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Optimal sub- or supersolutions in reaction-diffusion problems

R. Sperb

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Seminar für Angewandte Mathematik
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Abstract

The type of problem under consideration is

$$(*) \qquad \begin{cases} u_t = \Delta u + f(u) & \text{in} \quad \Omega \times (0,T) \\ \dfrac{\partial u}{\partial n} + g(u) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x)\,. \end{cases}$$

Here Ω is a finite domain of $\mathbb{R}^N$.

The solution of (*) is compared with a corresponding solution of the N-ball or a finite interval whose size depends on different quantities of an associated linear elliptic problem for Ω, such as e.g. the fixed membrane problem.

Possible applications include estimates for the blow-up or finite vanishing time.

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1 Introduction

Let Ω be a finite domain of R^N and consider the semilinear problem

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in} \quad \Omega \times (0,T) \\ \dfrac{\partial u}{\partial n} + g(u) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x)\,, \end{cases}$$

where n is the exterior normal on $\partial\Omega$. Concerning smoothness we will assume that Ω has a $C^{2+\epsilon}$ boundary and f and g have all the derivatives that are used in the assumptions of the theorems.

Sub- or supersolutions play an important role in proving existence theorems or solution bounds and in many other questions.

In this paper sub- or supersolutions are constructed which are optimal in the sense that they are the solution of (1.1) if Ω is the N-ball ($N \geq 1$) of an appropriate size. The corresponding construction for the steady state has been given in [6], [7] and was motivated by a paper of Payne [3].

In the parabolic case new features come in, and in particular the assumptions on $f(u)$, $g(u)$ are different from the elliptic case. The main idea can be used again and consists in considering two auxiliary problems:

a) the associated radially symmetric problem

$$(1.2) \qquad \begin{cases} \dfrac{\partial R}{\partial t} = \dfrac{1}{r^{N-1}}\, \dfrac{\partial}{\partial r}\left(r^{N-1} \cdot \dfrac{\partial R}{\partial r}\right) + f(R) \;\; \text{in} \;\; (0,r_0) \times (0,T_1) \\ \dfrac{\partial R}{\partial r}\,(0,t) = 0, \;\; \dfrac{\partial R}{\partial r}\,(r_0,t) + g(R(r_0,t)) = 0 \\ R(r,0) = R_0(r)\,, \end{cases}$$

and

b) a standard linear elliptic problem, for example the so-called torsion problem

$$(1.3) \qquad \begin{cases} \Delta\psi + 1 = 0 & \text{in} \quad \Omega\,, \\ \psi = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

Problem (1.3) serves to "transplant" the solution of (1.2) from an interval $(0, r_0)$ to the given domain Ω. This is motivated by the following observation.

For the N-ball one can write the solution of (1.3) as

$$\psi(r) = \frac{1}{2N}\,(N^2\tau^2 - r^2), \quad \tau = |\nabla\psi| \;\text{ on }\; \partial\Omega,\; N \geq 1$$

or as

$$\psi(x) = \psi_m - \frac{1}{2}\,x^2, \;\; \psi_m = \max_{\Omega}\,\psi(x),\; N = 1\,.$$

Hence for $N \geq 1$ one has

$$r = \sqrt{N^2\tau^2 - 2N\psi(r)}\,,$$

and for $N = 1$ we may also write

$$x = \sqrt{2(\psi_m - \psi(x))}\,.$$

These relations suggest the choice of sub- or supersolutions of the form

$$v(x,t) = R(r(x),t) \tag{1.4}$$

with $r(x) = \sqrt{N^2\tau^2 - 2N\psi(x)}$, $\tau = \max\limits_{\partial\Omega} |\nabla\psi|$ or else

$$v(x,t) = X(s(x),t) \tag{1.5}$$

with $s(x) = \sqrt{2(\psi_m - \psi(x))}$ and $X(s,t)$, being the solution of (1.1) for an interval $(0, s_0)$, i.e. $N = 1$ in (1.2).

Instead of the torsion problem one can select the clamped membrane problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in} \quad \Omega \\ \qquad\quad \varphi = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.6}$$

Then the solution for an interval now leads to the choice

$$s(x) = \frac{1}{\sqrt{\lambda_1}}\ \arccos\Big(\frac{\varphi(x)}{\varphi_m}\Big)\ , \tag{1.7}$$

with $\varphi_m = \max\limits_{\Omega} \varphi(x)$, λ_1 = first eigenvalue with associated eigenfunction $\varphi(x)$.

Another choice of an elliptic problem is $w(x)$, where

$$\begin{cases} \Delta w - c^2\, w(x) = 0 & \text{in} \quad \Omega \\ \qquad\qquad w = 1 & \text{on} \quad \partial\Omega\,. \end{cases}$$

This choice has been made in [6] already in the steady state case.

2 The N-ball as optimal domain

Let x be a point of Ω and set

$$r(x) = \sqrt{N^2\tau^2 - 2N\psi(x)}\,, \tag{2.1}$$

$\psi(x)$ being the solution of (1.2). The notation indicates that for the N-ball $r(x)$ = distance from the center. We denote by $R(r,t)$ the solution of (1.2) and use a prime for a derivative with respect to r or else a derivative with respect to R for $f(R)$, $g(R)$. Time derivatives will be denoted by a dot.

