Eidgenössische Technische Hochschule Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Coupled Problems for Viscous Incompressible Flow in Exterior Domains

M. Feistauer[1] and C. Schwab

Research Report No. 99-07
April 1999

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

[1]Charles University Prague, Faculty of Mathematics and Physics, Malostranské n. 25, 11800 Praha 1, Czech Republik, email: feist@ms.mff.cuni.cz

Coupled Problems for Viscous Incompressible Flow in Exterior Domains

M. Feistauer[1] and C. Schwab

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 99-07 April 1999

Dedicated to Prof. Jindřich Nečas on the occasion of his seventieth birthday

Abstract

The formulation of the *fluid flow in an unbounded exterior domain* Ω is not always convenient for computations and, therefore, the problem is often truncated to a bounded domain $\Omega^- \subset \Omega$ with an artificial exterior boundary Γ. Then the problem of the choice of suitable "transparent" boundary conditions on Γ appears. Another possibility is to simulate the presence of the fluid in the domain Ω^+ exterior to Γ with the use of a suitable (preferably linear) approximation of the equations describing the flow. The interior and exterior problems are coupled with the aid of *transmission conditions* on the interface Γ.

Here we briefly describe the formulation and analysis of the coupling of the interior Navier–Stokes problem and the exterior Stokes problem and Oseen problem. At the end we give the reformulation of the coupled problems with the aid of integral equations on the artificial interface.

[1] Charles University Prague, Faculty of Mathematics and Physics, Malostranské n. 25, 11800 Praha 1, Czech Republik, email: feist@ms.mff.cuni.cz

1 Coupling of interior Navier-Stokes problem with exterior Stokes problem

Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain which is the complement of the closure of a bounded open set (representing, e. g., a body imerged into a moving fluid). We set $\Gamma_0 = \partial\Omega$ and introduce an artificial interface Γ dividing Ω into two subdomains, a bounded interior domain Ω^- with boundary $\partial\Omega^- = \Gamma_0 \cup \Gamma$ and an unbounded domain Ω^+ with $\partial\Omega^+ = \Gamma$.

Classical formulation of the coupled problem: Find the velocity $\boldsymbol{u}^\pm = (u_1^\pm, u_2^\pm, u_3^\pm) : \overline{\Omega^\pm} \to \mathbb{R}^3$ and the pressure $p^\pm : \overline{\Omega^\pm} \to \mathbb{R}$ such that

$$
\begin{aligned}
&\text{a)} && u_i^\pm \in C^2(\overline{\Omega^\pm}), \quad i = 1, 2, 3, \quad p^\pm \in C^1(\overline{\Omega^\pm}),\\
&\text{b)} && -2\nu \sum_{j=1}^{3} \frac{\partial D_{ij}(\boldsymbol{u}^-)}{\partial x_j} + \sum_{j=1}^{3} u_j^- \frac{\partial u_i^-}{\partial x_j} + \frac{\partial p^-}{\partial x_i} = f_i, \; i = 1, 2, 3, \quad \text{in } \Omega^-,\\
&\text{c)} && \operatorname{div} \boldsymbol{u}^- = 0 \quad \text{in } \Omega^-,\\
&\text{d)} && \boldsymbol{u}^-|_{\Gamma_0} = 0,\\
(1.1)\quad &\text{e)} && -\nu\, \Delta\, \boldsymbol{u}^+ + \nabla p^+ = 0 \quad \text{in } \Omega^+,\\
&\text{f)} && \operatorname{div} \boldsymbol{u}^+ = 0 \quad \text{in } \Omega^+,\\
&\text{g)} && \lim_{|x|\to\infty} \boldsymbol{u}^+(x) = \boldsymbol{u}_\infty,\\
&\text{h)} && \boldsymbol{u}^- = \boldsymbol{u}^+ \quad \text{on } \Gamma,\\
&\text{i)} && \Big(p^- + \frac{1}{2}\, |\boldsymbol{u}^-|^2\Big)\boldsymbol{n} + 2\nu\, \mathbb{D}(\boldsymbol{u}^-)\, \boldsymbol{n} = \sigma_n(\boldsymbol{u}^+, p^+) \text{ on } \Gamma\,,\\
&\text{j)} && \sigma_n(\boldsymbol{u}^+, p^+) := \sigma[\boldsymbol{u}^+, p^+]\boldsymbol{n}, \quad \sigma[\boldsymbol{u}, p] = -p\mathbf{1} + 2\nu\mathbb{D}(\boldsymbol{u}).
\end{aligned}
$$

Here $\sigma[\boldsymbol{u}, p]$ denotes the hydrostatic stress tensor for the Stokes problem.

We prescribe the following data: $\nu > 0$ – constant viscosity, $\boldsymbol{f} = (f_1, f_2, f_3)$ – volume force with support $\operatorname{supp}\boldsymbol{f} \subset \Omega^-$, $\boldsymbol{u}_\infty \in \mathbb{R}^3$ – the free-stream velocity at ∞. By $\boldsymbol{n}$ we denote the unit outer normal to $\partial\Omega^-$ on Γ (pointing from Ω^- into Ω^+). $\mathbb{D}(\boldsymbol{u})$ is the velocity deformation tensor with components $\mathbb{D}_{ij}(\boldsymbol{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$.

In the domain Ω^- and Ω^+ the Navier–Stokes system and the Stokes system are considered, respectively. The coupling conditions on Γ representing the continuity of the velocity and the normal stress, augmented in Ω^- by the kinetic energy, were chosen in accordance with [16], [1].

1.1 Weak formulation

In order to reformulate the above problem in a weak sense, we introduce the following function spaces ([4], [6], [10], [12], [13]): $H^1(\Omega^-)$ – the Sobolev space equipped with the standard norm $\|\cdot\|_{1,\Omega^-}$, $H^{1/2}(\Gamma)$ – Sobolev–Slobodetskii space of traces $\gamma_0 u$ on Γ of functions $u \in H^1(\Omega^-)$ equipped with norm $\|\cdot\|_{1/2,\Gamma}$, $H^{-1/2}(\Gamma)$ – dual of $H^{1/2}(\Gamma)$, $W^1(\Omega^+)$

– weighted Sobolev space $= \{u; (1+|x|^2)^{-1/2}\, u \in L^2(\Omega^+),\ \partial u/\partial x_i \in L^2(\Omega^+),\ i=1,2,3\}$, equipped with the norm

$$\|u\|_{1,\Omega^+} = \Big\{ \int\limits_{\Omega^+} (1+|x|^2)^{-1/2}\, |u(x)|^2 \,\mathrm{d}x + |u|^2_{1,\Omega^+} \Big\}^{1/2},$$

where the seminorm

$$|u|_{1,\Omega^+} = \Big(\int\limits_{\Omega^+} |\nabla u|^2 \,\mathrm{d}x \Big)^{1/2}$$

is a norm equivalent to the norm $\|\cdot\|_{1,\Omega^+}$. We set $W_0^1(\Omega^+) = \{v \in W^1(\Omega^+);\ \gamma_0\, v = 0 \text{ on } \Gamma\}$. The space $H^{1/2}(\Gamma)$ can be interpreted as the space of traces $\gamma_0\, u$ of all $u \in W^1(\Omega^+)$ on Γ. By $\langle \cdot, \cdot \rangle$ we denote the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ induced by the $L^2(\Gamma)$ – scalar product.

If X is a Banach space with a norm $\|\cdot\|$, then we define the space $\boldsymbol{X} = X \times X \times X$ equipped with the norm

$$\|\boldsymbol{u}\| = \Big(\sum_{i=1}^{3} \|u_i\|^2 \Big)^{1/2}, \quad \boldsymbol{u} = (u_1, u_2, u_3) \in \boldsymbol{X}.$$

Now we set

$$\begin{aligned}
\boldsymbol{V}(\Omega^-) &= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^-);\ \ \boldsymbol{v}|_{\Gamma_0} = 0,\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega^-\},\\
\boldsymbol{W}_0^1(\Omega^+) &= \{\boldsymbol{v} \in \boldsymbol{W}^1(\Omega^+);\ \ \gamma_0 \boldsymbol{v} = 0 \text{ on } \Gamma\},\\
\boldsymbol{W}(\Omega^+) &= \{\boldsymbol{v} \in \boldsymbol{W}^1(\Omega^+);\ \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega^+\},\\
\boldsymbol{V}_0(\Omega^+) &= \{\boldsymbol{v} \in \boldsymbol{W}(\Omega^+);\ \ \gamma_0 \boldsymbol{v} = 0 \text{ on } \Gamma\},\\
\boldsymbol{H}_0^{1/2}(\Gamma) &= \{\boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma);\ \ \int\limits_{\Gamma} \boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}s = 0\}\,.
\end{aligned}$$

We have $\gamma_0\, \boldsymbol{v} \in \boldsymbol{H}_0^{1/2}(\Gamma)$ for $\boldsymbol{v} \in \boldsymbol{V}(\Omega^-)$.

It is possible to show that for any $\boldsymbol{u}_0 \in \boldsymbol{H}_0^{1/2}(\Gamma)$ there exists its extension $I\!R\, \boldsymbol{u}_0 \in \boldsymbol{W}(\Omega^+)$ such that $\gamma_0(I\!R\, \boldsymbol{u}_0) = \boldsymbol{u}_0$.

