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Numerical integration of differential algebraic systems and invariant manifolds

K. Nipp

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Abstract

The dynamics of a differential algebraic equation takes place on a lower dimensional manifold in phase space. Applying a numerical integration scheme, it is natural to ask if and how this geometric property is preserved by the discrete dynamical system. In the index-1 case answers to this question are obtained from the singularly perturbed case treated in [6] for Runge-Kutta methods and in [7] for linear multistep methods. As main result, it is shown that also for Runge-Kutta methods and linear multistep methods applied to an index-2 problem of Hessenberg form there is a (attractive) invariant manifold for the discrete dynamical system and this manifold is close to the manifold of the differential algebraic equation.

1 Introduction

The dynamics of a differential algebraic equation (DAE) is restricted to a manifold. When applying a numerical integration scheme to a DAE, does the discrete dynamical system preserve this geometric property of the continuous dynamical system? We investigate Runge-Kutta methods (RKMs) and linear multistep methods (LMMs) applied to DAEs of index 1 and index 2.

In Section 2 we show that the existence of invariant manifolds for RKMs and LMMs applied to DAEs of index 1 as well as convergence results and global error estimates are obtained from the singularly perturbed case (treated for RKMs in [6] and for LMMs in [7]) just by putting the singular perturbation parameter $\varepsilon = 0$. Indeed, we show that there is a commuting diagram for the two cases.

In Section 3 we consider index-2 problems of Hessenberg form. In Paragraph 3.1 we first deal with RKMs and LMMs applied to the index-1 formulation and show that at least for the case of a linear constraint the commuting diagram of Section 2 still exists also containing the additional 'index-2 submanifolds'. In the nonlinear case we prove a linear (in t) drift off the index-2 submanifold of the DAE. In Paragraph 3.2 we consider RKMs and LMMs applied to the index-2 formulation of the DAE which is preferred in practice (no reduction to index 1). Here, the question of interest is the existence of an attractive invariant 'index-1 manifold'. Again, for the case of a linear constraint it can easily be verified that there is such a manifold. In the nonlinear case, we prove the existence of such an invariant manifold and derive important additional properties for BDF-like RKMs and for LMMs.

For DAEs of index 2 we follow the lines of [2] and [4] where also a standard bibliography for DAEs may be found. It is to mention that in [1] invariant manifold techniques similar to ours have been applied to the index-1 formulation of index-2 DAEs in order to investigate stabilizations of the linear drift mentioned above. The results and invariant manifold techniques of this paper may also be applied to index-3 problems of Hessenberg form (cf. [2], [4]) which admit *three* types of manifolds that may or may not 'persist' under numerical approximation.

In this introductory section, in order to introduce the notation and to keep the paper mostly self-contained and legible, we first summarize the results for singularly perturbed ODEs and their approximations by RKMs and LMMs given in [6], [7]. There, RKMs and LMMs are applied to stiff systems of singular perturbation type of the form

$$(1)_\varepsilon \qquad \begin{aligned} \dot{x} &= f(x,y) \\ \varepsilon\dot{y} &= g(x,y) \end{aligned}$$

satisfying

Hypothesis H_ε

1) f and g are bounded and there is r with $r \geq 2$ such that $f \in C_b^r(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m)$, $g \in C_b^r(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$.

2) There is a function $s_0 \in C_b^r(\mathbb{R}^m, \mathbb{R}^n)$ such that $g(x, s_0(x)) = 0$ for $x \in \mathbb{R}^m$.

3) There is a positive constant b_0 such that all eigenvalues of the Jacobian $g_y(x, s_0(x))$ have real parts smaller than $-b_0$ for all $x \in \mathbb{R}^m$.

By C_b^r we denote spaces of functions of class C^r with bounded derivatives.

Under these assumptions Eq. $(1)_\varepsilon$ admits, for all $\varepsilon > 0$ small enough, an attractive invariant manifold M_ε which is the graph over x-space of a smooth function s

$$M_\varepsilon = \{(x, y) | x \in \mathbb{R}^m, \ y = s(x, \varepsilon)\}$$

with s of class C_b^r with respect to x *and* ε (for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, ε_0 small) and $s(x, 0) = s_0(x)$. M_ε is highly attractive, i.e.,

$$|y(t) - s(x(t), \varepsilon)| \leq K \, \chi_\varepsilon^t \, |y(0) - s(x(0), \varepsilon)| \,, \quad t \geq 0 \,,$$

where $\chi_\varepsilon^t := e^{-\beta t/\varepsilon}$ with $\beta \in (0, b_0)$.

M_ε possesses a stable foliation, i.e., there exists a positively invariant family of stable fibers which are smooth manifolds over y-space. The 'steepness' of the fibers is of the order of $L_{12}^\varepsilon = \varepsilon Lip_y(f)$, i.e., if (x, y) and $(\bar{x}, \bar{y})$ are two points on a fiber then $\bar{x} - x = O(\varepsilon)$. As a consequence, the property of 'asymptotic phase' holds: For every trajectory of Eq. $(1)_\varepsilon$ there exists a unique trajectory on M_ε such that the two trajectories tend to each other exponentially with rate χ_ε^t. The whole situation is sketched in Fig. 1.

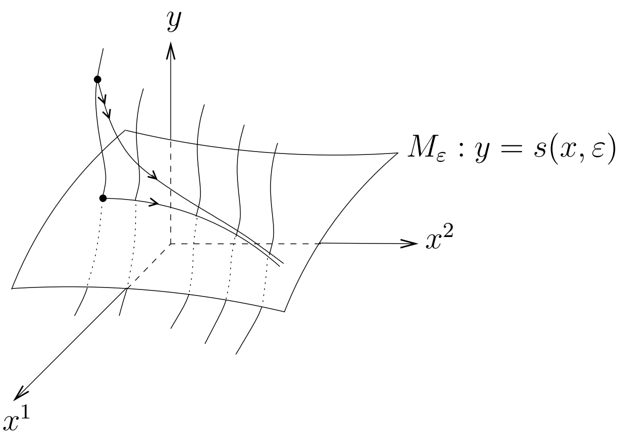

Fig. 1: The attractive invariant manifold M_ε of Eq. $(1)_\varepsilon$, stable foliation and 'asymptotic phase'

For RKMs applied to Eq. $(1)_\varepsilon$ it is assumed that the following assumptions hold.

Hypothesis $\mathrm{H_{RKM}}$

1) The RKM has order p and stage order $1 \leq q < p$.

2) The RKM-matrix A is invertible.

3) The stability function $R(z) := 1 + zb^T(I_s - zA)^{-1}\mathbb{1}_s$, $z \in \mathbb{C}$, where $\mathbb{1}_s := (1, \ldots, 1)^T \in \mathbb{R}^s$, satisfies $|R(\infty)| < 1$.

Here, s is the number of stages of the RKM. Since A is invertible, $R(\infty) = 1 - b^T A^{-1} 1\!\!1_s$ holds. For RKMs with $a_{si} = b_i, i = 1, \ldots, s$, this implies $R(\infty) = 0$. Under Hypothesis $\mathrm{H_{RKM}}$ the RKM

$$
\begin{aligned}
X &= 1\!\!1_s \otimes x + h(A \otimes I_m)\, f(X, Y) \\
Y &= 1\!\!1_s \otimes y + \frac{h}{\varepsilon}\,(A \otimes I_n)\, g(X, Y) \\
\bar{x} &= x + h(b^T \otimes I_m)\, f(X, Y) \\
\bar{y} &= y + \frac{h}{\varepsilon}\,(b^T \otimes I_n)\, g(X, Y)
\end{aligned}
$$

defines a smooth map $(\mathrm{I})_{\mathrm{h},\varepsilon} :\ (\mathrm{x}, \mathrm{y}) \longmapsto (\bar{\mathrm{x}}, \bar{\mathrm{y}})$ from $\mathbb{R}^m \times \mathbb{R}^n$ into itself. (We have used the notation $X := (X_1, \ldots, X_s)^T \in \mathbb{R}^{sm},\ f(X, Y) := (f(X_1, Y_1), \ldots, f(X_s, Y_s))^T \in \mathbb{R}^{sm}$, etc., $\otimes$ denotes the Kronecker product.) In coordinates measuring the difference to M_ε

$$
y = s(x, \varepsilon) + z, \quad Y = s(X, \varepsilon) + Z
$$

this map may be written as (we often suppress the dependence of $s(x, \varepsilon)$ on ε for short)

$$
(\widetilde{\mathrm{I}})_{\mathrm{h},\varepsilon} \qquad
\begin{aligned}
\bar{x} &= x + h(b^T \otimes I_m)\, f(X, s(X) + Z) \\
\bar{z} &= (R(\infty) I_n + O(\varepsilon/h)) z + ((b^T A^{-1} \otimes I_n) + O(\varepsilon/h))\, E - e
\end{aligned}
$$

where the stages X, Z satisfy

$$
\begin{aligned}
X &= 1\!\!1_s \otimes x + h(A \otimes I_m)\, f(X, s(X) + Z) \\
Z &= O(\varepsilon/h)\ .
\end{aligned}
$$

The functions E and e are defined as

$$
\begin{aligned}
E(x, X) \quad &:= \quad s(X) - 1\!\!1_s \otimes s(x) - \frac{h}{\varepsilon}\,(A \otimes I_n)\, g(X, s(X)) \\
e(x, \bar{x}, X) \quad &:= \quad s(\bar{x}) - s(x) - \frac{h}{\varepsilon}\,(b^T \otimes I_n)\, g(X, s(X))\ .
\end{aligned}
$$

Note that $g(X, s(X)) = O(\varepsilon)$.

The RKM-map $(\mathrm{I})_{\mathrm{h},\varepsilon}$ admits an attractive invariant manifold $M_{h,\varepsilon}$ which is the graph of a smooth function $\sigma(x, h, \varepsilon)$:

$$
M_{h,\varepsilon} = \{(x, y) | x \in \mathbb{R}^m,\ y = \sigma(x, h, \varepsilon)\}
$$

with σ of class C_b^r with respect to x *and* ε (also for $\varepsilon = 0$) and

$$
\sigma(x, h, \varepsilon) = s(x, \varepsilon) + \begin{cases} O(h^{q+1}) \\ O(\varepsilon h^q), \text{ if } \ b_i = a_{s_i},\ i = 1, \ldots, s\ . \end{cases}
$$

The attractivity rate is $\chi_{h,\varepsilon} = |R(\infty)| + c\,\varepsilon/h < 1$. $M_{h,\varepsilon}$ again has a stable foliation with fibers of 'steepness' $O(\varepsilon)$ (since the Lipschitz constant $L_{12}^{h,\varepsilon}$ of the right-hand side of the first equation of $(\widetilde{\mathrm{I}})_{\mathrm{h},\varepsilon}$ is $O(\varepsilon)$) and every RKM-orbit has an accompanying 'asymptotic phase' orbit on $M_{h,\varepsilon}$. In Fig. 2 a sketch of these results is given. For the global error of the RKM applied to Eq. $(1)_\varepsilon$ we have for h and ε/h small enough and for $jh \le Nh = T$ fixed

$$(\mathrm{GE})_{\mathrm{h},\varepsilon} \qquad \begin{aligned} x_j - x(jh) &= O(h^p) + O(\varepsilon h^{q+1}) + O(\varepsilon|y_0 - s(x_0,\varepsilon)|) \\ y_j - y(jh) &= O(h^{q+1}) + O((\varepsilon + \chi_{h,\varepsilon}^j)\,|y_0 - s(x_0,\varepsilon)|) \end{aligned}$$

where for $b_i = a_{s_i}$, $i = 1,\ldots,s$, the term $O(h^{q+1})$ in the y-equation is replaced by $O(h^p) + O(\varepsilon h^q)$.

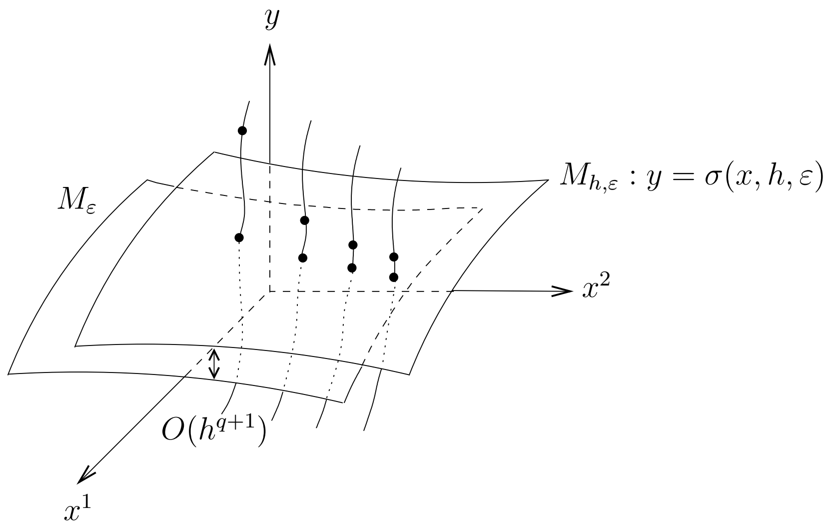

Fig. 2: The attractive invariant manifold $M_{h,\varepsilon}$ of the RKM-map $(\mathrm{I})_{\mathrm{h},\varepsilon}$, stable foliation and asymptotic phase and closeness to M_ε

For LMMs we have the analogous results although the situation is somewhat more complicated. This is due to the fact that LMMs cannot be considered as a map from phase space into itself. They are best described by a map in a high-dimensional space.