The first result can then be stated as

Theorem 1 *Suppose the following assumptions hold*

a) $g(R) \geq 0,\ g'(R) \geq 0,\ f''(R) \geq 0$ *and*

$$\Big(\frac{f(R)}{g(R)} + \frac{H \cdot N}{r_0}\ \log g(R)\Big)' \geq 0, \quad r_0 = N\tau\,.$$

b) *The initial distribution* $R_0(r)$ *of* (1.2) *satisfies for* $0 < r < r_0$,

$$\Big(\frac{R_0'(r)}{r}\Big)' \geq 0\,,$$

and

$$R_0(r(x)) \geq u_0(x)\,.$$

Then

$$\overline{u}(x,t) = R(r(x),\, t)$$

is a supersolution of (1.1) *for* $0 \leq t \leq T_1$.

Proof: From (2.1) we calculate

$$\nabla r = -\frac{N \nabla \psi}{r}\,, \tag{2.2}$$

$$\Delta r = \frac{N}{r}\,\Big(1 - \frac{N\,|\nabla \psi|^2}{r^2}\Big)\,. \tag{2.3}$$

For $\overline{u}(x,t) \equiv R(r(x),t)$ we then have

$$\overline{u}_t - \Delta \overline{u} - f(\overline{u}) = \dot{R} - \frac{N \cdot R'}{r}\,\Big(1 - \frac{N\,|\nabla\psi|^2}{r^2}\Big) - R'' \cdot \frac{N^2\,|\nabla\psi|^2}{r^2} - f(R)\,, \tag{2.4}$$

and using the differential equation for $R(r,t)$ to eliminate $\dot{R} - f(R)$, (2.4) takes the form

$$\overline{u}_t - \Delta \overline{u} - f(\overline{u}) = \Big(R'' - \frac{R'}{r}\Big)\Big\{1 - \frac{N^2\,|\nabla\psi^2|}{r^2}\Big\}\,. \tag{2.5}$$

It was proven by Payne [4] that

$$|\nabla\psi|^2 + \frac{2}{N}\,\psi \leq \tau^2\,,$$

and this inequality in turn implies that the bracket term { } is nonnegative because of the defining equation (2.1) for $r(x)$.

It remains therefore to investigate the sign of the other bracket term on the right of (2.5). We write the radially symmetric part of the Laplacian as Δ_r and set

$$h(r,t) = r^N\Big(R'' - \frac{R'}{r}\Big) = r^N\,\Delta_r\,R - N r^{N-1} \cdot R'\,. \tag{2.7}$$

After a routine calculation one finds that

$$\dot{h} - \Delta_r\, h + \frac{2N}{r}\, h' - f'(R) \cdot h = r^N \cdot f''(R) \cdot R'^2 \,. \tag{2.8}$$

At the end point $r = 0$ we have $h(0,t) = 0$ so that it remains to check the endpoint $r = r_0$. To this end, we form

$$h'(r_0, t) + g'(R) \cdot h(r_0, t)$$

and use that

$$h' = r^N (\Delta_r R)' = r^N (\dot{R} - f(R))' \,. \tag{2.10}$$

The expression $\dot{R}'$ can be eliminated by means of the time derivative of the boundary condition for R. A little manipulation shows then that ($\prime = \frac{d}{dR}$)

$$\frac{\partial h}{\partial r}\Big|_{r_0} + g'(R) \cdot h = r_0^N \cdot g^2 \left(\frac{f(R)}{g(R)} + \frac{N}{r_0} \, \log[g(R)] \right)' \geq 0 \,. \tag{2.11}$$

Since $h(r, 0) \geq 0$ by assumption, the maximum principle again implies that

$$h(r,t) \geq 0 \;\text{ in }\; (0, r_0) \times (0, T_1) \,,$$

and hence

$$\overline{u}_t - \Delta \overline{u} - f(\overline{u}) \geq 0 \;\text{ in }\; \Omega \times (0, T_1) \,. \tag{2.12}$$

On $\partial\Omega \times (0, T)$ we have

$$\frac{\partial \overline{u}}{\partial n} + g(\overline{u}) = R' \cdot \frac{\partial r}{\partial n} + g(R) = g(R) \left\{ 1 - N \, \frac{|\nabla \psi|}{r_0} \right\} \geq 0 \,, \tag{2.13}$$

if we choose $r_0 = N\tau = N \max_{\partial\Omega} |\nabla\psi|$. Finally $\overline{u}(x, 0) = R_0(r(x)) \geq u_0(x)$ by assumption and the proof is completed.

Remarks on Theorem 1

1) One can check that if all inequality signs except for g', are reversed in the assumptions of Theorem 1 then

$$\underline{u}(x,t) = R(r(x), t)$$

is a subsolution.

2) In the case of Dirichlet boundary conditions in (1.1) and (1.2) one can modify the arguments. It follows from the Maximum Principle that the solution $R(r,t)$ of (1.2) with $R(r_0, t) = 0$ now and $R_0(r) \geq 0$ remains nonnegative in $(0, r_0) \times (0, T_1)$ if $f \geq 0$. Hence $R'(r_0, t) \leq 0$. The differential equation for $R(r,t)$ evaluated at the end-point r_0 and the assumption $f(0) = 0$ then imply that $h(r_0, t) \geq 0$.

Hence for Dirichlet boundary conditions assumption a) has to be replaced by

$$(a*) \qquad f(0)=0,\ f(R)\geq 0,\ f''\geq 0\,.$$

Reversion of the inequality signs in (a^*), (b) again yields a subsolution.