For the weak formulation we introduce the following forms:

$$\begin{aligned}
a_0(\boldsymbol{u}, \boldsymbol{v}) &= 2\nu \int\limits_{\Omega^-} \sum_{i,j=1}^{3} D_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{v}) \,\mathrm{d}x,\\
a_1(\boldsymbol{u},\, \boldsymbol{w},\, \boldsymbol{v}) &= \int\limits_{\Omega^-} \sum_{i,j=1}^{3} u_j \frac{\partial w_i}{\partial x_j}\, v_i \,\mathrm{d}x,\\
a_2(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) &= -\frac{1}{2} \int\limits_{\Gamma} (\boldsymbol{u} \cdot \boldsymbol{w})\, (\boldsymbol{v} \cdot \boldsymbol{n}) \,\mathrm{d}s, \qquad (1.2)\\
a(\boldsymbol{u}, \boldsymbol{v}) &= a_0(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),\\
&\qquad \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{H}^1(\Omega^-),\\
a^+(\boldsymbol{z}, \boldsymbol{v}) &= \int\limits_{\Omega^+} \sum_{i,j=1}^{3} \frac{\partial z_i}{\partial x_j}\ \frac{\partial v_i}{\partial x_j} \,\mathrm{d}x, \quad \boldsymbol{z},\ \boldsymbol{v} \in \boldsymbol{W}^1(\Omega^+)\,.
\end{aligned}$$

Let us assume that $\boldsymbol{f} \in \boldsymbol{V}^*(\Omega^-)$ (= dual of $\boldsymbol{V}(\Omega^-)$) and denote by $\langle \cdot, \cdot \rangle_{\Omega^-}$ the duality between $\boldsymbol{V}^*(\Omega^-)$ and $\boldsymbol{V}(\Omega^-)$.

Starting from the classical formulation (1.1), using suitable (smooth) test functions (with compact supports) and Green's theorem, we arrive at the following weak formulations:

Weak formulation in Ω^-. Assume that $\sigma_n(\boldsymbol{u}^+, p^+) \in \boldsymbol{H}^{-1/2}(\Omega)$ is given.
Find $\boldsymbol{u}^- \in \boldsymbol{V}(\Omega^-)$ such that

$$a(\boldsymbol{u}^-, \boldsymbol{v}) - \langle \sigma_n(\boldsymbol{u}^+, p^+),\, \gamma_0\, \boldsymbol{v} \rangle = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{\Omega^-} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}(\Omega^-)\,. \tag{1.3}$$

Weak formulation in Ω^+. Assume that $\boldsymbol{u}_0 \in \boldsymbol{H}_0^{1/2}(\Gamma)$ is given.
Find $\boldsymbol{u}^+$ satisfying the following conditions:

$$\begin{aligned} &\text{a)} \qquad (\boldsymbol{u}^+ - \boldsymbol{u}_\infty) - I\!R(\boldsymbol{u}_0 - \boldsymbol{u}_\infty) \in \boldsymbol{V}_0(\Omega^+), \\ &\text{b)} \qquad a^+(\boldsymbol{u}^+ - \boldsymbol{u}_\infty,\, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0(\Omega^+). \end{aligned} \tag{1.4}$$

Using the Lax–Milgram lemma and results from [4] and [10], it is possible to establish

Theorem 1.1. *There exists a unique solution $\boldsymbol{u}^+$ of problem* (1.4)*. This solution is independent of the choice of the extension $I\!R(\boldsymbol{u}_0 - \boldsymbol{u}_\infty) \in \boldsymbol{W}(\Omega^+)$ of $\boldsymbol{u}_0 - \boldsymbol{u}_\infty$ from Γ onto Ω^+. The velocity $\boldsymbol{u}^+$ can be associated with a uniquely determined pressure $p^+ \in L^2(\Omega^+)$ such that*

$$a^+(\boldsymbol{u}^+ - \boldsymbol{u}_\infty,\, \boldsymbol{v}) - \int\limits_{\Omega^+} p^+ \operatorname{div} \boldsymbol{v}\, \mathrm{d}x = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_0^1(\Omega^+)\,. \tag{1.5}$$

□

Assuming that it is possible to define a generalization $\sigma_n(\boldsymbol{u}^+, p^+) \in \boldsymbol{H}^{-1/2}(\Gamma)$ of the normal stress for $\boldsymbol{u}^+$, p^+ from Theorem 1.1, we arrive at the **weak formulation of the coupled problem**:
Find $\boldsymbol{u}^-$, $\boldsymbol{u}^+$ satisfying (1.3) and (1.4) with

$$\boldsymbol{u}_0 = \gamma_0\, \boldsymbol{u}^- \quad \text{on } \Gamma\,. \tag{1.6}$$

1.2 Abstract problem

Let us assume for now that $\boldsymbol{u}^-$ is known. Then we solve problem (1.4) with the Dirichlet boundary condition (1.6). If the solution $\boldsymbol{u}^+$ and the pressure p^+ associated with $\boldsymbol{u}^+$ by Theorem 1.1 allow to express $\sigma_n(\boldsymbol{u}^+, p^+) \in \boldsymbol{H}^{-1/2}(\Gamma)$, we see that $\sigma_n(\boldsymbol{u}^+, p^+)$ is a function of $\boldsymbol{u}_0 = \gamma_0\, \boldsymbol{u}^-$:

$$\sigma_n(\boldsymbol{u}^+, p^+) = -\Lambda(\boldsymbol{u}_0). \tag{1.7}$$

The operator $\Lambda : \boldsymbol{H}_0^{1/2}(\Gamma) \to \boldsymbol{H}^{-1/2}(\Gamma)$ converting Dirichlet data into "Neumann" data via the solution of the exterior Stokes problem (1.4), is called the **Steklov–Poincaré**

operator. It allows us to reformulate the coupled problem as the **abstract problem**: Given $\Lambda : \boldsymbol{H}_0^{1/2}(\Gamma) \to \boldsymbol{H}^{-1/2}(\Gamma)$ and $\boldsymbol{f} \in \boldsymbol{V}^*(\Omega^-)$, find $\boldsymbol{u}^-$ such that

$$\text{(1.8)} \qquad \begin{aligned} &\text{a)} \quad \boldsymbol{u}^- \in \boldsymbol{V}(\Omega^-), \\ &\text{b)} \quad a(\boldsymbol{u}^-, \boldsymbol{v}) + \langle \Lambda(\gamma_0, \boldsymbol{u}^-), \, \gamma_0 \boldsymbol{v} \rangle + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega^-} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}(\Omega^-). \end{aligned}$$

The investigation of problem (1.8) yields the following result:

Theorem 1.2. *Let the operator* Λ *be* **weakly sequentially continuous** *and* **weakly noncoercive**, *i. e.,*

$$\text{(1.9)} \qquad \begin{aligned} &\boldsymbol{z}^n, \ \boldsymbol{z} \in \boldsymbol{H}_0^{1/2}(\Gamma), \ \boldsymbol{z}^n \to \boldsymbol{z} \ \text{weakly in } \boldsymbol{H}^{1/2}(\Gamma) \ \text{as } n \to \infty \implies \\ &\implies \langle \Lambda(\boldsymbol{z}^n), \boldsymbol{w} \rangle \longrightarrow \langle \Lambda(\boldsymbol{z}), \boldsymbol{w} \rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^{1/2}(\Gamma) \ \text{as } n \to \infty, \end{aligned}$$

and there exist constants $c_3 \in I\!\!R, \ c_4 \geq 0$ *such that*

$$\text{(1.10)} \qquad \langle \Lambda(\boldsymbol{z}), \boldsymbol{z} \rangle \geq c_3 - c_4 \|\boldsymbol{z}\|_{1/2,\Gamma} \quad \forall\, \boldsymbol{z} \in \boldsymbol{H}_0^{1/2}(\Gamma),$$

respectively. Then problem (1.8) *has at least one solution.*

Proof of this theorem is carried out by the Galerkin method, similarly as, e. g., in [6, Par. 8.4.20] or [12, Theorem 1.2, page 280] with the aid of the compact imbeddings $H^1(\Omega^-) \hookrightarrow\hookrightarrow L^2(\Omega^-)$, $H^1(\Omega^-) \hookrightarrow\hookrightarrow L^3(\Gamma)$, Korn's inequality and the relation $a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) + a_2(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{u}) = 0$ valid for all $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{V}(\Omega^-)$. □

1.3 Properties of the Steklov–Poincaré operator Λ

It remains to establish the existence of the operator Λ and its properties (1.9) and (1.10):

Theorem 1.3. *Let* $\boldsymbol{u}^+$ *be the solution of the exterior problem* (1.4) *and* p^+ *be the associated pressure by relation* (1.5). *Then, for all* $\boldsymbol{w} \in \boldsymbol{H}^{1/2}(\Gamma)$ *and* $\boldsymbol{v} \in \boldsymbol{W}^1(\Omega^+)$ *such that* $\gamma_0 \boldsymbol{v} = \boldsymbol{w}$, *the formula*

$$\text{(1.11)} \qquad \langle \sigma_n(\boldsymbol{u}^+, p^+), \, \boldsymbol{w} \rangle = -2\nu \int\limits_{\Omega^+} \sum_{i,j=1}^{3} D_{ij}(\boldsymbol{u}^+) \, D_{ij}(\boldsymbol{v}) \, \mathrm{d}x + \int\limits_{\Omega^+} p^+ \operatorname{div} \boldsymbol{v} \, \mathrm{d}x$$

determines a unique element $\sigma_n(\boldsymbol{u}^+, p^+) \in \boldsymbol{H}^{-1/2}(\Gamma)$. *If* $\boldsymbol{u}^+$ *and* p^+ *are sufficiently regular, then this element can be identified with the function* $\sigma_n(\boldsymbol{u}^+, p^+)$ *defined in* (1.1, j). *Further, the Steklov-Poincaré operator* Λ *defined by* (1.7) *has properties* (1.9) *and* (1.10). □

The results of Theorem 1.1 – 1.3 imply the existence of a weak solution of the coupled problem (1.1). All details can be found in [8].

