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Mixed *hp* Finite Element Methods for Stokes and Non-Newtonian Flow

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Abstract

We analyze the stability of hp finite elements for viscous incompressible flow. For the classical velocity-pressure formulation, we give new estimates for the discrete inf-sup constants on geometric meshes which are explicit in the polynomial degree k of the elements. In particular, we obtain new bounds for p-elements on triangles. For the three-field Stokes problem describing linearized non-Newtonian flow, we estimate discrete inf-sup constants explicit in both h and k for various subspace choices (continuous and discontinuous) for the extra-stress. We also give a stability analysis of the hp-version of an EVSS (Elastic-Viscous-Split-Stress) method and present elements that are **stable** and **optimal** in h **and** k. Finally, we present numerical results that show the exponential convergence of the hp version for Stokes flow over unsmooth domains.

Keywords: hp finite elements, mixed method, Stokes flow, non-Newtonian flow, EVSS method

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1 Introduction

Solutions of the boundary value problems of incompressible fluid flow in non-smooth domains Ω exhibit, as is well-known ([19], [21]), singularities at conical boundary points, even if the prescribed data are (piecewise) analytic. This is so for nonlinear problems such as the Navier-Stokes equations in the Newtonian case and, more so, for general non-Newtonian flows and their linearized versions (see, e.g., [20]). The situation is analogous to that for elliptic boundary value problems of potential and elasticity theory.

Our goal in this paper is to address the construction and mathematical validation of suitable *hp* finite element spaces that can be used to resolve the singularities encountered in such problems. The use of high order methods such as the p and hp[1] FEM is already well accepted in elasticity problems, for which it has been shown (see e.g. [15]) that

(1) for smooth solutions, **exponential** convergence is obtained by the pure p version and

(2) for singular solutions (such as those encountered in linear elliptic PDEs over polygonal domains), the exponential convergence can be preserved by the *hp* version over strongly graded, **geometric** meshes.

Our motivation here is the application of these versions to fluid flow problems, in particular the linearized versions of incompressible flows, both Newtonian and non-Newtonian. The p version has already been analyzed for Stokes flow, for instance in the form of the **spectral element method** (see e.g. [2]). Moreover, p and *hp* finite elements have been formulated and analyzed for Stokes flow in e.g. [6, 7, 29], but not in the context of exponential convergence, i.e. not in the context of *hp* spaces over geometric meshes, which we describe in Section 2. The first of our goals here is to extend previous analyses to this case, thereby establishing exponential convergence in the presence of singularities for the Stokes problem. (Note that it is *not* possible to prove or attain exponential convergence in the context of the spectral element method over a fixed mesh [2] when singularities are present.)

We do this in Section 3. The key concern here (not found in elasticity problems) is one of **stability**, since we now have a saddle point problem involving both the velocity and pressure unknowns, and an inf-sup (or Babuška-Brezzi (BB)) condition must be satisfied by the spaces chosen. We establish such an inf-sup condition for geometrical meshes containing **both** parallelograms and triangles. The theoretical treatment of p-stability for triangles is a necessary (and new) result, since previous results (e.g. in [2, 6, 29]) only established the inf-sup condition for the special case of parallelograms, and most *hp* meshes contain triangles as well. See Remark 3.8 in this context.

In Section 4, we extend our results to a three-field Stokes formulation which contains an extra field: the stress tensor $\boldsymbol{\sigma}$. This arises from the linearization of some differential models of non-Newtonian flows (see e.g. [8], [10]). As in [10], we consider the stability of this limiting linear problem as a means of formulating FE space combinations that can be used in the non-linear case as well. The difference is that we are interested in identifying spaces that are stable in terms of **both** h and p. We are motivated here by recent computations that show the superiority of p and *hp* FEM over the classical h FEM for non-Newtonian flows (see e.g. [17, 30, 31, 32]). Our work

[1] By the p version, we mean a FEM where convergence is achieved by increasing the degree p on a fixed mesh. This is in contrast to the classical h version, where p is fixed and the mesh is refined. The *hp* version combines both strategies.

here is a first step towards choosing such *hp* space combinations on the basis of mathematical principles. Using our analysis, we once again establish exponential convergence for the linearized version of such problems.

Our results in Section 4 indicate that to ensure stability of the limiting three-field Stokes problem, we may need spaces for $\boldsymbol{\sigma}$ that have higher polynomial degree than those for the velocity when continuous stresses are desired. This is consistent with computational observations (see e.g. [30]). In Section 4.5, we show that if *discontinuous* stresses are used, then the same (or even, for triangles, lower-order) spaces can be used for the stresses as for the velocities. In particular, we present the $\mathcal{Q}^{k+1} - \mathcal{Q}^{k+1} - \mathcal{P}^k$ combination over parallelograms that is **fully stable both** in terms of h **and** p.

If, however, *continuous* stresses are required, then a variant of the EVSS (**E**lastic **V**iscous **S**plit **S**tress) methods ([11], [12]) again allows the same (or lower) order space to be used for the stress as for the velocity. This is considered in Section 4.6, where we again establish exponential convergence for the *hp* method, and show that the $\mathcal{Q}^{k'} - \mathcal{Q}^{k+1} - \mathcal{P}^k$ combination over parallelograms is **stable** and **optimal** in **both** h **and** p.

The exponential convergence proven here agrees well with numerical results for problems such as the 4-to-1 contraction problem and the driven cavity problem. In Section 5, we present one such test, for Stokes flow over an unsmooth domain, where the *hp* exponential convergence is clearly observed. Let us remark that although our results have only been established for linearized problems, computational results in Newtonian and non-Newtonian flow indicate that the *hp* version again results in exponential convergence for the singularities in these more general cases.

Our conclusions are summarized in Section 6.

2 Domains and meshes

Throughout, $\Omega \subset \mathbb{R}^2$ will denote a bounded polygon with M vertices A_i, $i = 1, \ldots, M$, and straight, open sides Γ_i, $i = 1, \ldots, M$ (see Figure 1). For $i \geq M$, we will use $\Gamma_i = \Gamma_{i \bmod M}$, $A_i = A_{i \bmod M}$

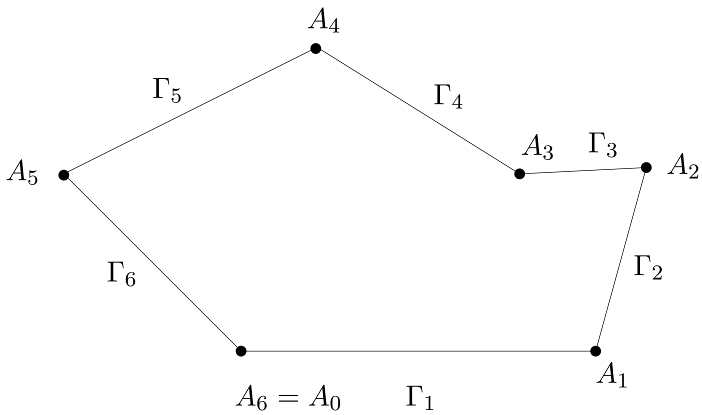

Figure 1: The polygon Ω and the notation $\Gamma_i = \Gamma_{i \bmod M}$, $A_i = A_{i \bmod M}$

In Ω, we define a family of geometric meshes $\{\Omega^{n,\sigma}\}$, where $0 < \sigma < 1$ and $k \in \mathbb{N}$, as follows: we partition Ω into a corner part Ω_1 and an interior part Ω_2 as shown in Figure 2.

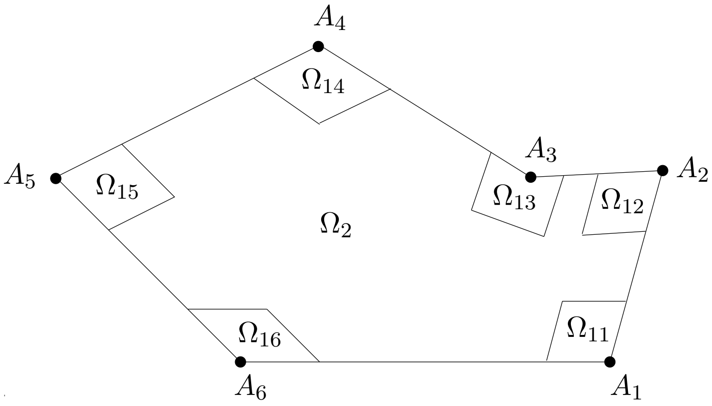

Figure 2: The subdomain Ω_2 and the corner patches Ω_{1i}

The corner part Ω_1 consists of M open corner patches Ω_{1i} which are assumed to be disjoint, i.e.

$$A_i \in \overline{\Omega}_{1i}, \quad A_j \notin \overline{\Omega}_{1i} \quad j \neq i .$$

Each corner patch is now subdivided geometrically into triangular or parallelogram elements K which are affinely equivalent to either the **reference square** $\hat{K} = \hat{Q} = (-1,1)^2$ or the **reference triangle** $\hat{K} = \hat{T} = \{(\hat{x}_1, \hat{x}_2) : |\hat{x}_1| < 1,\, 0 < \hat{x}_2 < \sqrt{3}(1 - |\hat{x}_1|)\}$, i.e.

$$K = F_K(\hat{K}), \quad F_K(\hat{x}) = A_K\, \hat{x} + b_K \ , \tag{2.1}$$

which implies in particular $J_K = \partial F_K / \partial \hat{x} = A_K = \text{constant}$.

A **geometric subdivision** $\Omega_{1j}^{n,\sigma}$ of Ω_{1j} with n layers and grading factor $0 < \sigma < 1$ is a collection of elements K as above that satisfies

i) $\Omega_{1j}^{n,\sigma}$ is shape regular,

ii) $\Omega_{1j}^{n,\sigma}$ is conforming, i.e. for any $K, K' \in \Omega_{1j}^{n,\sigma}$, $K \neq K'$, $\overline{K} \cap \overline{K}'$ is either empty, a vertex or an entire side,

iii) either: $A_j \in \overline{K}$; then $\operatorname{diam} \overline{K} \leq c_1\, \sigma^n$ for some $0 < \sigma < 1$, or:

$$0 < c_2(1-\sigma) \leq \operatorname{dist}(K, A_j)/\operatorname{diam}(K) \leq c_3(1-\sigma) < \infty \tag{2.2}$$

and there exists $K \in \Omega_{1j}^{n,\sigma}$ such that

$$\operatorname{diam}(K) \geq c_4 \operatorname{diam}\,(\Omega_{1j}) \tag{2.3}$$

where $c_\ell, \ell = 1, 2, 3, 4$ are independent of n and σ.

Note that (2.2) and (2.3) can be satisfied by constructing $\Omega_{1j}^{n,\sigma}$ by recursive subdivision (Figure 3).

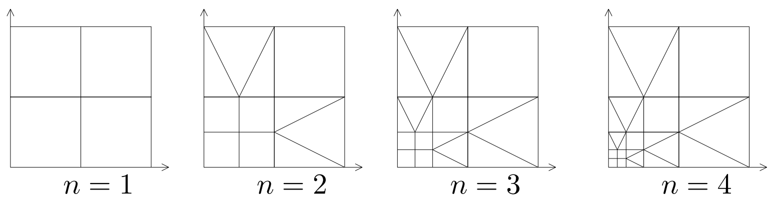

Figure 3: Sequence $\{\Omega_{1j}^{\sigma,n}\}_{n=1}^{4}$ for $\sigma = 1/2$ on corner patch $\Omega_{1j} = (0,1)^2$ with vertex $A_j = (0,0)$.

The remaining interior domain Ω_2 in Figure 2 is triangulated with a **fixed**, quasi-uniform partition Ω_2^σ consisting of triangles and/or parallelograms such that $\Omega_2^\sigma \cup \Omega_{11}^{n,\sigma} \cup \Omega_{12}^{n,\sigma} \cup \cdots \cup \Omega_{1M}^{n,\sigma}$ is a conforming shape regular partition of Ω for every n. Note that Ω_2^σ depends on σ but is independent of n. The resulting geometric mesh on Ω will be referred to as $\Omega^{n,\sigma}$.

Remark 2.1. We confine ourselves here for brevity to straight polygons. All results carry over, however, to polygons with smoothly curved sides, if the elements are mapped by blending maps and the pressure spaces are redefined using the Piola-transform (see [5], [6] for more).

We will use $H^k(S)$ to denote the Sobolev space of functions with k generalized derivatives on a domain $S \subset \mathbb{R}^n$, with $L^2(S) = H^0(S)$. The norm will be denoted by $\|\cdot\|_{H^k(S)}$, or simply $\|\cdot\|_k$ if $S = \Omega$. We also define $H_0^1(S) = \{u \in H^1(S),\ u = 0 \text{ on } \partial S\}$ and $L_0^2(S) = \{u \in L^2(S), \int_S u\,dx = 0\}$. Finally, $[H]_{\mathrm{sym}}^{2\times 2}$ denotes the space of all symmetric 2×2 tensors with each component in H.