For the steady states of (1.1) and (1.2), denoted by $u_s(x)$ or $R_s(r)$ respectively, the proof of Theorem 1 needs only a slight adjustment to show that one has

Corollary 1 *Let $u_s(x)$ and $R_s(r)$ denote steady states of* (1.1) *and* (1.2) *and suppose that $f\geq 0,\ f'\geq 0$ and $g\geq 0$. Then*

$$\overline{u}_s(x)=R_s(r(x))$$

is a supersolution of the steady state case of (1.1).

Proof: The calculations leading to (2.5) now show that

$$\Delta\overline{u}_s+f(\overline{u}_s)=-\Big(R_s''-\frac{R_s'}{r}\Big)\Big\{1-\frac{N^2\,|\nabla\psi|^2}{r^2}\Big\}\,. \tag{2.14}$$

The function

$$h(r)=r^N\Big(R_s''-\frac{1}{r}\,R_s'\Big)=r^N\Delta_r\,R_s-Nr^{N-1}\cdot R_s'$$

satisfies

$$h(0)=0$$

and

$$h'(r)=r^Nf(R_s)\cdot R_s'=:r\cdot f(R_s)\cdot v(r)\,.$$

But, if $f(R_s)\geq 0$, then we have

$$v(r)'=(r^{N-1}\cdot R_s')'\leq 0\,.$$

Since $v(0)=0$, it follows that $v(r)\leq 0$ and therefore $h'(r)\geq 0$, so that $h(r)\geq 0$. Hence one has

$$\Delta\overline{u}_s+f(\overline{u}_s)\leq 0\ \text{ in }\ \Omega\,, \tag{2.15}$$

and since (2.13) also holds for $\overline{u}_s$ the proof of Corollary 1 is completed.

Remark on Corollary 1:

If the inequality signs are reversed in Corollary 1 one obtains a subsolution.

3 The slab as optimal domain

As mentioned in the introduction there is another possibility of using the auxiliary problem (1.3). Let $X(s,t)$ be the solution of

$$(3.1)\qquad \begin{cases} \dot{X} = X'' + f(X) \text{ in } (0,s_0)\times(0,T_1)\,, \\ X'(0,t) = 0,\ X'(s_0,t) + g(X(s_0,t)) = 0 \\ X(s,0) = X_0(s)\,, \end{cases}$$

with a prime denoting a derivative with respect to s. We select now

$$(3.2)\qquad s(x) = \sqrt{2(\psi_m - \psi(x))},\ \psi_m = \max_{\Omega} \psi(x)\,.$$

The analogue of Theorem 1 is then

Theorem 2 *Suppose one has*

a) $f'' \geq 0,\ g \geq 0,\ g' \geq 0$ *and*

$$\Big(\frac{f}{g} + \frac{1}{s_0}\ \log g\Big)' \geq 0$$

for $s_0 = \sqrt{2\psi_m}$.

b) $\Big(\frac{X_0'(s)}{s}\Big)' \geq 0,\ X_0'(0) = 0,$ *and*

$$X_0(s(x)) \geq u_0(x)\,.$$

c) *The mean curvature of* $\partial\Omega$ *is nonnegative everywhere. Then*

$$\overline{u}(x,t) = X(s(x),t)$$

is a supersolution of (1.1).

Proof: Straightforward calculation gives

$$(3.3)\qquad \nabla s = -\frac{\nabla\psi}{s}\,,$$

$$(3.4)\qquad \Delta s = \frac{1}{s}\Big(1 - \frac{|\nabla\psi|^2}{s^2}\Big)$$

and

$$(3.5)\qquad \overline{u}_t - \Delta\overline{u} - f(\overline{u}) = \Big(X'' - \frac{1}{s}\,X'\Big)\Big\{1 - \frac{|\nabla\psi|^2}{s^2}\Big\}\,.$$

It was shown by Payne [4] that the {} term is nonnegative if the mean curvature of $\partial\Omega$ is nonnegative.

We can now just repeat the calculations from (2.7) on to (2.13) with $N = 1$ there and r_0 replaced by s_0. This proves Theorem 2.

The remarks (1) and (2) on Theorem 1 also apply to Theorem 2, with the appropriate changes: $s(x)$ in the place of $r(x)$, $N = 1$, s_0 instead of r_0. In particular one has

Corollary 2 *Let $u_s(x)$, $X_s(x)$ be the steady states of* (1.1) *and* (3.1) *and assume that $f \geq 0$, $f' \geq 0$ and $g \geq 0$. Then*

$$\overline{u}_s(x) = X_s(s(x))$$

with $s(x) = \sqrt{2(\psi_m - \psi(x))}$ is a supersolution of (1.1).

If the membrane problem (1.6) is used in the place of the torsion problem one is led to

Theorem 3 *Assume that the following assumptions hold:*

a) $f'' \geq 0$, $g \geq 0$, $g' \geq 0$ *and* $\left(\frac{f}{g}\right)' - \frac{\lambda_1}{g} \geq 0$.

b) $\left(\frac{X_0'(s)}{\sin(\sqrt{\lambda_1}s)}\right)' \geq 0$, $X_0'(0) = 0$, $X_0(s(x)) \geq u_0(x)$, *with* $s(x) = \frac{1}{\sqrt{\lambda_1}} \arccos\left(\frac{\varphi(x)}{\varphi_m}\right)$, $s_0 = \frac{\pi}{2\sqrt{\lambda_1}}$ *as defined in* (1.6), (1.7).