2 Coupling of interior Navier-Stokes problem with exterior Oseen problem

In this section we are concerned with the modelling of viscous incompressible flow in an unbounded exterior domain with the aid of the coupling of the nonlinear Navier–Stokes equations considered in a bounded domain with the linear Oseen system in an exterior domain.

Similarly as in the case of the coupling of the Navier–Stokes problem with the Stokes problem, an important question is the choice of transmission conditions on the artificial interface Γ. The transmission condition used in Section 1 is not suitable in the case of the exterior Oseen problem and, therefore, we propose its modification resembling a 'natural" boundary condition from [3]. We arrive then at the following **classical formulation of the coupled problem:**
Find $\boldsymbol{u}^\pm = (u_1^\pm, \ldots, u_N^\pm) : \overline{\Omega}^\pm \to I\!R^N,\ p^\pm : \overline{\Omega}^\pm \to I\!R$ such that

$$
\text{(2.1)}\qquad
\begin{aligned}
&\text{a)} \quad u_i^\pm \in C^2(\overline{\Omega}^\pm),\ i = 1, \ldots, N,\ p^\pm \in C^1(\overline{\Omega}^\pm),\\
&\text{b)} \quad -\nu\,\Delta\,\boldsymbol{u}^- + (\boldsymbol{u}^- \cdot \nabla)\,\boldsymbol{u}^- + \nabla\,p^- = \boldsymbol{f} \ \text{ in } \Omega^-,\\
&\text{c)} \quad \operatorname{div}\boldsymbol{u}^- = 0 \ \text{ in } \Omega^-,\\
&\text{d)} \quad \boldsymbol{u}^-|_{\Gamma_0} = 0,\\
&\text{e)} \quad -\nu\,\Delta\,\boldsymbol{u}^+ + (\boldsymbol{u}_\infty \cdot \nabla)\,\boldsymbol{u}^+ + \nabla\,p^+ = 0 \ \text{ in } \Omega^+,\\
&\text{f)} \quad \operatorname{div}\boldsymbol{u}^+ = 0 \ \text{ in } \Omega^+,\\
&\text{g)} \quad \lim_{|x|\to\infty} \boldsymbol{u}^+(x) = \boldsymbol{u}_\infty,\\
&\text{i)} \quad \boldsymbol{u}^- = \boldsymbol{u}^+ \ \text{ on } \Gamma,\\
&\text{j)} \quad -p^-\,\boldsymbol{n} + \nu\,\frac{\partial \boldsymbol{u}^-}{\partial \boldsymbol{n}} - \frac{1}{2}\,(\boldsymbol{u}^- \cdot \boldsymbol{n})\,\boldsymbol{u}^- = \sigma_n(\boldsymbol{u}^+,\, p^+) \text{ on } \Gamma\,.
\end{aligned}
$$

Here, and throughout we understand $\sigma_n(\boldsymbol{u}, p)$ in the context of the Oseen problem as

$$\sigma_n(\boldsymbol{u}^+,\, p^+) := \sigma[\boldsymbol{u}^+,\, p^+]\boldsymbol{n} \text{ where } \sigma[\boldsymbol{u},\, p] := -p\mathbf{1} + 2\nu I\!D(\boldsymbol{u}) - \frac{1}{2}\,\boldsymbol{u}\boldsymbol{u}_\infty^\top$$

denotes the hydrostatic stress tensor for the Oseen problem.

Other than that, we use the same notation as in Section 1.

Remark 2.1. For simplicity we consider the terms $\partial\boldsymbol{u}^\pm/\partial\boldsymbol{n}$ in (2.1,j), corresponding naturally to equations (2.1,b) and e). If we use the relations

$$\Delta u_i = \sum_{i=1}^{N} \frac{\partial D_{ij}(\boldsymbol{u})}{\partial x_j}, \qquad D_{ij}(\boldsymbol{u}) = \frac{1}{2}\,\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

valid for $\boldsymbol{u} \in C^2(\Omega^\pm)$ with $\operatorname{div}\boldsymbol{u} = 0$, then $\partial\boldsymbol{u}^\pm/\partial\boldsymbol{n}$ can be replaced by $\sum_{j=1}^N D_{ij}(\boldsymbol{u}^\pm)\,n_j$ as in Section 1.

2.1 Weak formulation

In what follows **we will assume** that $\partial\Omega^- = \Gamma_0 \cup \Gamma$ is Lipschitz-continuous. If $\widetilde{\Omega} \subset \Omega$ is a domain, then by $L^p(\widetilde{\Omega})$ and $W^{k,p}(\widetilde{\Omega})$ we denote the Lebesgue and Sobolev spaces, respectively, defined over $\widetilde{\Omega}$ (cf., [13]). For a bounded domain $\widetilde{\Omega}$ we set $W_0^{1,2}(\widetilde{\Omega}) = \{v \in W^{1,2}(\Omega);\ v|_{\partial\widetilde{\Omega}} = \text{the trace of } v \text{ on } \partial\widetilde{\Omega} = 0\}$. In $W_0^{1,2}(\widetilde{\Omega})$ we can use two equivalent norms

$$\|v\|_{W_0^{1,2}(\widetilde{\Omega})} = \Big(\int\limits_{\widetilde{\Omega}} (|v|^2 + |\nabla v|^2)\,\mathrm{d}x\Big)^{1/2}$$

and

$$|v|_{W_0^{1,2}(\widetilde{\Omega})} = \Big(\int\limits_{\Omega} |\nabla v|^2\,\mathrm{d}x\Big)^{1/2}.$$

It is well-known that

$$W_0^{1,2}(\widetilde{\Omega}) = \text{closure of } C_0^\infty(\widetilde{\Omega}) \text{ in } W^{1,2}(\widetilde{\Omega}),$$

where $C_0^\infty(\widetilde{\Omega})$ is the space of all infinitely continuously differentiable functions with compact supports in $\widetilde{\Omega}$: $\operatorname{supp} v \subset \widetilde{\Omega}$ for $v \in C_0^\infty(\widetilde{\Omega})$.

For the unbounded domain Ω we define the weighted Sobolev space

$$W^1(\Omega) = \Big\{u; (1+|x|^2)^{-1/2}\,\sigma_N\,u \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega)\Big\},$$

where $\sigma_N(x) = 1$ for $N = 3$ and $\sigma_N(x) = |\ln(1+|x|)|^{-1}$ for $N = 2$, equipped with the norm

$$\|u\|_{W^1(\Omega)} = \Big\{\int\limits_{\Omega} [(1+|x|^2)^{-1}\sigma_N^2\,|u|^2 + |\nabla u|^2]\,\mathrm{d}x\Big\}^{1/2},$$

which is equivalent to the seminorm

$$|u|_{W^1(\Omega)} = \Big\{\int\limits_{\Omega} |\nabla u|^2\,\mathrm{d}x\Big\}^{1/2}.$$

(See, e. g., [4, Theorem 1, page 118] or [10, Vol. I, page 60].)

Further, we put

$$W_0^1(\Omega) = \text{closure of } C_0^\infty(\Omega) \text{ in } W^1(\Omega).$$

Then

$$W_0^1(\Omega) = \Big\{v \in W^1(\Omega); v|_{\Gamma_0} = 0\Big\}.$$

We write $v \in W_{\mathrm{loc}}^{k,p}(\Omega)$, if $v|_{\widetilde{\Omega}} \in W^{k,p}(\widetilde{\Omega})$ for every bounded domain $\widetilde{\Omega} \subset \Omega$.

Let us define subspaces of $\boldsymbol{W}^1(\Omega)$:

$$\begin{aligned}\boldsymbol{\mathcal{V}}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}, \\ \boldsymbol{V}(\Omega) &= \text{closure of } \boldsymbol{\mathcal{V}}(\Omega) \text{ in } \boldsymbol{W}^1(\Omega).\end{aligned}$$

For functions $\boldsymbol{v}$ from subspaces of Sobolev spaces, the restrictions $\boldsymbol{v}|_\Gamma$, $\boldsymbol{v}|_{\Gamma_0}$ etc. will be understood in the sense of traces.