A k-step method applied to Eq. $(1)_\varepsilon$ is defined by

$$\left.\begin{aligned} \sum_{i=0}^{k} \alpha_i\, x_i &= h \sum_{i=0}^{k} \beta_i\, f(x_i,y_i) \\ \sum_{i=0}^{k} \alpha_i\, y_i &= \frac{h}{\varepsilon} \sum_{i=0}^{k} \beta_i\, g(x_i,y_i) \end{aligned}\right. \quad ,\ \alpha_k = 1\,,$$

where (x_i,y_i), $i = 0,\ldots,k-1$, are given starting values. We make the following assumption.

Hypothesis $\mathrm{H_{LMM}}$

1) The LMM is an irreducible k-step method of order $p \geq 1$.

2) The LMM is ρ_1-strictly stable, i.e., the polynomial $\rho(z) := \sum\limits_{j=0}^{k} \alpha_j z^j$ has 1 as a simple zero and all other zeros have modulus smaller than $\rho_1 < 1$.

3) The LMM is σ_1-stiffly stable, i.e., $\beta_k \neq 0$ and all zeros of the polynomial $\sigma(z) := \sum\limits_{j=0}^{k} \beta_j z^j$ have modulus smaller than $\sigma_1 < 1$.

Note that $\sum_{i=0}^{k} \alpha_i = 0$ and since $\alpha_k = 1$ this implies $\sum_{i=0}^{k-1} \alpha_i = -1$. Under the above assumptions, the LMM defines a smooth map $(\mathrm{I})_{\mathrm{h},\varepsilon}$ from $\mathbb{R}^{km} \times \mathbb{R}^{kn}$ into itself. Defining $\alpha := (\alpha_0, \dots, \alpha_{k-1})^T \in \mathbb{R}^k$, $\beta := (\beta_0, \dots, \beta_{k-1})^T \in \mathbb{R}^k$ and

$$R := \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & 1 \\ 0 & & & \\ & & & 0 \end{pmatrix}, \quad X_i := \begin{pmatrix} x_i \\ \vdots \\ x_{i+k-1} \end{pmatrix}, \quad L_\alpha := e_k\, \alpha^T = \begin{pmatrix} & 0 & \\ & & \\ \alpha_0 & \dots & \alpha_{k-1} \end{pmatrix},$$

etc., and again measuring the difference to M_ε by the change of coordinates $y = s(x, \varepsilon) + z$ this map has the form

$$(\widetilde{\mathrm{I}})_{\mathrm{h},\varepsilon} \qquad \begin{aligned} X_1 &= ((R - L_\alpha) \otimes I_m)\, X_0 + h(L_\beta \otimes I_m)\, f(X_0, s(X_0) + Z_0) \\ &\quad + h\beta_k(e_k \otimes f(x_k, s(x_k) + z_k)) \\ Z_1 &= \Big[(R \otimes I_n) - \frac{1}{\beta_k}\, (L_\beta \otimes I_n) + |\beta|\, O(\max_{0 \leq i < k} |x_i - x_0| \\ &\quad + h + d + \varepsilon/h) + O(\varepsilon/h)\Big]\, Z_0 + O(\varepsilon/h) \end{aligned}$$

mapping $(X_0, Z_0) \in \mathbb{R}^{km} \times \mathbb{R}^{kn} \cap \{|Z_0|_\infty \leq d\}$ to $(X_1, Z_1) \in \mathbb{R}^{km} \times \mathbb{R}^{kn}$.

The LMM-map $(\mathrm{I})_{\mathrm{h},\varepsilon}$ admits an m-dimensional attractive invariant manifold $S_{h,\varepsilon}$ in $\mathbb{R}^{km} \times \mathbb{R}^{kn}$

$$\begin{aligned} S_{h,\varepsilon} = \; & \{x_0, \dots, x_{k-1},\, y_0, \dots, y_{k-1} \,|\, x_0 \in \mathbb{R}^m, \\ & x_i = \Phi^i(x_0, h, \varepsilon), \;\; y_i = \sigma(x_i, h, \varepsilon), \;\; i = 0, \dots, k-1\}\,. \end{aligned}$$

The function Φ is a one-step method of order p for the differential equation $\dot{x} = f(x, s(x, \varepsilon))$ (by Φ^i we denote the i-th iterate) and

$$\sigma(x, h, \varepsilon) = s(x, \varepsilon) + O(\varepsilon h^p)\,.$$

If started appropriately, i.e., $x_i = \Phi^i(x_0, h, \varepsilon)$, $i = 0, \dots, k-1$, the manifold

$$M_{h,\varepsilon} = \{(x, y) \,|\, x \in \mathbb{R}^m, \;\; y = \sigma(x, h, \varepsilon)\}$$

is invariant under the map $(\mathrm{I})_{\mathrm{h},\varepsilon}$ and is $O(\varepsilon h^p)$-close to M_ε. The attractivity rate in y-direction is $\chi_{h,\varepsilon} = \max\{\rho_1, \sigma_1\} + O(h + \varepsilon/h + d) < 1$ and for $\beta = 0$ one has $\chi_{h,\varepsilon} = \rho_1 + O(h+\varepsilon/h) < 1$ and $y_j - \sigma(x_j, h, \varepsilon) = z_j + O(\varepsilon h^p) = O(\varepsilon/h)$ for $j \geq k$ as seen from $(\tilde{\mathrm{I}})_{\mathrm{h},\varepsilon}$ giving the y-attractivity $const \cdot \varepsilon/h \cdot \chi^j_{h,\varepsilon}$, $j \geq k$. Again there exists a stable foliation ($L^{h,\varepsilon}_{12} = O(\varepsilon)$, i.e., 'steepness' of the fibers $O(\varepsilon)$) implying the property of asymptotic phase. Hence, Fig. 2 again gives the right picture also for LMMs (after the k-th step). For the global error of the LMM applied to Eq. $(1)_\varepsilon$ we have for h and ε/h small enough and for $jh \leq T$ fixed:

$$(\mathrm{GE})_{\mathrm{h},\varepsilon} \qquad \begin{aligned} x_j - x(jh) &= O(h^p) + O(\max_{0\leq i<k}\{|x_i - x(ih)|\}) \\ &+ O(h\,[\max_{0\leq i<k}\{|y_i - y(ih)|\} + |y_0 - s(x_0,\varepsilon)|]) \\ y_j - y(jh) &= O(h^p) + O(\max_{0\leq i<k}\{|x_i - x(ih)|\}) \\ &+ O((h + \chi^j_{h,\varepsilon})[\max_{0\leq i<k}\{|y_i - y(ih)|\} + |y_0 - s(x_0,\varepsilon)|]) \,. \end{aligned}$$

For $\beta = 0$ the factor h in the x-equation is replaced by ε and the factor $(h + \chi^j_{h,\varepsilon})$ in the y-equation is replaced by the factor $(\varepsilon + O(\varepsilon/h)^{[j/k]})$.

Notation: Throughout this paper, we denote the continuous dynamical system of singular perturbation type by $(1)_\varepsilon$ (satisfying Hypothesis H_ε), its invariant manifold by M_ε with attractivity χ^t_ε and 'steepness' of the stable fibers L^ε_{12}. Similarly, for *both* RKMs *and* LMMs applied to $(1)_\varepsilon$ we denote the discrete dynamical system by $(\mathrm{I})_{\mathrm{h},\varepsilon}$, its invariant manifold by $M_{h,\varepsilon}$ with attractivity $\chi_{h,\varepsilon}$ and 'steepness' of the stable fibers $L^{h,\varepsilon}_{12}$, and the global error estimate by $(\mathrm{GE})_{\mathrm{h},\varepsilon}$. If we consider the variable z (instead of y) measuring the distance to M_ε we put a $\tilde{}$ on $(\mathrm{I})_{\mathrm{h},\varepsilon}$, i.e., we write $(\tilde{\mathrm{I}})_{\mathrm{h},\varepsilon}$.

In the DAE case, we replace the ε by 0 for index 1: $(1)_0$ (satisfying Hypothesis H_0), M_0, χ^t_0, L^0_{12}; $(\mathrm{I})_{\mathrm{h},0}$, $M_{h,0}$, $\chi_{h,0}$, $L^{h,0}_{12}$, $(\mathrm{GE})_{\mathrm{h},0}$ and $(\tilde{\mathrm{I}})_{\mathrm{h},0}$. For index 2 we keep the notation but put a $\bar{}$ on top, i.e., $(\bar{1})_0$ (satisfying Hypothesis $\bar{\mathrm{H}}_0$), $\overline{M}_0$; $(\bar{\mathrm{I}})_{\mathrm{h},0}$, $\overline{M}_{h,0}$, etc. ⊣

2 Systems of index 1

Consider the DAE

$$(1)_0 \qquad \begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y) \end{aligned}$$

satisfying

Hypothesis H_0

1) f and g are bounded and $f \in C^r_b(R^m \times \mathbb{R}^n, \mathbb{R}^m)$, $g \in C^r_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$ for $r \geq 2$.

2) There is a function $s_0 \in C^r_b(\mathbb{R}^m, \mathbb{R}^n)$ such that $g(x, s_0(x)) = 0$ for $x \in \mathbb{R}^m$.

3) The matrix $g_y(x, s_0(x))$ is invertible and $g_y(x, s_0(x))^{-1}$ is bounded for $x \in \mathbb{R}^m$.

Under these assumptions, Eq. $(1)_0$ is of index 1 since $g(x,y)=0$ has a unique solution $y=s_0(x)$ for $(x,y)\in\Omega_d:=\{x\in\mathbb{R}^m,\ |y-s_0(x)|\le d\}$ if d is small enough. All solutions of the DAE $(1)_0$ in Ω_d lie on the m-dimensional surface

$$M_0=\{(x,y)|\,x\in\mathbb{R}^m,\ y=s_0(x)\}\subset\mathbb{R}^m\times\mathbb{R}^n\ .$$

Remark 1): i) If, for $(x_0,y_0)\in\Omega_d$, we define the 'impulse solution' $(x(t),y(t))$ with $x(0)=x_0,\ y(0)=s_0(x_0)$ the set M_0 may be viewed as 'infinitely attractive invariant manifold' of $(1)_0$ (there is no dynamics off M_0) with a vertical stable foliation ($L^0_{12}=0$, i.e., all points on a fiber have the same x-coordinate). Note that the index-1 case is just the limit case ($\varepsilon=0$) of the singularly perturbed case (except for Hypothesis H_0 3)).

ii) Note that in Hypothesis H_0 3) we do not assume that $g_y(x,s_0(x))$ has eigenvalues with negative real part as we did in Hypothesis H_ε 3). M_0 is 'infinitely attractive' independently of the sign of the real part of the eigenvalues as is the invariant manifold $M_{h,\varepsilon}$ (and $M_{h,0}$ below) of the discrete dynamical system (since $|R(\infty)|<1$). Hence, since there is also an invariant manifold of Eq. $(1)_\varepsilon$ in the case of H_0 3) which is hyperbolic, however, all assertions of Section 2 hold under Hypothesis H_0 3). In this case, for the global error of RKMs and LMMs applied to Eq. $(1)_\varepsilon$ one has to consider solutions of $(1)_\varepsilon$ on M_ε, however. ⊣

2.1 RKMs

We follow the lines of [2] and [4].

a) The indirect approach. The RKM

$$\begin{aligned}
X &= 1\!\!1_s\otimes x+h(A\otimes I_m)\,f(X,Y)\\
0 &= g(X,Y)\\
\bar{x} &= x+h(b^T\otimes I_m)\,f(X,Y)\\
0 &= g(\bar{x},\bar{y})
\end{aligned}$$

is, due to Hypothesis H_0, equivalent to applying a RKM to the m-dimensional system $\dot{x}=f(x,s_0(x))$ and defining $y_k=s_0(x_k)$. If the method is of order p the global error is $O(h^p)$ (for a nonstiff vector field f).

b) The direct approach. Here, the RKM is derived via $(1)_\varepsilon$ and $(\mathrm{I})_{\mathrm{h},\varepsilon}$ and putting $\varepsilon=0$. For the RKM assume Hypothesis $\mathrm{H}_{\mathrm{RKM}}$. If we put $\varepsilon=0$ in Hypothesis H_ε, in the differential equation $(1)_\varepsilon$ and in the map $(\widetilde{\mathrm{I}})_{\mathrm{h},\varepsilon}$, in the invariant manifolds M_ε and $M_{h,\varepsilon}$, in the attractivity constants χ^t_ε and $\chi_{h,\varepsilon}$, and in the Lipschitz constants L^ε_{12} and $L^{h,\varepsilon}_{12}$ we obtain Hypothesis H_0, Eq. $(1)_0$, the map

$$(\widetilde{\mathrm{I}})_{\mathrm{h},0}\qquad \begin{aligned}
X &= 1\!\!1_s\otimes x+h(A\otimes I_m)\,f(X,s_0(X)+Z)\\
0 &= Z\\
\bar{x} &= x+h(b^T\otimes I_m)\,f(X,s_0(X)+Z)\\
\bar{z} &= (1-b^T A^{-1}1\!\!1_s)\,z+(b^T A^{-1}\otimes I_n)E_0-e_0
\end{aligned}$$

where $E_0 := s_0(X) - \mathbb{1}_s \otimes s_0(x)$, $e_0 := s_0(\bar{x}) - s_0(x)$, and the invariant manifolds M_0 with attractivity $\chi_0^t = 0$ and

$$M_{h,0} = \{(x,y) \,|\, x \in \mathbb{R}^m,\ y = \sigma_0(x,h) := \sigma(x,h,0)\}$$

with attractivity $\chi_{h,0} = |R(\infty)|$, and $L_{12}^{h,0} = 0$. From [6] we know that $(b^T A^{-1} \otimes I_n)E - e = O(h^{q+1}) + O(\varepsilon)$ and $\sigma(x,h,\varepsilon) - s(x,\varepsilon) = O(h^{q+1})$ implying $(b^T A^{-1} \otimes I_n)E_0 - e_0 = O(h^{q+1})$ and $\sigma_0(x,h) - s_0(x) = O(h^{q+1})$. Note that in the x, y-variables (in Ω_d) the RKM-map has the form

$$(\mathrm{I})_{\mathrm{h,0}} \qquad \begin{array}{rcl} X &=& \mathbb{1}_s \otimes x + h(A \otimes I_m)\, f(X,Y) \\ O &=& g(X,Y) \\ \bar{x} &=& x + h(b^T \otimes I_m)\, f(X,Y) \\ \bar{y} &=& (1 - b^T A^{-1}\, \mathbb{1}_s)\, y + (b^T A^{-1} \otimes I_n)\, Y \end{array}$$

considered in [2] and [4]. The global error $(\mathrm{GE})_{\mathrm{h,0}}$ of $(\mathrm{I})_{\mathrm{h,0}}$ is also obtained from $(\mathrm{GE})_{\mathrm{h},\varepsilon}$ by putting $\varepsilon = 0$.