c) *The mean curvature of $\partial\Omega$ is nonnegative everywhere. Then*

$$\overline{u}(x,t) = X(s(x),t)$$

is a supersolution of (1.1) *for $x \in \Omega$, $0 \leq t \leq T_1$.*

Proof: A routine calculation shows that $\overline{u}$ satisfies

$$\overline{u}_t - \Delta\overline{u} - f(\overline{u}) = \left(\frac{X''}{\lambda_1} - \frac{\cot(\sqrt{\lambda_1}s)}{\sqrt{\lambda_1}}\, X'\right)\left\{1 - \frac{|\nabla\varphi|^2}{\lambda_1(\varphi_m^2 - \varphi^2)}\right\}. \tag{3.6}$$

By a result of Payne & Stakgold [8] the bracket term {} is nonnegative if the mean curvature of $\partial\Omega$ is nonnegative. We have to find conditions to ensure the sign of the other bracket term in (3.6). To this end we now set

$$h(s,t) = X'' \cdot \sin(\sqrt{\lambda_1}s) - \cos(\sqrt{\lambda_1}s) \cdot X' \cdot \sqrt{\lambda_1}\,. \tag{3.7}$$

After some manipulation one obtains the parabolic equation

$$\dot{h} - h'' + 2\sqrt{\lambda_1}\cot(\sqrt{\lambda_1}s) \cdot h' - (f' - \lambda_1)\, h = \sin(\sqrt{\lambda_1}s) \cdot f'' \cdot X'^2\,, \tag{3.8}$$

where the prime is used for derivatives with respect to s and with respect to X (for $f(X)$). For $0 < s \leq s_0 = \frac{\pi}{2\sqrt{\lambda_1}}$ the right side of (3.8) is nonnegative. For $s = 0$ we have $h = 0$ and we therefore check the endpoint $s = s_0$. If we use the time derivative of the boundary condition for $X(s,t)$ and the differential equation in (3.1) we see that for $s = s_0$ one has

$$h' + g'(X) \cdot h = g^2\left[\left(\frac{f}{g}\right)' + \frac{\lambda_1}{g}\right] \geq 0\,. \tag{3.9}$$

Finally $h(s,0) \geq 0$ by the first inequality of assumption and therefore $h(s,t) \geq 0$ in $(0,s_0) \times (0,T_1)$ by the maximum principle. On the boundary $\partial\Omega$ one has

$$\frac{\partial \overline{u}}{\partial n} + g(\overline{u}) = X' \cdot \frac{\partial s}{\partial n} + g(X) = g(X)\,\Big\{1 - \frac{|\nabla \varphi|}{\sqrt{\lambda_1(\varphi_m^2 - \varphi^2)}}\Big\} \geq 0\,, \tag{3.10}$$

since $g \geq 0$ and the bracket term is nonnegative by the result of Payne-Stakgold [8].

By assumption $\overline{u}(x,0) = X_0(s(x)) \geq u_0(x)$ which completes the proof.

Remarks on Theorem 3

1. One can check that for zero Dirichlet boundary date assumption a) reduces to $f(0) = 0,\ f \geq 0,\ f'' \geq 0$.
2. If all inequality signs, except for g', are reversed in assumptions a), b) then one obtains a subsolution.
3. A possible choice for $X_0(s)$ is e.g. for Dirichlet boundary conditions $X_0(s) = \cos(\sqrt{\lambda_1} s)$ if φ_m can be chosen such that
$$\frac{\varphi(x)}{\varphi_m} \geq u_0(x)\,.$$
4. In the steady state situation a corresponding result can be proven:

Corollary 3 *Let $u_s(x)$ and $X_s(x)$ be steady state solutions of* (1.1) *and* (3.1) *respectively. Assume that $g \geq 0$, $f \geq 0$ and $f'(X_s) \geq \lambda_1$. Then*

$$\overline{u}_s(x) := X_s\,\Big(\frac{1}{\sqrt{\lambda_1}}\ \arccos\Big(\frac{\varphi(x)}{\varphi_m}\Big)\Big) \geq u_s(x)\,.$$

Proof: From (3.6) we deduce that now

$$\Delta \overline{u}_s + f(\overline{u}_s) = \Big(\frac{-X_s''}{\lambda_1} + \frac{\cot(\sqrt{\lambda,s})}{\sqrt{\lambda_1}}\ X'\Big) \cdot \Big\{1 - \frac{|\nabla\varphi|^2}{\lambda_1(\varphi_m^2 - \varphi^2)}\Big\}\,. \tag{3.11}$$

Since we know already that $\{\} \geq 0$ it remains to check

$$h(s) = f(X_s) \cdot \sin(\sqrt{\lambda_1} s) + \cos(\sqrt{\lambda_1} s) \cdot X_s' \sqrt{\lambda_1}\,. \tag{3.12}$$

But $h(0) = 0$ and

$$h'(s) = X_s' \sin(\sqrt{\lambda_1} s)(f'(X_s) - \lambda_1) \leq 0 \tag{3.13}$$

since

$$X_s' \leq 0$$

if $f \geq 0$ and $X_s'(0) = 0$. Hence one has

$$\Delta \overline{u}_s + f(\overline{u}_s) \leq 0 \text{ in } \Omega .$$

In addition the boundary inequality (3.10) still holds for $\overline{u}_s$ which shows that $\overline{u}_s$ is a supersolution.

As a last possibility we select (1.8) as an auxiliary problem and let $X(\sigma, t)$ be the solution of the one-dimensional case of (3.1) for the interval $(0, \sigma_0)$. For given value $c > 0$ in problem (1.8) let $w_0 = \min_\Omega w(x)$.