For $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, the limit at ∞ is zero and $\boldsymbol{v}|_{\Gamma_0} = 0$. In order to realize condition (2.1, g) in the weak formulation, we introduce a function $\boldsymbol{\phi}_\infty$ defined in the following way. Let $\mathcal{B}$ be a sufficiently large ball with centre at the origin such that $\overline{\Omega}^- \subset \mathcal{B}$. Then $\Omega^* := (\mathcal{B} \cap \Omega) - \overline{\Omega}^- \subset \Omega^+$ and $\partial\Omega^* = \Gamma \cup \Gamma^*$, where Γ and Γ^* is the interior and exterior component of $\partial\Omega^*$, respectively. Since $\int_\Gamma \boldsymbol{u}_\infty \cdot \boldsymbol{n} \, \mathrm{d}S = 0$, in virtue of [12, Lemma 2.2, page 24], there exists a function $\boldsymbol{\phi}^*$ such that

$$\boldsymbol{\phi}^* \in \boldsymbol{W}^{1,2}(\Omega^*), \quad \boldsymbol{\phi}^*|_\Gamma = 0, \quad \boldsymbol{\phi}^*|_{\Gamma^*} = \boldsymbol{u}_\infty, \quad \operatorname{div} \boldsymbol{\phi}^* = 0 \text{ in } \Omega^*.$$

Now we define $\boldsymbol{\phi}_\infty : \overline{\Omega} \to I\!R^N$:

$$\boldsymbol{\phi}_\infty = \begin{cases} 0 & \text{in } \overline{\Omega}^-, \\ \boldsymbol{\phi}^* & \text{in } \Omega^*, \\ \boldsymbol{u}_\infty & \text{in } \Omega^+ - \Omega^*. \end{cases}$$

Obviously, $\boldsymbol{\phi}_\infty \in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\widetilde{\Omega})$ and $\operatorname{div} \boldsymbol{\phi}_\infty = 0$ a. e. (= almost everywhere) in Ω.

Let us assume that $\boldsymbol{u}^\pm$, $p^\pm$ form a classical solution of the coupled problem (2.1). Let $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$. Multiplying equation (2.1, b) by $\boldsymbol{v}|_{\Omega^-}$ and (2.1, e) by $\boldsymbol{v}|_{\Omega^+}$, integrating over Ω^- and Ω^+, respectively, summing these integrals, applying Green's theorem and using the fact that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v}|_{\Gamma_0} = 0$, and putting

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}^- & \text{in } \overline{\Omega}^-, \\ \boldsymbol{u}^+ & \text{in } \overline{\Omega}^+. \end{cases}$$

we obtain the identity

$$\nu \int\limits_{\Omega^-} \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x + \nu \int\limits_{\Omega^+} \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x + \int\limits_{\Omega^-} \sum_{i,j=1}^N u_j \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}x$$
$$+ \int\limits_{\Omega^+} \sum_{i,j=1}^N u_{\infty j} \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}x - \frac{1}{2} \int\limits_\Gamma [(\boldsymbol{u} - \boldsymbol{u}_\infty) \cdot \boldsymbol{n}] \, [\boldsymbol{u} \cdot \boldsymbol{v}] \, \mathrm{d}s = \int\limits_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \, .$$

Let us introduce the forms

$$
(2.2)\qquad
\begin{aligned}
a_0(\boldsymbol{u},\boldsymbol{v}) &= \nu \int\limits_{\Omega^-} \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x,\\
a_1(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) &= \int\limits_{\Omega^-} \sum_{i,j=1}^{N} u_j\,\frac{\partial w_i}{\partial x_j}\,v_i\,\mathrm{d}x,\\
a_2(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) &= -\frac{1}{2}\int\limits_{\Gamma} [(\boldsymbol{u}-\boldsymbol{u}_\infty)\cdot\boldsymbol{n}]\,[\boldsymbol{w}\cdot\boldsymbol{v}]\,\mathrm{d}s,\\
b_0(\boldsymbol{u},\boldsymbol{v}) &= \nu \int\limits_{\Omega^+} \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x,\\
b_1(\boldsymbol{u},\boldsymbol{v}) &= \int\limits_{\Omega^+} \sum_{i,j=1}^{N} u_{\infty j}\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}x,\\
L(\boldsymbol{v}) &= \int\limits_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x,\\
a(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) &= a_0(\boldsymbol{u},\boldsymbol{v}) + a_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})\\
b(\boldsymbol{u},\boldsymbol{v}) &= b_0(\boldsymbol{u},\boldsymbol{v}) + b_1(\boldsymbol{u},\boldsymbol{v}),\\
&\quad \text{for } \boldsymbol{u},\,\boldsymbol{v}:\Omega\to I\!R^N,\ \ \boldsymbol{u},\,\boldsymbol{w}\in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega),\ \ \boldsymbol{v}\in \boldsymbol{C}_0^\infty(\Omega).
\end{aligned}
$$

On the basis of the above considerations we come to the following concept:

Definition 2.2. We call a vector valued function $\boldsymbol{u}:\Omega\to I\!R^N$ a **weak solution** of the coupled problem (2.1), if the following conditions are satisfied:

$$
(2.3)\qquad
\begin{aligned}
&\text{a)}\quad \boldsymbol{u}-\boldsymbol{\phi}_\infty \in \boldsymbol{V}(\Omega),\\
&\text{b)}\quad a(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in \boldsymbol{\mathcal{V}}(\Omega).
\end{aligned}
$$

Remark 2.3. From above it follows that the classical solution yields the weak solution. In (2.2, a), conditions (2.1, c, d, f, g) are hidden and $\boldsymbol{u}\in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$. Since $\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\Omega)$ has compact support, all integrals over Ω in (2.2) have sense. Moreover, also the form a_2 is well defined as follows from the trace theorem for functions from $\boldsymbol{W}^{1,2}(\widetilde{\Omega})$, where $\widetilde{\Omega}\subset\Omega$ is a bounded domain with $\Gamma\subset\partial\widetilde{\Omega}$. However, it is not possible to use $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ as test functions in (2.3, b), because the form $b_1(\boldsymbol{u},\boldsymbol{v})$ is not defined for $\boldsymbol{u}\in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ in general (cf. [10]). This is the reason that we cannot carry out the existence treatment as in Section 1. We apply now a completely different approach for proving the existence of a solution of problem (2.3). In fact, this new technique can also be applied to the coupling of the interior Navier–Stokes problem with the exterior Stokes problem. (Details will appear in [9].)

Remark 2.4. On the basis of results from [10], Chap. VII, the weak solution $\boldsymbol{u}$ of problem (2.3) can be associated with the pressure $p\in L^2_{\mathrm{loc}}(\Omega)$ such that

$$
(2.4)\qquad a(\boldsymbol{u},\boldsymbol{v}) - (p,\operatorname{div}\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v}\in \boldsymbol{C}_0^\infty(\Omega).
$$

2.2 Existence of a weak solution

First we prove some important properties of the forms a_0, a_1, a_2 defined in (2.2). These forms have sense, of course, also for functions from the space $\boldsymbol{W}^{1,2}(\Omega^-)$, as follows from the continuous imbedding $\boldsymbol{W}^{1,2}(\Omega^-) \hookrightarrow \boldsymbol{L}^4(\Omega^-)$ and the continuity of the trace operator from the space $\boldsymbol{W}^{1,2}(\Omega^-)$ into $\boldsymbol{L}^3(\Gamma)$. (We simply write $\boldsymbol{W}^{1,2}(\Omega^-) \hookrightarrow \boldsymbol{L}^3(\Gamma)$.)

Let us set

$$\begin{aligned} \boldsymbol{V}(\Omega^-) &= \Big\{ \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega^-); v|_{\Gamma_0} = 0, \operatorname{div} \boldsymbol{v} = 0 \ \text{ a.e. in } \Omega^- \Big\}, \\ \boldsymbol{\mathcal{V}}_0(\Omega^-) &= \Big\{ \boldsymbol{v} \in C^\infty(\overline{\Omega}^-); \operatorname{supp} \boldsymbol{v} \subset \Omega^- \cup \Gamma, \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega^- \Big\}, \\ \tilde{a}(\boldsymbol{u}, \boldsymbol{v}) &= -\frac{1}{2} \int_\Gamma (\boldsymbol{u} \cdot \boldsymbol{n}) \, |\boldsymbol{v}|^2 \, \mathrm{d}s, \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega^-) \, . \end{aligned}$$

Lemma 2.5. *a_0 is a continuous bilinear form on $\boldsymbol{W}^{1,2}(\Omega^-)$. Further, a_1 and a_2 are continuous trilinear forms on $\boldsymbol{W}^{1,2}(\Omega^-)$.*

For $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{V}(\Omega^-)$ we have

$$a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -a_1(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - \tilde{a}(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}) + \tilde{a}(\boldsymbol{u}, \boldsymbol{v}) + \tilde{a}(\boldsymbol{u}, \boldsymbol{w}) \, . \tag{2.5}$$

Let us define the form

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega^-) \, . \tag{2.6}$$

Then it holds: If $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{z}_n \in \boldsymbol{V}_0(\Omega^-)$, $n = 1, 2, \dots,$ and if

$$\begin{aligned} &\text{a)} \quad |\boldsymbol{z}_n|_{\mathbf{W}^{1,2}(\Omega^-)} \le C, \quad n = 1, 2, \dots, \\ &\text{b)} \quad \boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \textit{strongly in } \boldsymbol{L}^2(\Omega^-) \\ &\text{c)} \quad \boldsymbol{z}_n|_\Gamma \longrightarrow \boldsymbol{z}|_\Gamma \quad \textit{strongly in } \boldsymbol{L}^3(\Gamma) \ \textit{ as } n \to \infty, \end{aligned} \tag{2.7}$$

then

$$d(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow d(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \textit{as } n \to \infty. \tag{2.8}$$

□

The solvability of the coupled problem in the unbounded domain Ω is established with the aid of coupled problems considered on a monotone sequence of bounded subdomains. For any positive integer n we denote by $\mathcal{B}_n$ the ball with radius n and centre at the origin. We will consider $n \ge n_0$ with fixed n_0 such that $\mathcal{B} \subset \mathcal{B}_{n_0}$ ($\subset \mathcal{B}_n$), where $\mathcal{B}$ is the ball used in the definition of the function $\boldsymbol{\phi}_\infty$. Hence, $\partial\mathcal{B}_n \subset \Omega^+$ and $\boldsymbol{\phi}_\infty|_{\partial\mathcal{B}_n} = \boldsymbol{u}_\infty$ for $n \ge n_0$. We set $\Omega_n = \Omega \cap \mathcal{B}_n$ and $\Omega_n^+ = \Omega^+ \cap \mathcal{B}_n$. Then for $n \ge n_0$, we have $\Omega^- \subset \Omega_n$, $\Omega_n = \Omega^- \cup \Gamma \cup \Omega_n^+$, $\partial\Omega_n = \Gamma_0 \cup \Gamma_n$ and $\partial\Omega_n^+ = \Gamma \cup \Gamma_n$. Moreover, $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=n_0}^\infty \Omega_n = \Omega$. Γ_n is the exterior component of $\partial\Omega_n$ and $\partial\Omega_n^+$.

