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Exact Nonreflecting Boundary Condition For Elastic Waves

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Seminar für Angewandte Mathematik
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Abstract

An exact nonreflecting boundary condition is derived for the time dependent elastic wave equation in three space dimensions. This condition holds on a spherical surface $\mathcal{B}$, outside of which the medium is assumed to be linear, homogeneous, isotropic, and source-free. It is local in time, nonlocal on $\mathcal{B}$, and involves only first derivatives of the solution. Therefore it can be combined easily with any numerical method in the interior region.

Keywords: elastic waves, wave propagation, nonreflecting boundary conditions, absorbing boundary conditions, scattering theory

AMS Subject Classification: 35L15, 65M99, 73D15, 73D30

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1 Introduction

We consider elastic wave scattering by a complicated scattering region in an unbounded elastic medium in three dimensions. The scatterer may contain cracks, holes, inhomogeneities, and nonlinearities. To treat the scattering problem numerically, we surround the scatterer by an artificial boundary $\mathcal{B}$, and we denote by Ω the computational domain inside $\mathcal{B}$. On $\mathcal{B}$ we seek a boundary condition which ensures that the solution of the problem in Ω coincides with the restriction to Ω of the solution of the original problem in the unbounded region. Such an exact boundary condition is nonreflecting because it ensures that $\mathcal{B}$ will not introduce any spurious reflections. See Grote and Keller [1], [2], [3] for nonreflecting boundary conditions for the scalar wave equation and for Maxwell's equations, and Keller and Givoli [4] and Givoli and Keller [5] for the Helmholtz equation and time-harmonic two-dimensional elastodynamics.

Usually various approximate boundary conditions are used on $\mathcal{B}$, which are local differential operators on $\mathcal{B}$ (Givoli [6]). Well-known examples are the Clayton and Engquist [7], Engquist and Majda [8] [9], and the Bayliss and Turkel [10] conditions. Earlier Lindman [11] devised a non-local absorbing boundary condition for the scalar wave equation. It requires solving the inhomogeneous wave equation on the artificial boundary a number of times. Randall [12] [13] extended it to the elastic wave equation. Higdon [14] showed that any local boundary condition involving a differential operator eliminates spurious reflections at certain angles of incidence but not at others. Wolf and Song [15] developed an improved version of Higdon's boundary condition specifically geared to the finite element method.

A different approach to eliminating reflection has been to append an artificial transition layer outside $\mathcal{B}$, which is supposed to absorb outgoing waves. Two popular methods for doing this, the mapping technique [16] and the perfectly matched layer method [17], were adapted recently to the absorbtion of elastic waves, and they yielded comparable results [18].

Neither the local boundary conditions nor the use of absorbing layers leads to complete absorption of waves at all angles of incidence. Although most approximate boundary conditions perform well at nearly normal incidence, their performance degrades rapidly as grazing incidence is approached. In complex situations the scattered waves arrive at the artificial boundary from all interior angles and at all frequencies, so these methods then yield some spurious reflection. Moreover, errors due to spurious reflection accumulate with time and prevent accurate long-time integration. Thus it is often necessary to move $\mathcal{B}$ far from the region of interest, or to use a thick absorbing layer, to reduce the amount of reflection below a few percent and to achieve high accuracy. Unless accuracy as low as two significant digits is acceptable, both procedures become expensive in computer storage and in execution time. Moreover, due to limitations in available memory it may not be possible to achieve a desired accuracy. In the presence of nonlinearity, such as plastic deformation, any type of spurious reflection can lead to dramatic changes in the qualitative behavior of the solution inside the computational domain. To be sure that numerical results are valid in such situations, it is imperative to use a numerical method which ensures convergence to the true solution as the underlying mesh is refined. Another difficulty is that when discretized and combined with a numerical scheme in the interior, approximate boundary conditions can result in ill-posed formulations [19].

Some of these difficulties are avoided by an exact nonreflecting boundary condition for the wave equation proposed by Ting and Miksis [20]. It is based on a Kirchhoff integral representation of the solution on $\mathcal{B}$ and requires storing the solution at a surface inside $\mathcal{B}$ for the length of time it takes a wave to propagate across Ω. To update the solution value at any point on the two-dimensional artificial boundary $\mathcal{B}$ requires a two-dimensional convolution in time and space. Therefore using this boundary condition may be more expensive than using the numerical scheme itself inside Ω.

It is to avoid the various difficulties mentioned above that we have derived a new exact nonreflecting boundary condition. It applies in the special case when $\mathcal{B}$ is a sphere. First, in Section 2 we decompose the wave field into a compressional wave and two shear waves. Next, we show in Section 2.1 how the boundary condition we derived previously for the scalar wave equation [1] extends to one of the purely tangential shear waves. It does not extend to the two other waves, which are not orthogonal to each other and which do not travel at the same speed. In Sections 2.2 and 2.3 we show how to circumvent this difficulty. In Section 2.4 this leads to a novel coupled system of ordinary differential equations, which determines certain required auxilliary quantities. In Section 2.5 we state the final form of the exact nonreflecting boundary condition. It is local in time and involves only first derivatives of the displacement; hence it is well-suited for numerical implementation. To conclude we show that the solution inside Ω with this boundary condition imposed on $\mathcal{B}$ has a unique solution, which coincides with the restriction to Ω of the solution in the unbounded domain.

2 Derivation of an exact nonreflecting boundary condition

We choose $\mathcal{B}$ to be a sphere of radius R. In $\mathcal{B}^{ext}$, the region outside $\mathcal{B}$, the medium is assumed to be linear, homogeneous, and isotropic, with constant density ρ and Lamé constants λ and μ. In addition, we assume that at $t = 0$ the scattered field is confined to the computational domain Ω, which is in the interior of $\mathcal{B}$. In $\mathcal{B}^{ext}$ the displacement $\boldsymbol{u}$ satisfies the elastic wave equation [21] [22],

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c_p^2 \nabla\nabla \cdot \boldsymbol{u} + c_s^2 \nabla \times \nabla \times \boldsymbol{u} = 0, \tag{2.1}$$

with initial conditions

$$\boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = 0, \qquad t = 0. \tag{2.2}$$

Here c_p and c_s are the propagation speeds of compressional waves and shear waves, respectively,

$$c_p^2 = \frac{\lambda + 2\mu}{\rho}, \qquad c_s^2 = \frac{\mu}{\rho}, \qquad c_p > c_s. \tag{2.3}$$

In $\mathcal{B}^{ext}$ compressional waves and shear waves propagate independently of each other and at different speeds, so we decompose $\boldsymbol{u}$ into fields of these two types. Such a decomposition occurs naturally when we split $\boldsymbol{u}$ into a field with vanishing curl and a field with vanishing divergence:

$$\boldsymbol{u} = \nabla\varphi + \boldsymbol{\Psi}, \qquad \nabla \cdot \boldsymbol{\Psi} = 0. \tag{2.4}$$

We introduce (2.4) into (2.1) and conclude that

$$\frac{\partial^2 \varphi}{\partial t^2} - c_p^2\, \Delta\varphi = 0, \tag{2.5}$$

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} + c_s^2\, \nabla\times\nabla\times\boldsymbol{\Psi} = 0. \tag{2.6}$$

The potential φ of the irrotational wave field $\nabla\varphi$ satifies the scalar wave equation (2.5); it describes compressional waves propagating with speed c_p. The solenoidal field $\boldsymbol{\Psi}$ satisfies the vector wave equation (2.6); it describes shear waves propagating with speed c_s. From (2.6) we observe that if $\nabla\cdot\boldsymbol{\Psi}$ and $\nabla\cdot\partial_t\boldsymbol{\Psi}$ vanish at $t=0$, $\nabla\cdot\boldsymbol{\Psi}$ will remain zero for all time. Since $\boldsymbol{u}$ and $\partial_t\boldsymbol{u}$ vanish outside $\mathcal{B}$ at $t=0$, $\nabla\cdot\boldsymbol{\Psi}$ is identically zero outside $\mathcal{B}$ for all time.