One then has

Theorem 4 *Assume that the following assumption hold:*

a) $f'' \geq 0,\ g \geq 0,\ g' \geq 0$ *for positive arguments and*

$$\Big(\frac{f}{g}\Big)' + \frac{c}{\sqrt{1 - w_0^2}}\, (\log g)' + \frac{c^2}{g} \geq 0 .$$

b) *The initial distribution* $X_0(\sigma)$ *of (3.1) satisfies*

$$\Big(\frac{X_0'}{\mathrm{Sinh}(c\sigma)}\Big)' \geq 0 \;\; \text{and} \;\; X_0(\sigma(x)) \geq u_0(x)$$

with $\sigma(x) = \frac{1}{c}\ \mathrm{Arch}\ (\frac{w(x)}{w_0})$, $\sigma_0 = \frac{1}{c}\ \mathrm{Arch}\ (\frac{1}{w_0})$.

c) *The mean curvature of* $\partial\Omega$ *is nonnegative everywhere. Then*

$$\overline{u}(x, t) = X(\sigma(x), t)$$

is a supersolution of (1.1) *for* $0 \leq t \leq T_1$.

Proof: A calculation shows that

$$\overline{u}_t - \Delta \overline{u} - f(\overline{u}) = [X'' - c \cdot X' \cdot \mathrm{Coth}(c\sigma)] \Big\{1 - \frac{|\nabla w|^2}{c^2(w^2 - w_0^2)}\Big\} , \tag{3.14}$$

where the prime here denotes a derivative with respect to the variable σ. Again by the result of Payne & Stakgold [8] the bracket term $\{\}$ is nonnegative if c) holds.

One has to ensure again that the other bracket term in (3.14) is nonpositive. To show this, set

$$h(\sigma, t) = X'' \cdot \mathrm{Sinh}(c\sigma) - \mathrm{Cosh}(c \cdot \sigma) \cdot X' . \tag{3.15}$$

A straightforward calculation shows that

$$\frac{\partial h}{\partial t} - h'' + 2c \cdot \mathrm{Coth}(c\sigma) \cdot h' - (f' + c^2)h = f'' \cdot X'^2 \cdot \mathrm{Sinh}(c\sigma)\,. \tag{3.17}$$

Here f', f'' again denote derivatives of $f(X)$ with respect to X.

By assumption the right hand side of (3.17) is nonnegative. For $\sigma = 0$ we have $h = 0$ and we therefore investigate the endpoint $\sigma_0 = \frac{1}{c}$ Arch $(\frac{1}{w_0})$. There we use the boundary condition for $X(\sigma, t)$, the differential equation and their derivatives with respect to t. After some routine steps one obtains

$$h'(\sigma_0, t) + g'(X(\sigma_0, t))h(\sigma_0, t) = \mathrm{Sinh}(c\sigma_0)\, g^2\Big[\Big(\frac{f}{g}\Big)' + \mathrm{Coth}(c\sigma_0) \cdot (\log g)' + \frac{c^2}{g}\Big]\,. \tag{3.18}$$

The relation $\mathrm{Cosh}(c\sigma_0) = \frac{1}{w_0}$ and assumption a) allow to apply the maximum principle. Together with the fact that $h(\sigma, 0) \geq 0$ if the first inequality of assumption b) is satisfied we can then deduce that $h(\sigma, t) \geq 0$ in $(0, \sigma_0) \times (0, T_1)$. On $\partial\Omega$ we have

$$\frac{\partial \overline{u}}{\partial n} + g(\overline{u}) = X' \cdot \frac{\partial \sigma}{\partial n} + g(X) = g(X)\Big\{1 - \frac{|\nabla w|}{c\sqrt{w^2 - w_0^2}}\Big\} \geq 0\,, \tag{3.19}$$

again as a consequence of Payne-Stakgold [8].

Finally

$$\overline{u}(x, 0) = X_0(\sigma(x)) \geq u_0(x)$$

is assumed to hold so that all properties of a supersolution are as required.

Remarks on Theorem 4

1. In the case of homogeneous Dirichlet boundary conditions one can check again as before that assumption a) has to be replaced by

$$f(0) = 0,\ f \geq 0 \ \text{ and } \ f'' \geq 0\,. \tag{a^*}$$

2. If all inequality signs except for g', are reversed in assumptions a) and b) then $\underline{u}(x, t) = X(\sigma_0(x), t)$ is a subsolution.

3. The analogue of Corollary 3 can be deduced as well and is stated as

Corollary 4 *Let $u_s(x)$ and $X_s(\sigma)$ be the steady state solutions of* (1.1) *and* (3.1) *respectively. Assume that $g \geq 0$, $f \geq 0$ and $f'(X_s) \geq -c^2$ for some $c > 0$.*

Then

$$\overline{u}_s(x) = X_s\Big(\frac{1}{c}\ Arch\ \Big(\frac{w(x)}{w_0}\Big)\Big) \geq u_s(x)\,.$$

Proof: From (3.14) we see that $\overline{u}_s(x)$ satisfies

$$\Delta\overline{u}_s + f(\overline{u}_s) = [f(X_s) + cX_s' \cdot \mathrm{Coth}(c\sigma)] \left\{1 - \frac{|\nabla w|^2}{c^2(w^2 - w_0^2)}\right\}. \tag{3.20}$$

Furthermore the function

$$h(\sigma) = \mathrm{Sinh}(c\sigma)\, f(X_s(\sigma)) + cX_s'(\sigma) \cdot \mathrm{Cosh}(c\sigma) \tag{3.21}$$

satisfies $h(0) = 0$ and

$$h'(\sigma) = \mathrm{Sinh}(c\sigma) \cdot X_s'(\sigma)(f'(X_s(\sigma)) + c^2). \tag{3.22}$$

But if $f \geq 0$ and $g \geq 0$, then $X_s'(\sigma) \leq 0$ so that $h'(\sigma)$ and therefore $h(\sigma) \leq 0$ for $\sigma \geq 0$, implying that the right side of (3.20) is nonnegative.