For $n \geq n_0$ and $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega_n)$ we define the forms

$$(2.9)\qquad \begin{aligned} b_0^n(\boldsymbol{u},\boldsymbol{v}) &= \nu \int_{\Omega_n^+} \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x, \\ b_1^n(\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega_n^+} \sum_{i,j=1}^N \phi_{\infty j}\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}x, \\ a^n(\boldsymbol{u},\boldsymbol{v}) &= a_0(\boldsymbol{u},\boldsymbol{v}) + a_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b_0^n(\boldsymbol{u},\boldsymbol{v}) + b_1^n(\boldsymbol{u},\boldsymbol{v})\,. \end{aligned}$$

For every $n \geq n_0$ we introduce the spaces

$$\begin{aligned} \boldsymbol{\mathcal{V}}(\Omega_n) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega_n);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega_n\}\,, \\ \boldsymbol{V}(\Omega_n) &= \text{closure of } \boldsymbol{\mathcal{V}}(\Omega_n) \text{ in } \boldsymbol{W}^{1,2}(\Omega_n) \\ &= \left\{\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega_n);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega_n\right\}, \end{aligned}$$

and consider the following **auxiliary problem** in Ω_n: Find $\boldsymbol{u}_n : \Omega_n \to I\!\!R^N$ such that

$$(2.10)\qquad \begin{aligned} &\text{a)} \quad \boldsymbol{u}_n - \boldsymbol{\phi}_\infty|_{\Omega_n} \in \boldsymbol{V}(\Omega_n), \\ &\text{b)} \quad a^n(\boldsymbol{u}_n,\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}(\Omega_n) \end{aligned}$$

(the form $L(\boldsymbol{v})$ has sense for $\boldsymbol{v} \in \boldsymbol{V}(\Omega_n)$ extended by zero on Ω). Conditions (2.10) represent the weak formulation of a coupled "Navier–Stokes – Oseen" problem in the bounded domain $\Omega_n = \Omega^- \cup \Gamma \cup \Omega_n^+$.

The solution of problem (2.3) can be written in the form

$$(2.11)\qquad \boldsymbol{u} = \boldsymbol{\phi}_\infty + \boldsymbol{z}, \quad \boldsymbol{z} \in \boldsymbol{V}(\Omega).$$

Hence, (2.3) is equivalent to finding $\boldsymbol{z} : \Omega \to I\!\!R^N$ such that

$$(2.12)\qquad \begin{aligned} &\text{a)} \quad \boldsymbol{z} \in \boldsymbol{V}(\Omega), \\ &\text{b)} \quad a(\boldsymbol{\phi}_\infty + \boldsymbol{z},\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega). \end{aligned}$$

Similarly we can reformulate problem (2.10): Find $\boldsymbol{z}_n : \Omega_n \to I\!\!R^N$ such that

$$(2.13)\qquad \begin{aligned} &\text{a)} \quad \boldsymbol{z}_n \in \boldsymbol{V}(\Omega_n), \\ &\text{b)} \quad a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n,\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}(\Omega_n). \end{aligned}$$

Then $\boldsymbol{u}_n = \boldsymbol{\phi}_\infty + \boldsymbol{z}_n$. From the definition of $\boldsymbol{\phi}_\infty$ it follows that $\boldsymbol{u}_n = \boldsymbol{z}_n$ in $\overline{\Omega}^-$.

The solvability of the above auxiliary problems is proved with the aid of the following results:

Lemma 2.6. *For each $\boldsymbol{z} \in \boldsymbol{V}(\Omega_n)$ we have*

$$a_1(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + b_1^n(\boldsymbol{\phi}_\infty + \boldsymbol{z},\boldsymbol{z}) = \int_{\Omega_n^+} \sum_{i,j=1}^N u_{\infty j}\frac{\partial \phi_{\infty i}}{\partial x_j}\,z_i\,\mathrm{d}x\,.$$

□

Theorem 2.7. *For each $n \geq n_0$ problem* (2.13) *has at least one solution $\boldsymbol{z}_n$. There exists a constant $K > 0$ independent of n such that*

$$|\boldsymbol{z}_n|_{\mathbf{W}^{1,2}(\Omega_n)} \leq K, \quad n \geq n_0. \tag{2.14}$$

Proof is carried out with the aid of the Galerkin method in a standard way as, e. g., in [12], Theorem 1.2, page 280, [17], Chap. II, or [6], Par. 8.4.20. □

The main result of this section reads:

Theorem 2.8. *There exists at least one solution $\boldsymbol{u}$ of problem* (2.3). *This $\boldsymbol{u}$ is a weak solution of the coupled problem* (2.1).

Proof. As was stated above, problem (2.3) is equivalent to problem (2.12). In order to prove the solvability of problem (2.12), we extend the solution $\boldsymbol{z}_n$ of problem (2.13) ($n \geq n_0$) by zero from the domain Ω_n onto Ω. For simplicity, we will denote this extension again by $\boldsymbol{z}_n$. Hence, we have a sequence $\{\boldsymbol{z}_n\}_{n=n_0}^{\infty}$ such that

$$\boldsymbol{z}_n \in \boldsymbol{V}(\Omega), \quad n \geq n_0, \;\; |\boldsymbol{z}_n|_{\mathbf{W}^1(\Omega)} = |\boldsymbol{z}_n|_{\mathbf{W}^{1,2}(\Omega_n)} \leq K, \quad n \geq n_0\,. \tag{2.15}$$

Since the space $\boldsymbol{V}(\Omega)$ is reflexive and the sequence $\{\boldsymbol{z}_n\}_{n=n_0}^{\infty}$ is bounded in $\boldsymbol{V}(\Omega)$, there exists $\boldsymbol{z} \in \boldsymbol{V}(\Omega)$ and a subsequence of $\{\boldsymbol{z}_n\}_{n=n_0}^{\infty}$ (let us denote it again by $\{\boldsymbol{z}_n\}$) such that

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{weakly in } \boldsymbol{V}(\Omega) \text{ as } n \to \infty. \tag{2.16}$$

Our goal is to show that $\boldsymbol{z}$ is a solution of problem (2.12).

Let $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$. Then there exists $n^* \geq n_0$ such that $\operatorname{supp} \boldsymbol{v} \subset \Omega_{n^*}$ and, in virtue of (2.13), (2.2) and (2.9) we have $\boldsymbol{v}|_{\Omega_n} \in \boldsymbol{V}(\Omega_n)$ for $n \geq n^*$ and

$$a(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a^n(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = L(\boldsymbol{v}), \quad n \geq n^*. \tag{2.17}$$

Taking into account that $|\boldsymbol{z}_n|_{\mathbf{W}^{1,2}(\Omega_{n^*})} \leq |\boldsymbol{z}_n|_{\mathbf{W}^1(\Omega)}$, from (2.15) we see that the sequence $\{\boldsymbol{z}_n|_{\Omega_{n^*}}\}$ is bounded in $\boldsymbol{W}^{1,2}(\Omega_{n^*})$. Thus, we can suppose that

$$\boldsymbol{z}_n|_{\Omega_{n^*}} \longrightarrow \boldsymbol{z}|_{\Omega_{n^*}} \quad \text{weakly in } \boldsymbol{W}^{1,2}(\Omega_{n^*}) \quad \text{as } n \to \infty. \tag{2.18}$$

This and the compact imbeddings $\boldsymbol{W}^{1,2}(\Omega_{n^*}) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega_{n^*})$ and $\boldsymbol{W}^{1,2}(\Omega_{n^*}) \hookrightarrow\hookrightarrow \boldsymbol{L}^3(\Gamma)$ imply that

$$\begin{aligned} \boldsymbol{z}_n|_{\Omega_{n^*}} &\longrightarrow \boldsymbol{z}|_{\Omega_{n^*}} \text{ strongly in } \boldsymbol{L}^2(\Omega_{n^*}), \\ \boldsymbol{z}_n|_{\Gamma} &\longrightarrow \boldsymbol{z}|_{\Gamma} \text{ strongly in } \boldsymbol{L}^3(\Gamma), \;\; \text{as } n \to \infty. \end{aligned} \tag{2.19}$$