3 Stokes Problem (Classical Formulation)

3.1 Variational Formulation and FE discretization

The variational form of the Stokes problem in Ω reads:

Find $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\boldsymbol{v}) \quad \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \\
b(\boldsymbol{u},q) &= 0 \qquad\quad q \in L_0^2(\Omega)\ .
\end{aligned} \tag{3.1}
$$

Here

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &= \nu(\mathbf{grad}\,\boldsymbol{u},\ \mathbf{grad}\,\boldsymbol{v})\ , \\
b(\boldsymbol{v},p) &= -(p,\ \mathrm{div}\,\boldsymbol{v})\ .
\end{aligned}
$$

As is well-known (see, e.g., [13]), we have the continuous **inf-sup condition**

$$
\forall p \in L_0^2(\Omega): \sup_{0 \neq v \in \mathbf{H}_0^1(\Omega)} \frac{b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_1} \geq \delta\,\|p\|_0 \tag{3.2}
$$

with some $\delta > 0$. The condition (3.2) implies, together with

$$a(\boldsymbol{u},\boldsymbol{u}) \geq C\,\|\boldsymbol{u}\|_1^2 \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)\ , \tag{3.3}$$

that (3.1) admits, for every $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, a unique solution $(\boldsymbol{u},p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$.

The FE-discretization of (3.1) proceeds in the usual fashion: given families $\{\boldsymbol{V}_N\} \subset \boldsymbol{H}_0^1(\Omega)$ and $\{M_N\} \subset L_0^2(\Omega)$ of finite-dimensional subspaces with dimensions proportional to N, find $(\boldsymbol{u}_N, p_N) \in \boldsymbol{V}_N \times M_N$ such that

$$\begin{aligned} a(\boldsymbol{u}_N,\boldsymbol{v}) + b(\boldsymbol{v},p_N) &= (f,\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_N\ , \\ b(\boldsymbol{u}_N,q) &= 0 \qquad\quad \forall q \in M_N\ . \end{aligned} \tag{3.4}$$

If the **discrete inf-sup condition**

$$\forall q \in M_N :\ \sup_{0\neq v\in \mathbf{V}_N} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_1} \geq \delta(N)\,\|q\|_0 \tag{3.5}$$

holds for some $\delta(N) > 0$, the problem (3.4) admits a unique solution $(\boldsymbol{u}_N, p_N)$ which satisfies the **a priori error estimate** (see, e.g. [13], [27])

$$\|\boldsymbol{u}-\boldsymbol{u}_N\|_1 \leq \frac{C_1}{\delta(N)} \inf_{\boldsymbol{v}\in\mathbf{V}_N} \|\boldsymbol{u}-\boldsymbol{v}\|_1 + C_2 \inf_{q\in M_N} \|p-q\|_0\ . \tag{3.6}$$

$$\|p-p_N\|_0 \leq \frac{C_1}{(\delta(N))^2} \inf_{\boldsymbol{v}\in\mathbf{V}_N} \|\boldsymbol{u}-\boldsymbol{v}\|_1 + \frac{C_2}{\delta(N)} \inf_{q\in M_N} \|p-q\|_0\ . \tag{3.7}$$

We thus see the two basic ingredients of the convergence analysis: **stability** (via the constant $\delta(N)$) and **consistency**, which is governed by the approximability of the exact solution $(\boldsymbol{u},p)$ from $\{\boldsymbol{V}_N \times M_N\}$. To quantify the latter, we must specify the regularity of $(\boldsymbol{u},p)$ and the design of the FE spaces $(\boldsymbol{V}_N, M_N)$. We next describe the spaces $\boldsymbol{V}_N, M_N$.

3.2 The *hp* FE spaces $S_0^{\mathbf{k},\ell}(\Omega^{n,\sigma})$

The spaces $\boldsymbol{V}_N, M_N$ will be, as usual, spaces of piecewise polynomials on the geometric meshes $\Omega^{n,\sigma}$ defined in Section 2. The finite element approximations $\boldsymbol{u}_N, p_N$ will be piecewise polynomials on $\Omega^{n,\sigma}$. We associate with each $K \in \Omega^{n,\sigma}$ a polynomial degree $k_K \geq 2$ and combine all degrees k_K in the **degree vector**

$$\boldsymbol{k} = \{k_K : K \in \Omega^{n,\sigma}\},\ \ |\boldsymbol{k}| = \max_{K\in\Omega^{n,\sigma}} \{k_K\}\ . \tag{3.8}$$

Then

$$\boldsymbol{V}_N = [S_0^{\mathbf{k},1}(\Omega^{n,\sigma})]^2 := \{v \in H_0^1(\Omega) : v|_K \circ F_K \in V^{k_K}(\hat{K}),\ \ K \in \Omega^{n,\sigma}\}^2\ , \tag{3.9}$$

$$M_N = S_0^{\mathbf{k},0}(\Omega^{n,\sigma}) := \{q \in L_0^2(\Omega)\ : q|_K \circ F_K \in W^{k_K}(\hat{K}),\ \ K \in \Omega^{n,\sigma}\}\ . \tag{3.10}$$

Here, $V^k(\hat{K})$, $W^k(\hat{K})$ are spaces defined on the reference element in terms of $\mathcal{Q}^k = \mathrm{span}\{\hat{x}_1^{\alpha_1}\,\hat{x}_2^{\alpha_2} : 0 \le \alpha_1, \alpha_2 \le k\}$ and $\mathcal{P}^k = \mathrm{span}\{\hat{x}_1^{\alpha_1}\,\hat{x}_2^{\alpha_2} \colon 0 \le \alpha_1, \alpha_2,\ \alpha_1 + \alpha_2 \le k\}$, the sets of polynomials of separate, respectively, total degree k, as follows.

If $\hat{K} = \hat{T}$, we choose

$$V^k(\hat{T}) = \mathcal{P}^{k+1}(\hat{T}), \quad W^k(\hat{T}) = \mathcal{P}^{k-1}(\hat{T}),\ k \ge 2\ . \tag{3.11}$$

An alternate choice, discussed in [6], is to take $W^k(\hat{T})$ as above, with

$$V^k(\hat{T}) = \mathcal{P}^k(\hat{T}) \cup (\mathcal{P}^{k+1}(\hat{T}) \cap H_0^1(\hat{T}))\ . \tag{3.12}$$

This is an optimized version of (3.11), and leads to the same stability estimates.

For quadrilateral elements ($\hat{K} = \hat{Q}$), there are several choices; we present three of them and refer to [29] for alternative ones. The analog of (3.11) is to choose

$$V^k(\hat{Q}) = \mathcal{Q}^{k+1}(\hat{Q}),\ W^k(\hat{Q}) = \mathcal{Q}^{k-1}(\hat{Q})\ . \tag{3.13}$$

This is the choice used most often in spectral element methods [2], but, as shown in [7], [29], it is sub-optimal with respect to h convergence. An optimal choice (in terms of h) is to choose, instead ([7], [29])

$$V^k(\hat{Q}) = \mathcal{Q}^{k+1}(\hat{Q}),\ W^k(\hat{Q}) = \mathcal{P}^k(\hat{Q})\ . \tag{3.14}$$

(Recently [3] this choice has been shown to be completely stable with respect to k as well.)

Another alternative is to keep W^k as in (3.13), but minimize V^k. For this, we introduce the **space $\mathcal{E}^k(\hat{Q})$ of external degrees of freedom on $\hat{Q}$**:

$$\mathcal{E}^k(\hat{Q}) = \mathcal{P}^1(\hat{I}_1) \otimes \mathcal{P}^k(\hat{I}_2) \cup \mathcal{P}^k(\hat{I}_1) \otimes \mathcal{P}^1(\hat{I}_2)\ , \tag{3.15}$$

where $\hat{I}_i = \{\hat{x}_i : |\hat{x}_i| < 1\}$ is the unit interval and $\mathcal{P}^k(\hat{I}_i)$ denotes the space of polynomials of degree $\le k$ on $\hat{I}_i$. The **space $\mathcal{J}^k(\hat{Q})$ of internal degrees of freedom on $\hat{Q}$** is

$$\mathcal{J}^k(\hat{Q}) = \{b_{\hat{Q}} v : v \in Q^{k-2}(\hat{Q})\} \tag{3.16}$$

where $b_{\hat{Q}}$ is the basic bubble function on $\hat{Q}$, i.e.

$$b_{\hat{Q}}(\hat{x}_1, \hat{x}_2) = (1 - \hat{x}_1^2)(1 - \hat{x}_2^2)\ . \tag{3.17}$$

With these notations we define

$$V^k(\hat{Q}) = \mathcal{E}^k(\hat{Q}) \oplus \mathcal{J}^{k+1}(\hat{Q}), \quad W^k(\hat{Q}) = \mathcal{Q}^{k-1}(\hat{Q})\ . \tag{3.18}$$

For a comparison of these choices, see [6, 7].

For $V^k(\hat{K})$ let us define $V_0^k(\hat{K}) := V^k(\hat{K}) \cap H_0^1(\hat{K})$. Then we have

$$V_0^k(\hat{K}) = \{v \in V^k(\hat{K}) : v = b_{\hat{K}} w,\ w \in X^k(\hat{K})\} \tag{3.19}$$

where

$$X^k(\hat{K}) := \begin{cases} \mathcal{P}^{k-2}(\hat{T}) & \text{if} \quad \hat{K} = \hat{T}\ , \\ \mathcal{Q}^{k-1}(\hat{Q}) & \text{if} \quad \hat{K} = \hat{Q}\ . \end{cases} \tag{3.20}$$

Here, the bubble function $b_{\hat{T}}$ on $\hat{T}$ is defined by

$$b_{\hat{T}}(\hat{x}_1, \hat{x}_2) = \hat{x}_2(1 - \hat{x}_1 - \hat{x}_2/\sqrt{3})(1 + \hat{x}_1 - \hat{x}_2/\sqrt{3})\ . \tag{3.21}$$

We observe that our selections are such that

$$\nabla q \in [X^k(\hat{K})]^2 \quad \forall q \in W^k(\hat{K})\ . \tag{3.22}$$

The spaces $\boldsymbol{V}_N, M_N$ have been defined mainly with the discrete inf-sup condition (3.5) in mind, as we will see in the next paragraph. However, they also have decisive advantages in terms of approximability.

3.3 Approximation Properties of $S^{\mathbf{k},\ell}(\Omega^{n,\sigma})$

To state the approximation properties of $S^{\mathbf{k},\ell}(\Omega^{n,\sigma})$, we assume that the right hand side f in (3.1) is analytic in Ω. Then it follows from standard elliptic regularity that $\boldsymbol{u}$ and p are also analytic in Ω and even on $\overline{\Omega} \backslash \bigcup_{i=1}^M A_i$. There are, however, corner singularities at the vertices A_i. It was shown by Babuška and Guo for closely related elasticity and potential problems in polygonal domains that in this case the solutions belong to **countably normed spaces** $\mathcal{B}_\beta^\ell(\Omega)$ (see [15], [16], [1] for the definition and properties of these spaces). We make the following analogous **regularity hypothesis** for the Stokes problem (3.1):

$$\boldsymbol{u} \in \mathcal{B}_\beta^2(\Omega), \quad p \in \mathcal{B}_\beta^1(\Omega) \ \text{ for some } 0 < \beta < 1 \tag{3.23}$$

(the values of β depends on the corner angles of Ω).

We also assume the following about the spaces $S^{\mathbf{k},\ell}(\Omega^{n,\sigma})$. **Either** the polynomial degree is constant, $k_K = k$ for all $K \in \Omega^{n,\sigma}$, and proportional to n, the number of layer refinements in the geometric meshes $\Omega^{n,\sigma}$, **or** the polynomial degrees p_K are equal to 1 in the elements with A_j as a vertex and increase linearly away from $A_j : k_K = [\mu(\ell + 1)]$ where ℓ denotes the number of elements between $K \subset \Omega_{1j}$ and A_j. The (sufficiently large) constant $\mu(\sigma) > 0$ is the **slope of the degree vector** k and the degrees k_K for $K \subset \Omega_2$ in Figure 2 are equal to $\lfloor \mu(n + 1) \rfloor$.

Under these assumptions, the spaces $\boldsymbol{V}_N, M_N$ defined above have the following approximation properties (see [15], [27] for details).

Theorem 3.1. *Assume the regularity (3.23) and the choice of $\boldsymbol{k}$ described above. Then, for V_N, M_N defined as in (3.9), (3.10), there hold the approximability estimates*

$$\inf_{\boldsymbol{v} \in \mathbf{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_1 \leq C \exp(-bN^{1/3})\ , \tag{3.24}$$

$$\inf_{q \in M_N} \|p - q\|_0 \leq C \exp(-bN^{1/3})\ , \tag{3.25}$$

where the constants C and b are positive and independent of N.

Combining the convergence rates (3.24), (3.25) with (3.6), (3.7), we deduce that the mixed hp-FEM (3.4) converges **exponentially** whenever the inf-sup constant $\delta(N)$ in (3.5) approaches zero at most **algebraically**, i.e. whenever

$$\delta(N) \geq CN^{-\lambda}, \quad C, \lambda \text{ independent of } N \,. \tag{3.26}$$

Of course, if $\lambda > 0$, this means that strictly speaking the method is unstable. However, for methods with $\lambda > 0$ (e.g. using $[\mathcal{Q}^{k+1}]^2 - \mathcal{Q}^{k-1}$), the exponential convergence rates (3.24), (3.25) usually make this instability difficult to detect. Nevertheless, for large values of λ in (3.26), numerical difficulties may result as well, from ill-conditioning of the discrete problem. It is therefore essential that we obtain (3.26) with $\lambda = 0$ or with moderate $\lambda > 0$.

3.4 Divergence Stability of $[S_0^{k,1}(\Omega^{n,\sigma})]^2 - S_0^{k,0}(\Omega^{n,\sigma})$

Here we prove our main result regarding divergence-stability of the pairs of spaces $[S_0^{k,1}(\Omega^{n,\sigma})]^2 - S_0^{k,0}(\Omega^{n,\sigma})$ for any of the choices (3.11), (3.12), (3.13), (3.14), (3.18).