Hence, we have shown that the RKM-map $(\mathrm{I})_{\mathrm{h,0}}$ admits an m-dimensional attractive invariant manifold $M_{h,0}$ which is $O(h^{q+1})$-close to M_0. The precise results are given in

Theorem 1 *Let Eq.* $(1)_0$ *satisfy Hypothesis* H_0*, apply a RKM satisfying Hypothesis* $\mathrm{H}_{\mathrm{RKM}}$ *and assume* $r < p$.

Then there exist positive constants d, h_0, K *and a function* $\sigma_0 : \mathbb{R}^m \times (0, h_0] \to \mathbb{R}^n$*, of class* C_b^r *with respect to* x*, such that for* $h \leq h_0$ *and for* $(x,y) \in \Omega_d := \{(x,y) \,|\, x \in \mathbb{R}^m,$ $|y - s_0(x)| \leq d\} \subset \mathbb{R}^m \times \mathbb{R}^n$ *the following assertions hold.*

i) The set $M_{h,0} = \{(x,y) \,|\, x \in \mathbb{R}^m,\ y = \sigma_0(x,h)\}$ *is invariant under the RKM-map* $(\mathrm{I})_{\mathrm{h,0}}$.

ii) For (x,y) *in* Ω_d *the invariant manifold* $M_{h,0}$ *is attractive with attractivity constant* $\chi_{h,0} = |R(\infty)|$*, i.e.,*

$$|\bar{y} - \sigma_0(\bar{x},h)| \leq \chi_{h,0}\, |y - \sigma_0(x,h)|$$

holds.

iii) $M_{h,0}$ *has a stable (vertical) foliation of the form*

$$W^s(x,y) = \{(\xi,\eta) \,|\, \xi = x,\ \ |\eta - s_0(\xi)| \leq d\}\ ,$$

implying the property of asymptotic phase, i.e., for $(x_0,y_0) \in \Omega_d$ *there is* $(\hat{x},\hat{y}) \in M_{h,0}$ *such that the RKM-orbits* (x_j,y_j) *and* $(\hat{x}_j,\hat{y}_j)$ *satisfy*

$$\begin{array}{l} \hat{x}_j = x_j \\ |\hat{y}_j - y_j| \leq K\, \chi_{h,0}^j\, |y_0 - \sigma_0(x_0,h)| \end{array}$$

for $j \in \mathbb{N}_0$.

iv)
$$\sigma_0(x,h) = s_0(x) + h^{q+1}\, Q(x,h)\ ,$$

with σ_0 of class C_b^r with respect to x and $|Q| \le K$, for all $x \in \mathbb{R}^m$.

v) Let $(x(t), y(t))$ be a solution of Eq. $(1)_0$ with $x(0) = x_0$ and let $(x_0, y_0) \in \Omega_d$. Then for every $T > 0$ fixed, there is $K > 0$ such that the global error satisfies

$$(\mathrm{GE})_{\mathrm{h,0}} \qquad \begin{aligned} |x_j - x(jh)| &\le K\, h^p \\ |y_j - y(jh)| &\le K(h^{q+1} + \chi_{h,0}^j\, |y_0 - s_0(x_0)|) \end{aligned}$$

for $jh \le T$.

The assertions of Theorem 1 are illustrated in Fig. 3.

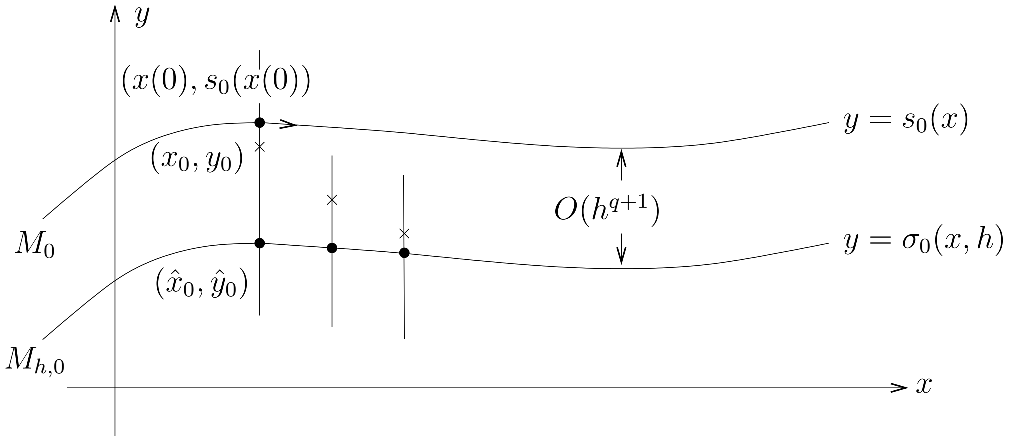

Fig. 3: The invariant manifolds for the DAE $(1)_0$ and the RKM-map $(\mathrm{I})_{\mathrm{h,0}}$

Remark 2): In the case where the RKM satisfies $a_{s_i} = b_i$, $i = 1, \ldots, s$ (implying $R(\infty) = 0$, since A is invertible), we have $\bar{y} = s_0(\bar{x})$ in $(\mathrm{I})_{\mathrm{h,0}}$ and therefore $\sigma_0(x,h) = s_0(x)$ and $M_{h,0} = M_0$, $\chi_{h,0} = 0$ (infinite attractivity). The global error satisfies

$$(\mathrm{GE})_{\mathrm{h,0}} \qquad \begin{aligned} |x_j - x(jh)| &\le K\, h^p \\ |y_j - y(jh)| &\le K\, h^p \end{aligned}\ ,\ jh \le T\ .$$

For $a_{s_i} = b_i$, $i = 1, \ldots, s$, the direct and the indirect approach are identical. ⊣

Summarizing, we have shown for the direct approach that the diagram of Fig. 4 commutes, i.e., the results for the RKM applied to Eq. $(1)_0$ are obtained from the results of the RKM applied to Eq. $(1)_\varepsilon$ just by putting $\varepsilon = 0$. Of course, the DAE-results could be derived by directly applying the invariant manifold theory for maps (cf. [5]) to the map $(\widetilde{\mathrm{I}})_{\mathrm{h,0}}$. However, on the one hand, the results for the singularly perturbed case have been derived before (cf. [6]) and, on the other hand, the diagram of Fig. 4 gives additional insight.

$$
\begin{array}{ccc}
(1)_\varepsilon; & \varepsilon\,\dot y & (1)_0\,; \\
M_\varepsilon, \chi_\varepsilon^t, L_{12}^\varepsilon & \varepsilon = 0 - - \rightarrow & M_0, \chi_0^t, L_{12}^0 \\
\text{RKM}\downarrow & & \downarrow\text{RKM} \\
(\mathrm{I})_{\mathrm{h},\varepsilon}; & \varepsilon = 0 & (\mathrm{I})_{\mathrm{h},0}; \\
M_{h,\varepsilon}, \chi_{h,\varepsilon}, L_{12}^{h,\varepsilon};\ (\mathrm{GE})_{\mathrm{h},\varepsilon} & & M_{h,0}, \chi_{h,0}, L_{12}^{h,0};\ (\mathrm{GE})_{\mathrm{h},0}
\end{array}
$$

Fig. 4: The correspondence between the DAE and the singularly perturbed differential equation and their RKM-maps

2.2 LMMs

For a LMM satisfying Hypothesis $\mathrm{H}_{\mathrm{LMM}}$ applied to Eq. $(1)_0$ we obtain analogous results as for the RKMs of Paragraph 2.1, in particular, there is again a commuting diagram. There is one major difference, however, the 'discrete' manifold $M_{h,0}$ is always equal to the 'continuous' manifold M_0. More precisely, by putting $\varepsilon = 0$ in $(\widetilde{\mathrm{I}})_{\mathrm{h},\varepsilon}$ we obtain the LMM-map in $\mathbb{R}^{km} \times \mathbb{R}^{kn}$ ($Y = s_0(X) + Z$, $|Z|_\infty \le d$, d small enough)

$$
(\widetilde{\mathrm{I}})_{\mathrm{h},0} \qquad
\begin{aligned}
X_1 &= ((R - L_\alpha) \otimes I_m)\, X_0 + h(L_\beta \otimes I_m)\, f(X_0, s_0(X_0) + Z_0) \\
&\quad + h(\beta_k \otimes f(x_k, s_0(x_k) + z_k)) \\
Z_1 &= \{(R \otimes I_n) + (L_\beta \otimes C(x_k, z_k)^{-1})\, \mathrm{diag}[B_0(X_0) + \widehat{B}(X_0, Z_0)]\}\, Z_0
\end{aligned}
$$

where $C(x_k, z_k) := -\beta_k(B_0(x_k) + \widehat{B}(x_k, z_k))$, $B_0(x) := g_y(x, s_0(x))$, $\widehat{B}(x, 0) = 0$. (Note that it can be seen directly that $Z = 0$ is an invariant manifold of $(\widetilde{\mathrm{I}})_{\mathrm{h},0}$).

Again, all results are inherited from the singularly perturbed case by putting $\varepsilon = 0$:

- Invariant manifold of $(\mathrm{I})_{\mathrm{h},0}$: $\quad M_{h,0} = M_0$.
- Attractivity in y-direction: $\quad \chi_{h,0} = \max\{\rho_1, \sigma_1\} + O(h + |\beta| d)$.
- Stable (vertical) foliation and asymptotic phase.
- Global error for $j \le N$:

$$
(\mathrm{GE})_{\mathrm{h},0} \qquad
\begin{aligned}
|x_j - x(jh)| &\le K_N[\max_{0 \le \ell < k}\{|x_\ell - x(\ell h)|\} + h(\max_{0 \le \ell < k}\{|y_\ell - y(\ell h)|\} \\
&\qquad + |y_0 - s_0(x_0)|) + h^p] \\
|y_j - y(jh)| &\le K_N[\max_{0 \le \ell < k}\{|x_\ell - x(\ell h)|\} + (h + \chi_{h,0}^j)(\max_{0 \le \ell < k}\{|y_\ell - y(\ell h)|\} \\
&\qquad + |y_0 - s_0(x_0)|) + h^p]\, .
\end{aligned}
$$

Remark 3): For $\beta = 0$ (BDF-like methods, corresponding to $a_{s_i} = b_i$, $i = 1, \ldots, s$, in the RKM case) we have:

- $\chi^k_{h,0} = 0$ (infinite attractivity in y-direction), i.e., $y_j = s_0(x_j)$ for $j \geq k$ independently of the starting values $y_0, \ldots, y_{k-1}$ (this is easily seen from $(\widetilde{\mathrm{I}})_{\mathrm{h},0}$ with $\beta = 0$).
- Global error for $j \leq N$:

$$(\mathrm{GE})_{\mathrm{h},0} \qquad \begin{aligned} |x_j - x(jh)| \;&\leq\; K_N[\max_{0\leq\ell<k}\{|x_\ell - x(\ell h)|\} + h^p] \\ |y_j - y(jh)| \;&\leq\; K_N[\max_{0\leq\ell<k}\{|x_\ell - x(\ell h)|\} + h^p \\ &\qquad + \chi^{[j/k]}_{h,0}(\max_{0\leq\ell<k}\{|y_\ell - y(\ell h)|\} + |y_0 - s_0(x_0)|)]\ . \end{aligned}$$

⊣

3 Systems of index 2

We consider the semi-explicit problem of so-called Hessenberg form

$$(\bar{1})_0 \qquad \begin{aligned} \dot{x} &= f(x,y) \\ 0 &= G(x) \end{aligned}$$

satisfying

Hypothesis $\overline{\mathrm{H}}_0$

1) f is bounded and $f \in C_b^{r+1}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m)$, $G \in C_b^{r+1}(\mathbb{R}^m, \mathbb{R}^n)$, $r \geq 2$.