Now we are ready to carry out the limit process in (2.17) for $n \to \infty$. Linearity and continuity of the forms $a_0(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v}) = a_0(\cdot, \boldsymbol{v})$, $b_0(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v})$ and $b_1^{n^*}(\boldsymbol{\phi}_\infty + \cdot, \boldsymbol{v})$ (let us remind that $\boldsymbol{\phi}_\infty = 0$ in Ω^-) imply that

$$
\begin{aligned}
a_0(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) = a_0(\boldsymbol{z}_n, \boldsymbol{v}) \quad &\longrightarrow \quad a_0(\boldsymbol{z}, \boldsymbol{v}) = a_0(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}),\\
b_0^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) \quad &\longrightarrow \quad b_0^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}),\\
b_1^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}_n, \boldsymbol{v}) \quad &\longrightarrow \quad b_1^{n^*}(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) \ \text{ as } n \to \infty.
\end{aligned}
\tag{2.20}
$$

From (2.15) and (2.19) we see that the sequence $\{z_n\}_{n=n_0}^\infty$ satisfies conditions (2.7, a–c). This and Lemma 2.5 imply that

$$
a_1(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow a_1(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \text{as } n \to \infty.
\tag{2.21}
$$

Now, from (2.17), (2.20) and (2.21) we conclude that the function $\boldsymbol{z} \in \boldsymbol{V}(\Omega)$ satisfies the identity

$$
a(\boldsymbol{\phi}_\infty + \boldsymbol{z}, \boldsymbol{v}) = L(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega),
$$

which means that $\boldsymbol{z}$ is a solution of problem (2.12) and $\boldsymbol{u} = \boldsymbol{\phi}_\infty + \boldsymbol{z}$ is a solution of problem (2.3), which we wanted to prove. □

3 Formulation of the coupled problems with the aid of boundary integral equations

The fact that the Stokes equations as well as the Oseen equations possess fundamental solutions allows us to reformulate the exterior Stokes and Oseen problem as integral equation on the coupling interface Γ. This may be used to reduce the coupled problems on the unbounded domain analyzed above to equivalent problems in the bounded domain Ω^- which are equipped with nonlocal boundary conditions on Γ. In this section, we derive explicit representations of the nonlocal boundary operators in terms of the Calderón Projector of the linear exterior problem which describes the far-field. The nonlocal boundary operators for the Navier-Stokes equations coupled with the exterior Stokes and Oseen problems will turn out to be strongly elliptic boundary integral operators which can be discretized by Galerkin boundary element methods. This approach was used for the solution of a number of elliptic problems in exterior domains in e.g., [2, 7, 11, 16].

As it is well-known, there are generally many possible approaches to reformulate exterior boundary value problems in terms of boundary integral equations. Correspondingly there are many ways to represent the Poincaré-Steklov operators. For the exterior Stokes problem of Section 1, we present a formulation in terms of single layer potentials based on indirect boundary reduction by potentials. The resulting representation of the Poincaré Steklov operator requires the inversion of a coercive, self-adjoint boundary integral operator of order -1 on Γ.

For the Oseen problem, there is an analogous formulation; however, the coercivity of the first kind boundary operator to be inverted is open - only a weaker Gårding-Inequality can be established then. Therefore, we present a different formulation based on a pure double layer ansatz for the exterior velocity field $\boldsymbol{u}^+$ in the Oseen problem [5]. Contrary to the Stokes problem, this is admissible in the Oseen case due to the different decay behaviour of the Oseen fundamental solution as $|x| \to\infty$. Here, the boundary reduction is direct, via the Faxén-formulas on Γ.

3.1 Exterior Stokes Problem

For the integral equation of the exterior Stokes problem, we shall require hydrodynamic potentials that are defined in terms of fundamental solutions of the Stokes operator (1.1, e). We shall in particular require the **velocity fundamental tensor $\boldsymbol{E}(z)$** given by

(3.1) $$E_{ij}(z) = (\delta_{ij}\,\Delta - \partial_i\partial_j)\,\Phi(z), \quad z \in I\!R^3\backslash\{0\}$$

where $1 \leq i, j \leq 3$ and $\Phi(z) = \Phi_{St}(z) := |z|/(8\pi\nu)$.

Further, we shall also use the **pressure fundamental vector $\boldsymbol{e}(z)$** given by

(3.2) $$e_i(z) = -\frac{1}{4\pi}\,\partial_i\Big(\frac{1}{|z|}\Big) = \frac{1}{4\pi}\,\frac{z_i}{|z|^3} \quad \text{where } 1 \leq i \leq 3 \,.$$

To obtain an expression of Λ in (1.8) in terms of boundary integral operators, we require a certain factor space of $\boldsymbol{H}^{-1/2}(\Gamma)$: we set

(3.3) $$\boldsymbol{T} := \boldsymbol{H}^{-1/2}(\Gamma)/\mathcal{R}$$

where $\mathcal{R}$ denotes the equivalence relation

(3.4) $$\boldsymbol{t} \sim \boldsymbol{t}' \Longleftrightarrow \boldsymbol{t} = \boldsymbol{t}' + \lambda\boldsymbol{n}$$

for some $\lambda \in I\!R$ (recall that $\boldsymbol{n}$ denotes the exterior unit normal to Ω^-, pointing into Ω^+). Then there holds

Theorem 3.1 *Assume that the coupling boundary Γ is smooth. The solution of the exterior Stokes Problem* (1.1, e) - (1.1, h) *in Ω^+ can be represented in the form of the Odqvist hydrodynamic potentials*

(3.5, a) $$u_i^+(x) = u_{\infty,i} + \sum_{k=1}^{3} \int\limits_{y\in\Gamma} E_{ki}(x-y)t_k(y)\,ds_y, \quad x \in \Omega^+, \quad i = 1, 2, 3,$$

(3.5, b) $$p^+(x) = \sum_{k=1}^{3} \int\limits_{y\in\Gamma} e_k(x-y)\,t_k(y)ds_y, \quad x \in \Omega^+$$

for some boundary densities $\boldsymbol{t} \in \boldsymbol{H}^{-1/2}(\Gamma)$ which are the unique solutions of the first kind boundary integral equations:

(3.7) $$u_{\infty,i} + \sum_{k=1}^{3} \int\limits_{y\in\Gamma} t_k(y)\,E_{ki}(x-y)ds_y = u_i^+(x), \; i = 1, 2, 3, \; x \in \Gamma \,,$$

or, more precisely, in variational form: find $\boldsymbol{t} \in \boldsymbol{T}$ *such that*

$$b(\boldsymbol{t}, \boldsymbol{t}') = \langle \boldsymbol{u}^+ - \boldsymbol{u}_\infty, \boldsymbol{t}' \rangle \quad \forall \boldsymbol{t}' \in \boldsymbol{T} \tag{3.8}$$

where the bilinear form $b(\boldsymbol{t}, \boldsymbol{t}')$, *given by*

$$b(\boldsymbol{t}, \boldsymbol{t}') = \sum_{i,j=1}^{3} \int_\Gamma \int_\Gamma t_i(x)\, E_{ij}(x-y)\, t'_j(y) ds_y\, ds_x \;, \tag{3.9}$$

is symmetric and coercive on $\boldsymbol{T}$: *there exists* $\beta > 0$ *such that*

$$b(\boldsymbol{t}, \boldsymbol{t}) \geq \beta \, \|\boldsymbol{t}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \boldsymbol{t} \in \boldsymbol{T}. \tag{3.10}$$

□

For the proof, we refer to [4], Chap. VI, Theorem 1. We remark that the symmetry and coercivity of the bilinear form $b(\cdot,\cdot)$ in (3.9) gives, upon discretization with a Galerkin boundary element method on Γ, a symmetric and positive-definite stiffness matrix corresponding to the hydrodynamic single layer operator $\boldsymbol{S}$ on the left hand side of (3.7). The operator $\boldsymbol{S}$ is continuous from $\boldsymbol{H}^{-1/2}(\Gamma) \to \boldsymbol{H}^{1/2}(\Gamma)$. Using (3.7) and (1.6), we get the nonlocal boundary condition

$$\boldsymbol{t} \in \boldsymbol{T} : \quad \langle \boldsymbol{t}', \boldsymbol{S}\, \boldsymbol{t} \rangle = \langle \gamma_0 \boldsymbol{u}^- - \boldsymbol{u}_\infty, \boldsymbol{t}' \rangle \quad \forall \boldsymbol{t}' \in \boldsymbol{T} \tag{3.11}$$

where $\langle \cdot,\cdot \rangle$ denotes the $\boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)$ duality pairing. By (3.10), $\boldsymbol{S}$ is invertible on $\boldsymbol{T}$ and we get that

$$\boldsymbol{t} = \boldsymbol{S}^{-1}(\boldsymbol{u}^- - \boldsymbol{u}_\infty)\;. \tag{3.12}$$

