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The Method of Transport for the Euler Equations Written as a Kinetic Scheme[1]

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Abstract

The Method of Transport is a genuinely multi-dimensional scheme to solve nonlinear systems of hyperbolic equations numerically. It is based on the framework of conservation laws. Here, we will consider the Euler equations. We will present an alternative formulation of the first order method based on kinetic theory. This will allow us to show that density and pressure of the numerical solution remain positive for all times. In addition, we can derive L^1-estimates for the numerical solution. We will also consider the second order method. This will give us more insight into the differences of the two formulations.

Keywords: Euler Equations, Multidimensional Schemes, Kinetic Schemes, Stability of Numerical Methods

AMS Subject Classification: 35L65 65M12 76P05 76N10

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1 Introduction

The compressible Euler equations describe the macroscopic flow of a gas. They are derived under the assumption that the gas behaves like a continuum in every control volume. On the microscopic level this means that the mean free path of a gas particle is small compared to other characteristic lengths of the problem. The equations express the conservation of the macroscopic quantities density, momentum, and energy, which, together with an equation of state, describe the state of a gas completely.

Alternatively, a gas can be described by a microscopic distribution function stating the probability that a particle can be found at a certain position in phase space. The relevant macroscopic quantities can be recovered by calculating the moments of the distribution function. The time evolution of the distribution function is determined by the Boltzmann equation. One side of the equation expresses the free flow of particles, whereas the other side takes collision of particles into account.

The above assumption that the mean free path of a particle is small compared to other characteristic lengths means that particles collide very often. The gas can be expected to come to local equilibrium rapidly. A gas in equilibrium solves the collision-less Boltzmann equation, and the solution is a Maxwellian distribution. Therefore, in the above situation, a suitable ansatz for the distribution function is a local Maxwellian. Its moments satisfy the Euler equations.

Accordingly, there are two approaches to solve the Euler equations numerically. The first (more common) approach is to attack the Euler equations in the framework of conservation laws. One difficulty that arises is the approximation of multi-dimensional phenomena. Schemes for problems in more than one space dimension often rely on one-dimensional operators. Thus dependencies on the underlying grid are introduced, which may cause unphysical solutions. One desirable feature of a numerical scheme is therefore that it be truly multi-dimensional, in the sense that the scheme does not show any dependencies on the underlying grid. The Method of Transport [2], a finite volume flux-vector splitting scheme, is such a scheme.

The second approach to solve the Euler equations numerically is based on the relation between the Euler equations and the collision-less Boltzmann equation. These so called kinetic schemes are truly multi-dimensional in nature. However, for performance reasons, simplifications introducing one-dimensional operators and grid dependencies often need to be made.

This paper is organized as follows: in section 2 we explain the main ideas of the Method of Transport. In section 3 we introduce a group of kinetic schemes that preserve positivity of density and pressure. In sections 4 and 5 we investigate the relation between the two first order, resp. second order schemes. Finally, in section 6, we give some conclusions.

2 The Method of Transport

The Method of Transport is based on the following formulation of the Euler equations:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\vec{\boldsymbol{u}}\\ E\end{pmatrix}+\nabla\cdot\left(\begin{pmatrix}\rho\\ \rho\vec{\boldsymbol{u}}\\ E\end{pmatrix}\vec{\boldsymbol{u}}^T+p\begin{pmatrix}\vec{\mathbf{0}}^T\\ \underline{\boldsymbol{I}}\\ \vec{\boldsymbol{u}}^T\end{pmatrix}\right)=0$$

where ρ denotes the density, $\rho\vec{\boldsymbol{u}}\in\mathbb{R}^N$ the moment, E the energy, and p the pressure of the gas. $\underline{\boldsymbol{I}}$ is the identity in $\mathbb{R}^{N\times N}$, and $\vec{\mathbf{0}}$ the zero vector in $\mathbb{R}^N$. The divergence acts on rows of the matrix. The equation of state is

$$p=(\gamma-1)\left(E-1/2\rho\vec{\boldsymbol{u}}^T\vec{\boldsymbol{u}}\right).$$

γ is the ratio of the heat capacities; it is $1+2/N$ for a monatomic gas, smaller than $1+2/N$ for a polytropic gas.

There are two main ideas to the Method of Transport: first, the Euler equations are decomposed into a finite number of advection equations, and then, each advection equation is solved separately by following characteristics.