2) There is a function $s_0 \in C_b^r(\mathbb{R}^m, \mathbb{R}^n)$ such that $G_x(x)\, f(x, s^0(x)) = 0$ for $x \in \mathbb{R}^m$.

3) The matrix $G_x(x)\, f_y(x, s_0(x))$ is invertible and the inverse is bounded for $x \in \mathbb{R}^m$.

Under these assumptions, Eq. $(\bar{1})_0$ is of index 2 since differentiating $0 = G(x)$ with respect to t yields $0 = G_x(x)\, f(x,y)$ which together with $\dot{x} = f(x,y)$ is an index-1 problem by Hypothesis $\overline{\mathrm{H}}_0$. The algebraic system $0 = G_x(x)\, f(x,y)$ has a unique solution $y = s_0(x)$ for $(x,y) \in \Omega_d := \{x \in \mathbb{R}^m,\ |y - s_0(x)| \leq d\}$, d small enough. All solutions $(x(t),\ y(t))$ in Ω_d of the index-1 DAE (the so-called index-1 formulation of the index-2 problem $(\bar{1})_0$)

$$(1)_0 \qquad \begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y) := G_x(x)\, f(x,y) \end{aligned}$$

lie in the m-dimensional surface

$$M_0 = \{(x,y)\,|\, x \in \mathbb{R}^m,\ y = s_0(x)\} \in \mathbb{R}^m \times \mathbb{R}^n\ .$$

In addition, they satisfy for all $t \in \mathbb{R}$

$$\int_0^t G_x(x(\tau))\, f(x(\tau),\ s_0(x(\tau))\, d\tau = 0$$

implying

$$G(x(t)) - G(x(0)) = 0\ .$$

This means that Eq. $(1)_0$ has G as first integral. The manifold

$$\overline{M}_0 = \{(x,y)\,|\,G(x) = 0,\ y = s_0(x)\} \subset M_0 \subset \mathbb{R}^m \times \mathbb{R}^n\ .$$

is invariant under $(\bar{1})_0$. Hence, we have shown that under Hypothesis $\overline{\mathrm{H}}_0$ all solutions of Eq. $(\bar{1})_0$ (in Ω_d) lie in the submanifold $\overline{M}_0$ of M_0 (cf. Fig. 5). Again the set $\overline{M}_0$ may be viewed as 'invariant manifold being infinitely attractive in y-direction with a vertical stable foliation' (cf. Remark 1)).

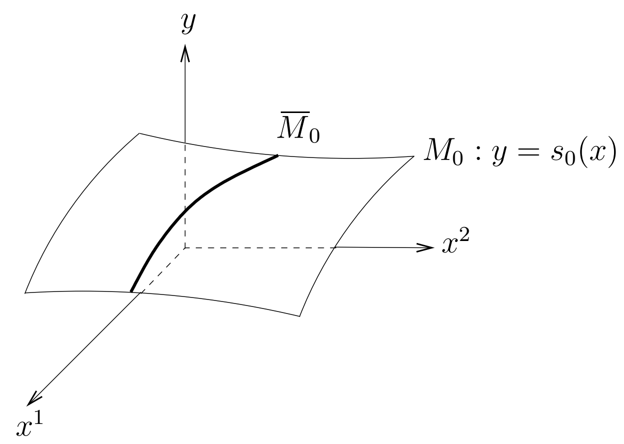

Fig. 5: The invariant manifold $\overline{M}_0$ of Eq. $(\bar{1})_0$

Remark 4): Note that every manifold $\overline{M}_0^{const} := \{(x,y)|G(x) = const,\ y = s_0(x)\}$ is invariant under Eq. $(1)_0$ and lies in M_0, i.e., the m-dimensional surface M_0 consists of submanifolds $\overline{M}_0^{const}$. And, of course, all solutions of $(1)_0$ satisfy $\dot{x} = f(x,y)$, $G(x) = G(x(0))$. ⊣

3.1 RKMs and LMMs applied to the index-1 formulation

Obviously, applying a RKM satisfying Hypothesis $\mathrm{H}_{\mathrm{RKM}}$ or a LMM satisfying Hypothesis $\mathrm{H}_{\mathrm{LMM}}$ to the index-1 formulation $(1)_0$ yields the results of Paragraphs 2.1 and 2.2, respectively, with the commuting diagram Fig. 4. However, the question of interest is if the dynamical systems $(1)_\varepsilon$, $(\mathrm{I})_{\mathrm{h},\varepsilon}$ and $(\mathrm{I})_{\mathrm{h},0}$, respectively, also possess a 'first integral', i.e., a submanifold of M_ε, $M_{h,\varepsilon}$ and $M_{h,0}$, respectively.

A) $(1)_\varepsilon$*:* The invariant manifold M_ε is the graph of the function $s(x,\varepsilon)$ which satisfies the invariance equation

$$\begin{aligned} \dot{x} &= f(x, s(x,\varepsilon)) \\ \varepsilon\, s_x(x,\varepsilon)\, f(x, s(x,\varepsilon)) &= g(x, s(x,\varepsilon))\ . \end{aligned}$$

The second equation is equivalent to $\varepsilon \dot{s} = G_x(x)\,\dot{x}$ implying

$$\varepsilon[s(x(t),\varepsilon) - s(x(0),\varepsilon)] = G(x(t)) - G(x(0))\ .$$

Hence, Eq. $(1)_\varepsilon$ with $g(x,y) = G_x(x) f(x,y)$ admits the invariant manifold

$$\overline{M}_\varepsilon = \{(x,y)\,|\,G(x) - \varepsilon\, s(x,\varepsilon) = 0,\ y = s(x,\varepsilon)\} \subset M_\varepsilon\ .$$

Of course, again any set

$$\overline{M}_\varepsilon^{const} = \{(x,y)\,|\,G(x) - \varepsilon\, s(x,\varepsilon) = const,\ y = s(x,\varepsilon)\} \subset M_\varepsilon$$

is an invariant manifold of Eq. $(1)_\varepsilon$. Putting $\varepsilon = 0$ one obtains $\overline{M}_0 \subset M_0$ or $\overline{M}_0^{const} \subset M_0$, respectively.

B.a) $(\mathrm{I})_{\mathrm{h},\varepsilon}$ *and* $(\mathrm{I})_{\mathrm{h},0}$ *for RKMs. i) The linear case:* We assume

$$G(x) = Bx + c$$

where B is a $n \times m$-matrix and c is a n-vector both independent of x. This implies $g(x,y) = G_x(x) f(x,y) = Bf(x,y)$. In the RKM case the discrete invariant manifold $M_{h,\varepsilon}$ is the graph of the function $\sigma(x,h,\varepsilon)$ satisfying the invariance equation

$$\begin{aligned} \bar{x} - x &= h \sum_{i=1}^{s} b_i\, f(X_i, Y_i) \\ \varepsilon[\sigma(\bar{x},h,\varepsilon) - \sigma(x,h,\varepsilon)] &= h \sum_{i=1}^{s} b_i\, g(X_i, Y_i) = B(\bar{x} - x)\ . \end{aligned}$$

This implies the existence of the invariant manifold

$$\overline{M}_{h,\varepsilon} = \{(x,y)\,|\,G(x) - \varepsilon\,\sigma(x,h,\varepsilon) = 0,\ y = \sigma(x,h,\varepsilon)\} \subset M_{h,\varepsilon}$$

for the RKM-map $(\mathrm{I})_{\mathrm{h},\varepsilon}$. Putting $\varepsilon = 0$ we obtain the invariant manifold

$$\overline{M}_{h,0} = \{(x,y)\,|\,G(x) = 0,\ y = \sigma_0(x,h)\} \subset M_{h,0}$$

for the RKM-map $(\mathrm{I})_{\mathrm{h},0}$. We know that $\sigma_0(x,h) = s_0(x) + O(h^{q+1})$. The situation for the DAE in the linear case is sketched in Fig. 6. Note that if $a_{s_i} = b_i,\ i = 1,\ldots,s$, we have $M_{h,0} = M_0$ and $\overline{M}_{h,0} = \overline{M}_0$, i.e., the continuous and the discrete manifolds are the same.

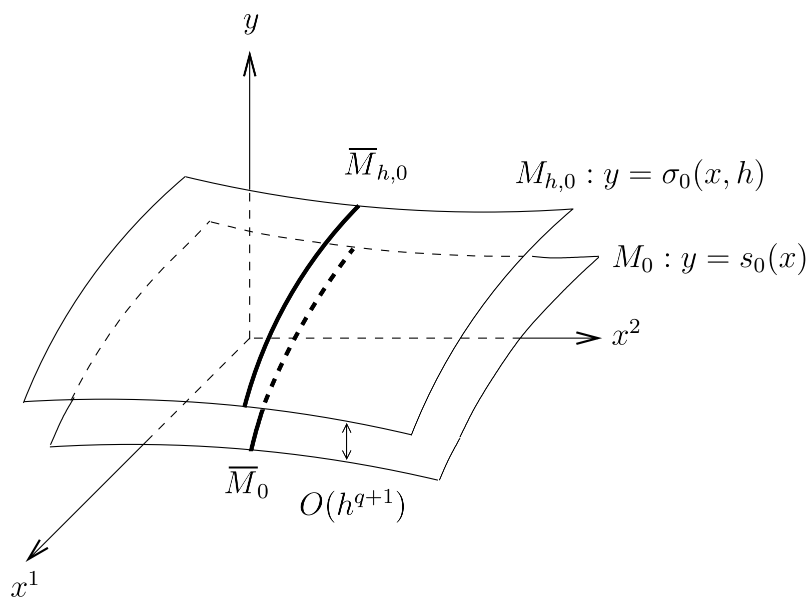

Fig. 6: The manifolds $\overline{M}_0$ and M_0 of Eq. $(1)_0$ and the invariant manifolds $\overline{M}_{h,0}$ and $M_{h,0}$ of the RKM-map $(\mathrm{I})_{\mathrm{h},0}$ for $G(x) = Bx + c$

For RKMs applied to the index-1 formulation of Eq. $(\bar{1})_0$ we have shown that in the linear case the commuting diagram of Fig. 4 also holds if the submanifolds $\overline{M}_0$, $\overline{M}_\varepsilon$, $\overline{M}_{h,\varepsilon}$ and $\overline{M}_{h,0}$, respectively, are added into the picture.

Remark 5): The invariant manifolds $\overline{M}_0$, $\overline{M}_\varepsilon$, $\overline{M}_{h,\varepsilon}$ and $\overline{M}_{h,0}$, respectively, inherit the attractivity of the invariant manifolds M_0, M_ε, $M_{h,\varepsilon}$ and $M_{h,0}$, respectively, in y-direction. Moreover, every 'shifted manifold' $\overline{M}_0^{const} := \{(x,y) \,|\, G(x) = \text{const},\ y = s_0(x)\}$, $\overline{M}_\varepsilon^{const}$, $\overline{M}_{h,\varepsilon}^{const}$ and $\overline{M}_{h,0}^{const}$ (defined analogously) is also invariant. ⊣

ii) The nonlinear case: For RKMs applied to Eq. $(1)_\varepsilon$ we have the following invariance equation for $M_{h,\varepsilon}$

(2)
$$\begin{aligned} \bar{x} - x &= h \sum_{i=1}^{s} b_i\, f(X_i, Y_i) \\ \varepsilon\,[\sigma(\bar{x}) - \sigma(x)] &= h \sum_{i=1}^{s} b_i\, G_x(X_i)\, f(X_i, Y_i) \end{aligned}$$

where we have suppressed the dependence of σ on h, ε. The stages X_i, Y_i, $i = 1, \ldots, s$, are functions of x, h and ε (smooth with respect to x, ε). We define the function $Q(x; h, \varepsilon)$ by

(3)
$$Q(x; h, \varepsilon) := G(\bar{x}) - G(x) - h \sum_{i=1}^{s} b_i\, G_x(X_i)\, f(X_i, Y_i)\ .$$

Starting the RKM with $G(x_0) - \varepsilon\sigma(x_0) = \Delta_0$ we obtain using the definition of Q and Eq. (2) that for $j \geq 1$

(4)
$$G(x_j) - \varepsilon\, \sigma(x_j, h, \varepsilon) = \sum_{\ell=0}^{j-1} Q(x_\ell; h, \varepsilon) + \Delta_0\ .$$

We want to estimate the function $Q(x_\ell; h, \varepsilon)$. Let $(x(t),\, y(t))$ be a solution of Eq. $(1)_\varepsilon$ with initial values $x(0) = x_\ell$, $y(0) = y_\ell = \sigma(x_\ell)$. Integrating $\frac{d}{dt}\, G(x(t)) = G_x(x(t))\, f(x(t),\, y(t))$ between 0 and h we get

$$G(x(h)) - G(x_\ell) = \int_0^h G_x(x(\tau))\, f(x(\tau),\, y(\tau))\, d\tau\ . \tag{5}$$

From the definition of Q we have

$$G(x_{\ell+1}) - G(x_\ell) = h \sum_{i=0}^{s} b_i\, G_x(X_i)\, f(X_i, Y_i) + Q(x_\ell; h, \varepsilon) \tag{6}$$

where X_i, Y_i are functions of x_ℓ, h and ε. From Eqs. (3) and (6) we obtain

$$Q(x_\ell; h, \varepsilon) = G(x_{\ell+1}) - G(x(h)) - [P_1 - P_2]$$

with $P_1 := h\, \sum_{i=0}^{s} b_i\, G_x(X_i)\, f(X_i, Y_i)$ and $P_2 := \int_0^h\, G_x(x(\tau))\, f(x(\tau),\, y(\tau)) d\tau$. From [6], Theorem 2 and Theorem 3 (with proof) we know that

$$\begin{aligned} x_{\ell+1} - x(h) &= O(h^{p+1}) + O(\varepsilon h^{q+1}) + O(\varepsilon |\sigma(x_\ell) - s(x_\ell)|) \\ &= O(h^{p+1}) + O(\varepsilon h^{q+1})\ . \end{aligned}$$

Since G is of class C_b^1 this implies $G(x_{\ell+1}) - G(x(h)) = O(h^{p+1}) + O(\varepsilon h^{q+1})$.