Having obtained $\boldsymbol{t}$ by (3.8), the exterior Stokes flow $(\boldsymbol{u}^+, p^+)$ is given by (3.5). In particular, we get with normal stress operator $\sigma_n(\boldsymbol{u}^+, p^+)$, applied to (3.5), (3.6), for a point $x_0 \in \Gamma$ with the jump relations of the Odqvist potentials that

$$\begin{aligned} \sigma_n(\boldsymbol{u}^+, p^+)(x_0) &= \lim_{\varepsilon \to 0^+} \sigma_n(\boldsymbol{u}^+, p^+)(x_0 + \varepsilon \boldsymbol{n}) \\ &= \frac{1}{2}\, \boldsymbol{t}(x_0) + \text{p.v.} \int_{y \in \Gamma} \boldsymbol{n}(x_0)\, \sigma_{x_0}[\boldsymbol{E}, \boldsymbol{e}](x_0 - y)\, \boldsymbol{t}(y)\, ds_y \\ &= \Big(\Big(\frac{1}{2}\, \boldsymbol{I} + \boldsymbol{K}' \Big)\, \boldsymbol{t} \Big)(x_0) \end{aligned} \tag{3.13}$$

where the integral over Γ has to be understood in the Cauchy principal value sense and the subscript x_0 indicates that the differentiations are with respect to x_0. The expression $\boldsymbol{n}\, \sigma[\boldsymbol{E}, \boldsymbol{e}]\, \boldsymbol{t}$ is interpreted as the vector with components $\sum_{j,k=1}^3 n_j\, \sigma_{ijk}\, t_k$, $i = 1, 2, 3$, where σ_{ijk}, $i, j = 1, 2, 3$, are components of the tensor $\sigma(\boldsymbol{E}_k, e_k)$, using the notation $\boldsymbol{E}_k$ and e_k for the $k-$th row of $\boldsymbol{E}$ and the $k-$th component of $\boldsymbol{e}$, respectively.

We therefore obtain with the weak formulation (1.3) in Ω^- the following, nonlocal boundary problem in $\Omega^- \cup \Gamma$ which is equivalent to the weak formulation of the coupled problem (1.8):
Find $\boldsymbol{u}^- \in \boldsymbol{V}(\Omega^-)$, $\boldsymbol{t} \in \boldsymbol{T}$ such that

$$\begin{aligned} a(\boldsymbol{u}^-, \boldsymbol{v}) - \Big\langle \Big(\frac{1}{2}\, \boldsymbol{I} + \boldsymbol{K}' \Big)\, \boldsymbol{t},\, \gamma_0\, \boldsymbol{v} \Big\rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega^-} && \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega^-) \\ -\langle \gamma_0\, \boldsymbol{u}^- - \boldsymbol{u}_\infty, \boldsymbol{t}' \rangle + \langle \boldsymbol{S} \boldsymbol{t}, \boldsymbol{t}' \rangle &= 0 && \forall \boldsymbol{t}' \in \boldsymbol{T}\;. \end{aligned} \tag{3.14}$$

With (3.12) and (3.13) we obtain the representation of the Steklov–Poincaré operator in terms of boundary integral operators

$$\Lambda(\boldsymbol{u}^-) = -\sigma_n(\boldsymbol{u}^+, p^+) = -\frac{1}{2}\,(\boldsymbol{I} + \boldsymbol{K}')\,\boldsymbol{S}^{-1}(\boldsymbol{u}^- - \boldsymbol{u}_\infty)\ . \tag{3.15}$$

Naturally, in a numerical implementation of the nonlocal boundary condition (1.1, i) in (1.3), the discrete inverse of $\boldsymbol{S}$ should not be explicitly calculated, but rather realized numerically by a fast algorithm.

3.2 Exterior Oseen Problem

We consider now the exterior Oseen Problem (2.1, e) - (2.1, j). We will use once again the Odqvist hydrodynamic potentials to reduce the coupled problem (2.1) to a nonlocal boundary value problem in $\Omega^- \cup \Gamma$. We shall now, however, not use a single layer ansatz (the so-called "indirect" method of boundary reduction), but rather the "direct" method based on the Faxén representation formula on Γ, leading to the "one integral equation" approach of [11].

To do so, we require once more for the exterior Oseen problem the velocity fundamental tensor $\boldsymbol{E}(z)$ and the pressure fundamental vector $\boldsymbol{e}(z)$. To define them, we assume without loss of generality that

$$\boldsymbol{u}_\infty = (u_\infty, 0, 0)^\top\ . \tag{3.16}$$

Then $\boldsymbol{E}$ and $\boldsymbol{e}$ are once more defined by (3.1), (3.2), however now with $\Phi(z)$ given by [15]

$$\Phi_{Os}(z) := \frac{1}{4\pi u_\infty} \int_0^{u_\infty s(z)/2\nu} (1 - e^{-\alpha})\alpha^{-1} d\alpha, \quad s(z) := |z| - z_1\ . \tag{3.17}$$

We recall further that for the Oseen problem the hydrostatic stress is given by

$$\sigma[\boldsymbol{u}, p] := -p\,\mathbf{1} + 2\nu\, I\!D(\boldsymbol{u}) - \frac{1}{2}\,\boldsymbol{u}\boldsymbol{u}_\infty^\top\ . \tag{3.18}$$

We shall also require the **adjoint stress operator**

$$\sigma^*[\boldsymbol{v}, q] := q\,\mathbf{1} + 2\nu\, I\!D(\boldsymbol{v}) + \frac{1}{2}\,\boldsymbol{v}\boldsymbol{u}_\infty^\top\ . \tag{3.19}$$

Then there holds the **Faxén representation formula**:
Any $(\boldsymbol{u}^+ - \boldsymbol{u}_\infty, p^+) \in \boldsymbol{H}^2_{\mathrm{loc}}(\Omega^+) \times H^1_{\mathrm{loc}}(\Omega^+)$ solving (2.1, e) - (2.1, j) can be represented in the form: for any $x \in \Omega^+$

$$\boldsymbol{u}^+(x) - \boldsymbol{u}_\infty = \int_{y\in\Gamma} \{(\boldsymbol{u}^+(y) - \boldsymbol{u}_\infty)\,\sigma^*_y[\boldsymbol{E}, \boldsymbol{e}](x-y)\,\boldsymbol{n} - \boldsymbol{E}(x-y)\sigma_y[\boldsymbol{u}^+ - \boldsymbol{u}_\infty, p]\,\boldsymbol{n})\} ds_y, \tag{3.20}$$

$$(3.21)\qquad p^+(x) = \int_{y\in\Gamma} \{(\boldsymbol{u}^+ - \boldsymbol{u}_\infty)\,\sigma_y^*[\boldsymbol{e}, e^*](x-y)\,\boldsymbol{n} - \boldsymbol{e}(x-y)\,\sigma_y[\boldsymbol{u}^+ - \boldsymbol{u}_\infty, p]\,\boldsymbol{n})\}ds_y,$$

where $p(x)$ is determined only mod $I\!R$, σ_y, σ_y^* are as in (3.18), (3.19), with the subscript y indicating that the differentiations are with respect to y and where

$$(3.22)\qquad e^*(z) = \frac{u_\infty}{4\pi}\,\partial_1\Big(\frac{1}{|z|}\Big) = -u_\infty\, e_1(z)$$

is the pressure corresponding to the velocity field $\boldsymbol{e}(z)$. The expression $(\boldsymbol{u}^+ - \boldsymbol{u}_\infty)\,\sigma^*[\boldsymbol{E}, \boldsymbol{e}]\,\boldsymbol{n}$ is interpreted in an analogous way as $\boldsymbol{n}\,\sigma[\boldsymbol{E}, \boldsymbol{e}]\,\boldsymbol{t}$ in Par. 3.1.

We observe that the leading singularities of $\boldsymbol{E}(z)$ and of $\boldsymbol{e}(z)$ at $|z| = 0$ in the Oseen and the Stokes case are identical. More precisely, for small $|z|$

$$(3.23)\qquad \boldsymbol{E}(z) = (\delta_{ij}\Delta - \partial_i\partial_j)\,\Phi_{Os}(z) = \frac{1}{8\pi\nu}\,(\delta_{ij}\Delta - \partial_i\partial_j)\,|z| + O(1)\ .$$

The hydrodynamic potentials admit therefore the same jump relations in the Stokes and the Oseen case. We reduce the exterior Oseen problem to Γ by passing with x in (3.20) to $x_0 \in \Gamma$: For any $x_0 = \lim_{\varepsilon\to 0^+} x_0 + \varepsilon\,\boldsymbol{n}(x_0) \in \Gamma$, the jump relations give

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \boldsymbol{u}^+(x_0 + \varepsilon\boldsymbol{n}) - \boldsymbol{u}_\infty = \boldsymbol{u}^+(x_0) - \boldsymbol{u}_\infty = \\ &-\int_{y\in\Gamma} \boldsymbol{E}(x_0 - y)\sigma_n(\boldsymbol{u}^+ - \boldsymbol{u}_\infty, p^+)(y)ds_y \\ &-\frac{1}{2}\,(\boldsymbol{u}^+(x_0) - \boldsymbol{u}_\infty) - \text{p.v.}\int_{y\in\Gamma} \{(\boldsymbol{u}^+(y) - \boldsymbol{u}_\infty)\cdot\sigma_y^*[\boldsymbol{E}, \boldsymbol{e}](x_0 - y)\,\boldsymbol{n}(y)\}ds_y\end{aligned}$$

or, symbolically,

$$(3.24)\qquad \gamma_0\boldsymbol{u}^+(x_0) - \boldsymbol{u}_\infty = -(\boldsymbol{S}\,\sigma_n(\boldsymbol{u}^+ - \boldsymbol{u}_\infty, p^+))(x_0) + \Big(\frac{1}{2}\,\boldsymbol{I} + \boldsymbol{K}\Big)(\boldsymbol{u}^+ - \boldsymbol{u}_\infty)(x_0)\ ,$$

where $\boldsymbol{K}$ denotes the hydrodynamic double layer operator, or, equivalently

$$(3.25)\qquad \Big(\frac{1}{2}\,\boldsymbol{I} - \boldsymbol{K}\Big)(\boldsymbol{u}^+ - \boldsymbol{u}_\infty) = -\boldsymbol{S}\,\sigma_n\ .$$