On the boundary we have as in (3.19)

$$\frac{\partial \overline{u}_s}{\partial n} + g(\overline{u}_s) \geq 0$$

so that $\overline{u}_s$ is a supersolution, which is the statement of Corollary 4.

4 Examples

4.1 Finite blow-up for nonlinear reaction

Consider the problem

$$\begin{cases} u_t = \Delta u + u^2 + \gamma u \text{ in } \Omega = \text{ball in } \mathbb{R}^3 \text{ of radius } 1,\ u = 0 \text{ on } \partial\Omega \\ u(x,0) = \varphi_1(x) = \text{first eigenfunction} = \frac{1}{2}\frac{\sin(\pi r)}{r}\ (r = |x| \leq 1). \end{cases} \tag{4.1}$$

It is well known that the solution of (4.1) blows up in finite time T.

Let us first mention some known bounds for T. Recall first Kaplan's method [2] which consists in considering the function

$$z(t) = \int_\Omega u(x,t)\varphi_1(x)\, dx. \tag{4.2}$$

Using Jensen's inequality and the scaling $\int_\Omega \varphi_1\, dx = 1$ one finds

$$\dot{z} \geq z^2 + (\gamma - \lambda_1)\, z,\ z(0) = \int_\Omega \varphi_1^2\, dx = z_0, \tag{4.3}$$

and therefore one has the estimate

$$T \leq \int_{z_0}^\infty \frac{dz}{z^2 + (\gamma - \lambda_1)z},\ \ (\gamma > \lambda_1 - z_0). \tag{4.4}$$

A different bound was given in [5], p. 161, namely

$$T \leq \frac{\gamma}{\gamma - \lambda_1} \int_{\varphi_m}^{\infty} \frac{dz}{z^2 + \gamma z}\,, \quad \varphi_m = \max_{\Omega} \varphi_1(x)\,. \tag{4.5}$$

By Theorem 3 we have the lower bound

$$T \geq T_1\,, \tag{4.6}$$

where T_1 is the blow-up time of the one-dimensional case, i.e. problem (1.2) with $N = 1$, Dirichlet boundary conditions and $r_0 = \frac{\pi}{2\sqrt{\lambda_1}} = \frac{1}{2}$, so that $R_0(r) = \frac{\pi}{2}\cos(\pi r)$.

In the next table we list a few values:

γ	Exact value T	lower bound (4.6)	upper bounds (4.4)	(4.5)
10	0.921	0.710	1.178	$15 \cdot 3$
20	0.226	0.210	0.260	0.259
30	0.141	0.135	0.163	0.149

4.2 Finite vanishing time

We consider now the problem

$$\begin{cases} u_t &= \Delta u - \mu \cdot u^p & \text{in } \Omega = \text{ball of radius 1 in } \mathbb{R}^3 \\ u &= 0 & \text{on } \partial\Omega \\ u(r,0) &= \varphi_1(r) = \dfrac{1}{2}\,\dfrac{\sin(\pi r)}{r}\,. \end{cases} \tag{4.7}$$

It is well known that for $0 < p < 1$ the solution vanishes identically in Ω if $t \to T_0 < \infty$.

Kaplan's method also works in this case and a similar reasoning yields the bound

$$T_0 \geq \int_0^{z_0} \frac{dz}{\mu z^p + \lambda_1 z}\,, \tag{4.8}$$

with the same meaning of z_0 and λ_1 as in (4.4). The reasoning leading to (4.5) now gives the alternative lower bound

$$T_0 \geq \frac{1}{\mu + \lambda_1 \varphi_m^{1-p}} \int_0^{\varphi_m} \frac{dz}{z^p}\,. \tag{4.9}$$

The application of Theorem 3 as in Section 4.1 leads to the bound

$$T_0 \leq T_1 \tag{4.10}$$

where T_1 is the vanishing time of problem (1.2) with $N = 1$, Dirichlet boundary conditions, and as before, $r_0 = \frac{1}{2}$, $R_0(r) = \frac{\pi}{2}\ \cos(\pi r)$.

In the next table some numerical values obtained by (4.8), (4.9), (4.10) are compared with the exact values.

$p = \frac{1}{2}$		lower bounds		upper bound
μ	Exact value T_0	(4.8)	(4.9)	(4.10)
5	0.220	0.205	0.144	0.238
10	0.141	0.127	0.112	0.153
30	0.061	0.052	0.059	0.065

4.3 Steady state in a degradation-absorption process

Consider a linear degradation reaction whose steady state concentration u is modeled by the equation

(4.11) $$\Delta u - \gamma^2 u = 0 \ \text{ in } \ \Omega$$

and the absorption through the boundary is described by

(4.12) $$\frac{\partial u}{\partial n} = \sigma(1-u)^p \ \text{ on } \ \partial\Omega\,.$$

Here γ, σ, p are given positive parameters and the exterior concentration is 1.