Theorem 3.2. *The pairs $\{\boldsymbol{V}_N, M_N\}$ of spaces, defined by (3.9) - (3.10) satisfy the inf-sup condition (3.5) with*

$$\delta(N) = \delta(|\boldsymbol{k}|) \geq C\,|\boldsymbol{k}|^{-3} \geq CN^{-1} \tag{3.27}$$

where $C > 0$ is independent of N, but may depend on σ.

Theorem 3.2 shows that the global inf-sup constant does not depend on the mesh, but only the choice of polynomial degree, for all the elements considered, even when geometric meshes are used. As shown in Theorem 3.3 of [29], the global stability follows whenever

(A1) the global spaces contain a combination of low-order subspaces that is h-stable for Stokes problem (e.g. $[\mathcal{Q}^2]^2 - \mathcal{Q}^0$ for parallelograms, $[\mathcal{P}^2]^2 - \mathcal{P}^0$ for triangles) and

(A2) the reference combination $(V^k(\hat{K}), W^k(\hat{K}))$ satisfies a **local** inf-sup condition:

$$\forall q \in W^k(\hat{K}) : \sup_{0 \neq \mathbf{v} \in \mathbf{V}^k(\hat{K})} \frac{\int_{\hat{K}} q \operatorname{div} \boldsymbol{v}\, dx}{\|\boldsymbol{v}\|_{H^1(\hat{K})}} \geq \delta(k)\, \|q\|_{L^2(\hat{K})} \,. \tag{3.28}$$

For the case of parallelograms, we have the following

Theorem 3.3. *If $\hat{K} = \hat{\mathcal{Q}}$ is a square, then (3.28) holds with $\delta(k) = C$ for the choice (3.14) and $\delta(k) = Ck^{-\frac{1}{2}}$ for the choices (3.13), (3.18).*

The proof for the $O(k^{-\frac{1}{2}})$ estimate may be found in [29], this was recently improved to $O(1)$ for the choice (3.14) in [3].

For triangles, we will prove the following.

Theorem 3.4. *If $\hat{K} = \hat{T}$ is a triangle, then (3.28) holds with $\delta(k) = Ck^{-3}$ for the choices (3.11), (3.12).*

Since with Theorem 3.3 and 3.4, our spaces $\boldsymbol{V}_N, M_N$ satisfy the above requirements (A1) and (A2), Theorem 3.2 follows by using the argument of Theorem 3.3 of [29].

It remains to establish Theorem 3.4. We do this in the same framework that the stability result for parallelograms was established in [29]. Accordingly, we define the projection $\Pi_k : H_0^1(\hat{K}) \to V_0^k(\hat{K})$ (for k large enough)

$$\int_{\hat{K}} (v - \Pi_k v) w \, d\xi = 0 \quad \forall w \in X^k(\hat{K}) \,. \tag{3.29}$$

Here $X^k(\hat{K})$ is defined in (3.19) and (3.20). We see that Π_k is a weighted $L^2(\hat{K})$-projection. Then, as shown in [29], the local condition (3.28) follows provided we can show

$$\|\Pi_k u\|_{H^1(\hat{K})} \leq \hat{C} \, (\delta(k))^{-1} \, \|u\|_{H^1(\hat{K})} \quad \forall u \in H_0^1(\hat{K}) \,. \tag{3.30}$$

We need a series of lemmas to prove (3.29) for triangles with $\delta(k) = Ck^{-3}$.

Lemma 3.5. *Let $\hat{T}$ be the reference triangle and let $b_{\hat{T}} \in \mathcal{P}^3(\hat{T})$ denote the basic bubble function (3.24) on $\hat{T}$. Assume further that for some θ,*

$$\Lambda(k) := \inf_{v \in \mathcal{P}^k(\hat{T})} \frac{\int_{\hat{T}} b_{\hat{T}}(\xi)(v(\xi))^2 \, d\xi}{\left(\int_{\hat{T}} (v(\xi))^2 \, d\xi\right)^{\frac{1}{2}} \left(\int_{\hat{T}} (b_{\hat{T}}(\xi))^2 (v(\xi))^2 \, d\xi\right)^{\frac{1}{2}}} \geq C \, k^{1-\theta} \tag{3.31}$$

with $C > 0$ independent of k. Then the projection Π_k in (3.29) satisfies

$$\|\Pi_k v\|_{H^1(\hat{T})} \leq C k^{\max\{2,\theta\}} \, \|v\|_{H^1(\hat{T})} \quad \forall v \in H_0^1(\hat{T}) \,.$$

Proof: Recall that for any $v \in H_0^1(\hat{T})$, $\Pi_k v$ is defined by $v_k = \Pi_k v \in V_0^k(\hat{T}) = \mathcal{P}^{k+1}(\hat{T}) \cap H_0^1(\hat{T})$ such that

$$\int_{\hat{T}} v_k w \, d\xi = \int_{\hat{T}} v w \, d\xi \quad \forall w \in \mathcal{P}^{k-2}(\hat{T}) \,. \tag{3.32}$$

Define $\widetilde{v}_k$ to be the H_0^1-projection of v onto $V_0^k(\hat{T})$, i.e. $\widetilde{v}_k \in V_0^k(\hat{T})$ such that

$$\int_{\hat{T}} \nabla \widetilde{v}_k \cdot \nabla w \, d\xi = \int_{\hat{T}} \nabla v \cdot \nabla w \, d\xi \quad \forall w \in V_0^k(\hat{T}) \,.$$

Then from the approximation property of the p-version FEM we have

$$\|v - \widetilde{v}_k\|_{L^2(\hat{T})} \leq C_\ell \, k^{-\ell} \, \|v\|_{H^\ell(\hat{T})}, \quad \ell \geq 0 \,. \tag{3.33}$$

We observe further that $v_k - \widetilde{v}_k \in V_0^k(\hat{T})$, so that by (3.19)

$$(v_k - \widetilde{v}_k)(\xi) = b_{\hat{T}}(\xi) \, w_k(\xi) \tag{3.34}$$

for some $w_k \in \mathcal{P}^{k-2}(\hat{T})$ whence it follows that

$$\int_{\hat{T}} (v_k - \widetilde{v}_k) \, w \, d\xi = \int_{\hat{T}} b_{\hat{T}} \, w_k \, w \, d\xi \quad \forall w \in \mathcal{P}^{k-2}(\hat{T}) \,. \tag{3.35}$$

Selecting $w = w_k$ in (3.35) we get with (3.32) that

$$\begin{aligned} \int\limits_{\hat{T}} b_{\hat{T}}(w_k)^2 d\xi = \int\limits_{\hat{T}} (v_k - \widetilde{v}_k)\, w_k\, d\xi \quad &= \quad \int\limits_{\hat{T}} (v - \widetilde{v}_k)\, w_k\, d\xi \\ &\leq \quad \|v - \widetilde{v}_k\|_{L^2(\hat{T})} \, \|w_k\|_{L^2(\hat{T})} \,. \end{aligned} \tag{3.36}$$

Hence it follows that

$$\frac{\int_{\hat{T}} b_{\hat{T}}(w_k)^2 d\xi}{\left(\int_{\hat{T}} (w_k)^2 d\xi \right)^{\frac{1}{2}}} \leq \|v - \widetilde{v}_k\|_{L^2(\hat{T})} \,.$$

By (3.31) (with $v = w_k$) we get from this

$$C \cdot \Big(\int\limits_{\hat{T}} (b_{\hat{T}})^2 (w_k)^2 \; d\xi \Big)^{1/2} \; k^{1-\theta} \leq \|v - \widetilde{v}_k\|_{L^2(\hat{T})} \,,$$

i.e. by (3.34)

$$\|v_k - \widetilde{v}_k\|_{L^2(\hat{T})} \leq C\, k^{\theta - 1} \, \|v - \widetilde{v}_k\|_{L^2(\hat{T})} \,. \tag{3.37}$$

The triangle inequality with (3.37) gives

$$\|v - v_k\|_{L^2(\hat{T})} \leq \|v - \widetilde{v}_k\|_{L^2(\hat{T})} + \|\widetilde{v}_k - v_k\|_{L^2(\hat{T})} \leq C\,(1 + k^{\theta-1})\, \|v - \widetilde{v}_k\|_{L^2(\hat{T})} \,.$$

Hence, by (3.33) with $\ell = 1$, we get that $v_k = \Pi_k v$ satisfies

$$\|v_k\|_{L^2(\hat{T})} \leq \|v\|_{L^2(\hat{T})} + \|v - v_k\|_{L^2(\hat{T})} \leq \|v\|_{L^2(\hat{T})} + C\,(1 + k^{\theta-1})\, k^{-1} \, \|v\|_{H^1(\hat{T})} \,.$$

Using the inverse inequality

$$\|v_k\|_{H^1(\hat{T})} \leq C\, k^2 \, \|v_k\|_{L^2(\hat{T})} \,,$$

we obtain the assertion. □

To prove Theorem 3.4, it remains therefore to prove the bound (3.31) with $\theta = 3$. This is the purpose of the following two lemmas.

Lemma 3.6. *For any $\alpha > 0$, there exists $C = C(\alpha) > 0$ such that*

$$\int\limits_{-1}^{1} (1 - x^2)^{\alpha} \, (\pi(x))^2 dx \geq C\, k^{-2\alpha} \int\limits_{-1}^{1} (\pi(x))^2 \, dx \tag{3.38}$$

for every $\pi \in \mathcal{P}^k(-1,1)$.

The proof of this lemma can be found, for example, in [23].

Lemma 3.7. *The estimate (3.31) holds with $\theta = 3$.*

Proof: We begin with the observation that with $\Lambda(k)$ as in (3.31), we have for every $v_k \in \mathcal{P}^k(\hat{T})$

$$1 \geq (\Lambda(k))^2 \geq \frac{1}{\|b_{\hat{T}}\|_{L^\infty(\hat{T})}} \, \frac{\displaystyle\int_{\hat{T}} b_{\hat{T}}\, v_k^2\, d\hat{x}}{\displaystyle\int_{\hat{T}} v_k^2\, d\hat{x}} \, .$$

Therefore it is sufficient to prove

$$\int_{\hat{T}} b_{\hat{T}}\, v_k^2\, d\hat{x} \geq C k^{-4} \int_{\hat{T}} v_k^2\, d\hat{x} \quad \forall v_k \in \mathcal{P}^k(\hat{T}) \, . \tag{3.39}$$

Let $0 < a \leq 1/\sqrt{3}$. We decompose $\hat{T}$ into 6 overlapping parallelograms $\hat{S}_i$, $i = 1, \dots 6$ and one triangle $\hat{T}_0$ as shown in Figure 4.

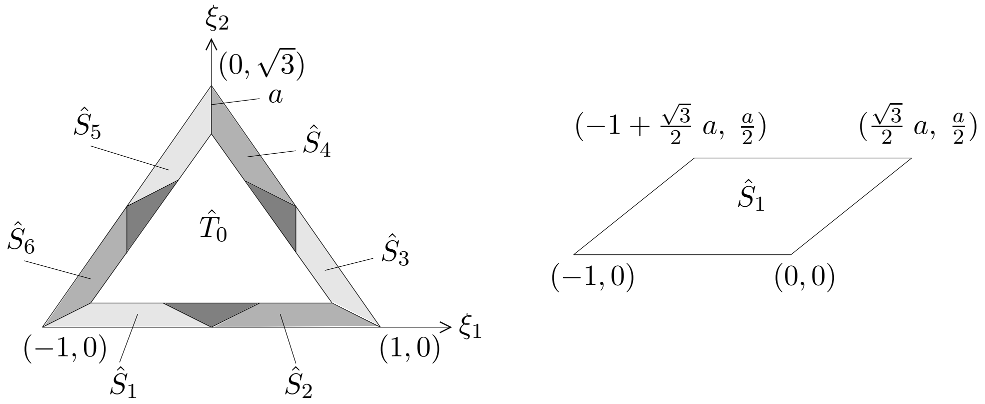

Figure 4: Reference triangle $\hat{T}$ and the notation

Consider the parallelogram $\hat{S}_1$. We map it into the standard square $\hat{S} = (-1, 1)^2$ by the linear map

$$\eta_1 = 2\xi_1 - 2\sqrt{3}\,\xi_2 + 1, \quad \eta_2 = \frac{4\xi_2}{a} - 1 \, .$$

Under this map, we have $b_{\hat{T}}(\xi) \to \hat{b}(\eta)$, where it may be verified

$$\hat{b}(\eta) \geq \widetilde{C}_1(a)(\eta_2 + 1)^2 + \widetilde{C}_2(a)(\eta_2 + 1)(\eta_1 + 1) \, .$$

with $\widetilde{C}_1(a) > 0$, $\widetilde{C}_2(a) > 0$. Transforming coordinates from (ξ_1, ξ_2) to (η_1, η_2), we therefore obtain, with $\widetilde{v}_k(\eta) = v_k(\xi)$,

$$\begin{aligned}
&\int_{\hat{S}_1} b_{\hat{T}}(\xi)\, v_k^2(\xi)\, d\xi = C(a) \int_{\hat{S}} \hat{b}(\eta)\, \widetilde{v}_k^2(\eta)\, d\eta \\
&\geq C_1(a) \int_{-1}^{1}\int_{-1}^{1} (\eta_2 + 1)^2\, \widetilde{v}_k^2(\eta)\, d\eta + C_2(a) \int_{-1}^{1}\int_{-1}^{1} (\eta_2 + 1)(\eta_1 + 1)\, \widetilde{v}_k^2(\eta)\, d\eta \\
&\geq C(a)\, k^{-4} \int_{-1}^{1}\int_{-1}^{1} \widetilde{v}_k^2(\eta)\, d\eta \quad \text{(using Lemma 3.6 with } \alpha = 2) \\
&\geq C'\, k^{-4} \int_{\hat{S}_1} v_k^2(\xi)\, d\xi \, .
\end{aligned}$$