Next, we introduce the polar coordinates r, ϑ, ϕ and the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\vartheta}}$, $\hat{\boldsymbol{\phi}}$. We let Y_{nm} denote the nm-th spherical harmonic

$$Y_{nm}(\vartheta,\phi) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}}\, P_n^{|m|}(\cos\vartheta)e^{im\phi}, \qquad n\geq 0,\ |m|\leq n. \tag{2.7}$$

The Y_{nm} are orthonormal with respect to the L_2 inner product on the *unit* sphere. If the problem considered is real, it is advantageous to use the real spherical harmonics, given by the real and imaginary parts of (2.19) with a modified normalization constant. In $\mathcal{B}^{ext}$ the general solution of (2.5) is

$$\varphi(r,\vartheta,\phi,t) = \sum_{n\geq 0}\sum_{|m|\leq n} h_{nm}(r,t)\, Y_{nm}(\vartheta,\phi), \quad r\geq R. \tag{2.8}$$

Here the Fourier coefficients h_{nm} satisfy

$$L_n[h_{nm};c_p] \equiv \left(\frac{1}{c_p^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{n(n+1)}{r^2}\right) h_{nm} = 0. \tag{2.9}$$

Next, we need to construct the general solution of (2.6) in $\mathcal{B}^{ext}$. Following [23], p. 170, we let $\boldsymbol{U}_{nm}$ and $\boldsymbol{V}_{nm}$ denote the vector spherical harmonics

$$\boldsymbol{U}_{nm}(\vartheta,\phi) = \frac{r\nabla Y_{nm}}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n(n+1)}}\left[\frac{\partial Y_{nm}}{\partial\vartheta}\hat{\boldsymbol{\vartheta}} + \frac{1}{\sin\vartheta}\frac{\partial Y_{nm}}{\partial\phi}\hat{\boldsymbol{\phi}}\right], \quad n\geq 1 \tag{2.10}$$

$$\boldsymbol{V}_{nm}(\vartheta,\phi) = \hat{\boldsymbol{r}}\times\boldsymbol{U}_{nm} = \frac{1}{\sqrt{n(n+1)}}\left[\frac{-1}{\sin\vartheta}\frac{\partial Y_{nm}}{\partial\phi}\hat{\boldsymbol{\vartheta}} + \frac{\partial Y_{nm}}{\partial\vartheta}\hat{\boldsymbol{\phi}}\right], \quad n\geq 1. \tag{2.11}$$

They form an orthonormal basis for the space of tangential L_2 fields on the unit sphere with respect to the L_2 inner product [23]. They also satisfy the following useful equations for any $f(r)\in\mathcal{C}^1$:

$$\nabla\times(f(r)\boldsymbol{V}_{nm}) = -\frac{\sqrt{n(n+1)}f(r)}{r}Y_{nm}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial(rf)}{\partial r}\boldsymbol{U}_{nm}, \tag{2.12}$$

$$\hat{\boldsymbol{r}} \times \nabla \times (f(r)\boldsymbol{V}_{nm}) = -\frac{1}{r}\frac{\partial(rf(r))}{\partial r}\boldsymbol{V}_{nm}, \tag{2.13}$$

$$\hat{\boldsymbol{r}} \times \nabla \times (f(r)Y_{nm}\hat{\boldsymbol{r}}) = \frac{\sqrt{n(n+1)}f(r)}{r}\boldsymbol{U}_{nm}, \tag{2.14}$$

$$\nabla \cdot (f(r)\boldsymbol{U}_{nm}) = -\frac{\sqrt{n(n+1)}f(r)}{r}Y_{nm}. \tag{2.15}$$

In $\mathcal{B}^{ext}$ the general solution of (2.6) consists of a sum of independent, orthogonal, divergence-free shear modes,

$$\boldsymbol{\Psi}(r,\vartheta,\phi,t) = \sum_{n\geq 1}\sum_{|m|\leq n}\left\{f_{nm}(r,t)\boldsymbol{V}_{nm}(\vartheta,\phi) + \nabla \times [g_{nm}(r,t)\boldsymbol{V}_{nm}(\vartheta,\phi)]\right\}. \tag{2.16}$$

Here the Fourier coefficients f_{nm} and g_{nm} satisfy the scalar wave equation

$$L_n[f_{nm};c_s] = 0, \qquad L_n[g_{nm};c_s] = 0, \qquad r \geq R, \tag{2.17}$$

with L_n defined by (2.9) with c_p replaced by c_s.

By adding the two solutions (2.8) and (2.16) we obtain the general solution of (2.1) in $\mathcal{B}^{ext}$:

$$\boldsymbol{u}(r,\vartheta,\phi,t) = \sum_{n\geq 0}\sum_{|m|\leq n}\boldsymbol{u}_{nm}(r,\vartheta,\phi,t), \qquad r \geq R. \tag{2.18}$$

Each multipole field $\boldsymbol{u}_{nm}$ is given by

$$\begin{aligned}\boldsymbol{u}_{nm}(r,\vartheta,\phi,t) = \quad & f_{nm}(r,t)\boldsymbol{V}_{nm}(\vartheta,\phi) + \nabla \times [g_{nm}(r,t)\boldsymbol{V}_{nm}(\vartheta,\phi)] \\ + \quad & \nabla[h_{nm}(r,t)Y_{nm}(\vartheta,\phi)].\end{aligned} \tag{2.19}$$

Thus f_{nm} and g_{nm} determine the two independent shear modes, and h_{nm} determines the compressional mode. For $n = 0$, both $\boldsymbol{V}_{nm}$ and $\boldsymbol{U}_{nm}$ vanish, so a solution independent of ϑ and ϕ consists of a single compressional mode $\boldsymbol{u}(r,t) = \partial_r h_{00}(r,t)\hat{\boldsymbol{r}}$.

Next, we use (2.12) in (2.19) to calculate the three orthogonal components of $\boldsymbol{u}_{nm}$,

$$\begin{aligned}\boldsymbol{u}_{nm} = \quad & f_{nm}\boldsymbol{V}_{nm} + \left(\frac{\sqrt{n(n+1)}h_{nm}}{r} - \frac{1}{r}\frac{\partial(rg_{nm})}{\partial r}\right)\boldsymbol{U}_{nm} \\ + \quad & \left(\frac{\partial h_{nm}}{\partial r} - \frac{\sqrt{n(n+1)}g_{nm}}{r}\right)Y_{nm}\hat{\boldsymbol{r}}.\end{aligned} \tag{2.20}$$

Equation (2.20) shows that $f_{nm}\boldsymbol{V}_{nm}$ is orthogonal to the two other modes; thus, f_{nm} can be computed easily from $\boldsymbol{u}$ via $f_{nm} = (\boldsymbol{u}, \boldsymbol{V}_{nm})$. In addition, if either g_{nm} or h_{nm} is zero, the other coefficient can be obtained by computing either $(\boldsymbol{u}, \boldsymbol{U}_{nm})$ or $(\hat{\boldsymbol{r}} \cdot \boldsymbol{u}, Y_{nm})$. In general, however, both g_{nm} and h_{nm} are nonzero. Therefore knowledge of $\boldsymbol{u}$ on $\mathcal{B}$ does not suffice to determine f_{nm} and h_{nm} through inner products over $\mathcal{B}$ because of the presence of (unknown) radial derivatives in (2.20). In contrast, in the absence of compressional modes ($h_{nm} = 0$), which corresponds to Maxwell's equations, knowledge of the field on $\mathcal{B}$ is sufficient to compute its representation in terms of potentials via inner products over $\mathcal{B}$ [3].

For time dependent elastic waves in three space dimensions, the shear mode g_{nm} and the compressional mode h_{nm} are inextricably linked and are not orthogonal to each other. Only in very special situations, such as (single frequency) harmonic or static waves ([24], Chapter 13.3), is it possible to construct mutually orthogonal modes through linear combinations of shear and compressional waves.