Let us first write down the solution of (4.11), (4.12) for an interval, a disk and a ball: For an interval $(-s_0, s_0)$ the solution is

(4.13) $$X(s) = \alpha_1 \cdot \mathrm{Cosh}(\gamma s)$$

where α_1 is the unique solution of

(4.14) $$\alpha_1 \gamma\, \mathrm{Sinh}(\gamma s_0) = \sigma(1 - \alpha_1\, \mathrm{Cosh}(\gamma s_0))^p\,.$$

For a disk of radius r_0 the solution is

(4.15) $$R_2(r) = \alpha_2\, I_0(\gamma r),\ I_0 = \text{Besselfunction}\,,$$

and α_2 is the unique solution of

(4.16) $$\alpha_2 \gamma I_1(\gamma r_0) = (1 - \alpha_2 I_0(\gamma r_0))^p\,.$$

Finally for a ball a radius r_0 one obtains the solution

(4.17) $$R_3(r) = \alpha_3\, \frac{\mathrm{Sinh}(\gamma r)}{r}$$

with α_3 being the solution of

$$\alpha_3 \frac{1}{r_0^2} \left(\gamma \operatorname{Cosh}(\gamma r_0) \cdot r_0 - \operatorname{Sinh}(\gamma r_0)\right) = \sigma\Big(1 - \alpha_3 \frac{\operatorname{Sinh}(\gamma r_0)}{r_0}\Big)^p . \tag{4.18}$$

Let us denote the minimum value of the concentration $u(x)$ by μ. Then it is not hard to see from (4.13), (4.14) that μ is the unique solution in $(0,1)$ of the equation

$$\mu\gamma \operatorname{Sinh}(\gamma s_0) = \sigma(1 - \mu \operatorname{Cosh}(\gamma s_0))^p \tag{4.19}$$

if Ω is the intervall $(-s_0, s_0)$. It is easy to see that μ is decreasing with increasing s_0. Hence one would like to have s_0 as small as possible.

Now Corollary 1 gives $s_0 = \tau$ $(N = 1)$, Corollary 2 has $s_0 = \sqrt{2\psi_m}$ and Corollary 3 uses $s_0 = \frac{\pi}{2\sqrt{\lambda_1}}$.

The difference between Corollary 1 and Corollaries 2, 3 is that the first needs no assumption on $\partial\Omega$, but the latter need a boundary whose mean curvature is nonnegative. One has (see [4]) for any geometry of Ω

$$2\psi_m \leq N\tau^2 \tag{4.20}$$

and also (see [5])

$$\lambda_1 \geq \frac{\pi^2}{8\psi_m} \tag{4.21}$$

if the mean curvature of $\partial\Omega$ is nonnegative. Clearly (4.20), (4.21) show that Corollary 3 gives the best value for s_0. For a general domain one will have to use bounds for τ, λ_1 or ψ_m and then it is no longer clear which bound is best. Hence all three Corollaries may be useful.

A typical result one could derive by combining e.g. Corollary 3 with the inequality (see [5])

$$\lambda_1 \geq \frac{\pi^2}{4\rho^2}\,, \quad \rho = \text{radius of largest ball contained in } \Omega\,, \tag{4.22}$$

is stated as

Corollary 5 *Assume that the mean curvature of $\partial\Omega$ is nonnegative. Then the minimum of the concentration u of* (4.11), (4.12) *is bounded below by the unique solution μ in (0,1) of the equation*

$$\mu \cdot \gamma \operatorname{Sinh}(\gamma\rho) = \sigma(1 - \mu \cdot \operatorname{Cosh}(\gamma\rho))^p\,,$$

$$\rho = \textit{radius of largest ball contained in } \Omega\,.$$

In order to get an idea of how close the bounds for $u_{\min} = \mu$ derived e.g. from Corollary 5 are, we compare in the following table the exact value μ with the lower bound. We take $\gamma = \sigma = 1$, Ω a disk or a ball of radius 1 and different values of p.

$\Omega =$ disk, $\lambda_1 = 5.78$	p	$\mu =$	$\mu \geq$
	0.5	0.675	0.649
	2	0.410	0.391
	4	0.288	0.271
$\Omega =$ ball, $\lambda_1 = \pi^2$			
	0.5	0.781	0.751
	2	0.490	0.455
	4	0.344	0.320

4.4 Gelfand problem

The problem under consideration is

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.23}$$

It is well known that this problem has a positive solution only for $0 < \lambda \leq \lambda^* < \infty$. For a disk of radius r_0 one has the solution

$$R(r) = R_m - 2\log\Big[1 + (e^{R_{m/2}} - 1)\Big(\frac{r}{r_0}\Big)^2\Big]\,, \tag{4.24}$$

where $R_m = \max_{0<r<r_0} R(r)$ and

$$\lambda = \frac{8}{r_0^2}\,(e^{-R_{m/2}} - e^{-R_m}) \leq \frac{2}{r_0^2} = \lambda^*\,, \tag{4.25}$$

where λ^* is attained for $R_m = \log 4$ and for $\lambda < \lambda^*$ one has two solutions with values

$$R_m = -2\log\Big[\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\lambda r_0^2}{8}}\Big]\,. \tag{4.26}$$

For the interval $(-L, L)$ the solution can be written as

$$X(s) = X_m - 2\log\Big[\mathrm{Cosh}\Big(\frac{s}{L}\,\mathrm{Arch}(e^{X_{m/2}})\Big)\Big],\; X_m = \max_s\, X(s)\,. \tag{4.27}$$

In this case the relation between λ and X_m is

$$\lambda = \frac{2}{L^2}\,e^{-X_m} \cdot \mathrm{Arch}^2[e^{X_{m/2}}] \leq \frac{0.8785}{L^2} = \lambda^*\,, \tag{4.28}$$

and the maximum value of λ is attained for the solution of

$$m = 2\log\Big[\mathrm{Cosh}\Big(\sqrt{\frac{e^m}{e^m-1}}\Big)\Big] \cong 1.1868\,. \tag{4.29}$$

Some implications of Corollaries 2 and 3 are now considered. Since $\underline{u} \equiv 0$ is a subsolution to (4.23) it suffices to find a positive supersolution.