Decomposition of the Euler equations. The decomposition is based on properties of the Euler equations, namely that the characteristics lie on the Mach cone. This motivates the decomposition of the state vector $\mathbf{U}=(\rho,\rho\vec{\boldsymbol{u}},E)^T$ into

$$\mathbf{U}=\mathbf{R}_2+\frac{1}{k}\sum_{i=1}^{k}(\mathbf{R}_1+\underline{\boldsymbol{L}}\vec{\boldsymbol{n}}_i) \tag{1}$$

with

$$\mathbf{R}_1:=\frac{1}{\gamma}\rho\begin{pmatrix}1\\ \vec{\boldsymbol{u}}\\ H\end{pmatrix},\quad \mathbf{R}_2:=\frac{\gamma-1}{\gamma}\rho\begin{pmatrix}1\\ \vec{\boldsymbol{u}}\\ \vec{\boldsymbol{u}}^T\vec{\boldsymbol{u}}/2\end{pmatrix},\quad \underline{\boldsymbol{L}}:=\frac{\rho c}{\gamma}\begin{pmatrix}\vec{\mathbf{0}}^T\\ \underline{\boldsymbol{I}}\\ \vec{\boldsymbol{u}}^T\end{pmatrix}.$$

c is the speed of sound and H the total enthalpy. The vector $\mathbf{S}_0:=\mathbf{R}_2$ is transported along the streamline $\vec{\boldsymbol{u}}$, and the vectors $\mathbf{S}_i:=\mathbf{R}_1+\underline{\boldsymbol{L}}\vec{\boldsymbol{n}}_i$ are transported along the Mach cone. The vectors $\vec{\boldsymbol{n}}_i$ are chosen such that the velocities $\vec{\boldsymbol{u}}+c\vec{\boldsymbol{n}}_i$, $i=1\ldots k$, approximate the Mach cone. Some conditions need to be imposed on the $\vec{\boldsymbol{n}}_i$ to guarantee consistency, in particular $\vec{\boldsymbol{n}}_i^T\vec{\boldsymbol{n}}_i=N$. The Euler equations can then be written as a finite sum of advection equations:

$$\phi_0+\frac{1}{k}\sum_{i=1}^{k}\phi_i=0$$

where

$$\begin{aligned}\phi_0 &:= \frac{\partial}{\partial t}\mathbf{S}_0+\nabla\cdot(\mathbf{S}_0\vec{\boldsymbol{u}}^T)\\ \phi_i &:= \frac{\partial}{\partial t}\mathbf{S}_i+\nabla\cdot(\mathbf{S}_i(\vec{\boldsymbol{u}}^T+c\vec{\boldsymbol{n}}_i^T)).\end{aligned}$$

Each component of the advection equations $\phi_i = 0$, $i = 0 \ldots k$, is solved separately. Thus $(k+1)(N+2)$ scalar advection equations need to be solved. This decomposition is first order accurate.

Solving the advection equations. The equations of the type

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{\boldsymbol{a}}^T) = 0$$

are solved by a finite volume scheme. The value in a cell Ω_j at time $n\Delta t$, h_j^n, is updated by adding all in-going and subtracting all outgoing flows from the neighboring cells:

$$\begin{aligned} h_j^0 &= \frac{1}{|\Omega_j|}\int_{\Omega_j} h(\vec{\boldsymbol{x}}, 0) d\vec{\boldsymbol{x}} \\ h_j^{n+1} &= h_j^n - \frac{1}{|\Omega_j|}\sum_{k\neq j}(F_{\Omega_j\Omega_k} - F_{\Omega_k\Omega_j}); \end{aligned}$$

the contributions $F_{\Omega_j\Omega_k}$ approximate the physical flux from the cell Ω_j into the cell Ω_k. They are calculated by following characteristic curves $\vec{\boldsymbol{z}}$:

$$F_{\Omega_j\Omega_k} = \int_{\Omega_k}\int_{\Omega_j} h(\vec{\boldsymbol{y}}, t)\delta[\vec{\boldsymbol{z}}(t+\Delta t, \vec{\boldsymbol{y}}) - \vec{\boldsymbol{x}}] d\vec{\boldsymbol{y}} d\vec{\boldsymbol{x}},$$

where $\vec{\boldsymbol{z}}(\tau, \vec{\boldsymbol{y}})$ is the solution of

$$\dot{\vec{\boldsymbol{z}}}(\tau) = \vec{\boldsymbol{a}}(\vec{\boldsymbol{z}}(\tau)), \quad \vec{\boldsymbol{z}}(t) = \vec{\boldsymbol{y}}. \tag{2}$$

For a first order scheme one assumes that h and $\vec{\boldsymbol{a}}$ are constant in every cell. Thus:

$$F_{\Omega_j\Omega_k} = \int_{\Omega_k} h(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}_j\Delta t, t)\chi_{\Omega_j}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}_j\Delta t) d\vec{\boldsymbol{x}}, \tag{3}$$

where χ_Ω is the characteristic function

$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}.$$

This method is multi-dimensional because the physical propagation directions $\vec{\boldsymbol{a}}$ are followed independently of the grid.