2.1 Exact boundary condition for $(\boldsymbol{u}, \boldsymbol{V}_{nm})$

In this section we derive the exact nonreflecting boundary condition for the mode amplitude $f_{nm}(r,t) = (\boldsymbol{u}, \boldsymbol{V}_{nm})$. The analysis follows closely that for the scalar wave equation (see [1],[2]). The real difficulty in the elastic case is the determination of exact boundary conditions for g_{nm} and h_{nm}, which we do in Sections 2.2 and 2.3.

In Lemma 3.2 [1] it was shown that if f_{nm} satisfies the scalar wave equation (2.17), then $G_n[f_{nm}]$ satisfies the one-dimensional wave equation,

$$\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}G_n[f_{nm}] - \frac{\partial^2}{\partial r^2}G_n[f_{nm}] = 0, \qquad r \geq R. \tag{2.21}$$

Here the integral operator $G_n[v]$ is defined as

$$G_n[v](r,t) \equiv \begin{cases} rv(r,t) & \text{if } n = 0 \\ r\displaystyle\int_r^\infty \frac{(s^2-r^2)^{n-1}v(s,t)}{(2s)^{n-1}(n-1)!}\,ds & \text{if } n \geq 1. \end{cases} \tag{2.22}$$

Since the initial data f_{nm} and $\partial_t f_{nm}$ have compact support and since the speed of propagation is finite, at any fixed time $f_{nm}(r,t)$ vanishes for r sufficiently large. Therefore the integral in (2.22) with v replaced by f_{nm} exists. Moreover, at $t = 0$, both $G_n[f_{nm}]$ and $\partial_t G_n[f_{nm}]$ vanish outside $\mathcal{B}$, which together with (2.21) imply that

$$\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right) G_n[f_{nm}] = 0, \qquad r \geq R. \tag{2.23}$$

Since $G_n[f_{nm}](r,t)$ is not known, (2.23) cannot be used directly in a numerical scheme. Therefore we shall recast the exact boundary condition (2.23) in terms of f_{nm}, and ultimately in terms of the displacement field $\boldsymbol{u}$.

To derive the exact boundary condition for f_{nm}, we multiply (2.20) by r, take the inner product with $\boldsymbol{V}_{nm}$, and use Lemma 1 in the Appendix to obtain

$$r\, f_{nm}(r,t) = \sum_{j=0}^{n} \frac{\gamma_{nj}}{r^j}\left(-\frac{\partial}{\partial r}\right)^{n-j} G_n[f_{nm}](r,t). \tag{2.24}$$

Next, we apply $\partial_r + c_s^{-1}\partial_t$ to (2.24) and use (2.23) to replace $-\partial_r G_n[f_{nm}]$ by $c_s^{-1}\partial_t G_n[f_{nm}]$. This yields

$$\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right)[r\, f_{nm}(r,t)] = -\frac{1}{r}\sum_{j=1}^{n} \frac{j\,\gamma_{nj}}{r^j\, c_s^{n-j}}\left(\frac{\partial}{\partial t}\right)^{n-j} G_n[f_{nm}](r,t), \qquad r = R. \tag{2.25}$$

Equation (2.25) is an exact nonreflecting boundary condition for f_{nm} on $\mathcal{B}$, but it involves time derivatives of $G_n[f_{nm}]$ up to order $n-1$. We note that a crucial consequence of applying $\partial_r + c_s^{-1}\partial_t$ to (2.24) is the elimination of the n-th derivative of $G_n[f_{nm}]$, the term with $j=0$ in (2.24). To compute the time derivatives of $G_n[f_{nm}]$ up to order $n-1$ at $r=R$, we again use (2.23) in (2.24). The result is

$$(2.26)\quad \frac{1}{c_s^n}\frac{\partial^n}{\partial t^n}G_n[f_{nm}](r,t) = -\sum_{j=1}^{n}\frac{\gamma_{nj}}{r^j\, c_s^{n-j}}\left(\frac{\partial}{\partial t}\right)^{n-j} G_n[f_{nm}](r,t) + r\, f_{nm}(r,t), \qquad r=R.$$

Here we have used the fact that $\gamma_{n0}=1$. Equation (2.26) is an n-th order ordinary differential equation for $G_n[f_{nm}](R,t)$.

To simplify the notation, we define the n-component vector function $\boldsymbol{\psi}^f_{nm}(t) = \{\psi^{f,j}_{nm}(t)\}$ by

$$(2.27)\qquad \psi^{f,j}_{nm}(t) = \frac{\gamma_{nj}}{R\,\gamma_{n1}c_s^{n-j}}\left(\frac{\partial}{\partial t}\right)^{n-j} G_n[f_{nm}](R,t), \qquad j=1,\dots,n.$$

In addition, we let $\boldsymbol{d}_n = \{d_n^j\}$ denote the *constant* n-component vector

$$(2.28)\qquad d_n^j = \frac{n(n+1)j}{2R^j}, \qquad j=1,\dots,n.$$

With these new variables the exact nonreflecting boundary condition (2.25) reduces to

$$(2.29)\qquad \left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right)(r\,\boldsymbol{u}, \boldsymbol{V}_{nm})\big|_{r=R} = -\boldsymbol{d}_n\cdot\boldsymbol{\psi}^f_{nm}(t).$$

Next, we note that by definition of $\psi^{f,j}_{nm}(t)$ we have

$$(2.30)\qquad \frac{1}{c_s}\frac{d}{dt}\psi^{f,j}_{nm} = \frac{\gamma_{nj}}{\gamma_{n,j-1}}\,\psi^{f,j-1}_{nm} = \frac{(n+j)(n+1-j)}{2j}\,\psi^{f,j-1}_{nm}, \qquad 2\le j\le n.$$

Since f_{nm} and $\partial_t f_{nm}$ vanish identically for $r\ge R$ at $t=0$, so do $G_n[f_{nm}]$ and all its time derivatives up to order $n-1$. This implies that $\boldsymbol{\psi}^f_{nm}$ is equal to zero at $t=0$. Therefore we rewrite (2.26) as the *linear first-order ordinary* differential equation

$$(2.31)\qquad \frac{1}{c_s}\frac{d}{dt}\boldsymbol{\psi}^f_{nm}(t) = \boldsymbol{A}_n\boldsymbol{\psi}^f_{nm}(t) + (u\,|_{r=R}\,, \boldsymbol{V}_{nm})\boldsymbol{e}_n, \qquad \boldsymbol{\psi}^f_{nm}(0)=0.$$

Here $\boldsymbol{e}_n = \{e_n^j\}$ is the constant n-component unit vector

$$(2.32)\qquad \boldsymbol{e}_n = [1,0,\dots,0]^\top,$$

and $\boldsymbol{A}_n = \{A_n^{ij}\}$ is the constant $n\times n$ matrix

$$(2.33)\qquad A_n^{ij} = \begin{cases} -n(n+1)/(2\,R^j) & \text{if } i=1,\\ (n+i)(n+1-i)/(2i) & \text{if } i=j+1,\\ 0 & \text{otherwise.}\end{cases}$$

The exact nonreflecting boundary condition on $(\boldsymbol{u}, \boldsymbol{V}_{nm})$ at $\mathcal{B}$ is given by (2.29). It involves only first derivatives of $\boldsymbol{u}$ and the function $\boldsymbol{\psi}^f_{nm}(t)$. This function can be computed by solving the linear first-order differential equation (2.31) concurrently with solving the problem for $\boldsymbol{u}$. The boundary condition (2.29) is local in time since past values of $\boldsymbol{u}$ are not required to apply it at time t. The necessary information from the past is contained implicitly in $\boldsymbol{\psi}^f_{nm}(t)$. The scaling of $G_n[f_{nm}]$ by γ_{nj} in (2.27) has removed the large coefficients which appear in (2.25) and (2.26).