Similarly, we may establish that

$$\int_{\hat{S}_i} b_{\hat{T}}\, v_k^2(\xi)\, d\xi \geq Ck^{-4} \int_{\hat{S}_i} v_k^2(\xi)\, d\xi \tag{3.40}$$

for $i = 2, \ldots, 6$. Since $b_{\hat{T}} \geq C$ on $\hat{T}_0$, we have also

$$\int_{\hat{T}_0} b_{\hat{T}}\, v_k^2(\xi)\, d\xi \geq C \int_{\hat{T}_0} v_k^2(\xi)\, d\xi \ . \tag{3.41}$$

The lemma now follows from (3.40) and (3.41) and the observation that for any non-negative function $f(\xi)$ on $\hat{T}$,

$$C\Big(\sum_{i=1}^{6} \int_{\hat{S}_i} f(\xi)\, d\xi + \int_{\hat{T}_0} f(\xi)\, d\xi\Big) \leq \int_{\hat{T}} f(\xi)\, d\xi \leq \sum_{i=1}^{6} \int_{\hat{S}_i} f(\xi)\, d\xi + \int_{\hat{T}_0} f(\xi)\, d\xi \ .$$

(We remark that in the above argument, the parameter a is chosen small enough, so that when any triangle T in the triangulation is mapped into $\hat{T}$, the pre-image of each $\hat{S}_i$ is still contained in T. By shape regularity, this is always possible.) □

Lemmas 3.7 and 3.5 imply Theorem 3.4 and hence, also Theorem 3.2. Note that for the geometric meshes $\Omega^{n,\sigma}$, we have $N \sim |k|^3$ which is why $\delta(N) \geq CN^{-1}$. Geometrically refined meshes consisting of triangles such as the mesh in Figure 5 (obtained by subdividing all quadrilaterals $K \in \Omega^{n,\sigma}$ along a diagonal) are admissible in Theorem 3.2.

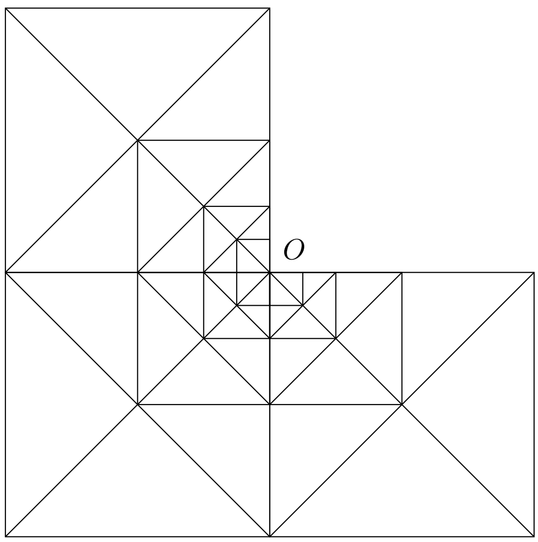

Figure 5: Geometric mesh consisting of triangles in L-shaped domain

Remark 3.8. The stability results for triangles established in this section are necessary to treat the meshes described in Section 2, as well as in Figure 5. If one is restricted to parallelogram elements (as in [29]), then exponential hp convergence can be obtained if one uses tensor product meshes (in which case there are issues regarding the aspect ratio of the elements) or hanging nodes (in which case the macroelement technique in [28] is applicable, using a Clément operator existing on meshes with hanging nodes [26]). Also, in [6], a new formulation for hp elements on quadrilaterals is analyzed, for which stability results are obtained for curved elements as well. Using these elements, strongly graded meshes can once again be created, and exponential convergence established.

Remark 3.9. (On continuous pressures)

Theorem 3.2 immediately implies a corresponding result for continuous pressures, i.e. for the case of subspaces $\boldsymbol{V}_N = [S_0^{\mathbf{k},1}(\Omega^{n,\sigma})]^2$, $M_N = S^{\mathbf{k},1}(\Omega^{n,\sigma}) \cap L_0^2(\Omega)$, since in this case the pressure space has been reduced. The inf-sup condition (3.5) also holds here, with constant $\delta(N)$ satisfying (3.27). We remark that for the p version, a better stability estimate is available for Taylor-Hood elements in [4]. On quasiuniform meshes, it is shown that the discrete inf-sup constant $\delta(N)$ in (3.5) is bounded as

$$\delta(N) \geq C/(|\boldsymbol{k}|^2 \sqrt{\log |\boldsymbol{k}|}) \ .$$

Theorems 3.1, 3.2 together with (3.6), (3.7) show that

$$\begin{aligned} \|u - u_N\|_1 &\leq CN \exp(-bN^{1/3}) \leq C \exp(-b' N^{1/3}) \\ \|p - p_N\|_0 &\leq CN \exp(-bN^{1/3}) \leq C \exp(-b'' N^{1/3}) \end{aligned}$$

i.e. we get exponential convergence for the Stokes problem.

Remark 3.10. (On the hp version with quasiuniform meshes)

The results in [3] show that (3.14) gives optimal convergence in $\boldsymbol{u}$ and p. For (3.13), (3.18), it is shown in [29] that if the exact solution $(\boldsymbol{u}, p) \in [H^m(\Omega)]^2 \times H^{m-1}(\Omega)$ and the triangulation is quasiuniform with meshwidth h, then

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_1 + \delta(N)\,\|p - p_N\|_0 \leq C(\epsilon)\, h^\ell\, k^{-(m-1)+\epsilon}(\|\boldsymbol{u}\|_m + \|p\|_{m-1}) \tag{3.42}$$

where $\ell = \min\{m-1, k\}$. This shows that the convergence in k is optimal (modulo k^ϵ) in $\boldsymbol{u}$ and shows deterioration of no worse than $O(k^{1/2+\epsilon})$ in p. Equation (3.42) is proved by interpolating the estimate (see [29])

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_1 + \delta(N)\|p - p_N\|_0 \leq C(\|\boldsymbol{u}\|_1 + \|p\|_0) \tag{3.43}$$

with the estimate obtained from (3.6) - (3.7). It will also hold for triangles, with $\delta(N) \approx k^{-3}$.

Remark 3.11. The stability of $O(k^{-3})$ proved here for triangles may be pessimistic. In [6], it is shown computationally that the inf-sup constant behaves like k^{-1} for the range of k commonly used in practice.

4 The Upper Convected Maxwell Model and its Limit

4.1 Motivation

Several problems of non-Newtonian fluid mechanics involve three unknown fields $\boldsymbol{u}, p, \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ (called the extra-stress tensor) arises as an additional unknown due to the nonlinear constitutive laws characterizing such fluids. The example we consider here is the Upper Convected Maxwell Model (UCM), which is one of the simplest (differential) non-Newtonian fluids. This is given by

$$-\operatorname{div}\boldsymbol{\sigma} + \operatorname{grad} p = \boldsymbol{f},\ \operatorname{div}\boldsymbol{u} = 0,\ \frac{\lambda\delta\boldsymbol{\sigma}}{\delta t} = 2\nu\,\boldsymbol{D}(\boldsymbol{u}) \tag{4.1}$$

where $\boldsymbol{D}(\boldsymbol{u})$ denotes the symmetric part of the gradient of $\boldsymbol{u}$, $\delta\boldsymbol{\sigma}/\delta t$ is the upper convected derivative [10] and λ is a parameter called the relaxation time of the viscoelastic material.

FE discretizations of problems like (4.1) must provide subspaces $\boldsymbol{V}_N$, M_N and Σ_N for each of these quantities. The question then arises: how to choose these subspaces so that the resulting discretization satisfies the correct **stability** and **approximation** properties.

For stability, we use the criterion proposed in [10] by Fortin and Pierre, which they used to mathematically validate the 4×4 stress element of Crochet and Marchal [8]. If one allows the relaxation time parameter λ in (4.1) to go to zero, the limit gives the so-called **three-field Stokes problem**

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma} + \operatorname{grad} p &= \boldsymbol{f} && \text{in } \Omega\,, \\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega\,, \\ \boldsymbol{\sigma} &= 2\nu\,\boldsymbol{D}(\boldsymbol{u}) && \text{in } \Omega\,, \\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{4.2}$$

The choice of spaces must be such that they are stable for (4.2), since if they don't work for the limit $\lambda = 0$, they cannot be expected to work for $\lambda > 0$. Another reason to consider this three-field Stokes problem is that linearization of the upper convected derivative in (4.1) also results in a system similar to (4.2), see [11].

Although low-order (h version) elements have been derived and validated with the above criterion, our concern here is with high order p and hp type FEM. These have been recently developed for non-Newtonian flow problems in e.g. [17], [32]. Our goal here is to mathematically validate some combinations of spaces, using the stability (in both h and k) of the limiting problem (4.2) as our criterion. As in [10], a basic building block will be Theorem 3.2, i.e. the inf-sup condition for the classical Stokes problem.

For these new p/hp FEMs, comparison with methods based on lower-order elements have indicated good performance and high accuracy for several benchmark problems [17]. In particular, the so-called "4 to 1 contraction problem" is posed on what in solid mechanics is called an "L-shaped domain". The solution then contains a radial singularity emanating from the reentrant corner the precise nature of which is not yet known for many nonlinear constitutive laws, (see, however [20]). Exponential rates of convergence for the approximation of the singularity can once again be observed if a geometric mesh and high polynomial degree is used ([17], [18]), provided suitable stabilization techniques are used in the nonlinear case. The hp stability analysis we give here (valid also for meshes containing triangles) establishes this exponential convergence, for the limiting three-field Stokes case.

4.2 Mixed Method

The **mixed variational formulation** of (4.2) we consider is given by:

Find $((\boldsymbol{\sigma}, p), \boldsymbol{u}) \in \boldsymbol{\Sigma} \times M \times \boldsymbol{V}$ such that

$$
\begin{aligned}
a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) - b((\boldsymbol{\tau}, q), \boldsymbol{u}) &= 0 \quad \forall (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma} \times M \ , \\
b((\boldsymbol{\sigma}, p), \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} \ .
\end{aligned} \tag{4.3}
$$

Here

$$
\begin{aligned}
a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, \boldsymbol{q})) &:= \frac{1}{2\nu} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx \ , \\
b((\boldsymbol{\tau}, q), \boldsymbol{u}) &:= \int_\Omega (\boldsymbol{\tau} - q\boldsymbol{I}) : \boldsymbol{D}(\boldsymbol{u}) \, dx \ ,
\end{aligned}
$$

and the function spaces are given by

$$
\boldsymbol{V} = [H_0^1(\Omega)]^2, \ \boldsymbol{\Sigma} = [L^2(\Omega)]^{2\times 2}_{\text{sym}} \ , \ M = L_0^2(\Omega) \ . \tag{4.4}
$$

Existence and uniqueness of a weak solution to (4.3) are ensured by the following coercivity properties of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ [10]: Let

$$
Z = \{(\boldsymbol{\sigma}, p) \in \boldsymbol{\Sigma} \times M : \ b((\boldsymbol{\sigma}, p), \ \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\} \ . \tag{4.5}
$$

Then $a(\cdot,\cdot)$ is coercive on Z, i.e. there exists $\alpha > 0$ with

$$
a((\boldsymbol{\sigma}, p), \ (\boldsymbol{\sigma}, p)) \geq \alpha(\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2) \quad \forall (\boldsymbol{\sigma}, p) \in Z \ , \tag{4.6}
$$

and $b(\cdot,\cdot)$ satisfies an inf-sup condition,

$$
\inf_{v \in V} \ \sup_{(\boldsymbol{\sigma}, p) \in \boldsymbol{\Sigma} \times M} \frac{b((\boldsymbol{\sigma}, p), \boldsymbol{v})}{(\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2)^{1/2} \ \|\boldsymbol{v}\|_1} \geq \beta > 0 \ . \tag{4.7}
$$

4.3 General FE discretization

For the FE-discretization of problem (4.2), we choose families $\boldsymbol{\Sigma}_N \times M_N \times \boldsymbol{V}_N$ of subspaces of $\boldsymbol{\Sigma} \times M \times \boldsymbol{V}$ as follows: For $\boldsymbol{V}_N$ and M_N, we choose as for the Stokes problem

$$
\boldsymbol{V}_N = [S_0^{\mathbf{k},1}(\Omega^{n,\sigma})]^2, \quad M_N = S_0^{\mathbf{k},0}(\Omega^{n,\sigma}) \tag{4.8}
$$

where $\Omega^{n,\sigma}$ is a geometric mesh with n layers and grading factor $\sigma \in (0,1)$ in the polygon Ω. For the extra stress approximation, we choose a continuous approximation which is customary in simulations of non-Newtonian flow

$$
\begin{aligned}
\boldsymbol{\Sigma}_N = \Big\{ \boldsymbol{\sigma} \in [C^0(\Omega)]^{2\times 2}_{\text{sym}} : \ & \sigma_{\alpha\beta}|_K \circ F_K \in \mathcal{Q}^{k_K+3}(\hat{K}) \quad \text{if} \ \hat{K} = \hat{Q} \ , \\
& \sigma_{\alpha\beta}|_K \circ F_K \in \mathcal{P}^{k_K+3}(\hat{K}) \quad \text{if} \ \hat{K} = \hat{T} \Big\} \ .
\end{aligned} \tag{4.9}
$$

So, on a single element $K \in \Omega^{n,\sigma}$, we have the combinations shown in Table 1

$\boldsymbol{\sigma}$	p	$\boldsymbol{u}$
$\mathcal{Q}^{k+3}(\hat{Q})$	$\mathcal{P}^k(\hat{Q}) \, / \, \mathcal{Q}^{k-1}(\hat{Q})$	$\mathcal{Q}^{k+1}(\hat{Q})$
$\mathcal{P}^{k+3}(\hat{T})$	$\mathcal{P}^{k-1}(\hat{T})$	$\mathcal{P}^{k+1}(\hat{T})$

Table 1: Element polynomial spaces in the hp-FEM for three-field Stokes problem based on $\mathbf{\Sigma}_N \times M_N \times \boldsymbol{V}_N$ ($k \geq 2$), continuous stresses

We remark that the spaces $\boldsymbol{V}_N$ in Table 1 can be optimized (see (3.12), (3.18)). This in turn leads to corresponding optimizations of $\mathbf{\Sigma}_N$, which will be proper subsets of the spaces shown above, containing fewer "side" shape functions (the stability analysis will be the same). Moreover, continuous pressure spaces could be used as in Taylor-Hood elements for triangles (Remark 3.10), giving better stability estimates and reducing the total degrees of freedom.