It remains to estimate $P_1 - P_2$. From nonstiff RKM-theory (cf. [3]) we know that for a RKM applied to $\dot{u} = f(u, s(u))$, $u(0) = u_0 := x_\ell$ the relations

$$\begin{aligned} u(c_i h) - u_0 &= h \sum_{j=1}^{s} a_{ij}\, f(u(c_j h),\, s(u(c_j h))) + O(h^{q+1})\ , \quad i = 1, \ldots s, \\ u(h) - u_0 &= h \sum_{i=1}^{s} b_i\, f(u(c_i h),\, s(u(c_i h))) + O(h^{p+1}) \end{aligned} \tag{7}$$

hold. Moreover, for the function $G_x(u(t))\, f(u(t),\, s(u(t)))$ the RKM is a quadrature formula of order p, i.e.,

$$\begin{aligned} &\int_0^h G_x(u(\tau))\, f(u(\tau),\, s(u(\tau))) d\tau = \\ &= h \sum_{i=1}^{s} b_i\, G_x(u(c_i h))\, f(u(c_i h),\, s(u(c_i h))) + O(h^{p+1})\ . \end{aligned} \tag{8}$$

We introduce $Z_i, i = 1, \ldots, s$, and $z(t)$ by $Y_i = s(X_i) + Z_i$ and $y(t) = s(x(t)) + z(t)$ and we estimate

$$P_1 - P_2 = P_1 - \overline{P}_1 + \overline{P}_1 - \overline{P}_2 + \overline{P}_2 - P_2$$

where

$$\begin{aligned} \overline{P}_1 &:= h \sum_{i=1}^{s} b_i\, G_x(U_i)\, f(U_i,\, s(U_i)) \\ \overline{P}_2 &:= h \sum_{i=1}^{s} b_i\, G_x(u(c_i h))\, f(u(c_i h),\, s(u(c_i h)))\ . \end{aligned}$$

We have

$$\begin{aligned} P_1 - \overline{P}_1 &:= h\,O\Big(\max_{1\le i\le s}\ |X_i - U_i| + \max_{1\le i\le s}\ |Z_i|\Big)\,,\\ \overline{P}_1 - \overline{P}_2 &:= h\,O\Big(\max_{1\le i\le s}\ |U_i - u(c_i h)|\Big) \end{aligned}$$

and with (8)

$$\overline{P}_2 - P_2 = O(h^{p+1}) + h\,O\Big(\max_{0\le t\le h}\ |x(t)-u(t)| + \max_{0\le t\le h}\ |z(t)|\Big)\ .$$

From [6] we know that

$$Z_i = O(\varepsilon h^q)\,, \quad X_i - U_i = O(\varepsilon h^{q+1})\,, \quad i = 1,\dots,s\ .$$

The first relation of (7) implies

$$U_i - u(c_i h) = O(h^{q+1})\,,\ \ i=1,\dots,s\ ,$$

and a simple Gronwall type argument yields

$$\max_{0\le t\le h}\ |x(t)-u(t)| = h\,O(\max_{0\le t\le h}\ |z(t)|)\ .$$

For $z(t)$ we apply Theorems 1 and 3 of [6]:

$$\begin{aligned} z(t) &= x(t) - s(x(t)) = O(e^{-\mathrm{const}\cdot t/\varepsilon}|y(0)-s(x(0))|)\\ &= O(|\sigma(x_\ell) - s(x_\ell)|) = O(h^{q+1})\ . \end{aligned}$$

Adding up, we finally have

$$Q(x_\ell; h, \varepsilon) = O(h^{q+2})\ .$$

(We have used the fact that the $\varepsilon \ll h$ and $q < p$ which cancels out the $O(h^{p+1})$- and $O(\varepsilon h^{q+1})$-terms). Hence, Eq. (4) implies for all $j \in \mathbb{N}$

$$G(x_j) - \varepsilon\sigma(x_j; h, \varepsilon) = \Delta_0 + jh\cdot O(h^{q+1}) \tag{9}$$

where the constant in $O(h^{q+1})$ is independent of j. This means a RKM-orbit starting on $M_{h,\varepsilon}$ and $O(h^{q+1})$-close to $\overline{M}_{h,\varepsilon}$ has at worst a linear (in $t = jh$) drift off in the x-component. (Of course, starting off $M_{h,\varepsilon}$ there is the attractivity in the y-component with $\chi_{h,\varepsilon}$).

Remark 6): Using the global error estimate $(\mathrm{GE})_{\mathrm{h},\varepsilon}$ it is simple to get the estimate

$$G(x_j) - \varepsilon\sigma(x_j, h, \varepsilon) = \Delta_0 + O(h^p) + O(\varepsilon h^{q+1})\ .$$

However, here the constants in the O-terms are of the form Konst $\cdot\, e^{\mathrm{const}\cdot jh}$ which is a much weaker result than the one of Eq. (9).

The above results also hold for $\varepsilon = 0$. We state the DAE case in

Theorem 2 *Let Eq.* $(\overline{1})_0$ *satisfy Hypothesis* $\overline{\mathrm{H}}_0$*, apply a* RKM *satisfying Hypothesis* $\mathrm{H}_{\mathrm{RKM}}$ *to its 'index-1 formulation'* $(1)_0$ *and assume* $r < p$*. Consider for* $\Delta_0 \in \mathbb{R}^n$ *the set*

$$\overline{M}_{h,0}^{\Delta_0} = \{(x, y) \,|\, G(x) = \Delta_0,\ y = \sigma_0(x, h)\} \subset M_{h,0}$$

with σ_0 *and* $M_{h,0}$ *from Theorem 1. Then the following assertions hold.*

i) A RKM*-orbit* (x_j, y_j)*,* $j \in \mathbb{N}$*, with* $G(x_0) = \Delta_0, y_0 = \sigma(x_0, h)$ *satisfies*

$$|G(x_j) - \Delta_0| \leq jh \cdot Kh^{q+1}\,, \qquad y_j = \sigma_0(x_j, h)$$

for some constant K *independent of* j*.*

ii) If G *is linear,* $G(x) = Bx+c$*, the set* $\overline{M}_{h,0}^{\Delta_0}$ *is an invariant manifold of the* RKM*-map* $(\mathrm{I})_{\mathrm{h,0}}$*.*

Remark 7): i) Note, that for $a_{s_i} = b_i$, $i = 1, \ldots, s$, $\sigma_0(x, h) = s_0(x)$ and hence $\overline{M}_{h,0}^{\Delta_0} = \overline{M}_0^{\Delta_0} = \{(x, y) \,|\, G(x) = \Delta_0,\ y = s_0(x)\} \subset M_0$.

ii) If the RKM-orbit is started such that $|y_0 - s_0(x_0)| \leq d$ small enough, one obtains due to the vertical stable foliation of $M_{h,0}$ the same x-estimates as in Theorem 2 and exponential attractivity to $\overline{M}_{h,0}^{\Delta_0}$ in y-direction (cf. Theorem 1). ⊣

iii) The global error is $(\mathrm{GE})_{\mathrm{h,0}}$ of the index-1 problem as given in Theorem 1 v) and in Remark 2).

B.b) $(\mathrm{I})_{\mathrm{h},\varepsilon}$ *and* $(\mathrm{I})_{\mathrm{h,0}}$ *for LMMs.* The invariance equation for $M_{h,\varepsilon}$ is

$$\begin{aligned} \sum_{i=0}^{k} \alpha_i\, x_i &= h \sum_{i=0}^{k} \beta_i\, f(x_i, \sigma(x_i)) \\ \varepsilon \sum_{i=0}^{k} \alpha_i\, \sigma(x_i) &= h \sum_{i=0}^{k} \beta_i\, G_x(x_i)\, f(x_i, \sigma(x_i)) \end{aligned} \tag{10}$$

together with $x_i = \Phi^i(x_0, h, \varepsilon)$, $i = 1, 2, \ldots$. Here, we again have suppressed the dependence of σ on h and ε. We define the function

$$Q(x_0; h, \varepsilon) := \sum_{i=0}^{k} \alpha_i\, G(x_i) \;-\; h \sum_{i=0}^{k} \beta_i\, G_x(x_i)\, f(x_i, \sigma(x_i))\,. \tag{11}$$

We will also need the fact that Φ is a one-step method of order p for $\dot{u} = f(u, s(u, \varepsilon))$ (cf. [7], Theorem 2). For $\Delta_0 \in \mathbb{R}^n$ we take x_0 such that $G(x_0) - \varepsilon\sigma(x_0) = \Delta_0$ and we define for $j = 0, 1, 2, \ldots$

$$\Delta_j := G(x_j) - \varepsilon\sigma(x_j)$$

and

$$Q_j := Q(x_j; h, \varepsilon)$$

where $Q(x_j; h, \varepsilon)$ is defined as in (11) with x_i replaced by x_{j+i}. From Eqs. (10) and (11) we have for $j \geq 0$

$$\sum_{i=0}^{k} \alpha_i \, \Delta_{j+i} = Q_j$$

or with $\delta^0 := (\Delta_0, \dots, \Delta_{k-1})^T \in \mathbb{R}^{kn}$, $\delta^1 := (\Delta_1, \dots, \Delta_k)^T$, $r^0 := (0, \dots, 0, Q_0)^T \in \mathbb{R}^{kn}$, $r^1 := (0, \dots, 0, Q_1)^T$, etc.,

$$\delta^j = ((R - L_\alpha) \otimes I_n) \, \delta^{j-1} + r^{j-1} , \quad j \geq 1 ,$$

and, therefore,

$$\delta^j = ((R - L_\alpha) \otimes I_n)^j \, \delta^0 + \sum_{\ell=0}^{j-1} ((R - L_\alpha) \otimes I_n)^\ell \, r^{j-\ell-1} , \quad j \geq 1 .$$

By Hypothesis $\mathrm{H_{LMM}}$ the $k \times k$-matrix $R - L_\alpha$ has one eigenvalue 1 and all others have modulus smaller than $\rho_1 < 1$. Hence, we have

$$|\Delta_{j+k-1}| \leq |\delta^j|_\infty \leq K_0 \, |\delta^0|_\infty + j \cdot K_0 \, K_1 , \ \ j = 1, 2, \dots ,$$

if $K_0 > 0$ is such that $|((R - L_\alpha) \otimes I_n)^j| \leq K_0$ for $j > 0$ and if we assume that $|Q_j| \leq K_1$ for all j. This second assumption has to be verified. In fact, with similar techniques as for RKMs using the results in [7], one finds $K_1 = \overline{K}_1 \, h^{p+1}$ and $\Delta_i = \Delta_0 + O(h^{p+1})$, $i = 1, \dots, k-1$. The role of Eq. (7) is taken by the relation

$$\begin{aligned} L(G(u), t; h) &= \sum_{i=0}^{k} [\alpha_i \, G(u(ih)) - h\beta_i \, G_x(u(ih)) \, f(u(ih), \, s(u(ih))] \\ &= O(h^{p+1}) \end{aligned}$$

(cf. [3]).

Summarizing, we have for $j \in \mathbb{N}$

$$|G(x_j) - \varepsilon\sigma(x_j, h, \varepsilon)| \leq jh \cdot Kh^p + K_0 \max_{0 \leq i < k} |\Delta_i|$$

with constants K, K_0 independent of j. This, of course, again means at worst a linear drift (in $t = jh$) off $G(x) - \varepsilon\sigma(x, h, \varepsilon) = \Delta_0$. For G linear, $G(x) = Bx + c$, it is easy to see that the set $G(x) - \varepsilon\sigma(x, h, \varepsilon) = \Delta_0$ is an invariant set of the LMM-map.

In the DAE case ($\varepsilon = 0$), we thus have analogously to Theorem 2 in the RKM case:

- The LMM-orbit (x_j, y_j), $j \in \mathbb{N}$, with $G(x_0) = \Delta_0$ and lying in $M_{h,0} = M_0 = \{(x,y) \,|\, x \in \mathbb{R}^m,\, y = s_0(x)\}$ satisfies

(12) $$|G(x_j) - \Delta_0| \le jh \cdot Kh^p\,, \quad y_j = s_0(x_j)$$

  for some constant K independent of j.