2.2 Exact boundary condition for $(\boldsymbol{u}, \boldsymbol{U}_{nm})$

We shall now derive the exact boundary condition for the second tangential component, $(\boldsymbol{u}, \boldsymbol{U}_{nm})$. To begin we multiply (2.20) by r, take the inner product with $\boldsymbol{U}_{nm}$, and use Lemmas 1 and 2 (Appendix) to obtain

$$\begin{aligned}(r\,\boldsymbol{u}, \boldsymbol{U}_{nm}) &= \sqrt{n(n+1)}h_{nm} - \partial_r(r g_{nm}), \\ &= \sum_{j=0}^{n+1} \frac{\beta_{nj}}{r^j}(-\partial_r)^{n+1-j} G_n[g_{nm}] + \sqrt{n(n+1)} \sum_{j=0}^{n} \frac{\gamma_{nj}}{r^{j+1}}(-\partial_r)^{n-j} G_n[h_{nm}]. \end{aligned} \tag{2.34}$$

Both g_{nm} and h_{nm} satisfy the scalar wave equation (2.9) with propagation speeds c_s and c_p respectively. Therefore an argument similar to that used in the previous section for $G_n[f_{nm}]$ yields the exact boundary conditions for g_{nm} and h_{nm}:

$$\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right) G_n[g_{nm}] = 0, \tag{2.35}$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{c_p}\frac{\partial}{\partial t}\right) G_n[h_{nm}] = 0, \qquad r \geq R. \tag{2.36}$$

Now we shall construct a differential operator which annihilates the highest derivatives of $G_n[g_{nm}]$ and $G_n[h_{nm}]$ in (2.34). The two different speeds of propagation preclude a straightforward extension of the argument used before. Again we apply $\partial_r + c_s^{-1}\partial_t$ to (2.34). We also use (2.36) to get

$$\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right) G_n[h_{nm}] = \left(\frac{c_p}{c_s} - 1\right) \frac{1}{c_p}\frac{\partial}{\partial t} G_n[h_{nm}]. \tag{2.37}$$

Then we use (2.37) and (2.35) to obtain

$$\begin{aligned}\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right)(r\,\boldsymbol{u}, \boldsymbol{U}_{nm}) &= -\sum_{j=1}^{n+1} \frac{j\,\beta_{nj}}{r^{j+1}c_s^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[g_{nm}] \\ &+ \sqrt{n(n+1)} \sum_{j=0}^{n+1} \frac{\gamma_{nj}(c_p/c_s - 1) - j\,\gamma_{n,j-1}}{r^{j+1}c_p^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[h_{nm}]. \end{aligned} \tag{2.38}$$

Here we have defined $\gamma_{n,n+1} = 0$.

In (2.38) we have succeeded in removing the $(n+1)$-st derivative of $G_n[g_{nm}]$, but we have introduced an $(n+1)$-st derivative of $G_n[h_{nm}]$. We now seek a differential operator which will annihilate it. This operator must not affect the $\boldsymbol{V}_{nm}$-component of $\boldsymbol{u}_{nm}$, must not depend on n, and must produce a term in $\sqrt{n(n+1)}\boldsymbol{U}_{nm}$ when applied to $\boldsymbol{u}_{nm}$. To find such a differential operator, we recall (2.14) and apply it to the r-component of $\boldsymbol{u}_{nm}$:

$$r\,\hat{\boldsymbol{r}} \times \nabla \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{u}_{nm})\hat{\boldsymbol{r}}) = \left[\sqrt{n(n+1)}\,\partial_r h_{nm} - n(n+1)\frac{g_{nm}}{r}\right] \boldsymbol{U}_{nm}. \tag{2.39}$$

The differential operator on the left of (2.39) fulfills all the requirements listed above; for instance, it does not affect the $\boldsymbol{V}_{nm}$-component of $\boldsymbol{u}_{nm}$ since it only acts upon the radial component. Next, we take the inner product of (2.39) with $\boldsymbol{U}_{nm}$ and evaluate the right side of the resulting expression by using Lemmas 1 and 3 (Appendix). After replacing radial by time derivatives according to (2.35) and (2.36), we obtain

$$\begin{aligned} r\,(\hat{\boldsymbol{r}} \times \nabla \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{u})\hat{\boldsymbol{r}}), \boldsymbol{U}_{nm}) \;&=\; -\sqrt{n(n+1)}\sum_{j=0}^{n+1} \frac{\beta_{nj} + \gamma_{n,j-1}}{r^{j+1}\, c_p^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[h_{nm}] \\ &\quad - \; n(n+1)\sum_{j=1}^{n+1} \frac{\gamma_{n,j-1}}{r^{j+1}\, c_s^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[g_{nm}]. \end{aligned} \tag{2.40}$$

We recall that by definition $\gamma_{n,-1} = 0$, and $\gamma_{n0} = \beta_{n0} = 1$. Thus, upon comparison of the terms with $j = 0$ in (2.38) and (2.40), we find that they are identical but for a factor of $-(c_p/c_s - 1)$. Therefore, when (2.40) is multiplied by $c_p/c_s - 1$ and added to (2.38), the two $(n+1)$-st derivatives of $G_n[h_{nm}]$ cancel. This results in

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right)(r\,\boldsymbol{u}, \boldsymbol{U}_{nm}) \;&+\; r\left(\frac{c_p}{c_s} - 1\right)(\hat{\boldsymbol{r}} \times \nabla \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{u})\hat{\boldsymbol{r}}), \boldsymbol{U}_{nm}) = \\ &- \sqrt{n(n+1)}\sum_{j=1}^{n+1} \frac{j\gamma_{n,j-1}c_p/c_s}{r^{j+1}\, c_p^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[h_{nm}] \\ &- \sum_{j=1}^{n+1} \frac{j\beta_{nj} + (c_p/c_s - 1)n(n+1)\gamma_{n,j-1}}{r^{j+1}\, c_s^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[g_{nm}]. \end{aligned} \tag{2.41}$$

The exact boundary condition on $(\boldsymbol{u}, \boldsymbol{U}_{nm})$ is obtained by setting $r = R$ in (2.41). It involves the time derivatives of $G_n[g_{nm}]$ and $G_n[h_{nm}]$ on $\mathcal{B}$ up to order n. In Section 2.4 we shall derive ordinary differential equations which determine them.

To simplify the notation, we define the $n+1$ component vector functions $\boldsymbol{\psi}^g_{nm}(t) = \{\psi^{g,j}_{nm}(t)\}$ and $\boldsymbol{\psi}^h_{nm}(t) = \{\psi^{h,j}_{nm}(t)\}$ by

$$\psi^{g,j}_{nm}(t) = \frac{\beta_{nj}}{R\,\beta_{n1} c_s^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[g_{nm}](R,t), \qquad j = 1, \ldots, n+1, \tag{2.42}$$

$$\psi^{h,j}_{nm}(t) = \frac{\beta_{nj}}{R\,\beta_{n1} c_p^{n+1-j}} \left(\frac{\partial}{\partial t}\right)^{n+1-j} G_n[h_{nm}](R,t), \qquad j = 1, \ldots, n+1. \tag{2.43}$$

In addition, we let $\boldsymbol{p}_n = \{p^j_n\}$ and $\boldsymbol{q}_n = \{q^j_n\}$ denote the *constant* $n+1$ component vectors

$$p^j_n = \frac{\sqrt{n(n+1)}\, j\, \gamma_{n,j-1}\beta_{n1}c_p/c_s}{\beta_{nj}R^j}, \qquad j = 1, \ldots, n+1, \tag{2.44}$$

$$(2.45)\qquad q_n^j = \frac{(c_p/c_s-1)n(n+1)\gamma_{n,j-1}+j\beta_{nj}}{R^j\beta_{nj}}\beta_{n1}, \qquad j=1,\dots,n+1.$$

With these new variables we can write the boundary condition (2.41) with $r=R$ as

$$(2.46)\qquad \begin{aligned}\left(\frac{\partial}{\partial r}+\frac{1}{c_s}\frac{\partial}{\partial t}\right)(r\,\boldsymbol{u},\boldsymbol{U}_{nm}) \;&+\; R\left(\frac{c_p}{c_s}-1\right)(\hat{\boldsymbol{r}}\times\nabla\times((\hat{\boldsymbol{r}}\cdot\boldsymbol{u})\hat{\boldsymbol{r}}),\boldsymbol{U}_{nm})\\ &= -\boldsymbol{p}_n\cdot\boldsymbol{\psi}^h_{nm}(t)-\boldsymbol{q}_n\cdot\boldsymbol{\psi}^g_{nm}(t), \qquad r=R.\end{aligned}$$