With the above selection of the subspaces, the hp-FE discretization of (4.3) reads: Find $((\boldsymbol{\sigma}_N, p_N), \boldsymbol{u}_N) \in \mathbf{\Sigma}_N \times M_N \times \boldsymbol{V}_N$ such that

$$
\begin{aligned}
a((\boldsymbol{\sigma}_N, p_N), (\boldsymbol{\tau}, q)) - b((\boldsymbol{\tau}, q), \boldsymbol{u}_N) &= 0 \quad \forall (\boldsymbol{\tau}, q) \in \mathbf{\Sigma} \times M_N \ , \\
b((\boldsymbol{\sigma}_N, p_N), \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_N \ .
\end{aligned} \tag{4.10}
$$

From the standard theory of FE-discretization of saddle-point problems (see, e.g., [27], Chap. V or [5]), we have the following: Define the discrete kernel

$$
Z_N = \{(\boldsymbol{\sigma}, p) \in \mathbf{\Sigma}_N \times M_N : b((\boldsymbol{\sigma}, p), \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_N\} \tag{4.11}
$$

and assume the discrete stability conditions

$$
a((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p)) \geq \alpha(N)(\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2) \quad \forall (\boldsymbol{\sigma}, p) \in Z_N \ , \tag{4.12}
$$

$$
\forall \boldsymbol{v} \in \boldsymbol{V}_N \quad \sup_{(\boldsymbol{\sigma}, p) \in \mathbf{\Sigma}_N \times M_N} \frac{b((\boldsymbol{\sigma}, p), \boldsymbol{v})}{(\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2)^{1/2}} \geq \beta(N) \, \|\boldsymbol{v}\|_1 \ . \tag{4.13}
$$

Then the FE approximations $(\boldsymbol{\sigma}_N, p_N, \boldsymbol{u}_N)$ in (4.10) exist and satisfy the **a-priori error estimates**

$$
\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 + \|p - p_N\|_0 &\leq \frac{C_1}{\alpha(N)\,\beta(N)} \inf_{\substack{\boldsymbol{\tau} \in \mathbf{\Sigma}_N \\ q \in M_N}} (\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|p - q\|_0) \\
&\quad + \frac{C_2}{\alpha(N)} \inf_{\mathbf{v} \in \mathbf{V}_N} \|\boldsymbol{u} - \mathbf{v}\|_1
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_N\|_1 &\leq \frac{C_1}{(\alpha(N)(\beta(N))^2} \inf_{\substack{\boldsymbol{\tau} \in \mathbf{\Sigma}_N \\ q \in M_N}} (\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|p - q\|_0) \\
&\quad + \frac{C_2}{\alpha(N)\,\beta(N)} \inf_{\mathbf{v} \in \mathbf{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_1 \ .
\end{aligned} \tag{4.15}
$$

As in Section 3, (4.14) and (4.15) again imply an exponential convergence result under suitable regularity assumptions on $\boldsymbol{\sigma}$, p, $\boldsymbol{u}$ and algebraic behaviour of $\alpha(N)$, $\beta(N)$ in (4.12), (4.13).

4.4 Proof of the inf-sup conditions for continuous stresses

Here we establish the inf-sup conditions (4.12), (4.13) for the *hp* FE subspaces $\boldsymbol{V}_N$, M_N and $\boldsymbol{\Sigma}_N$ defined in (4.8), (4.9). It turns out that $\alpha(N)$ in (4.12) can be bounded independently of the choice of $\boldsymbol{\Sigma}_N$ when the pair $(\boldsymbol{V}_N, M_N)$ satisfies (3.5).

Theorem 4.1. *Let $(\boldsymbol{V}_N, M_N)$ satisfy (3.5) with inf-sup constant $\delta(N)$. Then for any $\boldsymbol{\Sigma}_N$, (4.12) holds with*

$$\alpha(N) \geq (\delta(N))^2 .$$

Proof: Let $(\boldsymbol{\sigma}, p) \in Z_N$ be given. Then for any $\boldsymbol{v} \in \boldsymbol{V}_N$,

$$(\boldsymbol{\sigma}, \boldsymbol{D}(\boldsymbol{v})) = (p, \operatorname{div} \boldsymbol{v}) .$$

By (3.5) we have

$$\begin{aligned} \|p\|_0 &\leq C\,(\delta(N))^{-1} \sup_{\mathbf{v}\in\mathbf{V}_N} \frac{(p, \operatorname{div}\boldsymbol{v})}{\|\boldsymbol{v}\|_1} = C\,(\delta(N))^{-1} \sup_{\boldsymbol{v}\in\mathbf{V}_N} \frac{(\boldsymbol{\sigma}, \boldsymbol{D}(\boldsymbol{v}))}{\|\boldsymbol{v}\|_1} \\ &\leq C\,(\delta(N))^{-1}\, \|\boldsymbol{\sigma}\|_0 . \end{aligned}$$

Hence, for any $(\boldsymbol{\sigma}, p) \in Z_N$,

$$a((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p)) = \frac{1}{2\nu}\, \|\boldsymbol{\sigma}\|_0^2 \geq \frac{C}{\nu}\ (\delta(N))^2\ (\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2) .$$

□

Corollary 4.2. *For $\boldsymbol{V}_N$, M_N, $\boldsymbol{\Sigma}_N$ defined as in (4.8), (4.9),*

$$\alpha(N) = \frac{C}{\nu}\, |\boldsymbol{k}|^{-6} \geq \frac{C}{\nu}\, N^{-2} .$$

We turn next to the second inf-sup condition (4.13).

Theorem 4.3. *Let $\boldsymbol{V}_N, M_N$ and $\boldsymbol{\Sigma}_N$ be defined as in (4.8), (4.9). Then the stability condition (4.13) holds with*

$$\beta(N) \geq C\beta/|\boldsymbol{k}|^2 = C\beta N^{-2/3} . \tag{4.16}$$

To prove this theorem, we use the following lemma, due to Fortin (see, e.g. [10]):

Lemma 4.4. *Let $\boldsymbol{V}_N \subset \boldsymbol{V}$, $\boldsymbol{\Sigma}_N \subset \boldsymbol{\Sigma}$ and $M_N \subset M$ be closed subspaces and assume the continuous inf-sup condition (4.7). If there exists a projection $\Pi_N : \boldsymbol{\Sigma} \times M \to \boldsymbol{\Sigma}_N \times M_N$ such that*

$$\|\Pi_N(\boldsymbol{\sigma}, p)\|_0 \leq \mu(N)(\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2)^{1/2} , \tag{4.17}$$

and such that

$$b((\boldsymbol{\sigma}, p) - \Pi_N(\boldsymbol{\sigma}, p), \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_N , \tag{4.18}$$

then the discrete inf-sup condition (4.13) holds with $\beta(N) = \beta/\mu(N)$.

The next lemmas will be tools for verifying (4.17) and (4.18) in our case. For any polynomial space $S(\Omega)$, $\Omega \subset \mathbb{R}^1$ or $\mathbb{R}^2$, $S_0(\Omega)$ will denote the subset of polynomials vanishing on $\partial\Omega$.

Lemma 4.5. *Let $I = (-1, 1)$. Let $\Pi_k^x : L^2(I) \to \mathcal{P}_0^{k+2}(I)$ be such that for any $\tau \in L^2(I)$, $\sigma_k = \Pi_k^x \tau \in \mathcal{P}_0^{k+2}(I)$ is defined by*

$$\int_I (\tau - \sigma_k)\, \tau_k \, dx = 0 \quad \forall \tau_k \in \mathcal{P}^k(I) \; . \tag{4.19}$$

Then

$$\|\Pi_k^x \tau\|_{L^2(I)} \le C\sqrt{k}\, \|\tau\|_{L^2(I)} \quad \forall \tau \in L^2(I) \; . \tag{4.20}$$

Proof: Since $\mathcal{P}_0^{k+2}(I) = B_2\, \mathcal{P}^k(I)$ where $B_2(x) = 1 - x^2$, (4.19) defines σ_k uniquely. Let $L_i(x)$ be the Legendre polynomial of degree i and let

$$U_i(x) = \gamma_i(L_{i+1}(x) - L_{i-1}(x)), \quad \gamma_i = (2i+1)^{-1} \; .$$

Then we may write

$$\tau(x) = \sum_{i=0}^{\infty} b_i\, L_i(x), \quad \sigma_k(x) = \sum_{i=1}^{k+1} a_i\, U_i(x) \; .$$

We have

$$\begin{aligned} \sigma_k \;&=\; -a_1\gamma_1 L_0 - a_2\gamma_2 L_1 + (a_1\gamma_1 - a_3\gamma_3)L_2 + (a_2\gamma_2 - a_4\gamma_4)L_3 + \\ &\quad + \cdots + (a_{k-1}\gamma_{k-1} - a_{k+1}\gamma_{k+1})L_k + a_k\gamma_k L_{k+1} + a_{k+1}\gamma_{k+1} L_{k+2} \end{aligned}$$

whence it follows that

$$\begin{aligned} \|\sigma_k\|_{0,I}^2 \;&=\; 2[(a_1\gamma_1)^2\, \gamma_0 + (a_2\gamma_2)^2\, \gamma_1 + (a_1\gamma_1 - a_3\gamma_3)^2\, \gamma_2 \\ &\quad + \cdots + (a_k\gamma_k)^2\, \gamma_{k+1} + (a_{k+1}\gamma_{k+1})^2\, \gamma_{k+2}] \; . \end{aligned} \tag{4.21}$$

From (4.19) we get the equations

$$\begin{aligned} -a_1\gamma_1 \;&=\; b_0 \; , \\ -a_2\gamma_2 \;&=\; b_1 \; , \\ a_1\gamma_1 - a_3\gamma_3 \;&=\; b_2 \; , \\ &\;\vdots \\ a_{k-2}\gamma_{k-2} - a_k\gamma_k \;&=\; b_{k-1} \; , \\ a_{k-1}\gamma_{k-1} - a_{k+1}\gamma_{k+1} \;&=\; b_k \; . \end{aligned} \tag{4.22}$$

We see that the first $k+1$ terms in the expression (4.20) for $\|\sigma_k\|_{L^2(I)}^2$ are bounded by

$$2\,[b_0^2\gamma_0 + b_1^2\gamma_1 + \cdots + b_k^2\gamma_k] \le \|\tau\|_{L^2(I)}^2 \; . \tag{4.23}$$

Furthermore we obtain from (4.22) that (assuming k is odd)

$$a_k\gamma_k = -(b_{k-1} + b_{k-3} + \cdots + b_0) \; .$$

Hence

$$
\begin{aligned}
(a_k\gamma_k)^2\,\gamma_{k+1} &= \gamma_{k+1}\Big(\sum_{\ell \text{ even}} b_\ell\Big)^2 \\
&\le \gamma_{k+1}\Big(\sum_{\ell=0}^{k}\gamma_\ell b_\ell^2\Big)\Big(\sum_{\ell=0}^{k}\gamma_\ell^{-1}\Big) \\
&\le Ck^{-1}\,\|\tau\|_{L^2(I)}^2\,k^2 = Ck\,\|\tau\|_{L^2(I)}^2\,.
\end{aligned}
$$

We can similarly bound the last term of (4.21). Combining these estimates, we get (4.20). ☐

Lemma 4.6. *Let $\hat{K} = (-1,1)^2$ be the reference square and let $\boldsymbol{\tau} \in [L^2(\hat{K})]^{2\times 2}_{\mathrm{sym}}$. Then, for $k \ge 0$ there exists $\boldsymbol{\sigma}_k \in [\mathcal{Q}_0^{k+2}(\hat{K})]^{2\times 2}_{\mathrm{sym}}$ such that*

$$
\int_{\hat{K}} \boldsymbol{\sigma}_k : \boldsymbol{\tau}_k\,d\hat{x} = \int_{\hat{K}} \boldsymbol{\tau} : \boldsymbol{\tau}_k\,d\hat{x} \quad \forall \boldsymbol{\tau}_k \in \mathcal{Q}^k(\hat{K})\,, \tag{4.24}
$$

and

$$
\|\boldsymbol{\sigma}_k\|_{L^2(\hat{K})} \le Ck\,\|\boldsymbol{\tau}\|_{L^2(\hat{K})}\,. \tag{4.25}
$$

Proof: Let $\hat{x} = (\hat{x}_1, \hat{x}_2)$. For any $\tau(\hat{x}) \in C^\infty(\hat{K})$, we define

$$
\widetilde{\sigma}_k(\hat{x}) = \Pi_k^{\hat{x}_1}\,\Pi_k^{\hat{x}_2}\,\tau(\hat{x})\,.
$$

Then using (4.19), we see that with $\tau_k = \tau_k^1(\hat{x}_1)\,\tau_k^2(\hat{x}_2)$, $\tau_k^i \in \mathcal{P}^k(I)$,

$$
\int_{\hat{K}} (\tau - \widetilde{\sigma}_k)\,\tau_k\,d\hat{x} = 0 \quad \forall \tau_k \in \mathcal{Q}^k(\hat{K})\,. \tag{4.26}
$$

Also, since each of $\Pi_k^{\hat{x}_1}$, $\Pi_k^{\hat{x}_2}$ satisfy (4.20), we may show

$$
\|\widetilde{\sigma}_k\|_{L^2(\hat{K})} \le Ck\,\|\tau\|_{L^2(\hat{K})}\,. \tag{4.27}
$$

The proof is completed by first using a density argument to deduce (4.26) and (4.27) for all $\tau \in L^2(\hat{K})$, then by defining the tensor $\boldsymbol{\sigma}_k$ in (4.24) componentwise in the same way as $\widetilde{\sigma}_k$ above. The symmetry of $\boldsymbol{\tau}$ and the uniqueness of $\widetilde{\sigma}_k$ constructed above imply that $\boldsymbol{\sigma}_k \in [\mathcal{Q}_0^{k+2}(\hat{K})]^{2\times 2}_{\mathrm{sym}}$ and that (4.24) and (4.25) hold. ☐

Lemma 4.5 addressed only the case of the reference square $\hat{Q}$, but a scaling argument shows that the result holds for any parallelogram $Q \in \Omega^{n,\sigma}$.