The second order scheme. For simplicity we will only consider the one-dimensional case. In order to obtain a second order decomposition of the Euler equations we substitute the vector $\mathbf{L}$ by $\mathbf{L} + \mathbf{K}$. $\mathbf{K}$ is a vector containing correction terms found by comparing the Taylor expansions of the exact and the numerical solutions (cf. [3]):

$$\begin{pmatrix} k^\rho \\ k^{\rho u} \\ k^E \end{pmatrix} = \begin{pmatrix} -\dfrac{\Delta t\rho}{2c}(\gamma u u_x + c c_x) \\ -\dfrac{\Delta t\rho}{2}((\gamma-2)c u_x + u c_x) + u k^\rho \\ -\dfrac{\Delta t\rho c}{2\gamma(\gamma-1)}(u u_x - c c_x) + u k^{\rho u} - \dfrac{u^2}{2} k^\rho \end{pmatrix}. \tag{4}$$

For second order in space we assume that a and h are linearly represented in every cell. We need a second order scheme to solve the advection equations. The characteristic equation (2) reads

$$\dot{z}(\tau) = a_0 + a_1 z(\tau), \quad z(t) = y.$$

The second order approximation of the solution $z(t + \Delta t, y)$ allows a second order approximation of $z^{-1}(t + \Delta t, x)$ (for details see [5]):

$$z(t + \Delta t, y) = y + \Delta t(a_0 + a_1 y) + \frac{\Delta t^2}{2}(a_1 a_0 + a_1^2 y) + O(\Delta t^3) \quad (5)$$

$$z^{-1}(t + \Delta t, x) = x - \Delta t(a_0 + a_1 x) + \frac{\Delta t^2}{2}(a_1 a_0 + a_1^2 x) + O(\Delta t^3). \quad (6)$$

3 Kinetic schemes

Kinetic schemes are based on the relation between the Boltzmann equation and the Euler equations. The unknown in the Boltzmann equation is a distribution function $f(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, t) : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$. It solves

$$f_t + \vec{\boldsymbol{v}} f_{\vec{\boldsymbol{x}}} = \left(\frac{\partial f}{\partial t}\right)_{coll}.$$

The macroscopic quantities of a monatomic gas are recovered by calculating moments of the distribution function:

$$(\rho, \rho\vec{\boldsymbol{u}}, E) = \int_{\mathbb{R}^N} (1, \vec{\boldsymbol{v}}, 1/2\ \vec{\boldsymbol{v}}^T \vec{\boldsymbol{v}}) f(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, t) d\vec{\boldsymbol{v}}. \quad (7)$$

A polytropic gas on the other hand consists of particles with additional degrees of freedom. One way to deal with them is to add a variable $I \in \mathbb{R}^+$ denoting the inner energy due to the additional motion of the particles. In this case the macroscopic quantities are recovered by

$$(\rho, \rho\vec{\boldsymbol{u}}, E) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^+} (1, \vec{\boldsymbol{v}}, 1/2\ \vec{\boldsymbol{v}}^T \vec{\boldsymbol{v}} + I) \tilde{f}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, I, t) dI d\vec{\boldsymbol{v}}.$$

One group of kinetic schemes takes advantage of the fact that the collision-less Boltzmann equation can be solved exactly:

$$f(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, t + \Delta t) = f(\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}\Delta t, \vec{\boldsymbol{v}}, t).$$

A finite volume method (in the case of a monatomic gas) is built as follows:

i. for given initial data ρ, $\rho\vec{\boldsymbol{u}}$, E construct a distribution function $f_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}) \geq 0$ representing the initial state

ii. advance f_0 in time by solving the collision-less Boltzmann equation (exactly or numerically)

iii. calculate the moments (7) of the advanced distribution function

Deshpande [1] introduced this type of schemes by proposing a Maxwellian ansatz for the distribution function f_0. Perthame [7] investigated various distribution functions with compact support.