2.3 Exact boundary condition for $(\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})$

To derive an exact boundary condition on $(\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})$ we parallel the procedure employed in the previous section. First we multiply (2.20) by r, take the inner product with $Y_{nm}\hat{\boldsymbol{r}}$, and use Lemmas 1 and 3 (Appendix) to evaluate the right side of the result:

$$(2.47)\qquad \begin{aligned}&(r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm}) = r\partial_r h_{nm}-\sqrt{n(n+1)}g_{nm}\\ &= -\sum_{j=0}^{n+1}\frac{\beta_{nj}+\gamma_{n,j-1}}{r^j}(-\partial_r)^{n+1-j}G_n[h_{nm}]-\sqrt{n(n+1)}\sum_{j=0}^{n}\frac{\gamma_{nj}}{r^{j+1}}(-\partial_r)^{n-j}G_n[g_{nm}].\end{aligned}$$

We apply $\partial_r+c_p^{-1}\partial_t$ to (2.47) to eliminate the highest derivative of $G_n[h_{nm}]$. To simplify the resulting expression, we use (2.35) to get

$$(2.48)\qquad \left(\frac{\partial}{\partial r}+\frac{1}{c_p}\frac{\partial}{\partial t}\right)G_n[g_{nm}] = -\left(1-\frac{c_s}{c_p}\right)\frac{1}{c_s}\frac{\partial}{\partial t}G_n[g_{nm}].$$

This relation together with (2.36) yields

$$(2.49)\qquad \begin{aligned}\left(\frac{\partial}{\partial r}+\frac{1}{c_p}\frac{\partial}{\partial t}\right)(r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm}) \;&=\; \sum_{j=1}^{n+1}\frac{j\,(\beta_{nj}+\gamma_{n,j-1})}{r^{j+1}c_p^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[h_{nm}]\\ &+\; \sqrt{n(n+1)}\sum_{j=0}^{n+1}\frac{\gamma_{nj}(1-c_s/c_p)+j\,\gamma_{n,j-1}}{r^{j+1}c_s^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[g_{nm}].\end{aligned}$$

We recall that by definition $\gamma_{n,n+1}=0$.

Although we have succeeded in removing the $(n+1)$-st derivative of $G_n[h_{nm}]$ from (2.49), we have introduced an $(n+1)$-st derivative of $G_n[g_{nm}]$, the term with $j=0$ in the second sum. To annihilate it we construct a differential operator which does not affect the $\boldsymbol{V}_{nm}$ component of $\boldsymbol{u}_{nm}$, does not depend on n, and introduces a term of the form $\sqrt{n(n+1)}\,Y_{nm}\hat{\boldsymbol{r}}$. To do so, we recall (2.15) and apply it to $\boldsymbol{u}^{\tan}=(0,u^\vartheta,u^\phi)$, the tangential part of $\boldsymbol{u}=(u^r,u^\vartheta,u^\phi)$. Since $\nabla\cdot\boldsymbol{V}_{nm}=0$, we immediately obtain

$$(2.50)\qquad r\,\nabla\cdot\boldsymbol{u}^{\tan}_{nm} = \left[\frac{\sqrt{n(n+1)}}{r}\partial_r(r g_{nm})-\frac{n(n+1)}{r}h_{nm}\right]Y_{nm}.$$

Next we take the inner product of (2.50) with Y_{nm} and evaluate the right side of the result by using Lemmas 1 and 2 in the Appendix. After replacing radial by time derivatives according to (2.35) and (2.36), we obtain

$$\begin{aligned}\left(r\,\nabla\cdot\boldsymbol{u}^{\tan},Y_{nm}\right) = \; & -\;\sqrt{n(n+1)}\sum_{j=0}^{n+1}\frac{\beta_{nj}}{r^{j+1}\,c_s^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[g_{nm}] \\ & -\;n(n+1)\sum_{j=1}^{n+1}\frac{\gamma_{n,j-1}}{r^{j+1}\,c_p^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[h_{nm}]. \end{aligned} \tag{2.51}$$

We recall that by definition $\gamma_{n,-1}=0$, and $\gamma_{n0}=\beta_{n0}=1$. Thus, upon comparison of the two terms with $j=0$ in (2.49) and (2.51), we find that they are identical but for a factor of $-(1-c_s/c_p)$. Therefore, when we multiply (2.51) by $1-c_s/c_p$ and add the resulting expression to (2.49), the two $(n+1)$-st order derivatives of $G_n[g_{nm}]$ cancel, yielding

$$\begin{aligned}\left(\frac{\partial}{\partial r}+\frac{1}{c_p}\frac{\partial}{\partial t}\right)&(r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})+r\left(1-\frac{c_s}{c_p}\right)\left(\nabla\cdot\boldsymbol{u}^{tan},Y_{nm}\right)= \\ &\sqrt{n(n+1)}\sum_{j=1}^{n+1}\frac{\gamma_{n,j-1}(1+(j-1)c_s/c_p)}{r^{j+1}\,c_s^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[g_{nm}] \\ &-\sum_{j=1}^{n+1}\frac{j\beta_{nj}-(n(n+1)(1-c_s/c_p)-j)\gamma_{n,j-1}}{r^{j+1}\,c_p^{n+1-j}}\left(\frac{\partial}{\partial t}\right)^{n+1-j}G_n[h_{nm}]. \end{aligned} \tag{2.52}$$

We set $r=R$ in (2.52); this yields an exact boundary condition on $(\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})$ at $\mathcal{B}$. It involves the time derivatives of $G_n[g_{nm}]$ and $G_n[h_{nm}]$ at $\mathcal{B}$ up to order n. In Section 2.4 we shall derive ordinary differential equations for them. To simplify (2.52) we make use of the $n+1$ component vector functions $\boldsymbol{\psi}^g_{nm}(t)$ and $\boldsymbol{\psi}^h_{nm}(t)$ defined by (2.42) and (2.43), respectively. In addition, we let $\boldsymbol{a}_n=\{a_n^j\}$ and $\boldsymbol{b}_n=\{b_n^j\}$ denote the *constant* $n+1$ component vectors

$$a_n^j \;=\; \frac{j-((1-c_s/c_p)n(n+1)-j)\gamma_{n,j-1}/\beta_{nj}}{R^j},\qquad j=1,\ldots,n+1, \tag{2.53}$$

$$b_n^j \;=\; \frac{\sqrt{n(n+1)}\,(1+(j-1)c_s/c_p)\gamma_{n,j-1}}{\beta_{nj}R^j},\qquad j=1,\ldots,n+1. \tag{2.54}$$

With these new variables we can write the boundary condition (2.52) with $r=R$ as

$$\begin{aligned}\left(\frac{\partial}{\partial r}+\frac{1}{c_p}\frac{\partial}{\partial t}\right)(r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})\;&+\;R\left(1-\frac{c_s}{c_p}\right)\left(\nabla\cdot\boldsymbol{u}^{tan},Y_{nm}\right) \\ &=\;\boldsymbol{a}_n\cdot\boldsymbol{\psi}^h_{nm}(t)+\boldsymbol{b}_n\cdot\boldsymbol{\psi}^g_{nm}(t),\qquad r=R. \end{aligned} \tag{2.55}$$

2.4 Ordinary differential equations for $\boldsymbol{\psi}^g_{nm}(t)$ and $\boldsymbol{\psi}^h_{nm}(t)$

The exact boundary conditions for $(\boldsymbol{u},\boldsymbol{U}_{nm})$ and $(\hat{\boldsymbol{r}}\cdot\boldsymbol{u},Y_{nm})$ in (2.46) and (2.55) involve $\boldsymbol{\psi}^g_{nm}(t)$ and $\boldsymbol{\psi}^h_{nm}(t)$. We wish to calculate them without using past values of $\boldsymbol{u}$. The modes with amplitudes g_{nm} and h_{nm} are linked and are not orthogonal. Therefore we cannot derive two independent ordinary differential equations for $\boldsymbol{\psi}^g_{nm}(t)$ and $\boldsymbol{\psi}^h_{nm}(t)$. Instead, we shall derive a *coupled* system of two ordinary differential equations.