To prove Theorem 4.3, we need an analog of Lemma 4.5 for triangles. Once again, it is sufficient to consider only the reference triangle $\hat{T}$.

Lemma 4.7. *Let $\boldsymbol{\tau} \in [L^2(\hat{T})]^{2\times 2}_{\text{sym}}$ be given. Then, for any $k \geq 0$, there exists $\boldsymbol{\sigma}_k \in [\mathcal{P}_0^{k+3}(\hat{T})]^{2\times 2}_{\text{sym}}$ such that*

$$(4.28) \qquad \int_{\hat{T}} \boldsymbol{\sigma}_k : \boldsymbol{\tau}_k \, d\hat{x} = \int_{\hat{T}} \boldsymbol{\tau} : \boldsymbol{\tau}_k \, d\hat{x} \quad \forall \boldsymbol{\tau}_k \in [\mathcal{P}^k(\hat{T})]^{2\times 2}$$

and

$$(4.29) \qquad \|\boldsymbol{\sigma}_k\|_{L^2(\hat{T})} \leq Ck^2 \, \|\boldsymbol{\tau}\|_{L^2(\hat{T})} \, .$$

Proof: Let $\tau \in L^2(\hat{T})$. Then, $\sigma_k \in \mathcal{P}_0^{k+3}(\hat{T})$ defined by

$$(4.30) \qquad \int_{\hat{T}} \sigma_k \tau_k d\hat{x} = \int_{\hat{T}} \tau \, \tau_k \, d\hat{x} \quad \forall \tau_k \in \mathcal{P}^k(\hat{T})$$

is uniquely defined, since $\mathcal{P}_0^{k+3}(\hat{T}) = b_{\hat{T}} \mathcal{P}^k(\hat{T})$ implies that the Gram matrix for the spaces $\mathcal{P}^k(\hat{T})$ and $\mathcal{P}_0^{k+3}(\hat{T})$ is non-singular for every $k \geq 0$. To prove (4.29), we write $\sigma_k = b_{\hat{T}} \, \hat{\sigma}_k$ where $\hat{\sigma}_k \in \mathcal{P}^k(\hat{T})$. Selecting in (4.30) $\tau_k = \hat{\sigma}_k$, we get

$$\begin{aligned} \int_{\hat{T}} b_{\hat{T}} (\hat{\sigma}_k)^2 \, d\hat{x} \quad &\leq \quad \|\tau\|_{L^2(\hat{T})} \|\hat{\sigma}_k\|_{L^2(\hat{T})} \, , \text{ or} \\ \|\tau\|_{L^2(\hat{T})} \quad &\geq \quad \frac{\int_{\hat{T}} b_{\hat{T}} (\hat{\sigma}_k)^2 \, d\hat{x}}{\left(\int_{\hat{T}} (\hat{\sigma}_k)^2 \, d\hat{x} \right)^{1/2}} \, . \end{aligned}$$

Now we use (3.31) and Lemma 3.7, to deduce

$$(4.31) \qquad \|\tau\|_{L^2(\hat{T})} \geq Ck^{-2} \, \|b_{\hat{T}} (\hat{\sigma}_k)\|_{L^2(\hat{T})} = Ck^{-2} \, \|\sigma_k\|_{L^2(\hat{T})} \, .$$

For given $\boldsymbol{\tau} \in [L^2(\hat{T})]^{2\times 2}_{\text{sym}}$, we define $\boldsymbol{\sigma}_k \in [\mathcal{P}_0^{k+3}(\hat{T})]^{2\times 2}$ via (4.30) component-wise. Then $\boldsymbol{\sigma}_k$ is symmetric and (4.29) follows from (4.31). $\square$

Now we turn to the **proof of Theorem 4.3**. We use Lemma 4.4 and need to construct $\Pi_N(\boldsymbol{\sigma}, p)$. To this end, denote by P_N^{Σ} the $L^2(\Omega)$ projection of $\boldsymbol{\Sigma}$ onto $\boldsymbol{\Sigma}_N$, and by P_N^M the $L^2(\Omega)$ projection of M onto M_N and define

$$(4.32) \qquad \boldsymbol{\tau} = (\boldsymbol{\sigma} - P_N^{\Sigma} \boldsymbol{\sigma}) - (p - P_N^M \, p) \, \boldsymbol{I} \, .$$

Since

$$\|P_N^{\Sigma} \boldsymbol{\sigma}\|_0 \leq \|\boldsymbol{\sigma}\|_0 \, , \; \|P_N^M \, p\|_0 \leq \|p\|_0$$

we have for $\boldsymbol{\tau}$ in (4.32) that

$$(4.33) \qquad \|\boldsymbol{\tau}\|_0 \leq (\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2)^{1/2} \, .$$

We now construct $\Pi_N(\boldsymbol{\sigma}, p)$ in (4.17).

Pick any $K \in \Omega^{n,\sigma}$. Applying a linear transformation and Lemmas 4.5 and 4.7 (depending on whether K is a quadrilateral or a triangle), we obtain $\boldsymbol{\tau}_k^K \in [Q_0^{p_k+2}(K)]^{2\times 2}_{\rm sym}$ (respectively $\boldsymbol{\tau}_k^K \in [\mathcal{P}_0^{p_K+3}(K)]^{2\times 2}_{\rm sym}$) such that

$$\|\boldsymbol{\tau}_k^K\|_{L^2(K)} \leq C_K\, k_K^2\, \|\boldsymbol{\tau}\|_{L^2(K)}\ . \tag{4.34}$$

Let $\boldsymbol{\tau}_k \in \boldsymbol{\Sigma}_N$ be defined by $\boldsymbol{\tau}_k|_K = \boldsymbol{\tau}_k^K$. Then from (4.34),

$$\|\boldsymbol{\tau}_k\|_0 \leq C(\Omega, \sigma)\ |\boldsymbol{k}|^2\ \|\boldsymbol{\tau}\|_0\ . \tag{4.35}$$

From the definition of $\boldsymbol{V}_N$, it follows with (4.24) and (4.28) that

$$\int\limits_K \boldsymbol{\tau}_k : \boldsymbol{D}(\boldsymbol{v})\, dx = \int\limits_K \boldsymbol{\tau} : \boldsymbol{D}(\boldsymbol{v})\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{V}_N\ .$$

This is because $\boldsymbol{v} \in [\mathcal{Q}^{k+1}(\hat{Q})]^2 \implies \boldsymbol{D}(\boldsymbol{v}) \in [\mathcal{Q}^{k+1}(\hat{Q})]^{2\times 2}_{\rm sym}$, $\boldsymbol{v} \in [\mathcal{P}^{k+1}(\hat{T})]^2 \implies \boldsymbol{D}(\boldsymbol{v}) \in [\mathcal{P}^{k}(\hat{T})]^{2\times 2}_{\rm sym}$. We set for $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, $p \in M$

$$\boldsymbol{\sigma}_N := P_N^{\Sigma}\, \boldsymbol{\sigma} + \boldsymbol{\tau}_k,\ \ p_N = P_N^M\, p,\ \ (\boldsymbol{\sigma}_N, p_N) = \Pi_N(\boldsymbol{\sigma}, p)\ .$$

Then $\boldsymbol{\sigma}_N \in \boldsymbol{\Sigma}_N$ and for any $\boldsymbol{v} \in \boldsymbol{V}_N$ we have

$$\int\limits_\Omega [(\boldsymbol{\sigma} - \boldsymbol{\sigma}_N) - (p - p_N)\, \boldsymbol{I}] : \boldsymbol{D}(\boldsymbol{v})\, dx = \sum_{K \in \Omega^{n,\sigma}} \int\limits_K [\boldsymbol{\tau} - \boldsymbol{\tau}_k] : \boldsymbol{D}(\boldsymbol{v})\, dx = 0$$

which is (4.18). Further, (4.33) and (4.35) imply

$$(\|\boldsymbol{\sigma}_N\|_0^2 + \|p_N\|_0^2)^{1/2} \leq C\, |\boldsymbol{k}|^2\ (\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2)^{1/2}$$

which is (4.17) with $\mu(N) = |\boldsymbol{k}|^2$. □

Based on Theorems 4.1 and 4.3, we may specify the abstract error estimates (4.14), (4.15) in the following way. (The estimate (4.39) follows by Theorem 3.1.)

Theorem 4.8. *Let $\boldsymbol{\Sigma}_N$, M_N and $\boldsymbol{V}_N$ be the hp-subspaces defined in (4.8), (4.9). Then for the hp-FE approximations $\boldsymbol{\sigma}_N, p_N$ and $\boldsymbol{u}_N$ of the three-field Stokes problem, there hold the error estimates*

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 + \|p - p_N\|_0 \ &\leq\ CN^{8/3} \inf_{\substack{\tau \in \boldsymbol{\Sigma}_N \\ q \in M_N}} \{\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|p - q\|_0\} \\ &+\ CN^2 \inf_{\boldsymbol{v} \in \mathbf{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_1\ , \end{aligned} \tag{4.36}$$

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_N\|_1 \ &\leq\ CN^{10/3} \inf_{\substack{\tau \in \boldsymbol{\Sigma}_N \\ q \in M_N}} \{\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|p - q\|_0\} \\ &+\ CN^{8/3} \inf_{\boldsymbol{v} \in \mathbf{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_1\ . \end{aligned} \tag{4.37}$$

If, in addition, $\boldsymbol{u}, \boldsymbol{\sigma}$ *and* p *have the regularity*

$$\boldsymbol{u} \in \mathcal{B}^2_\beta(\Omega), \quad \boldsymbol{\sigma}, p \in \mathcal{B}^1_\beta(\Omega) \tag{4.38}$$

for some $0 < \beta < 1$*, then the following exponential convergence estimate holds,*

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 + \|p - p_N\|_0 \leq C \, \exp(-b\sqrt[3]{N}) \,. \tag{4.39}$$

Remark 4.9. (On the *hp* version with quasiuniform meshes)

For the case of meshes consisting only of parallelograms, equations (4.37) - (4.38) can be improved to

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 + \|p - p_N\|_0 \quad &+ \quad k^{-1} \, \|\boldsymbol{u} - \boldsymbol{u}_N\|_1 \\ &\leq \quad C h^\ell \, k^{-(m-r)} (\|\boldsymbol{\sigma}\|_{m-1} + \|p\|_{m-1} + \|\boldsymbol{u}\|_m) \end{aligned} \tag{4.40}$$

where $\ell = \min\{m-1, k+1\}$ and $r = 2$ when the pressure space is $\mathcal{P}^k$, and $\ell = \min\{m-1, k\}$, $r = 3$ when it is $\mathcal{Q}^{k-1}$.

From (4.40), it can be clearly seen that we pay quite a bit for stability, since even though we are using polynomials of degree $k+3$ for $\boldsymbol{\sigma}$, we only realise $O(h^k)$ (or $O(h^{k+1})$) convergence instead of $O(h^{k+4})$ convergence in $\boldsymbol{\sigma}$. This can be improved in one of two ways, either by using **discontinuous** stresses or by using the modified EVSS method with **continuous** stresses. In each of these methods, discussed in the next two sections, the stability ends up only depending on the velocity-pressure inf-sup constant $\delta(N)$.