We emphasize that now $\boldsymbol{S}$ is neither symmetric nor coercive, generally. Using the continuity of the velocities (1.6), and casting (3.25) in weak form, we find the integral equation for the hydrodynamic normal stress σ_n corresponding to the exterior Oseen problem due to the velocity $\boldsymbol{u}^-$ on Γ:

$$(3.26)\qquad \boldsymbol{\sigma}_n \in \boldsymbol{H}^{-1/2}(\Gamma) : \langle\boldsymbol{\tau}, \boldsymbol{S}\boldsymbol{\sigma}_n\rangle + \Big\langle\boldsymbol{\tau}, \Big(\frac{1}{2}\,\boldsymbol{I} - \boldsymbol{K}\Big)(\boldsymbol{u}^- - \boldsymbol{u}_\infty)\Big\rangle = 0$$

for all $\boldsymbol{\tau} \in \boldsymbol{H}^{-1/2}(\Gamma)$.

The hydrodynamic single layer potential $\boldsymbol{S} : \boldsymbol{H}^{-1/2}(\Gamma) \to \boldsymbol{H}^{1/2}(\Gamma)$ is continuous and satisfies, in virtue of (3.23) and of (3.9), the Gårding inequality: there is $c > 0$ such that

$$\forall \boldsymbol{\tau} \in \boldsymbol{H}^{-1/2}(\Gamma) : \langle \boldsymbol{\tau}, \boldsymbol{S\tau} \rangle \geq c \, \|\boldsymbol{\tau}\|^2_{H^{-1/2}(\Gamma)} - k(\boldsymbol{\tau}, \boldsymbol{\tau})$$

where $k(\cdot, \cdot)$ is a compact form on $\boldsymbol{H}^{-1/2}(\Gamma)$. Equation (3.26) gives now, together with the (formal) weak form (1.3) of the Navier-Stokes system in Ω^- the desired nonlocal boundary value problem in $\Omega^- \cup \Gamma$: Find $\boldsymbol{u}^- \in \boldsymbol{V}(\Omega^-)$, $\boldsymbol{\sigma}_n \in \boldsymbol{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} a(\boldsymbol{u}^-, \boldsymbol{v}) - \langle \boldsymbol{\sigma}_n, \gamma_0 \boldsymbol{v} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega^-} \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega^-)\,, \\ \Big\langle \boldsymbol{\tau}, \Big(\frac{1}{2}\, \boldsymbol{I} - \boldsymbol{K}\Big)(\boldsymbol{u}^- - \boldsymbol{u}_\infty) \Big\rangle + \langle \boldsymbol{\tau}, \boldsymbol{S\sigma}_n \rangle &= 0 \qquad\qquad \forall \boldsymbol{\tau} \in \boldsymbol{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{3.27}$$

Here the nonlinear form $a(\cdot, \cdot)$ is as in (1.2).

Whereas the nonlocal problem (3.14) and the corresponding one (3.27) for the exterior Stokes equation are mathematically on solid ground due to Theorem 1.3 and 3.1, in the Oseen case research is in progress on the following questions:

a) Existence of solutions to the nonlocal problems (3.14), (3.27) in the exterior Oseen case,

b) Coercivity of $\boldsymbol{S}$ in the Oseen case,

c) Convergence of Galerkin-discretizations of (3.14), (3.27) in the Stokes and Oseen case (note that the nonlinearity is not of the type treated in [11]).

Acknowledgements. The research of M. Feistauer has been supported under the Grant No. 201/97/0217 of the Czech Grant Agency. The authors gratefully acknowledge this support.

References

[1] C. Bégue, E. Conca, F. Murat, O. Pironneau: The Stokes and Navier–Stokes equations with boundary conditions involving pressure. Publications du Laboratoire d'Analyse Numérique, Université P. et. M. Curie, Paris VI, R 88027, Novembre 1988.

[2] H. Berger, G. Warnecke, W. Wendland: Analysis of a FEM/BEM coupling method for transonic flow computations. *Math. Comp.*, **66** (1997), 1407–1440.

[3] Ch. Bruneau, P. Fabrie: Effective downstream boundary conditions for incompressible Navier–Stokes equations. *Int. J. Numer. Meth. Fluids*, **19** (1994), 693–705.

[4] R. Dautray, J. L. Lions: *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 4, Springer–Verlag, Berlin – Heidelberg – New York, 1993.

[5] R. Farwig: The stationary Navier–Stokes equations in a 3D–exterior domain. TH Darmstadt, Fachbereich Mathematik, Preprint - Nr. 1911, April 1997.

[6] M. Feistauer: *Mathematical Methods in Fluid Dynamics.* Pitman Monographs and Surveys in Pure and Applied Mathematics 67, Longman Scientific & Technical, Harlow, 1993.

[7] M. Feistauer, G. C. Hsiao, R. E. Kleinman, R. Tezaur: Analysis and numerical realization of coupled BEM and FEM for nonlinear exterior problems. In: Methoden und Verfahren der Mathematischen Physik, Band 40, "Inverse Scattering and Potential Problems in Mathematical Physics" (R. E. Kleinman, R. Kress, E. Martensen – eds.), P. Lang, Frankfurt am Main, 1995, 47–73.

[8] M. Feistauer, C. Schwab: On coupled problems for viscous flow in exterior domains. M^3AS, 8 (1998), 657–684.

[9] M. Feistauer, C. Schwab: Coupling of an interior Navier–Stokes problem with an exterior Oseen problem. Research Report No. 98-01, February 1998, SAM ETH Zürich (submitted).

[10] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations.* Springer Tracts in Natural Philosophy, Volume 38, Springer–Verlag, New York, 1994.

[11] G. N. Gatica, G. C. Hsiao: Boundary field equation methods a class of nonlinear problems, Pitman Res. Notes in Math. No. 331, Pitman Publ. Harlow, UK (1995).

[12] V. Girault, P. A. Raviart: *Finite Element Methods for Navier–Stokes Equations.* Springer Series in Computational Mathematics 5, Springer–Verlag, Berlin, 1986.

[13] A. Kufner, O. John, S. Fučík: *Function Spaces.* Academia, Praha, 1977.

[14] J. L. Lions: *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris, 1969.

[15] C.W. Oseen: *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akad. Verlagsgesellschaft, Leipzig, 1927.

[16] A. Sequeira: The coupling of boundary integral and finite element methods for the bidimensional exterior steady problems. *Math. Meth. in the Appl. Sci.,* **5** (1983), 356–375.

[17] R. Temam: *Navier–Stokes Equations.* North–Holland, Amsterdam – New York – Oxford, 1977.

Research Reports

No.	Authors	Title
99-07	M. Feistauer, C. Schwab	Coupled Problems for Viscous Incompressible Flow in Exterior Domains
99-06	J. Maurer, M. Fey	A Scale-Residual Model for Large-Eddy Simulation
99-05	M.J. Grote	Am Rande des Unendlichen: Numerische Verfahren für unbegrenzte Gebiete
99-04	D. Schötzau, C. Schwab	Time Discretization of Parabolic Problems by the hp-Version of the Discontinuous Galerkin Finite Element Method
99-03	S.A. Zimmermann	The Method of Transport for the Euler Equations Written as a Kinetic Scheme
99-02	M.J. Grote, A.J. Majda	Crude Closure for Flow with Topography Through Large Scale Statistical Theory
99-01	A.M. Matache, I. Babuška, C. Schwab	Generalized p-FEM in Homogenization
98-10	J.M. Melenk, C. Schwab	The *hp* Streamline Diffusion Finite Element Method for Convection Dominated Problems in one Space Dimension
98-09	M.J. Grote	Nonreflecting Boundary Conditions For Electromagnetic Scattering
98-08	M.J. Grote, J.B. Keller	Exact Nonreflecting Boundary Condition For Elastic Waves
98-07	C. Lage	Concept Oriented Design of Numerical Software
98-06	N.P. Hancke, J.M. Melenk, C. Schwab	A Spectral Galerkin Method for Hydrodynamic Stability Problems
98-05	J. Waldvogel	Long-Term Evolution of Coorbital Motion
98-04	R. Sperb	An alternative to Ewald sums, Part 2: The Coulomb potential in a periodic system
98-03	R. Sperb	The Coulomb energy for dense periodic systems
98-02	J.M. Melenk	On n-widths for Elliptic Problems
98-01	M. Feistauer, C. Schwab	Coupling of an Interior Navier–Stokes Problem with an Exterior Oseen Problem
97-20	R.L. Actis, B.A. Szabo, C. Schwab	Hierarchic Models for Laminated Plates and Shells
97-19	C. Schwab, M. Suri	Mixed *hp* Finite Element Methods for Stokes and Non-Newtonian Flow
97-18	K. Gerdes, D. Schötzau	*hp* FEM for incompressible fluid flow - stable and stabilized