- If G is linear, $G(x) = Bx + c$, and if in addition $Bx_i + c = \Delta_0$, $i = 1, \dots, k-1$, then the set $\overline{M}_0^{\Delta_0} = \{(x,y) \,|\, G(x) = \Delta_0,\, y = s_0(x)\} \in M_0$ is an invariant manifold of the LMM-map $(\mathrm{I})_{\mathrm{h},0}$.

If we take starting values off M_0 satisfying $x_i - x(ih) = O(h^{p+1})$, $i = 1, \dots, k-1$, where $x(t)$ is the solution of Eq. $(1)_0$ with $x(0) = x_0$, $G(x_0) = \Delta_0$ and/or $|y_i - s_0(x_i)| \le d$, $i = 0, \dots, k-1$, d small enough, then the x-estimate (12) still holds and in y-direction one has exponential attractivity to M_0. For $\beta = 0$, M_0 is 'infinitely y-attractive', i.e., $y_j = s_0(x_j)$ for $j \ge k$ (cf. Paragraph 2.2, Remark 3)).

3.2 RKMs and LMMs applied to the index-2 formulation

While appropriate numerical integration of the index-2 formulation preserves $G(x) = 0$, here, the point of interest is the existence of a y-attractive invariant index-1 manifold for the discrete dynamical system.

a) RKMs. We apply a RKM satisfying Hypothesis $\mathrm{H_{RKM}}$ to the DAE of index 2

$(\bar{1})_0$ $$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= G(x) \end{aligned}$$

satisfying Hypothesis $\overline{\mathrm{H}}_0$. Following [2] and [4] we have

(13) $$\begin{aligned} X &= \mathbb{1}_s \otimes x + h(A \otimes I_m)\, f(X,Y) \\ 0 &= G(X) \end{aligned}$$

for the stages and

$(\bar{\mathrm{I}})_{\mathrm{h},0}$ $$\begin{aligned} \bar{x} &= x + h(b^T \otimes I_m)\, f(X,Y) \\ \bar{y} &= (1 - b^T A^{-1}\, \mathbb{1}_s)\, y + (b^T A^{-1} \otimes I_n)\, Y \end{aligned}$$

for one step of the RKM. Introducing the variable z by means of $y = s_0(x) + z$ and similarly for $\bar{z}$ this is equivalent to

$(\tilde{\bar{\mathrm{I}}})_{\mathrm{h},0}$ $$\begin{aligned} \bar{x} &= x + h(b^T \otimes I_m)\, f(X,Y) \\ \bar{z} &= R(\infty)\, z + (b^T A^{-1} \otimes I_n)(Y - \mathbb{1}_s \otimes s_0(x)) + s_0(x) - s_0(\bar{x})\,. \end{aligned}$$

Of course, we have to show that $(\bar{\mathrm{I}})_{\mathrm{h},0}$ and $(\tilde{\bar{\mathrm{I}}})_{\mathrm{h},0}$, respectively, are well defined maps in phase space, i.e., we have to show that Eq. (13) has a unique solution (X, Y) given x appropriately in $\mathbb{R}^m$. This is done in Lemma 3 below. There, we consider an altered system (Eq. (14)) and show that for all $x \in \mathbb{R}^m$ it has a unique solution near $(\mathbb{1}_s \otimes x, \mathbb{1}_s \otimes s_0(x))$. For our original system (13) this implies the existence of unique solution not only for x-starting values satisfying $G(x) = 0$ but also for perturbed values in a $O(h^2)$-neighbourhood of $G(x) = 0$. Note that the solution does not depend on the y-starting value.

Lemma 3 *Let $\Delta : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ be a bounded function of class C_b^r.*

Then, there are positive constants h_1, K such that for $h \leq h_1$ and $x \in \mathbb{R}^m$ the non-linear system

$$
\begin{aligned}
X &= \mathbb{1}_s \otimes x + h(A \otimes I_m)\, f(X, Y) \\
G(X) &= \mathbb{1}_s \otimes (G(x) - h^2\, \Delta(x, h))
\end{aligned}
\tag{14}
$$

has a unique solution $(X, Y)(x, h)$, C_b^r with respect to x, satisfying

$$|X(x, h) - \mathbb{1}_s \otimes x| \leq Kh\,, \qquad |Y(x, h) - \mathbb{1}_s \otimes s_0(x)| \leq Kh\ .$$

Proof: If we introduce $Z \in \mathbb{R}^{sn}$ by $Y = s_0(X) + Z$ and add the term $h \sum_{j=1}^{s} a_{ij}\, G_x(X_j)\, f(X_j, s_0(X_j) + Z_j)$ on both sides of the second equation, Eq. (14) in components ($i = 1, \ldots, s$) takes the following form

$$
\begin{aligned}
X_i &= x + h \sum_{j=1}^{s} a_{ij}\, f(X_j,\, s_0(X_j) + Z_j) \\
h \sum_{j=1}^{s} a_{ij}\, G_x(X_j)\, f(X_j,\, s_0(X_j) + Z_j) &= G(x) - G(X_i) - h^2 \Delta(x, h) \\
&\quad + h \sum_{j=1}^{s} a_{ij}\, G_x(X_j)\, f(X_j,\, s_0(X_j) + Z_j)\ .
\end{aligned}
\tag{15}
$$

Using the first equation of Eq. (15) we find

$$
\begin{aligned}
G(X_i) - G(x) &= \int_0^1 G_x(x + \tau(X_i - x))d\tau \cdot (X_i - x) \\
&= h \sum_{j=1}^{s} a_{ij} \int_0^1 G_x(x + \tau(X_i - x))d\tau \cdot f(X_j,\, s_0(X_j) + Z_j)\ .
\end{aligned}
$$

Hence, the second equation of Eq. (15) is of the form

$$
\begin{aligned}
\frac{1}{h} \sum_{j=1}^{s} a_{ij} G_x(X_j) f(X_j, s_0(X_j) + Z_j) &= \frac{1}{h} \sum_{j=1}^{s} a_{ij} \int_0^1 [G_x(X_j) - G_x(x + \tau(X_i - x))]d\tau\ \cdot \\
&\quad \cdot f(X_j,\, s_0(X_j) + Z_j) - \Delta(x, h) \\
&=: Q_i(x, X, Z; h)\ .
\end{aligned}
$$

Since the norm of the integrand is bounded by $L_{G_x}(|X_j - x| + |X_i - x|) \leq \text{const} \cdot h$ due to the first equation of (15) and since Δ is bounded by assumption, the functions Q_i, $i = 1, \ldots, s$, are bounded for all h small. Hence, back to the 'big space' and since the Runge-Kutta matrix A is invertible by Hypothesis $\mathrm{H_{RKM}}$, Eq. (15) has the form

$$\begin{aligned} X &= 1\!\!1_s \otimes x + h(A \otimes I_m)\, f(X, s_0(X) + Z) \\ \text{diag}[G_x(X)]\, f(X, s_0(X) + Z) &= h(A^{-1} \otimes I_n)\, Q(x, X, Z; h) \end{aligned}$$

with Q bounded and of class C_b^r with respect to x, X, Z. By Hypothesis $\overline{\mathrm{H}}_0$ we know that $\text{diag}[G_x(X)]\, f(X, s_0(X)) = 0$ and that $\text{diag}[G_x(X)]\, f_y(X, s_0(X))$ is invertible with bounded inverse. Hence, for $|Z| \leq d$, d small enough, the second equation can be written as

$$Z = h\, C(X, Z)^{-1}(A^{-1} \otimes I_n)\, Q(x, X, Z; h)$$

with $C(X, Z) = \text{diag}[G_x(X)] f_y(X, s_0(X)) + O(|Z|)$ and $C(X, Z)^{-1}$ bounded. Considering the two equations as a fixed point equation (for some map) the contraction principle implies the existence of a unique solution $(X, Z)(x, h)$ satisfying $|X - 1\!\!1_s \otimes x| \leq \textit{const} \cdot h$, $|Z| \leq \textit{const} \cdot h$. The smoothness follows from the implicit function theorem. $\perp$

Remark 8): For an x-starting value such that $G(x) = h^2\, \Delta(x, h)$, Lemma 3 corresponds to Theorem 4.1 of [2] (cf. also [4], Theorem 7.1). We have given a different proof and we do not need their assumption $G_x(x)\, f(x, y) = O(h)$. $\dashv$

i) The linear case: We assume that $G(x) = Bx + c$ for the function G and that the starting values x and y are such that $G(x) = 0$ and $|y - s_0(x)| \leq d$, d small enough. From Lemma 3 with $\Delta \equiv 0$ it follows that Eq. (13) has a unique solution $(X, Y)(x, h)$ near $(1\!\!1_s \otimes x, 1\!\!1_s \otimes s_0(x))$. It is easy to see that by our assumptions $G(X) = 0$ implies $g(X, Y) := \text{diag}[G_x(X)]\, f(X, Y) = 0$ and vice versa. Hence, for $G(x) = 0$ and $|y - s_0(x)|$ small enough the maps $(\mathrm{I})_{\mathrm{h},0}$ (i.e., the RKM applied to the index-1 formulation of Eq. $(\bar{1})_0$) and $(\bar{\mathrm{I}})_{\mathrm{h},0}$ create the same $(\bar{x}, \bar{y})$ and therefore the same orbit. Moreover, $G(x_j) = 0$ for all $j \geq 0$. Summarizing (cf. Theorem 1), we have the existence of an invariant manifold for the map $(\bar{\mathrm{I}})_{\mathrm{h},0}$

$$\overline{M}_{h,0} = \{(x, y) \,|\, G(x) = 0,\ y = \sigma_0(x, h)\}$$

which is attractive in y-direction and satisfies $\sigma_0(x, h) = s_0(x) + O(h^{q+1})$. (Again, $\overline{M}_{h,0} = \overline{M}_0$ $(y = s_0(x))$ if $a_{s_i} = b_i$, $i = 1, \ldots, s$.) Hence, for linear G and x-starting value such that $G(x) = 0$ it does not matter if we approximate numerically by an appropriate RKM the index-2 problem $(\bar{1})_0$ or its index-1 formulation $(1)_0$.

ii) The general case: For RKMs applied to Eq. $(\bar{1})_0$ which satisfy Hypothesis $\mathrm{H_{RKM}}$ and $a_{s_i} = b_i$, $i = 1, \ldots, s$, we are also able to prove the existence of an attractive invariant ('index-1') manifold. This is done by first extending the RKM-map to all $\mathbb{R}^m$, applying the invariant manifold theorems of [5] to this altered map and then taking the restriction to the subspace $G(x) = 0$. The result is given in Theorem 4 below and is sketched in Fig. 7.

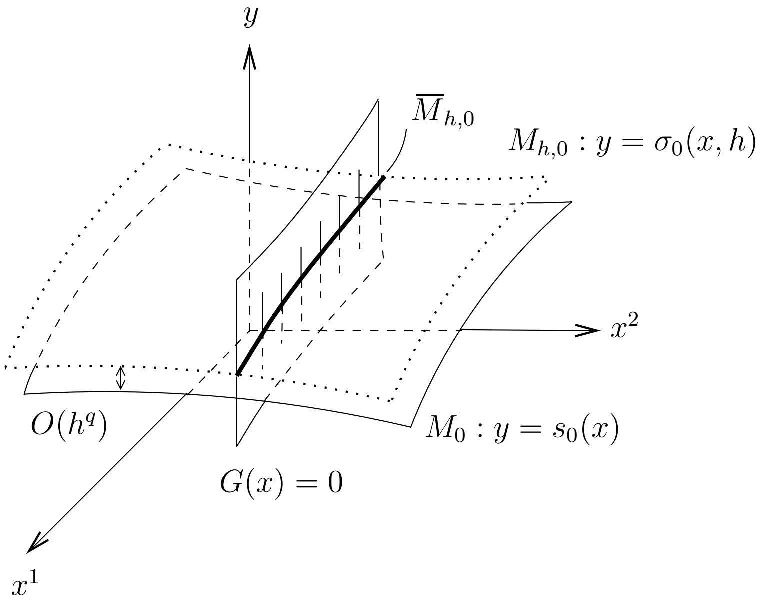

Fig. 7: The invariant manifolds of Eq. $(\bar{1})_0$ and of the RKM-map $(\bar{\mathrm{I}})_{\mathrm{h},0}$ with $a_{s_i} = b_i,\ i = 1, \ldots, s$

Theorem 4 *Let Eq.* $(\bar{1})_0$ *satisfy Hypothesis* $\overline{\mathrm{H}}_0$, *apply a RKM satisfying Hypothesis* $\mathrm{H}_{\mathrm{RKM}}$ *and* $a_{s_i} = b_i,\ i = 1, \ldots, s$, *and assume* $r < p$.