4.5 Discontinuous stress elements

So far, we investigated the stability for the *hp*-FEM (4.10) based on (4.9), the spaces $\boldsymbol{\Sigma}_N$ of continuous stresses. Here we had to increase the polynomial degree k for $\boldsymbol{\sigma}$ by 2 over that of $\boldsymbol{v}$ in order to ensure stability of $\boldsymbol{\Sigma}_N \times M_N \times \boldsymbol{V}_N$. It turns out that with discontinuous stresses, the degree increase can be avoided. Instead of (4.9), we select

$$\boldsymbol{\Sigma}_N = \begin{aligned} \Big\{ \boldsymbol{\sigma} \in [L^2(\Omega)]^{2\times 2}_{\text{sym}} : \quad & \sigma_{\alpha\beta}|_K \circ F_K \in \mathcal{Q}^{k_K+1}(\hat{K}) \quad \text{if } \hat{K} = \hat{Q} \,, \\ & \sigma_{\alpha\beta}|_K \circ F_K \in \mathcal{P}^{k_K}(\hat{K}) \qquad \text{if } \hat{K} = \hat{T} \Big\} \,, \end{aligned} \tag{4.41}$$

so that we have the combination shown in Table 2

$\boldsymbol{\sigma}$	p	$\boldsymbol{u}$
$\mathcal{Q}^{k+1}(\hat{Q})$	$\mathcal{P}^k(\hat{Q})/Q^{k-1}(\hat{Q})$	$Q^{k+1}(\hat{Q})$
$\mathcal{P}^k(\hat{T})$	$\mathcal{P}^{k-1}(\hat{T})$	$\mathcal{P}^{k+1}(\hat{T})$

Table 2: Element polynomial spaces in the *hp*-FEM for three-field Stokes problem based on discontinuous stresses ($\boldsymbol{\Sigma}_N$ as in (4.41)).

We have

Theorem 4.10. *Let $\boldsymbol{V}_N, M_N, \boldsymbol{\Sigma}_N$ be defined as in (4.8) and (4.41), respectively, and $|\boldsymbol{k}|$ be as in Theorem 4.1. Then the discrete inf-sup condition (4.13) holds with*

$$\beta(N) \geq C\beta\,. \tag{4.42}$$

For the proof of this result, we proceed exactly as in the proof of Theorem 4.3. However, using the fact that $\boldsymbol{\sigma}$ is discontinuous, in Lemmas 4.5 and 4.7 we may simply choose the sets $[\mathcal{Q}^{k+1}(\hat{Q})]^{2\times 2}_{\text{sym}}$ and $[\mathcal{P}^k(\hat{T})]^{2\times 2}_{\text{sym}}$, respectively, which contain $\boldsymbol{D}(\boldsymbol{v})$ for $\boldsymbol{v} \in \boldsymbol{V}_N$. The projections in Lemmas 4.5 and 4.7 are then $L^2(\hat{K})$-projections which have norms bounded by one, independently of k. $\square$

Using (4.42) and Theorem 4.1 in the abstract error estimates (4.14), (4.15) gives a-priori error bounds analogous to those in Theorem 4.8 resp. Remark 4.9, however now with a smaller loss in the orders of k. We leave their derivation to the reader. Note that now the stability only depends on the discrete velocity-pressure constant $\delta(N)$ in (3.5), since $\alpha(N) = (\delta(N))^2$.

Remark 4.11. (A completely stable combination)

For the case of parallelograms with the choice $\mathcal{Q}^{k+1}(Q)$ for $\boldsymbol{\sigma}$, $\mathcal{P}^k(Q)$ for p and $\mathcal{Q}^{k+1}(Q)$ for $\boldsymbol{u}$, we see now that $\alpha(N)$, $\beta(N)$ are both $O(1)$ when discontinuous stresses are used. Hence, this choice is **completely stable**, and we have, with $\ell = \min\{m-1, k+1\}$,

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 \;&+\; \|p - p_N\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_N\|_1 \\ &\leq\; C h^\ell\, k^{-(m-1)}(\|\boldsymbol{\sigma}\|_{m-1} + \|p\|_{m-1} + \|\boldsymbol{u}\|_m) \end{aligned} \tag{4.43}$$

which is the **optimal** convergence rate on quasiuniform meshes for $\boldsymbol{u}$ and p. (For $\boldsymbol{\sigma}$, we lose a power of h compared to the expected rate of $O(h^{k+2})$ using $\mathcal{Q}^{k+1}$ elements.)

4.6 The hp-FEM for the modified EVSS method

As mentioned in Section 4.4, in order to achieve stability, the hp-FE discretization of the three-field Stokes problem (4.2) required fairly large (continuous) spaces $\boldsymbol{\Sigma}_N$ for the approximation of the extra stress $\boldsymbol{\sigma}$. This is observed also for other (continuous) discretizations of (4.2) (see e.g. [5], where the stability of the so-called 4×4 stress element is shown, and also [30], where it is visible in practical computations). The large stress spaces are responsible for driving up the cost of the method, primarily for the sake of stability (see Remark 4.9).

The EVSS method introduced in [24] was designed to circumvent the need for large spaces $\boldsymbol{\Sigma}_N$ while still using continuous stresses. In the recent variant [11], it was shown that a suitable reformulation of (4.2) allows for a discretization that only depends on $\delta(N)$, i.e. the divergence stability of $\boldsymbol{V}_N, M_N$. Thus we may select here the hp-FE spaces analyzed in Section 3 and pick $\boldsymbol{\Sigma}_N$ based solely on approximability considerations. Let us describe the reformulation of (4.2) from [11] and its hp-FE discretization. The theorems and proofs in this section are essentially from [11], we include them here to get the explicit dependence of $\inf - \sup$ constants on N.

The key idea is to introduce into (4.2) the new variable

$$\boldsymbol{d} = \boldsymbol{D}(\boldsymbol{u}) \tag{4.44}$$

and let $\alpha > 0$ be a parameter at our disposal. We then write the equivalent problem

$$
\begin{aligned}
-\text{div}\, \boldsymbol{\sigma} + 2\alpha\, \text{div}\, \boldsymbol{d} - 2\alpha\, \text{div}\, \boldsymbol{D}(\boldsymbol{u}) + \nabla p &= \boldsymbol{f}\,, \\
\boldsymbol{d} - \boldsymbol{D}(\boldsymbol{u}) &= 0\,, \\
\text{div}\, \boldsymbol{u} &= 0\,, \\
\boldsymbol{\sigma} - 2\nu\, \boldsymbol{D}(\boldsymbol{u}) &= 0\,, \\
\boldsymbol{u} &= 0 \ \text{ on } \ \partial\Omega
\end{aligned}
\tag{4.45}
$$

or, in weak form: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{\Sigma} \times \boldsymbol{V},\ (\boldsymbol{d}, p) \in \boldsymbol{D} \times M$ such that

$$
\begin{aligned}
a((\boldsymbol{\sigma}, \boldsymbol{u}),\ (\boldsymbol{\tau}, \boldsymbol{v})) + b_1((\boldsymbol{\tau}, \boldsymbol{v}),\ (\boldsymbol{d}, p)) &= (\boldsymbol{f}, \boldsymbol{v}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma} \times \boldsymbol{V}\,, \\
b_2((\boldsymbol{\sigma}, \boldsymbol{u}),\ (\boldsymbol{e}, q)) &= 0 \qquad\quad \forall (\boldsymbol{e}, q) \in \boldsymbol{D} \times M\,.
\end{aligned}
\tag{4.46}
$$

Here

$$
\boldsymbol{D} = \boldsymbol{\Sigma} = (L^2(\Omega))^{2\times 2}_{\text{sym}}\,, \quad \boldsymbol{V} = (H^1_0(\Omega))^2,\ M = L^2_0(\Omega)
$$

and

$$
\begin{aligned}
a((\boldsymbol{\sigma}, \boldsymbol{u}),\ (\boldsymbol{\tau}, \boldsymbol{v})) &= \int_\Omega ([\boldsymbol{\sigma} + 2\alpha\, \boldsymbol{D}(\boldsymbol{u})] : \boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{\tau})\, dx\,, \\
b_1((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{e}, q)) &= \int_\Omega (\boldsymbol{e} : [\boldsymbol{\tau} - 2\alpha\, \boldsymbol{D}(\boldsymbol{v})] - q\boldsymbol{I} : \boldsymbol{D}(\boldsymbol{v}))\, dx\,, \\
b_2((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{e}, q)) &= \int_\Omega (\boldsymbol{e} : [\boldsymbol{\tau} - 2\nu\, \boldsymbol{D}(\boldsymbol{v})] - q\boldsymbol{I} : \boldsymbol{D}(\boldsymbol{v}))\, dx\,.
\end{aligned}
$$

We observe that $\alpha > 0$ adds a stabilization term and that (4.46) requires a generalization of the inf-sup conditions, since $b_1 \neq b_2$ if $\alpha \neq \nu$. So far, (4.46) is equivalent to (4.3), as is easily verified. This changes, however, once (4.46) is discretized. To this end, we select finite dimensional subspaces

$$
\boldsymbol{V}_N \subset \boldsymbol{V},\ M_N \subset M,\ \boldsymbol{\Sigma}_N = \boldsymbol{D}_N \subset [L^2(\Omega)]^{2\times 2}_{\text{sym}}
\tag{4.47}
$$

where $\boldsymbol{V}_N, M_N$ are as in Section 3 and $\boldsymbol{\Sigma}_N = \boldsymbol{D}_N$ is as yet unspecified. The **discrete inf-sup conditions** to be satisfied now by $a(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$ are as follows: let, for $i = 1, 2$,

$$
\boldsymbol{Z}^i_N := \{(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_N \times \boldsymbol{V}_N : b_i((\boldsymbol{\tau}, \boldsymbol{v}),\ (\boldsymbol{e}, q)) = 0 \quad \forall (\boldsymbol{e}, q) \in \boldsymbol{D}_N \times M_N\}\,.
\tag{4.48}
$$

Then we must have ([22])

$$
\inf_{(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{Z}^2_N} \ \sup_{(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{Z}^1_N} \ \frac{a((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v}))}{(\|\boldsymbol{\sigma}\|_0^2 + \|\boldsymbol{u}\|_1^2)^{1/2}\, (\|\boldsymbol{\tau}\|_0^2 + \|\boldsymbol{v}\|_1^2)^{1/2}} \geq \gamma(N) > 0
\tag{4.49}
$$

and, since $\boldsymbol{\Sigma}_N = \boldsymbol{D}_N$, for $i = 1, 2$,

$$
\inf_{(e,q) \in \Sigma_N \times Q_N} \ \sup_{(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_N \times \boldsymbol{V}_N} \ \frac{b_i((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{e}, q))}{(\|\boldsymbol{\tau}\|_0^2 + \|\boldsymbol{v}\|_1^2)^{1/2}\, (\|\boldsymbol{e}\|_0^2 + \|q\|_0^2)^{1/2}} \geq \beta_i(N) > 0\,.
\tag{4.50}
$$

Under these conditions, a discrete solution $(\boldsymbol{\sigma}_N, \boldsymbol{u}_N) \in \boldsymbol{\Sigma}_N \times \boldsymbol{V}_N$, $(\boldsymbol{d}_N, p_N) \in \boldsymbol{D}_N \times M_N$ of (4.46) exists and satisfies the a priori error estimates

$$\begin{aligned}&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_N\|_1 \le \\ &C\left(1 + \frac{1}{\gamma(N)}\right)\left(\left(1 + \frac{1}{\beta_2(N)}\right)(\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|\boldsymbol{u} - \boldsymbol{v}\|_1) + \|\boldsymbol{d} - \boldsymbol{e}\|_0 + \|p - q\|_0\right),\end{aligned} \tag{4.51}$$

$$\begin{aligned}&\|\boldsymbol{d} - \boldsymbol{d}_N\|_0 + \|p - p_N\|_0 \le C\left(1 + \frac{1}{\beta_1(N)}\right)\left(1 + \frac{1}{\gamma(N)}\right) \times \\ &\left\{\|\boldsymbol{d} - \boldsymbol{e}\|_0 + \ \|p - q\|_0 + \left(1 + \frac{1}{\beta_2(N)}\right)\left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_0 + \|\boldsymbol{u} - \boldsymbol{v}\|_1\right)\right\}\end{aligned} \tag{4.52}$$

for every $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_N, \boldsymbol{v} \in \boldsymbol{V}_N$, $\boldsymbol{e} \in \boldsymbol{D}_N$ and $q \in M_N$.

It remains to check (4.49), (4.50). Let us first estimate $\gamma(N)$ in (4.49). We note that

$$\boldsymbol{Z}_N^i = \{(2\lambda\, \boldsymbol{D}_N(\boldsymbol{v}),\, \boldsymbol{v}) \,\big|\, \boldsymbol{v} \in \boldsymbol{Z}_N\} \tag{4.53}$$

where $\lambda = \alpha$ if $i = 1$, $\lambda = \nu$ if $i = 2$,

$$\boldsymbol{Z}_N = \left\{\boldsymbol{v} \in \boldsymbol{V}_N : \int_\Omega q\boldsymbol{I} : \boldsymbol{D}(\boldsymbol{v})\, dx = (q, \operatorname{div} \boldsymbol{v}) = 0 \quad \forall q \in M_N\right\} \tag{4.54}$$

and $\boldsymbol{D}_N(\boldsymbol{u}) \in \boldsymbol{\Sigma}_N$ is the L^2 projection of $\boldsymbol{D}(\boldsymbol{u})$ onto $\boldsymbol{\Sigma}_N$, defined by

$$(\boldsymbol{D}_N(\boldsymbol{u}),\, \tau) = (\boldsymbol{D}(\boldsymbol{u}),\, \tau) \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_N .$$

On $\boldsymbol{Z}_N^1 \times \boldsymbol{Z}_N^2$, the bilinear form $a(\cdot, \cdot)$ in (4.46) is, by (4.53) - (4.54), equivalent to

$$\begin{aligned}\widetilde{a}(\boldsymbol{u}, \boldsymbol{v}) &= 2\alpha\,(\boldsymbol{D}(\boldsymbol{u}),\, \boldsymbol{D}(\boldsymbol{v})) - 2\alpha\,(\boldsymbol{D}_N(\boldsymbol{u}),\, \boldsymbol{D}_N(\boldsymbol{v})) \\ &+ 2\nu\,(\boldsymbol{D}_N(\boldsymbol{u}),\, \boldsymbol{D}_N(\boldsymbol{v}))\,,\end{aligned} \tag{4.55}$$

where $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{Z}_N$.