By Corollary 2 we can select the supersolution

$$\overline{u}(x) = X(\sqrt{2(\psi_m - \psi(x)}) \tag{4.30}$$

for any $\lambda \leq \frac{0.8785}{2\psi_m}$ and hence one has

Corollary 6 *Let Ω be a domain such that the mean curvature of $\partial\Omega$ is nonnegative. Then the critical value λ^* satisfies*

$$\lambda^* \geq \frac{0.4392}{\psi_m} \tag{4.31}$$

For given value of $\lambda \in (0, \frac{0.4392}{\psi_m})$ an upper bound for $u_m = \max_\Omega u$ can be derived ($u =$ minimal solution of (4.23))

$$u_m \leq M(\lambda) \tag{4.32}$$

where $M(\lambda)$ is the first positive solution of

$$\frac{1}{\psi_m}\, e^{-M} \cdot \mathrm{Arch}^2[e^{M/2}] = \lambda\,. \tag{4.33}$$

A lower bound for u_m can be given as well. It is easy to check from (3.13) that the function

$$\underline{u}(x) = X\Big(\frac{1}{\sqrt{\lambda_1}}\ \arccos\Big(\frac{\varphi(x)}{\varphi_m}\Big)\Big) \tag{4.34}$$

is a subsolution to (4.23) if

$$\lambda e^{X_m} \leq \lambda_1\,. \tag{4.33}$$

By (4.28) (with $L = \frac{\pi}{2\sqrt{\lambda_1}}$ there) a little manipulation shows that (4.32) holds provided

$$\lambda \leq \frac{\lambda_1}{\mathrm{Cosh}^2[\frac{\pi}{2\sqrt{2}}]} \cong \frac{\lambda_1}{2.83227} =: \lambda_0\,. \tag{4.34}$$

Therefore, for $\lambda \leq \lambda_0$ one has a lower bound for u_m given by the first positive solution $m(\lambda)$ of

$$\frac{8}{\pi^2}\, \lambda_1\, e^{-m} \cdot \mathrm{Arch}^2[e^{m/2}] = \lambda\,. \tag{4.35}$$

Remark: The bounds (4.33), (4.35) were proven in [3] by different methods.

Some general bounds for λ^*, u_m in problem (4.23) for two-dimensional domains can be found in [1]. One has for a two-dimensional region Ω

$$\frac{2\pi}{A} \leq \lambda^* \leq \frac{2}{\dot{r}^2}\,, \tag{4.36}$$

where A = area of Ω, $\dot{r}$ = maximal conformal radius of Ω. Equality holds in (4.36) if Ω is a disk. An alternative bound is

$$\lambda^* \leq \frac{\lambda_1}{e}\,. \tag{4.37}$$

For given λ with $\mu = \frac{\lambda A}{2\pi} \leq 1$ one has (see [1], p. 199)

$$u_m \leq \log 4 - 2\log\Big(\frac{\mu}{1-\sqrt{1-\mu}}\Big)\,. \tag{4.38}$$

In the next table we compare different bounds for λ^*, $u_m(\lambda)$ for the case that Ω is a square or a rectangle.

Domain	bounds for λ^*	$u_m(\lambda) \geq$ (4.35)	λ	$u_m^{(\lambda)} \leq$ (4.32)	(4.38)
Square	(4.36):	0.066	1	0.0787	0.085
side 1	$6.283 \leq \lambda^* \leq 6.875$	0.141	2	0.170	0.182
$\Psi_m = 0.07367$	(4.31):				
$\lambda_1 = 2\pi^2$	$5.96 \leq \lambda^*$	0.226	3	0.279	0.298
$\dot{r} = 0.5394$		0.329	4	0.418	0.443
		0.458	5	0.616	0.641
	(4.36):				
Rectangle	$3.14 \leq \lambda^* \leq 5$	0.110	1	0.127	0.182
sides 2,1					
	(4.31), (4.37):	0.245	2	0.290	0.443
$\psi_m = 0.11387$	$3.86 \leq \lambda^* \leq 4.538$	0.429	3	0.533	1.00
$\lambda_1 = \frac{5}{4}\,\pi^2$					
$\dot{r} = 0.63189$					

5 Extensions

5.1 Systems

For diffusion-reaction systems of the form

$$u_t^k = D_k \Delta u^k + f^k(uj) \;\text{ in }\; \Omega \times (0,T), \quad k,j = 1,\dots,n \tag{5.1}$$

there are possible extensions of Theorem 1 to 4, provided one has among other things

$$\frac{\partial f^k}{\partial uj} \geq 0 \text{ for } k \neq j \text{ and the matrices}$$

$$A^k_{\ell m} := \Big(\frac{\partial^2 f^k}{\partial u^\ell \partial u^m}\Big) \text{ are positive semidefinite for } k = 1, \dots, n\,.$$

Another version are systems with a mixed quasimonotone structure (see e.g. [9]).

5.2 Elliptic operator L instead of Δ

If the Laplacian is replaced by a general uniformly elliptic operator one has to use a generalisation of the result of Payne & Stakgold [8] applied in this paper. Such generalisations are discussed in [5]. An important case is that L is the Laplace-Beltrami operator. For a surface and an elliptic problem this is treated in [7].

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