Then there exist positive constants h_0, γ, K *and a function* $\sigma_0 : \mathbb{R}^m \times (0, h_0] \to \mathbb{R}^n$, *of class* C_b^r *with respect to* x, *such that for* $h \leq h_0$ *the following assertions hold.*

i) The set $\overline{M}_{h,0} = \{(x,y) \,|\, G(x) = 0,\ y = \sigma_0(x,h)\}$ *is an invariant manifold for the RKM-map* $(\bar{\mathrm{I}})_{\mathrm{h},0}$.

ii) $\overline{M}_{h,0}$ *is 'infinitely attractive', i.e.,* $(x,y) \in \{\,|G(x)| \leq \gamma h^2\} \times \mathbb{R}^n$ *implies for the image* $(\bar{x}, \bar{y})$ *under* $(\bar{\mathrm{I}})_{\mathrm{h},0}$

$$G(\bar{x}) = 0, \quad \bar{y} = \sigma_0(\bar{x}, h)\ .$$

iii) $\overline{M}_{h,0}$ *has a vertical stable foliation implying the existence of an 'asymptotic phase' orbit* $(\hat{x}_j, \hat{y}_j)$ *with* $\hat{x}_j = x_j,\ j \geq 0$, *and* $\hat{y}_j = y_j = \sigma_0(x_j, h),\ j \geq 1$.

iv) Closeness to $\overline{M}_0$ *:*

$$|\sigma_0(x,h) - s_0(x)| \leq K h^q\ .$$

Proof: i), ii), iii) For $x \in \mathbb{R}^m$ we consider

(16)
$$\begin{aligned} X &= \mathbb{1}_s \otimes x + h(A \otimes I_m)\, f(X,Y) \\ G(X) &= \mathbb{1}_s \otimes G(x) \\ \\ \bar{x} &= x + h(b^T \otimes I_m)\, f(X,Y) \\ \bar{y} &= (1 - b^T A^{-1} \mathbb{1}_s) y + (b^T A^{-1} \otimes I_n)\, Y\ . \end{aligned}$$

By means of Lemma 3 with $\Delta \equiv 0$ we know that Eq. (16) describes a well defined map from $\mathbb{R}^m \times \mathbb{R}^n$ into itself. Introducing z by $y = s_0(x) + z$ and Z by $Y = s_0(X) + Z$, this map has the form

$$
\begin{aligned}
\bar{x} &= x + h(b^T \otimes I_m)\, f(X, s_0(X) + Z) \\
\bar{z} &= R(\infty)\, z + (b^T A^{-1} \otimes I_n)\, Z + (b^T A^{-1} \otimes I_n)(s_0(X) - \mathbb{1}_s \otimes s_0(x)) \\
&\quad + s_0(x) - s_0(\bar{x})\ .
\end{aligned}
\tag{17}
$$

By means of Lemma 3 and since $R(\infty) = 0$ by Hypothesis $\mathrm{H_{RKM}}$ and $a_{s_i} = b_i$, $i = 1, \cdots, s$, we have for every $d > 0$ that for h small enough Eq. (17) defines a smooth map from $\mathbb{R}^m \times \mathbb{R}^n \cap \{|z| \le d\}$ into itself. We apply the invariant manifold theorems of [5] to this map of the form

$$
\begin{aligned}
\bar{x} &= x + hF_1(x, h) \\
\bar{z} &= F_2(x, h)\ .
\end{aligned}
$$

In the domain considered, F_1 is invertible with respect to x, i.e., for every $\bar{x}, z$ there is x such that $\bar{x} = F_1(x, z, h)$. For the 'lower' Lipschitz constant of F_1 with respect to x we find $\Gamma_{11} = 1 + O(h)$. For the Lipschitz constant of F_1 with respect to z we have $L_{12} = 0$. For F_2 we have $L_{21} = O(h)$ and $L_{22} = |R(\infty)| = 0$. Hence, the conditions $B1, B2, B3(r)$ of [5] are satisfied and we obtain the existence of an m-dimensional invariant manifold $M_{h,0} = \{(x, y) \,|\, x \in \mathbb{R}^m,\ y = \sigma_0(x, h)\}$ for the map (16).

Now restricting the map (16) to the subspace $G(x) = 0$ it follows immediately since $a_{s_i} = b_i$, $i = 1, \ldots, s$, that this subspace is positively invariant under (16). On the other hand, since in Eq. (16) $G(X) = \mathbb{1}_s \otimes G(x)$ there is for every $\bar{x}$ with $G(\bar{x}) = 0$ a pre-image x with $G(x) = 0$ obtained by iteration of $x^{\ell+1} = \bar{x} - h(b^T \otimes I_m)\, f(X(x^\ell, h),\ Y(x^\ell, h))$, $\ell \ge 0$, $x^0 = \bar{x}$. Hence, we have shown the existence of the invariant manifold

$$
\overline{M}_{h,0} = \{(x, y) \,|\, G(x) = 0,\ y = \sigma_0(x, h)\} \subset M_{h,0}\ .
$$

From Lemma 3 we know that for a starting value (x_0, y_0) with $G(x_0) = h^2\, \Delta_0$ the RKM-image (x_1, y_1) exists. Due to $a_{s_i} = b_i$, $i = 1, \ldots, s$, we have $G(x_1) = 0$. Due to $R(\infty) = 0$ and $L_{12} = 0$ we have $y_1 = \sigma_0(x_1, h)$. Together this gives an infinite attractivity to $\overline{M}_{h,0}$.

The smoothness of σ_0 as well as the foliation of $M_{h,0}$ follows from the invariant manifold results applied to Eq. (17). Due to $L_{12} = 0$ the foliation is vertical and the asymptotic phase orbit therefore satisfies $\hat{x}_j = x_j$, $j \ge 0$. Due to the infinite attractivity in y-direction we also have $\hat{y}_j = y_j = \sigma_0(x_j, h)$, $j \ge 1$. Hence, we have shown Assertions i), ii), iii) of the theorem.

iv) For the closeness of $M_{h,0}$ to M_0 we have to estimate the right-hand side of the z-equation of (17). We consider the nonstiff equation $\dot{u} = f(u, s_0(u))$ with initial condition $u(0) = x$. Applying the given RKM with $u_0 = u(0)$ we obtain

$$
\begin{aligned}
U &= \mathbb{1}_s \otimes x + h(A \otimes I_m)\, f(U, s_0(U)) \\
\bar{u} &= x + h(b^T \otimes I_m)\, f(U, s_0(U))\ .
\end{aligned}
$$

Taking into account the x-equation of (17) (with the corresponding X-equation) we therefore find

(18) $$X - U = O(h)Z \ \text{ and } \ \bar{x} - \bar{u} = O(h)\, Z\ .$$

Taking the z-equation of (17) and adding zero appropriately we have

$$\begin{aligned} \bar{z} &= R(\infty)z + (b^T A^{-1} \otimes I_n)Z \\ &\quad + (b^T A^{-1} \otimes I_n)(s_0(X) - s_0(U)) + s_0(\bar{u}) - s_0(\bar{x}) \\ &\quad + (b^T A^{-1} \otimes I_n)(s_0(U) - 1\!\!1_s \otimes s_0(x)) + s_0(x) - s_0(\bar{u})\ . \end{aligned}$$

As seen in Paragraph 2.1 the last two terms together can be studied via the singularly perturbed case of [6] and the limit $\varepsilon \to 0$ yields the estimate $O(h^{q+1})$. Thus, we have together with Eq. (18)

$$\bar{z} = R(\infty)z + O(1)Z + O(h^{q+1})\ .$$

It remains to estimate $|Z|$. Taking the components we have taking into account the solution $(x(t),\, y(t) = s_0(x(t)))$ of Eq. $(\bar{1}_0)$ at the t-values $c_i h$

$$Z_i = Y_i - s_0(X_i) = Y_i - s_0(x(c_i h)) + s_0(x(c_i h)) - s_0(X_i),\ \ i = 1, \ldots, s\ .$$

From [2], Lemma 4.3 and proof (cf. also [4], Lemma 7.4) we have the 'local error' estimates

$$X_i - x(c_i h) = O(h^{q+1})\,, \quad Y_i - y(c_i h) = O(h^q)\,, \quad i = 1, \ldots, s\ ,$$

implying $Z_i = O(h^q)$. Hence, we have $\bar{z} = R(\infty)z + O(h^q)$ and the invariant manifold results of [5] give the assertion claimed. ⊥

Remark 9): Having an estimate for the global error of the x-component which is at least $O(h^{q+1})$ for $jh \leq$ const (see, e.g., the results in [2] or [4]) we may estimate for the y-component for $j \geq 1$

$$\begin{aligned} |y_j - y(jh)| \leq\ & |y_j - \sigma_0(x_j, h)\,| + |\sigma_0(x_j, h) - s_0(x_j)| \\ & + |s_0(x_j) - s_0(x(jh))| + |s_0(x(jh)) - y(jh)| \end{aligned}$$

where $(x(t), y(t))$ is a solution of $(\bar{1})_0$ with $|x_0 - x(0)| \leq \gamma h^2$. The first and the last term on the right-hand side are zero due to the infinite attractivity in y-direction of $M_{h,0}$ and M_0, respectively, the third term is of the order of $|x_j - x(jh)|$. Hence, we have

$$|y_j - y(jh)| \leq |\sigma_0(x_j, h) - s_0(x_j)| + K\, |x_j - x(jh)|$$

and the x-estimate together with Assertion iv) of the theorem give an estimate $O(h^q)$ for the y-component. ⊣

In the case of general RKMs (i.e., $b_i \neq a_{si}$ for at least one $i \in \{1, \ldots, s\}$ and Hypothesis $\mathrm{H_{RKM}}$) we are not able to prove the existence of an index-2 manifold. The y-attractive index-1 manifold $M_{h,0}\,(y_0 = \sigma_0(x, h))$, however, exists in the following sense. If the RKM orbit is started wit x such that $G(x) = O(h^2)$ then the orbit (x_j, y_j) approaches exponentially the manifold $M_{h,0}$ which is $O(h^q)$-close to M_0. This holds for all $j > 0$ such that $|G(x_j)| \leq \gamma h^2$ for a given $\gamma > 0$.

b) LMMs. For LMMs applied to Eq. $(\overline{1})_0$ we have (cf. [4])

$$
\begin{aligned}
\sum_{i=0}^{k} \alpha_i\, x_i &= h \sum_{i=0}^{k} \beta_i\, f(x_i, y_i) \\
0 &= G(x_k)
\end{aligned}
\tag{19}
$$

with $\sum_{i=0}^{k} \alpha_i = 0$, $\alpha_k = 1$, $\beta_k \neq 0$ by Hypothesis $\mathrm{H}_{\mathrm{LMM}}$. The existence of a unique solution (x_k, y_k) for h small enough follows from Lemma 5 below. As in the RKM case (cf. Lemma 3) we again consider an altered system (Eq. (20)) and thus obtain the existence of a solution of Eq. (19) for x_i-values in an $O(h^2)$-neighbourhood of $G(x) = 0$.

Lemma 5 *Let the x-starting values satisfy $|x_i - x_\ell| \leq const \cdot h$ for $i, \ell = 0, \dots, k-1$, and let $\Delta : \mathbb{R}^{kn} \times \mathbb{R}_+ \to \mathbb{R}^n$ be a bounded function of class C_b^r.*

Then, there are positive constants h_1, d, K such that for $h \leq h_1$ and for $|y_i - s_0(x_i)| \leq d$, $i = 0, \dots, k-1$, the nonlinear system

$$
\begin{aligned}
\sum_{i=0}^{k} \alpha_i\, x_i &= h \sum_{i=0}^{k} \beta_i\, f(x_i, y_i) \\
G(x_k) &= -\sum_{i=0}^{k-1} G(x_i) - h^2 \Delta(x_0, \dots, x_{k-1}; h)
\end{aligned}
\tag{20}
$$

has a unique solution $(x_k, y_k)(x_0, \cdots, x_{k-1};\ y_0, \dots, y_{k-1}; h)$, C_b^r with respect to the x- and y-arguments, satisfying

$$
\Big| x_k + \sum_{i=0}^{k-1} \alpha_i\, x_i \Big| \leq K(h + |\beta| d)\,, \quad |y_k - s_0(-\sum_{i=0}^{k-1} \alpha_i\, x_i)| \leq K(h + |\beta| d)\,.
$$

Proof: The second equation of (20) is equivalent to

$$
0 = \sum_{i=0}^{k-1} \alpha_i [G(x_k) - G(x_i)] - h^2 \Delta
\tag{21}
$$

where we have skipped the arguments of Δ for short. We may write

$$
\sum_{i=0}^{k-1} \alpha_i [G(x_k) - G(x_i)] = \sum_{i=0}^{k-1} \alpha_i \int_0^1 G_x(x_i + \tau(x_k - x_1)) d\tau \cdot (x_k - x_i)\,.
$$

Moreover, multiplying the first equation of (20) from the left by $G_x(x_k)$ yields with $\sum_{i=0}^{k-1} \alpha_i = -1$

$$
\sum_{i=0}^{k-1} \alpha_i\, G_x(x_k) x_i - \sum_{i=0}^{k-1} \alpha_i\, G_x(x_k) x_k = h \sum_{i=0}^{k} \beta_i\, G_x(x_k)\, f(x_i, y_i)\,.
$$

Adding this equation to (21) and dividing by h^2 we thus obtain

$$\frac{1}{h}\sum_{i=0}^{k}\beta_i\,G_x(x_k)\,f(x_i,y_i) = \frac{1}{h^2}\sum_{i=0}^{k-1}\alpha_i\int_0^1[G_x(x_i+\tau(x_k-x_i))-G_x(x_k)]d\tau\cdot(x_k-x_i)-\Delta$$
$$=:\ Q(x_0,\ldots,x_k;h)\ .$$