When we use (2.35) and (2.36) to replace radial by time derivatives and set $r = R$ in (2.34), we obtain

$$\begin{aligned}\frac{1}{c_s^{n+1}}\frac{\partial^{n+1}}{\partial t^{n+1}}G_n[g_{nm}] &= -\sum_{j=1}^{n+1}\frac{\beta_{nj}}{R^j c_s^{n+1-j}}\frac{\partial^{n+1-j}}{\partial t^{n+1-j}}G_n[g_{nm}] \\ &- \sqrt{n(n+1)}\sum_{j=1}^{n+1}\frac{\gamma_{nj}}{R^j c_p^{n+1-j}}\frac{\partial^{n+1-j}}{\partial t^{n+1-j}}G_n[h_{nm}] + R(\boldsymbol{u}, \boldsymbol{U}_{nm}), \qquad r = R. \end{aligned} \tag{2.56}$$

Here we have used the fact that $\beta_{n0} = 1$. Equation (2.56) is an $(n+1)$-st order ordinary differential equation for $G_n[g_{nm}]$ at $r = R$. It determines $\psi^{g,1}_{nm}$, since by definition of $\boldsymbol{\psi}^g_{nm}(t)$ in (2.42),

$$\frac{1}{c_s}\frac{d}{dt}\psi^{g,1}_{nm}(t) = \frac{1}{R\,c_s^n}\frac{\partial^n}{\partial t^n}G_n[g_{nm}](R,t). \tag{2.57}$$

Moreover, we have

$$\frac{1}{c_s}\frac{d}{dt}\psi^{g,j}_{nm} = \frac{\beta_{nj}}{\beta_{n,j-1}}\,\psi^{g,j-1}_{nm}, \qquad 2 \le j \le n+1. \tag{2.58}$$

To further simplify the notation, we define the *constant* $n+1$ component vector $\boldsymbol{z}_n = \{z_n^j\}$ by

$$z_n^j = \frac{\sqrt{n(n+1)}\gamma_{n,j-1}\beta_{n1}}{\beta_{nj}R^j}, \qquad j = 1, \ldots, n+1. \tag{2.59}$$

Next, we note that $\beta_{n1} = n(n+1)/2$ and we let $\boldsymbol{S}_n = \{S_n^{ij}\}$ denote the constant $(n+1)\times(n+1)$ matrix

$$S_n^{ij} = \begin{cases} -n(n+1)/(2\,R^j) & \text{if } i = 1, \\ \beta_{ni}/\beta_{n,j} & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.60}$$

With this new notation and the definition of $\boldsymbol{\psi}^h_{nm}(t)$ in (2.43) we can combine (2.56) and (2.58) into the single first-order ordinary differential equation

$$\frac{1}{c_s}\frac{d}{dt}\boldsymbol{\psi}^g_{nm}(t) = \boldsymbol{S}_n\boldsymbol{\psi}^g_{nm}(t) + [(\boldsymbol{u}|_{r=R}\,, \boldsymbol{U}_{nm}) - \boldsymbol{z}_n \cdot \boldsymbol{\psi}^h_{nm}(t)]\boldsymbol{e}_{n+1}. \tag{2.61}$$

Here $\boldsymbol{e}_{n+1}$ is the $n+1$ constant unit vector defined in (2.32).

We now proceed in a similar fashion to derive an ordinary differential equation for $\boldsymbol{\psi}^h_{nm}(t)$. First, we use (2.35) and (2.36) to replace radial by time derivatives and set $r = R$ in (2.47). This yields

$$\begin{aligned}\frac{1}{c_p^{n+1}}\frac{\partial^{n+1}}{\partial t^{n+1}}G_n[h_{nm}] &= -\sum_{j=1}^{n+1}\frac{\beta_{nj}+\gamma_{n,j-1}}{R^j c_p^{n+1-j}}\frac{\partial^{n+1-j}}{\partial t^{n+1-j}}G_n[h_{nm}] \\ &- \sqrt{n(n+1)}\sum_{j=1}^{n+1}\frac{\gamma_{n,j-1}}{R^j c_s^{n+1-j}}\frac{\partial^{n+1-j}}{\partial t^{n+1-j}}G_n[g_{nm}] - R(\hat{\boldsymbol{r}}\cdot\boldsymbol{u}, Y_{nm}), \qquad r = R. \end{aligned} \tag{2.62}$$

Equation (2.62) is an $(n+1)$-st order ordinary differential equation for $G_n[h_{nm}]$ at $r = R$, which determines $\psi^{h,1}_{nm}$. Indeed, by definition of $\boldsymbol{\psi}^h_{nm}(t)$ in (2.43),

$$\frac{1}{c_p}\frac{d}{dt}\psi^{h,1}_{nm}(t) = \frac{1}{R\,c_p^n}\frac{\partial^n}{\partial t^n}G_n[h_{nm}](R,t). \tag{2.63}$$

The remaining n components of $\psi^{h,j}_{nm}(t)$, $2 \le j \le n+1$, are determined by (2.58) with c_s replaced by c_p, $G_n[g_{nm}]$ by $G_n[h_{nm}]$, and $\psi^{g,j}_{nm}$ by $\psi^{h,j}_{nm}$. Again we simplify the notation by introducing the constant $n+1 \times n+1$ matrix $\boldsymbol{T}_n = \{T_n^{ij}\}$

$$T_n^{ij} = \begin{cases} -n(n+1)(1+\gamma_{n,j-1}/\beta_{nj})/(2\,R^j) & \text{if } i = 1, \\ \beta_{ni}/\beta_{n,j} & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.64}$$

With this new notation and the definition of $\boldsymbol{\psi}^g_{nm}(t)$ in (2.42) we obtain the first-order ordinary differential equation

$$\frac{1}{c_p}\frac{d}{dt}\boldsymbol{\psi}^h_{nm}(t) = \boldsymbol{T}_n\boldsymbol{\psi}^h_{nm}(t) - [(\hat{\boldsymbol{r}}\cdot\boldsymbol{u}|_{r=R}, Y_{nm}) + \boldsymbol{z}_n\cdot\boldsymbol{\psi}^g_{nm}(t)]\boldsymbol{e}_{n+1}. \tag{2.65}$$

3 Exact nonreflecting boundary condition for $\boldsymbol{u}$

Here we gather the various boundary conditions obtained in Section 2 and combine them into an exact nonreflecting boundary condition for the full time-dependent three-dimensional displacement field $\boldsymbol{u}$. For the radial component of the boundary condition, we multiply (2.55) by Y_{nm} and sum over n and m to obtain

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \frac{1}{c_p}\frac{\partial}{\partial t}\right)(r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u}) \quad &+ \quad R\left(1-\frac{c_s}{c_p}\right)\nabla\cdot\boldsymbol{u}^{tan} \\ &= \sum_{n\ge 0}\sum_{|m|\le n}[\boldsymbol{a}_n\cdot\boldsymbol{\psi}^h_{nm}(t) + \boldsymbol{b}_n\cdot\boldsymbol{\psi}^g_{nm}(t)]Y_{nm}, \qquad r = R. \end{aligned} \tag{3.1}$$