The ansatz for the distribution function determines the order of the time-discretization of the scheme. Often, an ansatz leading to a first order scheme can be modified such that it leads to a second order scheme. First order in space is achieved by assuming constant values in every cell, second order by linear representations.

Again, modifications need to be made in case of a polytropic gas. One possibility is to propose two ansatzes $f_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}) \geq 0$ and $g_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}) \geq 0$ for

$$f(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, t) := \int\limits_{\mathbb{R}^+} \tilde{f}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, I, t) dI, \quad \text{and} \quad g(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, t) := \int\limits_{\mathbb{R}^+} I\tilde{f}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}, I, t) dI$$

The numerical scheme consists of applying i.-iii. to f_0 and g_0.

The resulting schemes are conservative and genuinely multi-dimensional. The following theorem is straightforward (cf. [7]):

Theorem 3.1 *Numerical solutions obtained by kinetic schemes as described above have the following properties:*

- *their density and pressure remain positive for all times:*

$$\begin{aligned} \rho_j^{n+1} &\geq 0, \\ E_j^{n+1} &\geq \frac{1}{2}\rho_j^{n+1}(\vec{\boldsymbol{u}}_j^T \vec{\boldsymbol{u}}_j)^{n+1}, \end{aligned}$$

- *they satisfy the L^1-estimates:*

$$\begin{aligned} \sum_j |\rho_j^{n+1}||\Omega_j| &= \|\rho(\vec{\boldsymbol{x}}, n\Delta t)\|_{L^1} \\ \sum_j |E_j^{n+1}||\Omega_j| &= \|E(\vec{\boldsymbol{x}}, n\Delta t)\|_{L^1} \\ \sum_j \rho_j^{n+1}|\vec{\boldsymbol{u}}_j^{n+1}||\Omega_j| &\leq (\|\rho(\vec{\boldsymbol{x}}, n\Delta t)\|_{L^1}\|E(\vec{\boldsymbol{x}}, n\Delta t)\|_{L^1})^{1/2}. \end{aligned} \tag{8}$$

For some distribution functions entropy inequalities can be found. For example, in the case of the Maxwellian ansatz, the numerical solution satisfies the physical entropy inequality (cf. [1], [6]).

4 The first order Method of Transport written as a kinetic scheme

In order to relate the Method of Transport to kinetic schemes, the characteristic propagation directions need to be transformed into the phase space of the distribution function. Each of the advection equations is equivalent to a deterministic motion. Thus the distribution function consists of several δ-functions that model

the proper advection processes. For the Euler equations we obtain the following functions (cf. [4]):

$$\begin{aligned} f_0(\vec{\boldsymbol{x}},\vec{\boldsymbol{v}},t) &= A(\vec{\boldsymbol{x}},t)\frac{1}{k}\sum_{i=1}^{k}\delta[\vec{\boldsymbol{v}}-(\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)+c(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{n}}_i)] \\ &\quad +B(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)] \qquad (9) \\ g_0(\vec{\boldsymbol{x}},\vec{\boldsymbol{v}},t) &= \frac{1}{k}\sum_{i=1}^{k}C_i(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-(\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)+c(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{n}}_i)]. \qquad (10) \end{aligned}$$

Note that f_0 and g_0 are not continuous nor do they solve $(\partial f/\partial t)_{coll}=0$. However, if A, B, and C_i are chosen appropriately, the resulting numerical scheme is consistent with the Euler equations.

The proper choices for A, B and C_i for first order accuracy in time are:

$$A(\vec{\boldsymbol{x}},t)=\frac{1}{\gamma}\rho(\vec{\boldsymbol{x}},t),\quad B(\vec{\boldsymbol{x}},t)=\frac{\gamma-1}{\gamma}\rho(\vec{\boldsymbol{x}},t),\quad C_i(\vec{\boldsymbol{x}},t)=\left(\frac{1}{\gamma-1}-\frac{N}{2}\right)p(\vec{\boldsymbol{x}},t) \qquad (11)$$

Note that the coefficients A and B are the same as in the decomposition (1) of the Euler equations.

To ensure positivity of f_0 and g_0, the coefficients A, B and C_i need to be positive. This is the case if the initial density and pressure are positive.