We estimate

$$x_k-x_i = x_k+\sum_{\ell=0}^{k-1}\alpha_\ell\,x_\ell+\sum_{\ell=0}^{k-1}\alpha_\ell[x_i-x_\ell] = O(h)$$

by means of the first equation (20) and by our assumption on $|x_i-x_\ell|$. We therefore conclude that the function Q is bounded. It remains to show that the nonlinear system

$$(22)\qquad \begin{aligned} x_k &= -\sum_{i=0}^{k-1}\alpha_i\,x_i+h\sum_{i=0}^{k}\beta_i\,f(x_i,y_i)\\ \beta_k\,G_x(x_k)\,f(x_k,y_k) &= -\sum_{i=0}^{k-1}\beta_i\,G_x(x_i)\,f(x_i,y_i)+hQ(x_0,\ldots,x_k;h)\end{aligned}$$

has a unique solution (x_k,y_k) near $(-\sum_{i=0}^{k-1}\alpha_i x_i, s_0(-\sum_{i=0}^{k-1}\alpha_i x_i))$. We introduce the variables z_i by $y_i = s_0(x_i)+z_i$, $i=0,\ldots,k$, with $|z_i|\le d$, $i=0,\ldots,k-1$. Since, by Hypothesis $\overline{\mathrm{H}}_0$, $G_x(x)f(x,s_0(x))=0$ and $G_x(x)f_y(x,s_0(x))$ is invertible and its inverse is bounded for $x\in\mathbb{R}^m$ we can apply, for $|z_k|\le d_k$ small enough, the Newton-Kantorovich theorem implying for h and $|\beta|d$ sufficiently small the existence of a unique solution $(x_k,y_k)(x_0,\ldots,x_{k-1};y_0,\ldots,y_{k-1};h)$ satisfying $|x_k+\sum_{i=0}^{k-1}\alpha_i\,x_i|\le const\cdot(h+|\beta|d)$, $|z_k|\le const\cdot(h+|\beta|d)$ (cf. [7]). The smoothness follows from the implicit function theorem.

⊥

Remark 10): i) For x_i, $i=0,\ldots,k-1$, such that $-\sum_{i=0}^{k-1}\alpha_i G(x_i) = h^2\Delta$, Lemma 5 corresponds to Theorem 6.1 of [4]. With our type of proof we do not need their assumption $y_i-y(ih)=O(h)$, $i=0,\ldots,k-1$, but only $|y_i-s_0(x_i)|\le d$, d small enough (if $\beta\ne 0$). If we choose the y_i such that $y_i-s_0(x_i)=O(h)$, $i=0,\ldots,k-1$, we have

$$\Big|x_k+\sum_{i=0}^{k-1}\alpha_i\,x_i\Big|\le Kh\ ,\qquad |y_k-s_0(x_k)|\le Kh$$

as in [4]. For BDF-like methods, i.e., $\beta=0$ in Eq. (19) this estimate holds independently of the y-starting values y_i. In this case, there is no restriction on the y_i and the solution (x_k,y_k) does not depend on the y-starting values y_i at all.

ii) In [4] also LMMs with $0=\sum_{i=0}^{k}\beta_i G(x_i)$ as second equation (in Eq. (19)) are considered. Lemma 5 yields a solution (x_k,y_k) of such a method for x-starting values x_i such that $-\sum_{i=0}^{k-1}(\alpha_i-\frac{\beta_i}{\beta_k})G(x_i)=h^2\Delta$. ⊣

In the linear case, it is again easy to verify that the maps $(\mathrm{I})_{\mathrm{h},0}$ and $(\bar{\mathrm{I}})_{\mathrm{h},0}$ create the same orbit if $G(x_i)=0$, $i=0,\ldots,k-1$, implying the existence of the y-attractive invariant manifold $\overline{M}_0 = \{(x,y) \,|\, G(x)=0,\ y=s_0(x)\}$ (for $\beta=0$ *and* $\beta \neq 0$) with the properties already derived in Paragraph 3.1 for $(\mathrm{I})_{\mathrm{h},0}$.

In the nonlinear case we consider for $\beta=0$ (BDF-like methods)

$$\begin{aligned} \sum_{i=0}^{k} \alpha_i\, x_i &= h\beta_k\, f(x_k,y_k) \\ G(x_k) &= -\sum_{i=0}^{k-1} \alpha_i\, G(x_i)\ . \end{aligned} \tag{23}$$

For z_k defined by $y_k = s_0(x_k)+z_k$ we know from the proof of Lemma 5 (with $\Delta \equiv 0$) that the solution $(x_k,z_k)(x_0,\ldots,x_{k-1};h)$ of Eq. (23) satisfies

$$\begin{aligned} x_k &= -\sum_{i=0}^{k-1} \alpha_i\, x_i + h\beta_k\, f(x_k, s_0(x_k)+z_k) \\ z_k &= h\overline{Q}(x_0,\ldots,x_{k-1};h) \end{aligned} \tag{24}$$

with $\overline{Q} = O(|Q|)$ (cf. Eq. (22) for $\beta=0$). For every $x_0 \in \mathbb{R}^m$ and $\gamma>0$ the function $\overline{Q}$ is bounded and of class C_b^r with respect to $x_0,\ldots,x_{k-1}$ if $|x_i - x_0| \le \gamma h$, $i=1,\ldots,k-1$. We extend $\overline{Q}$ to all $\mathbb{R}^{km}$ by modifying it in the following way: Inside the tube $\Omega_{\gamma/2} = \{X_0 \in \mathbb{R}^{km} \,|\, x_0 \in \mathbb{R}^m,\ |x_i - x_0| \le \frac{\gamma}{2}\ h,\ i=1,\ldots,k-1\}$ we put $\widehat{Q} := \overline{Q}$, outside the tube Ω_γ we put $\widehat{Q}=0$ and, in between, $\widehat{Q}$ is taken such that it is C_b^r. The first equation of (24) may be considered for all $X_0 \in \mathbb{R}^{km}$. Hence, for d and h small enough we have a smooth map from $\mathbb{R}^{km} \times \{Z \in \mathbb{R}^{kn} \,|\, |Z|_\infty \le d\}$ into itself of the form

$$\begin{aligned} X_1 &= ((R-L_\alpha)\otimes I_m)\, X_0 + h(e_k \otimes \widehat{P}(X_0;h)) \\ Z_1 &= (R \otimes I_n) Z_0 + h(e_k \otimes \widehat{Q}(X_0;h))\ . \end{aligned} \tag{25}$$

As done in [7] we apply the invariant manifold theory of [5] and the analogous construction yields the existence of an m-dimensional attractive invariant manifold for the modified map (25) (in the x,y-variables)

$$M_{h,0} = \{(x,y) \,|\, x \in \mathbb{R}^m,\ y = \sigma_0(x,h)\}$$

if started appropriately, i.e., $x_i = \Phi^i(x_0,h), i=0,\ldots,k-1$. The closeness to M_0 is $\sigma_0(x,h) - s_0(x) = O(h\,|\widehat{Q}|) = O(h|Q|)$, the attractivity in y-direction is $y_j - \sigma_0(x_j,h) = z_j + \sigma_0(x_j,h) - s_0(x_j) = O(h\,|\widehat{Q}|)$ for $j \ge k$ and $\chi_{h,0}^k = 0$ if $x_i = \Phi^i(x_0,h)$, $i=0,\ldots,k-1$, and $M_{h,0}$ has a vertical stable foliation ($L_{12}^{h,0}=0$, since the $O(h)$-terms in Eq. (25) do not depend on Z_0). Since on $M_{h,0}$ we have $x_i = \Phi^i(x_0,h)$, $x_i - x_0 = O(h)$, $i=0,\ldots,k-1$, it follows that for γ large enough, $M_{h,0}$ is an invariant manifold for the original map (24).

For the closeness of $M_{h,0}$ to M_0 we have to get a better estimate for the function Q established in the proof of Lemma 5. This is done using similar techniques as for RKMs. As seen in Paragraph 3.1 the solution $x(t)$ of Eq. $(\overline{\mathrm{I}})_0$ with $x(0)=x_0$ satisfies

$$\sum_{i=0}^{k} \alpha_i\, G(x(ih)) - h\,\beta_k\, G_x(x(kh))\, f(x(kh), s_0(x(kh))) = O(h^{p+1})\ .$$

Since $G(x(t)) = 0$, $t \geq 0$, we thus have together with

$$h\beta_k\, G_x(x_k)\, f(x_k, s_0(x_k) + z_k) = h^2\, Q(x_0, \ldots, x_k; h)$$

that

$$hQ = \beta_k\, [G_x(x_k)\, f(x_k, s_0(x_k) + z_k) - G_x(x(kh))\, f(x(kh), s_0(x(kh))] + O(h^p)\ .$$

The term in brackets is of order $O(|z_k|) + O(|x_k - x(kh)|)$. We estimate

$$\begin{aligned} z_k = y_k - s_0(x_k) &= y_k - s_0(x(kh)) + s_0(x(kh)) - s_0(x_k) \\ &= y_k - y(kh) + O(|x_k - x(kh)|)\ . \end{aligned}$$

Since on $M_{h,0}$ we have $x_i = \Phi^i(x_0, h)$ with $x_i - x(ih) = O(h^{p+1})$, $i = 0, \ldots, k-1$, it follows from the results in [4] that

$$x_k - x(kh) = O(h^p)\,, \quad y_k - y(kh) = O(h^p)\ ,$$

implying $hQ = O(h^p)$.

Taking the restriction to the subspace $G(x) = 0$, i.e., taking $G(x_i) = 0$, $i = 0, \ldots k-1$, it follows from Eq. (23) that $G(x_k) = 0$ and, hence, that this subspace is positively invariant under the LMM-map. It is easy to see that this subspace is indeed invariant. Summarizing we have:

- Invariant manifold of $(\bar{\mathrm{I}})_{\mathrm{h},0}$:

$$\overline{M}_{h,0} = \{(x, y)\,|\,G(x) = 0,\ \ y = \sigma_0(x, h))\} \subset M_{h,0}\ .$$

- Closeness to M_0:

$$\sigma_0(x, h) - s_0(x) = O(h^p)\ .$$

- Infinite attractivity in x-direction:

$$|-\sum_{i=0}^{k-1} \alpha_i\, G(x_i)| \leq const \cdot h^2 \quad \text{implies} \quad G(x_j) = 0 \ \text{ for } \ j \geq k\,.$$

- Attractivity in y-direction:

$$\text{infinite (i.e., } \chi^k_{h,0} = 0) \ \text{ if } \ x_i = \Phi^i(x_0, h) \ \text{ for } \ i = 0, \ldots, k-1\,;$$

  in any case,

$$|y_j - \sigma_0(x_j, h)| \leq Kh^p \ \text{ for } \ j \geq k\ .$$

- Stable (vertical) foliation and asymptotic phase.

Remark 11): i) We have shown that also for LMMs with $\beta = 0$ the situation of Fig. 7 holds (if started appropriately) but with closeness $O(h^p)$ instead of $O(h^q)$.

ii) As for RKMs the global error estimate for the x-component which is $O(h^p)$ (cf. [4]) immediately gives the estimate $O(h^p)$ for the y-component by means of

$$|y_j - y(jh)| \leq |\sigma_0(x_j, h) - s_0(x_j)| + K\,|x_j - x(jh)|\ .$$

$\dashv$

In the case of general LMMs (i.e., $\beta \neq 0$ in Eq. (19) and Hypothesis $\mathrm{H_{LMM}}$) the same invariant manifold result holds as in the BDF-like case (cf. also [7]). The attractivity in y-direction is $|y_j - \sigma(x_j, h)| \leq K\,\chi_{h,0}^j\ \max_{0\leq i<k}\{|y_i - \sigma_0(x_i, h)|\}$ for $j \geq k$ with $\chi_{h,0} = \max\{\rho_1, \sigma_1\} + O(|\beta| h) + O(h) < 1$ if $x_i = \Phi^i(x_0, h)$, $i = 0, \ldots k-1$. The stable fibers are not vertical anymore.

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References

[1] U. Ascher, H. Chin and S. Reich, *Stabilization of DAEs and invariant manifolds*, Numer. Math. 67 (1994), 131-149.

[2] E. Hairer, Ch. Lubich and M. Roche, *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta methods*, Lecture Notes in Mathematics 1409, Springer-Verlag, Berlin, 1989.

[3] E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, 2nd revised edition, Berlin, 1993.

[4] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*, Springer-Verlag, Berlin, 1991.

[5] K. Nipp and D. Stoffer, *Attractive invariant manifolds for maps: Existence, smoothness and continuous dependence on the map*, Report 92-11, SAM, ETH Zürich, 1992.

[6] K. Nipp and D. Stoffer, *Invariant manifolds and global error estimates of numerical integration schemes applied to stiff systems of singular perturbation type – Part I: RK-methods*, Numer. Math. 70 (1995), 245-257.

[7] K. Nipp and D. Stoffer, *Invariant manifolds and global error estimates of numerical integration schemes applied to stiff systems of singular perturbation type – Part II: Linear multistep methods*, Numer. Math. 74 (1996), 305-323.

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