For the tangential components, we first multiply (2.29) by $\boldsymbol{V}_{nm}$ and (2.46) by $\boldsymbol{U}_{nm}$ and sum over n and m. Then we add the two resulting expressions and conclude by orthogonality of $\boldsymbol{V}_{nm}$ and $\boldsymbol{U}_{nm}$ that

$$\begin{aligned} &\left(\frac{\partial}{\partial r} + \frac{1}{c_s}\frac{\partial}{\partial t}\right)(r\,\boldsymbol{u}^{\tan}) + R\left(\frac{c_p}{c_s}-1\right)(\hat{\boldsymbol{r}}\times\nabla\times((\hat{\boldsymbol{r}}\cdot\boldsymbol{u})\hat{\boldsymbol{r}})) \\ &= -\sum_{n\ge 1}\sum_{|m|\le n}\boldsymbol{d}_n\cdot\boldsymbol{\psi}^f_{nm}(t)\boldsymbol{V}_{nm} - \sum_{n\ge 1}\sum_{|m|\le n}[\boldsymbol{p}_n\cdot\boldsymbol{\psi}^h_{nm}(t) + \boldsymbol{q}_n\cdot\boldsymbol{\psi}^g_{nm}(t)]\boldsymbol{U}_{nm}, \qquad r = R. \end{aligned} \tag{3.2}$$

We recall that $\boldsymbol{u}^{\tan}$ and $\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\boldsymbol{u})$ stand for

$$\boldsymbol{u}^{\tan} = \begin{pmatrix} 0 \\ u^\vartheta \\ u^\phi \end{pmatrix}, \qquad (\hat{\boldsymbol{r}}\cdot\boldsymbol{u})\hat{\boldsymbol{r}} = \begin{pmatrix} u^r \\ 0 \\ 0 \end{pmatrix}, \tag{3.3}$$

where $\boldsymbol{u} = (u^r, u^\vartheta, u^\phi)$. The functions $\boldsymbol{\psi}^f_{nm}(t)$, $\boldsymbol{\psi}^g_{nm}(t)$, and $\boldsymbol{\psi}^h_{nm}(t)$, satisfy the following first order, linear, ordinary differential equations

$$\text{(3.4)} \quad \frac{1}{c_s}\frac{d}{dt}\boldsymbol{\psi}^f_{nm}(t) = \boldsymbol{A}_n\boldsymbol{\psi}^f_{nm}(t) + (\boldsymbol{u}\,|_{r=R}\,, \boldsymbol{V}_{nm})\boldsymbol{e}_n, \qquad \boldsymbol{\psi}^f_{nm}(0) = 0,$$

$$\text{(3.5)} \quad \frac{1}{c_s}\frac{d}{dt}\boldsymbol{\psi}^g_{nm}(t) = \boldsymbol{S}_n\boldsymbol{\psi}^g_{nm}(t) + [(\boldsymbol{u}|_{r=R}\,, \boldsymbol{U}_{nm}) - \boldsymbol{z}_n \cdot \boldsymbol{\psi}^h_{nm}(t)]\boldsymbol{e}_{n+1}, \qquad \boldsymbol{\psi}^g_{nm}(0) = 0,$$

$$\text{(3.6)} \quad \frac{1}{c_p}\frac{d}{dt}\boldsymbol{\psi}^h_{nm}(t) = \boldsymbol{T}_n\boldsymbol{\psi}^h_{nm}(t) - [(\hat{\boldsymbol{r}} \cdot \boldsymbol{u}|_{r=R}\,, Y_{nm}) + \boldsymbol{z}_n \cdot \boldsymbol{\psi}^g_{nm}(t)]\boldsymbol{e}_{n+1}, \qquad \boldsymbol{\psi}^h_{nm}(0) = 0.$$

Only inner products of $\boldsymbol{u}$ with spherical harmonics appear on the right sides of (3.4)–(3.6), and no derivatives of $\boldsymbol{u}$ normal to $\mathcal{B}$ appear. The boundary functions $\boldsymbol{\psi}^f_{nm}(t)$, $\boldsymbol{\psi}^g_{nm}(t)$, and $\boldsymbol{\psi}^h_{nm}(t)$, can be computed concurrently with the numerical solution $\boldsymbol{u}$ inside Ω. Since the ψ's satisfy ordinary differential equations, the boundary condition is local in time. It does not require saving past values of $\boldsymbol{u}$.

Finally, we combine (3.1) and (3.2) into a single exact nonreflecting boundary condition for $\boldsymbol{u}$ at $\mathcal{B}$:

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial r} + \frac{\boldsymbol{u}}{R} + \frac{1}{c_s}\frac{\partial \boldsymbol{u}^{\tan}}{\partial t} + \frac{\hat{\boldsymbol{r}}}{c_p}\frac{\partial(\hat{\boldsymbol{r}} \cdot \boldsymbol{u})}{\partial t} \\ &+ \left(\frac{c_p}{c_s} - 1\right)(\hat{\boldsymbol{r}} \times \nabla \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{u})\hat{\boldsymbol{r}})) + \left(1 - \frac{c_s}{c_p}\right)\nabla \cdot \boldsymbol{u}^{tan}\hat{\boldsymbol{r}} \\ \text{(3.7)} \qquad &= -\frac{1}{R}\sum_{n\geq 1}\sum_{|m|\leq n} \boldsymbol{d}_n \cdot \boldsymbol{\psi}^f_{nm}(t)\boldsymbol{V}_{nm} - \frac{1}{R}\sum_{n\geq 1}\sum_{|m|\leq n} [\boldsymbol{p}_n \cdot \boldsymbol{\psi}^h_{nm}(t) + \boldsymbol{q}_n \cdot \boldsymbol{\psi}^g_{nm}(t)]\boldsymbol{U}_{nm} \\ &+\frac{1}{R}\sum_{n\geq 0}\sum_{|m|\leq n} [\boldsymbol{a}_n \cdot \boldsymbol{\psi}^h_{nm}(t) + \boldsymbol{b}_n \cdot \boldsymbol{\psi}^g_{nm}(t)]Y_{nm}\hat{\boldsymbol{r}}, \qquad r = R. \end{aligned}$$

The boundary condition (3.7) relates normal derivatives of normal and tangential components of $\boldsymbol{u}$ to tangential and time derivatives of $\boldsymbol{u}$. In fact radial derivatives of the displacement $\boldsymbol{u}$ appear only in the first term in (3.7), namely $\partial_r \boldsymbol{u}$. All the other terms involve only tangential and time derivatives of $\boldsymbol{u}$.

For numerical computation the sums over n on the right of (3.7) need to be truncated at some finite value N. Then most of the work involved in applying the boundary condition results from computing the inner products over $\mathcal{B}$ in (3.4)–(3.6) and the right-hand sides of (3.7). To compute the inner products it is not necessary to compute $O(N^2)$ inner products over the entire sphere. Since the spherical harmonics Y_{nm} separate in θ and ϕ, it is sufficient to compute $O(N)$ inner products in ϕ, and then to compute $O(N^2)$ *one-dimensional* inner products in θ over $[0, \pi]$. The same method can be used to calculate the sums over n and m on the right of (3.7).

It remains to show that the displacement field $\boldsymbol{u}$ in Ω with the boundary condition (3.7) imposed at $\mathcal{B}$ coincides with the restriction to Ω of the solution in the unbounded domain. To be specific, we consider the following model problem:

Find $\boldsymbol{u}$ in $I\!R^3 \times [0, T)$ with $T > 0$ or $T = \infty$ satisfying

$$\text{(3.8)} \qquad \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c_p^2\,\nabla\nabla \cdot \boldsymbol{u} + c_s^2\,\nabla \times \nabla \times \boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } I\!R^3 \times (0, T),$$

with initial conditions

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), \qquad \frac{\partial}{\partial t}u(\boldsymbol{x},0)=\dot{\boldsymbol{u}}_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in I\!R^3. \tag{3.9}$$

Here $c_p = c_p(\boldsymbol{x})$, $c_s = c_s(\boldsymbol{x})$, and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{u}, \nabla \boldsymbol{u})$ may be nonlinear in Ω. However, outside $\mathcal{B}$, c_s and c_p are constant while $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0$, and $\boldsymbol{f}$ are identically zero. Hence in $\mathcal{B}^{ext}$ the scattered field $\boldsymbol{u}(\boldsymbol{x}, t)$ satisfies (2.1) with initial conditions (2.2), and $\boldsymbol{u}$ is continuous across $\mathcal{B}$. We shall now prove the following theorem:

Theorem 1 *Suppose that the initial value problem (3.8) and (3.9) has a unique smooth solution. Then so does the initial boundary value problem (3.8) and (3.9) in $\Omega \times [0, T)$ with (3.7) imposed on $\mathcal{B}$. The two solutions coincide in Ω.*

Proof

The existence of the solution to (3.8) and (3.9) in Ω with (3.7) imposed on $\mathcal{B}$ is immediate, since the restriction to Ω of the solution in the unbounded domain satisfies (3.8), (3.9), and (3.7) by construction. To prove uniqueness, we let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be two solutions of (3.8) and (3.9) in Ω with (3.7) imposed on $\mathcal{B}$. Next, let $\boldsymbol{v}_i$, $i = 1, 2$, be the unique solutions of (2.1)–(2.3) with the boundary conditions

$$\boldsymbol{v}_i(\boldsymbol{x},t)=\boldsymbol{u}_i(\boldsymbol{x},t) \qquad \text{on } \|\boldsymbol{x}\|=R, \qquad t>0, \quad i=1,2. \tag{3.10}$$

Let $\boldsymbol{w}_i = \boldsymbol{u}_i$ in Ω and $\boldsymbol{w}_i = \boldsymbol{v}_i$ outside Ω. Then by (3.10), both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are continous across $\mathcal{B}$, so that their time derivatives and tangential derivatives are continous on $\mathcal{B}$. Since $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ satisfy second order partial differential equations, we only need to show that the normal derivative of $\boldsymbol{w}_i$ is continuous across $\mathcal{B}$ to show that $\boldsymbol{w}_i$ is a smooth solution of (3.8) and (3.9) in the entire unbounded domain. In the derivation of the nonreflecting boundary condition (3.7), only the fact that the solution satisfied the homogeneous elastic wave equation (2.1) outside $\mathcal{B}$ with zero initial conditions was used. Thus, $\boldsymbol{v}_i$ satisfies (3.7), with $\boldsymbol{u}_i$ replaced by $\boldsymbol{v}_i$, and $\boldsymbol{\psi}^{f,i}_{nm}$, $\boldsymbol{\psi}^{g,i}_{nm}$, and $\boldsymbol{\psi}^{h,i}_{nm}$ replaced by other functions $\boldsymbol{\eta}^{f,i}_{nm}$, $\boldsymbol{\eta}^{g,i}_{nm}$, and $\boldsymbol{\eta}^{h,i}_{nm}$, respectively. Since $\boldsymbol{u}_i = \boldsymbol{v}_i$ at $r = R$, it follows that $\boldsymbol{\psi}^{f,i}_{nm}$, $\boldsymbol{\psi}^{g,i}_{nm}$, $\boldsymbol{\psi}^{h,i}_{nm}$ and $\boldsymbol{\eta}^{f,i}_{nm}$, $\boldsymbol{\eta}^{g,i}_{nm}$, $\boldsymbol{\eta}^{h,i}_{nm}$ satisfy the same ordinary differential equations (3.4)–(3.6) and initial conditions; therefore they coincide for all time. Moreover, all the terms in (3.7) but $\partial_r \boldsymbol{u}$ involve only tangential and time derivatives of $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$, which coincide. Therefore (3.7) implies that

$$\frac{\partial \boldsymbol{u}_i}{\partial r}\Big|_{r=R^-}=\frac{\partial \boldsymbol{v}_i}{\partial r}\Big|_{r=R^+}, \qquad i=1,2. \tag{3.11}$$

In other words, the normal derivative of $\boldsymbol{w}_i$ is continuous across $r = R$. This implies that both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are smooth solutions of the initial value problem in the infinite domain. By the hypothesis of the theorem, this problem has a unique solution. Therefore $\boldsymbol{w}_1 \equiv \boldsymbol{w}_2$, which completes the proof.

□

4 Conclusion

We have derived (3.7), an exact nonreflecting boundary condition for the time dependent elastic wave equation in three space dimensions, when the artificial boundary is a sphere. Because this boundary condition is exact, the artificial boundary can be brought in as close as desired to the scatterer without loss of accuracy. The condition is local in time. It contains first derivatives of $\boldsymbol{u}$ and inner products with spherical harmonics of the displacement on the artificial boundary. Therefore it can be combined easily with any standard numerical scheme in the interior, such as a finite difference or a finite element method.

The derivation of (3.7) employs the elastic wave equation and the initial conditions outside the spherical boundary $r = R$. It does not make use of any differential equation or initial condition in the interior region $r < R$. Therefore it applies no matter what equations or conditions are imposed upon the solution in the region $r < R$. In particular, it applies to scattering problems with obstacles, inhomogeneities, or nonlinearity confined to the interior region.

Appendix

Lemma 1 *Let $G_n[u](r,t)$ be defined by (2.22). Then*

$$r\,u(r,t) = \sum_{j=0}^{n} \frac{\gamma_{nj}}{r^j} \left(-\frac{\partial}{\partial r}\right)^{n-j} G_n[u](r,t), \tag{4.1}$$

where

$$\gamma_{nj} = \frac{(n+j)!}{(n-j)!j!2^j}, \qquad 0 \le j \le n. \tag{4.2}$$

Proof

See [1], Lemma 7.1. Note that the definition of G_n used in (2.22) differs from that used in [1] by a factor of $(-1)^n$.

□

Lemma 2 *Let $G_n[u](r,t)$ be defined by (2.22). Then*

$$\frac{\partial}{\partial r}(r\,u(r,t)) = -\sum_{j=0}^{n+1} \frac{\beta_{nj}}{r^j} \left(-\frac{\partial}{\partial r}\right)^{n+1-j} G_n[u](r,t), \tag{4.3}$$

where

$$\beta_{nj} = \begin{cases} 1, & \text{if } j = 0 \\ \gamma_{nj} + (j-1)\gamma_{n,j-1}, & \text{if } 1 \le j \le n, \\ n\,\gamma_{nn}, & \text{if } j = n+1. \end{cases} \tag{4.4}$$

Proof

For $n = 0$, (2.22) shows that $G_0[u] = ru$, and the result follows immediately from (4.4), since $\beta_{00} = 1$ and $\beta_{01} = 0$.

For $n \ge 1$, we use Lemma 1 above to obtain

$$\partial_r(ru) = \sum_{j=0}^{n} \frac{\partial}{\partial r} \left\{ \frac{\gamma_{nj}}{r^j}(-\partial_r)^{n-j} G_n[u] \right\}. \tag{4.5}$$

We evaluate (4.5), which yields

$$\begin{aligned} \partial_r(ru) &= -(-\partial_r)^{n+1} G_n[u] - \frac{n\,\gamma_{nn}}{r^{n+1}} G_n[u] \\ &- \sum_{j=1}^{n} \frac{\gamma_{nj} + (j-1)\gamma_{n,j-1}}{r^j} (-\partial_r)^{n+1-j} G_n[u]. \end{aligned} \tag{4.6}$$

The first two terms on the right of (4.6) correspond to the two terms $j = 0$ and $j = n+1$ of the sum in (4.3), whereas the sum in (4.6) correponds to those terms with $1 \le j \le n$. Now the result follows immediately by definition of β_{nj}.

□

A direct consequence of the two Lemmas above is:

Lemma 3 *Let* $G_n[u](r,t)$ *be defined by (2.22),* γ_{nj} *by (4.2),* β_{nj} *(4.4), and for all* $n \geq 0$ *set*

$$\gamma_{n,-1} = 0. \tag{4.7}$$

Then

$$\frac{\partial}{\partial r}(u(r,t)) = -\sum_{j=0}^{n+1} \frac{\beta_{n,j} + \gamma_{n,j-1}}{r^{j+1}} \left(-\frac{\partial}{\partial r}\right)^{n+1-j} G_n[u](r,t), \tag{4.8}$$

Proof

We rewrite $\partial_r u$ as $r^{-1}\partial_r(ru) - r^{-2}(ru)$ and apply Lemma 2 to the first term and Lemma 1 to the second term.

□

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