Each δ-function in the distribution function corresponds to one of the advection equations $\phi_i=0$. In fact, we can write

$$\begin{aligned} \mathbf{S}_0 &= \int_{\mathbb{R}^N}\begin{pmatrix}1\\ \vec{\boldsymbol{v}}\\ \vec{\boldsymbol{v}}^T\vec{\boldsymbol{v}}/2\end{pmatrix}B(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)]d\vec{\boldsymbol{v}} \\ \mathbf{S}_i &= \int_{\mathbb{R}^N}\begin{pmatrix}1\\ \vec{\boldsymbol{v}}\\ \vec{\boldsymbol{v}}^T\vec{\boldsymbol{v}}/2\end{pmatrix}A(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-(\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)+c(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{n}}_i)] \\ &\quad +\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}C_i(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-(\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)+c(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{n}}_i)]d\vec{\boldsymbol{v}},\quad i=1\ldots k \end{aligned}$$

Consider for example a quantitiy $h(\vec{\boldsymbol{x}},t)$ of the form

$$h(\vec{\boldsymbol{x}},t)=\int_{\mathbb{R}^N}D(\vec{\boldsymbol{v}})\tilde{h}(\vec{\boldsymbol{x}},t)\delta[\vec{\boldsymbol{v}}-\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t)]d\vec{\boldsymbol{v}},$$

where $\tilde{h}(\vec{\boldsymbol{x}},t)$ is one of $B(\vec{\boldsymbol{x}},t)$ and $A(\vec{\boldsymbol{x}},t)$, $D(\vec{\boldsymbol{v}})$ is one of 1 and $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t)$ is one of $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)$ and $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)+c(\vec{\boldsymbol{x}},t)\vec{\boldsymbol{n}}_i$. Then the value of h at the new time level is calculated by

$$h_k^{n+1}=\int_{\Omega_k}\int_{\mathbb{R}^N}D(\vec{\boldsymbol{v}})\tilde{h}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{v}}\Delta t,n\Delta t)\delta[\vec{\boldsymbol{v}}-\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{v}}\Delta t,n\Delta t)]d\vec{\boldsymbol{v}}d\vec{\boldsymbol{x}},$$

and the contribution from one cell to another is

$$F_{\Omega_j\Omega_k} = \int\limits_{\Omega_k} D(\vec{\boldsymbol{a}}_j)\tilde{h}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}_j\Delta t, n\Delta t)\chi_{\Omega_j}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}_j\Delta t)d\vec{\boldsymbol{x}}.$$

which is the same contribution as in (3). Thus the overall first order kinetic scheme defined by (11) is the same as the overall first order Method of Transport.

We obtain the following result:

Theorem 4.1 *The first order Method of Transport for the Euler equations as defined in [2] can be written as a kinetic scheme. Therefore, numerical solutions have the properties stated in theorem 3.1, i. e.:*

- *density and pressure remain positive for all times*
- *they satisfy the L^1-estimates (8)*

5 The second order scheme

While the overall first order Method of Transport is the same as the overall first order kinetic scheme, the second order extensions of the two schemes are different. This becomes obvious if we calculate the time accuracy of the first order kinetic scheme defined by (11) (with exact timeintegration): the density is not only first order accurate but even second order accurate.

The second order kinetic scheme in one dimension is based on the ansatz (9), (10) with the coefficients

$$\begin{aligned} A(x,t) &= \frac{1}{\gamma}\rho(x,t) - k_1 \\ B(x,t) &= \frac{\gamma-1}{\gamma}\rho(x,t) + k_1 \\ C_{1,2}(x,t) &= \frac{3-\gamma}{2(\gamma-1)}p(x,t) + k_2 \pm k_3, \end{aligned} \tag{12}$$

where

$$\begin{aligned} k_1 &= \frac{\Delta t}{2}\frac{\gamma-3}{\gamma}\rho(x,t)u_x(x,t) \\ k_2 &= \frac{\Delta t}{4}(\gamma-3)p(x,t)u_x(x,t) \\ k_3 &= \frac{\Delta t}{\gamma-1}p(x,t)c_x(x,t). \end{aligned}$$

The resulting correction terms differ considerably from those in (4).

For second order in space we need a second order approximation of $y = x - v\Delta t$, where v is the solution of $v = a(x - v\Delta t, t) = a_0 + a_1(x - v\Delta t)$. Thus we obtain second order approximations for x and y:

$$\begin{aligned} x &= y + (a_0 + a_1 y)\Delta t \\ y &= x - (a_0 + a_1 x)\Delta t + (a_0 + a_1 x)a_1\Delta t^2 + O(\Delta t^3). \end{aligned}$$

This time integration differs from that in (5) and (6) by terms of the order $O(\Delta t^2)$.

