$). The argument above is valid for each $i = 1, \dots, M$ and gives, with (5.15), that $(\underline{u}, p) \in H^{k+2,2}_\beta(\Omega)^2 \times H^{k+1,1}_\beta(\Omega)$, for any $k \ge 0$. Further, the analyticity of $(\underline{u}, p)$ in $\Omega_{\delta/2}$ gives, together with $(\underline{u}, p) \in B^2_{\beta_i}(S^i_{\delta/2})^2 \times B^1_{\beta_i}(S^i_{\delta/2})$, $i = 1, \dots, M$, the assertion. □

References

[1] M.S. Agranovich and M.K. Vishik: Elliptic problems with a parameter and parabolic problems of general type. *Uspekhi Mat. Nauk.*, **19**, No. 3 (1959), 626-727. (Russian Math. Surveys, **19**, No. 3 (1964), 53-157).

[2] I. Babuška and B.Q. Guo: Regularity of the solution of elliptic problems with piecewise analytic data. Part 1. Boundary value problems for linear elliptic equation of second order. *SIAM J. Math. Anal.*, **19** (1988), 172-203.

[3] I. Babuška and B.Q. Guo: Regularity of the solution of elliptic problems with piecewise analytic data. Part 2. The trace spaces and application to boundary value problems with non-homogeneous conditions. *SIAM J. Math. Anal.*, **20** (1989), 763-781.

[4] I. Babuška and B.Q. Guo: The *hp* version of the finite element method for problems with non-homogeneous essential boundary condition. *Comput. Methods Appl. Mech. Engrg.*, **74** (1989), 1-18.

[5] M. Baouendi and J. Sjöstrand: Analytic regularity for the Dirichlet problem in domains with conic singularities. *Ann. Scuola Norm. Sup. Pisa 4*, **4**, No. 3 (1977), 515-530.

[6] P. Bolley, M. Dauge and J. Camus: Regularité Gevrey pour le problème de Dirichlet dans des domaines à singularités coniques. *Comm. Partial Differential Equations*, **10**, No. 4 (1985), 391-431.

[7] F. Brezzi: On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *RAIRO Anal. Numer.*, **8** (1994), 129-151.

[8] M. Dauge: Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners I: linearized equations. *SIAM J. Math. Anal.*, **20** (1989), 74-97.

[9] C. Foias and R. Temam: Gevrey Class Regularity of the solutions of the Navier-Stokes equations. *J. Funct. Anal.*, **87** (1989), 359-369.

[10] G. Galdi: An introduction to the mathematical theory of the Navier-Stokes equations I. Springer-Verlag, Heidelberg, 1994.

[11] V. Girault and P.A. Raviart: Finite element methods for Navier Stokes equations. Springer-Verlag, Berlin, 1986.

[12] B.Q. Guo and I. Babuška: The $h-p$ version of the finite element methods. Part 1. The basic approximation results. *Comput. Mech.*, **1** (1986), 21-41; Part 2. General results and applications. *Comput. Mech.*, **1** (1986), 203-220.

[13] B.Q. Guo and I. Babuška: On the regularity of elasticity problems with piecewise analytic data. *Adv. Appl. Math.*, **14** (1993), 307-347.

[14] V.A. Kondratev: Boundary value problem for elliptic equations in domain with conic or angular points. *Trans. Moscow Math. Soc.*, (1967), 227-313.

[15] V.A. Kondratev: Boundary value problem for parabolic equations in corner domains. *Trans. Moscow Math. Soc.*, (1966), 450-504.

[16] V.A. Kondratev and O.A. Oleinik: Boundary value problems for partial differential equations in nonsmooth domains. *Russian Math. Surveys*, **38** (1983), 1-86.

[17] K. Masuda: On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation. *Proc. Japan Acad.*, **43** (1967), 827-832.

[18] V.G. Maz'ya and B.A. Plamenevskii: Estimates in L_p and Hölder class and the Miranda-Agmon maximum principle for solutions of elliptic boundary problems in domains with singular points on the boundary. *Amer. Math. Soc. Transl. (2)*, **123** (1984), 1-56.

[19] V.G. Maz'ya and B.A. Plamenevskii: The first boundary value problem for the classical equations of mathematical physics in domains with piecewise smooth boundary II. *Zeitschrift f. Analysis und ihre Anw.,* **2**(6) (1983), 523-551 (in russian).

[20] C.B. Morrey: Multiple Integrals in Calculus of Variations. Springer-Verlag, Berlin/Heidelberg/New York, 1966.

[21] M. Orlt: Regularitätsuntersuchungen und Fehlerabschätzungen für allgemeine Randwertprobleme der Navier-Stokes Gleichungen. Doctoral Dissertation, Stuttgart University, 1998.

[22] C. Schwab and M. Suri: Mixed *hp* finite element methods for Stokes and non-Newtonian flow. *Comp. Meth. Appl. Mech. Engg.,* **175** (1999), 217-241.

[23] C. Schwab: *p*- and *hp*-FEM. Oxford University Press, 1998.

Research Reports

No.	Authors	Title
00-18	B.Q. Guo, C. Schwab	Analytic regularity of Stokes flow in polygonal domains
00-17	M.H. Gutknecht, S. Röllin	Variations of Zhang's Lanczos-Type Product Method
00-16	M.H. Gutknecht, S. Röllin	The Chebyshev iteration revisited
00-15	G. Schmidlin, C. Schwab	Wavelet Galerkin BEM on unstructured meshes by aggregation
00-14	B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab	Local Discontinuous Galerkin methods for the Stokes system
00-13	O. Bröker, M.J. Grote, C. Mayer, A. Reusken	Robust Parallel Smoothing for Multigrid Via Sparse Approximate Inverses
00-12	C. Lasser, A. Toselli	An overlapping domain decomposition preconditioner for a class of discontinuous Galerkin approximations of advection-diffusion problems
00-11	J. Liesen, M. Rozložník, Z. Strakoš	On Convergence and Implementation of Minimal Residual Krylov Subspace Methods for Unsymmetric Linear Systems
00-10	C. Schwab, O. Sterz	A scalar boundary integrodifferential equation for eddy current problems using an impedance boundary condition
00-09	M.H. Gutknecht, M. Rozložník	By how much can residual minimization accelerate the convergence of orthogonal error methods?
00-08	M. Rozložník, V. Simoncini	Short-term recurrences for indefinite preconditioning of saddle point problems
00-07	P. Houston, C. Schwab, E. Süli	Discontinuous *hp*-Finite Element Methods for Advection-Diffusion Problems
00-06	W.P. Petersen	Estimation of Weak Lensing Parameters by Stochastic Integration
00-05	M.H. Gutknecht	A Matrix Interpretation of the Extended Euclidean Algorithm
00-04	M.J. Grote	Nonreflecting Boundary Conditions for Time Dependent Wave Propagation
00-03	M.H. Gutknecht	On Lanczos-type methods for Wilson fermions
00-02	R. Sperb, R. Strebel	An alternative to Ewald sums. Part 3: Implementation and results
00-01	T. Werder, K. Gerdes, D. Schötzau, C. Schwab	*hp* Discontinuous Galerkin Time Stepping for Parabolic Problems