The reason for these differences is that in the kinetic scheme the velocity is not decoupled from the other transported quantities. While calculating the density at time $t + \Delta t$ one also calculates the velocity at time $t + \Delta t$. On the other hand, in the Method of Transport one freezes the velocity at time t and does not use any information on its development in time. We would like to reintroduce this coupling into the method.

Modified Method of Transport. In (2) we assumed the advecting velocity a to be time independent. However, if we solve all the scalar equations of one equation $\phi_i = 0$ at the same time instead of solving them independently, we can calculate the time development of a. Two of the scalar equations to be solved in each $\phi_i = 0$ are

$$\rho_t + \nabla(\rho a) = 0 \quad \text{and} \quad (\rho a)_t + \nabla(\rho a^2) = 0,$$

and thus:

$$a_t = -a a_x. \tag{13}$$

Therefore we can assume a to be time dependent.

Theorem 5.1 *The second order Method of Transport based on a timeintegration using (13) can be written as a kinetic scheme. Its numerical solutions have the properties stated in theorem 3.1, but only under the following conditions:* ($\gamma \neq 3$)

$$-\frac{2}{\Delta t(3-\gamma)} \leq u_x \leq \frac{2(\gamma-1)}{\Delta t(3-\gamma)}, \tag{14}$$

$$\frac{\gamma-1}{4}(3-\gamma)u_x - \frac{3-\gamma}{2\Delta t} \leq c_x \leq \frac{3-\gamma}{2\Delta t} - \frac{\gamma-1}{4}(3-\gamma)u_x. \tag{15}$$

$$-\frac{2\rho}{\Delta x} \leq \rho_x \leq \frac{2\rho}{\Delta x} \tag{16}$$

$$-\frac{2p}{\Delta x} \leq p_x \leq \frac{2p}{\Delta x}. \tag{17}$$

Δx *is the length of an interval.*

Proof. We again consider the characteristic equation (2) but now a is assumed to be explicitly time dependent:

$$\dot{z}(\tau) = a(z(\tau), \tau), \quad z(t) = y$$

By using (13) in the taylor expansion of z we obtain the following approximations of $z(t + \Delta t, y)$ and $z^{-1}(t + \Delta t, x)$:

$$\begin{aligned} z(t+\Delta t, y) &= y + a(y)\Delta t + O(\Delta t^3) \\ z^{-1}(t+\Delta t, x) &= x - a(x)\Delta t + a(x)a_x(x)\Delta t^2 + O(\Delta t^3) \end{aligned} \tag{18}$$

They are the same as in the above kinetic scheme.

For second order accuracy in time we find correction terms for $\mathbf{L}$, $\mathbf{R}_1$, and $\mathbf{R}_2$:

$$\mathbf{K}_{\mathbf{R}_2} = \begin{pmatrix} k_1 \\ k_1 u \\ k_1 u^2/2 \end{pmatrix}, \qquad \mathbf{K}_{\mathbf{R}_1} = \begin{pmatrix} -k_1 \\ -k_1 u \\ -k_1(u^2+c^2)/2 + k_2 \end{pmatrix},$$

$$\mathbf{K_L} = \begin{pmatrix} -k_1 \\ -k_1 c \\ -k_1 uc + k_3 \end{pmatrix}. \tag{19}$$

These correction terms correspond to those of the kinetic scheme defined by the coefficients (12). Therefore the Method of Transport using the timeintegration (18) and the correction terms (19) is the same as the above second order kinetic scheme. To ensure the positivity of the coefficients (12) conditions (14) - (17) are sufficient. □

Note that no correction terms in the density are necessary for second order accuracy. The scheme without them would correspond to a combination of two kinetic schemes: for the density we use the first order distribution function with coefficients (11), whereas for the momentum and the energy we use the second order distribution function with coefficients (12).

6 Conclusions

We have seen that the first order Method of Transport can be written as a kinetic scheme. The consequences are stated in theorem 4.1: the scheme preserves positivity of density and pressure, and it allows the L^1-estimates (8). No conditions need to be imposed.

The second order Method of Transport however, cannot be written as a kinetic scheme. Yet, by introducing a coupling of the conservative variables into the method we obtain the same timeintegration as in the kinetic scheme. There seems to be no other difference between the two methods. The properties of the modified Method of Transport are stated in theorem 5.1: the method preserves positivity of density and pressure, and it allows the L^1-estimates (8).

Finally, note that the simple form of the distribution functions introduced here allows the exact integration of the moments. No simplifications need to be made in higher space dimensions.

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