For $\boldsymbol{u} = \boldsymbol{v} \in \boldsymbol{Z}_N$ in (4.55), we get

$$\widetilde{a}(\boldsymbol{u}, \boldsymbol{u}) = 2\alpha\, \|\boldsymbol{D}(\boldsymbol{u})\|_0^2 + 2(\nu - \alpha)\, \|\boldsymbol{D}_N(\boldsymbol{u})\|_0^2 .$$

If $\nu - \alpha \ge 0$, we are done, since $\|\boldsymbol{D}(\boldsymbol{u})\|_0^2 \ge C^2(\Omega)\, \|\boldsymbol{u}\|_1^2$ by Korn's inequality. If $\nu - \alpha \le 0$, $\alpha - \nu \ge 0$ and we write, using $\|\boldsymbol{D}_N(\boldsymbol{u})\|_0 \le \|\boldsymbol{D}(\boldsymbol{u})\|_0$,

$$\begin{aligned}\widetilde{a}(\boldsymbol{u}, \boldsymbol{u}) &= 2\nu\, \|\boldsymbol{D}(\boldsymbol{u})\|_0^2 + 2(\alpha - \nu)(\|\boldsymbol{D}(\boldsymbol{u})\|_0^2 - \|\boldsymbol{D}_N(\boldsymbol{u})\|_0^2) \\ &\ge 2\nu\, \|\boldsymbol{D}(\boldsymbol{u})\|_0^2 .\end{aligned}$$

Hence we get

$$\widetilde{a}(\boldsymbol{u}, \boldsymbol{u}) \ge 2\min(\alpha, \nu)\, \|\boldsymbol{D}(\boldsymbol{u})\|_0^2 \ge C^2(\Omega)\ \min(\alpha, \nu)\, \|\boldsymbol{u}\|_1^2 \quad \forall \boldsymbol{u} \in \boldsymbol{Z}_N . \tag{4.56}$$

Now using (4.53), we get for any $\boldsymbol{u} \in \boldsymbol{Z}_N$

$$\sup_{\mathbf{v}\in\mathbf{Z}_N} \frac{a((2\nu\,\boldsymbol{D}_N(\boldsymbol{u}),\boldsymbol{u}),(2\alpha\,\boldsymbol{D}_N(\boldsymbol{v}),\boldsymbol{v}))}{(\|2\alpha\,\boldsymbol{D}_N(\boldsymbol{v})\|_0^2+\|\boldsymbol{v}\|_1^2)^{1/2}} \geq \frac{\widetilde{a}(\boldsymbol{u},\boldsymbol{u})}{(\|2\alpha\,\boldsymbol{D}(\boldsymbol{u})\|_0^2+\|u\|_1^2)^{1/2}}$$

$$\geq C^2(\Omega)\ \min(\alpha,\nu)\ \frac{\|\boldsymbol{u}\|_1^2+1/(2C^2(\Omega))\,\|\boldsymbol{D}(\boldsymbol{u})\|_0^2}{(4\alpha^2\,\|\boldsymbol{D}(\boldsymbol{u})\|_0^2+\|\boldsymbol{u}\|_1^2)^{1/2}}$$

$$\geq KC^2(\Omega)\ \min(\alpha,\nu)\ \min\left(1,\ \frac{1}{\alpha\nu\,C(\Omega)}\right)(\|u\|_1^2+4\nu^2\,\|\boldsymbol{D}(\boldsymbol{u})\|_0^2)^{1/2}$$

which is (4.49) with $\gamma(N)\geq\gamma_0>0$ where γ_0 depends only on α, ν and the Korn constant $C(\Omega)$.

Let us now prove (4.50).

Lemma 4.12. *Assume $(\boldsymbol{V}_N, M_N)$ satisfies (3.5) with $\delta(N)>0$. Then (4.50) holds with*

$$\beta_i(N) \geq \frac{C\delta^2(N)}{\lambda^2+\delta(N)}\,,\quad i=1,2 \tag{4.57}$$

where $\lambda=\alpha$ if $i=1$, $\lambda=\nu$ if $i=2$.

Proof: By (3.5), for every $p\in M_N$, there exists $\boldsymbol{v}\in\boldsymbol{V}_N$ such that

$$-(p,\,\mathrm{div}\,\boldsymbol{v})=(p,q)\quad \forall q\in M_N,\quad \|\boldsymbol{v}\|_1\leq\frac{1}{\delta(N)}\,\|p\|_0\,. \tag{4.58}$$

For any $(\boldsymbol{d},p)\in\boldsymbol{D}_N\times M_N$, select $\boldsymbol{v}\in\boldsymbol{V}_N$ as in (4.58) and $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_N$ equal to $\eta\boldsymbol{d}$ where $\eta>0$ is a number to be selected. Then, by (4.58),

$$\begin{aligned} \|\boldsymbol{\sigma}\|_0^2+\|\boldsymbol{v}\|_1^2 &\leq \eta^2\,\|\boldsymbol{d}\|_0^2+\frac{1}{(\delta(N))^2}\,\|p\|_0^2 \\ &\leq \max(\eta^2,\,(\delta(N))^{-2})(\|\boldsymbol{d}\|_0^2+\|p\|_0^2)\,. \end{aligned} \tag{4.59}$$

Further, with $q=p$ in (4.58),

$$\begin{aligned} b_i(\boldsymbol{\sigma},\boldsymbol{v}),(\boldsymbol{d},p)) &= \eta\,\|\boldsymbol{d}\|_0^2-2\lambda(\boldsymbol{d},\boldsymbol{D}(\boldsymbol{v}))-(p,\,\mathrm{div}\,\boldsymbol{v}) \\ &\geq \eta\,\|\boldsymbol{d}\|_0^2-\tfrac{1}{\epsilon}\,\|\boldsymbol{d}\|_0^2-\lambda^2\epsilon\,\|\boldsymbol{D}(\boldsymbol{v})\|_0^2+\|p\|_0^2 \\ &\geq \left(\eta-\tfrac{1}{\epsilon}\right)\,\|\boldsymbol{d}\|_0^2+\left(1-\tfrac{2\lambda^2\,\epsilon}{(\delta(N))^2}\right)\,\|p\|_0^2\,. \end{aligned}$$

Select now $\epsilon=(\delta(N))^2/(4\lambda^2)$ and $\eta=\frac{1}{2}+\frac{1}{\epsilon}$. This yields

$$b_i((\boldsymbol{\sigma},\boldsymbol{v}),(\boldsymbol{d},p))\geq\frac{1}{2}\,(\|\boldsymbol{d}\|_0^2+\|p\|_0^2)\,.$$

Replacing $\max(\eta^2,(\delta(N))^{-2})$ by $(\eta^2+(\delta(N))^{-2})$ in (4.59) gives the lemma. □

Using the above stability estimates together with the estimate for $\delta(N)$ from (3.27), equations (4.51) - (4.52) give the following error bound for the modified EVSS method for the case $\alpha\geq\alpha_0>0$:

$$\begin{aligned} \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_N\|_0+\|\boldsymbol{u}-\boldsymbol{u}_N\|_1 &\leq C(N^2\,(\|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_0+\|\boldsymbol{u}-\boldsymbol{v}\|_1) \\ &\quad +\|\boldsymbol{d}-\boldsymbol{e}\|_0+\|p-q\|_0) \end{aligned} \tag{4.60}$$

$$\|\boldsymbol{d}-\boldsymbol{d}_N\|_0+\|p-p_N\|_0 \;\le\; CN^2\left(\|\boldsymbol{d}-\boldsymbol{e}\|_0+\|p-q\|_0+N^2(\|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_0 + \|\boldsymbol{u}-\boldsymbol{v}\|_1)\right). \tag{4.61}$$

for every $\boldsymbol{\tau}\in\boldsymbol{\Sigma}_N$, $\boldsymbol{v}\in\boldsymbol{V}_N$, $\boldsymbol{e}\in\boldsymbol{D}_N$ and $q\in M_N$. Each of the norms on the right can be bounded by $Ce^{-bN^{1/3}}$ for a geometric mesh, giving exponential convergence for any reasonable choice of $\boldsymbol{\Sigma}_N=\boldsymbol{D}_N$.

Remark 4.13. (A stable and optimal choice).

As seen from the above, the choice of $\boldsymbol{\Sigma}_N=\boldsymbol{D}_N$ can now be determined by approximability considerations. For parallelograms, suppose we choose $\boldsymbol{V}_N\times M_N$ to be the $[\mathcal{Q}^{k+1}(\hat{Q})]^2\times\mathcal{P}^k(\hat{Q})$ combination, for which $\delta(N)\sim C$. Then, since we want $\boldsymbol{\sigma}_N$ to be continuous, we can take (minimally) each component to be in $\mathcal{Q}^{k\prime}(\hat{Q})$ (the serendipity space for $k\ge 2$). For $\alpha\ge\alpha_0>0$, we then have $\gamma(N)\sim C$ and $\beta_i(N)\sim C$, so that we obtain the estimate ($\ell=\min\{m-1,k+1\}$)

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_N\|_0+\|\boldsymbol{u}-\boldsymbol{u}_N\|_1+\left(\|\boldsymbol{d}-\boldsymbol{d}_N\|_0+\|p-p_N\|_0\right.\\ &\le Ch^\ell\, k^{-(m-1)}\left(\|\boldsymbol{\sigma}\|_{m-1}+\|\boldsymbol{u}\|_m+\|p\|_{m-1}+\|\boldsymbol{d}\|_{m-1}\right).\end{aligned} \tag{4.62}$$

This is completely stable, like the element in Remark 4.9, but now is **optimal** in h and k for $\boldsymbol{\sigma}$ as well. However, we now have an additional unknown $\boldsymbol{d}_N$ and a non-symmetric problem.

5 Numerical Experiments

We report some numerical experiments on mixed *hp*-FEM with geometric meshes for stationary Newtonian flow in an L-shaped domain. These results are taken from [25], where several other related numerical investigations are presented. We have used geometric refinement towards the reentrant corner and depict a typical mesh and the domain in Figure 6. The problem was solved with exact solution

$$\boldsymbol{u}=\begin{pmatrix} r^\alpha\left[(1+\alpha)\sin(\varphi)\psi(\varphi)+\cos(\varphi)\psi'(\varphi)\right]\\ r^\alpha\left[\sin(\varphi)\psi'(\varphi)-(1+\alpha)\cos(\varphi)\psi(\varphi)\right]\end{pmatrix}$$

$$p=-r^{\alpha-1}\left[(1+\alpha)^2\psi'(\varphi)+\psi^{(3)}(\varphi)\right]/(1-\alpha)$$

where $\omega=3\pi/2$ and $\alpha=0.5444...$ and

$$\psi(\varphi)=\frac{\sin((1+\alpha)\varphi)\cos(\alpha\omega)}{1+\alpha}-\cos((1+\alpha)\varphi)-\frac{\sin((1-\alpha)\varphi)\cos(\alpha\omega)}{1-\alpha}+\cos((1-\alpha)\varphi)$$

This solution satisfies the homogenous Stokes equations exactly (i.e. (3.1) with $\boldsymbol{f}=0$), but has nonzero boundary values on the sides of Ω which do not abut at the reentrant vertex. It is a model for the singularities arising also in non-Newtonian flow at the reentrant corners in the so-called 4 : 1-contraction problem.

The implementation was accomplished in the general purpose *hp*-FE code $hp-90$, an object oriented FORTRAN90 *hp*-FE framework for elliptic systems [9]. The velocity pressure space pair $[\mathcal{Q}^{k+1}]^2\times\mathcal{Q}^k$ was used with discontinuous pressures. The (non-parallelogram) elements were mapped with bilinear element maps and, thus, do not completely fall into our theory (but see [6] in this connection). In the numerical experiments below, a uniform degree-vector $\mathbf{k}$ was chosen and the number n of layers in the geometric mesh was related to k by $n=k+2$. Geometric refinement towards the reentrant corner with several different grading factors ranging from 0.5 (corresponding to element bisection) to 0.1 (corresponding to strong refinement towards the reentrant corner) was used.

The error curves shown in Figure 7 show clearly the predicted convergence rate of $\exp(-b\sqrt[3]{N})$, both for the velocity as well as for the pressure error. We further see the importance of the proper choice of the grading factor in the geometric mesh: the error with choice 0.5 is an order of magnitude larger both for the velocity and the pressure, compared to that for 0.15, with the same number of degrees of freedom.

We recall the 1-d result stating that in *hp*-FEM for singular functions, the optimal geometric mesh has a grading factor of $(\sqrt{2}-1)^2 \sim 0.17...$ *independently* of the strength α of the singularity [14]. The same can be expected in higher dimensions for solutions containing radial singularities emanating from a point like the one considered here. This is of particular relevance for problems in non-Newtonian flow, where the strength of the singularity is not known, but where geometric refinement with this ratio could be prescribed to capture the singularity. We remark that in our example, a grading factor of 0.15 gives a relative percentage error for velocity and pressure of 0.1% at about 5000 DOF.

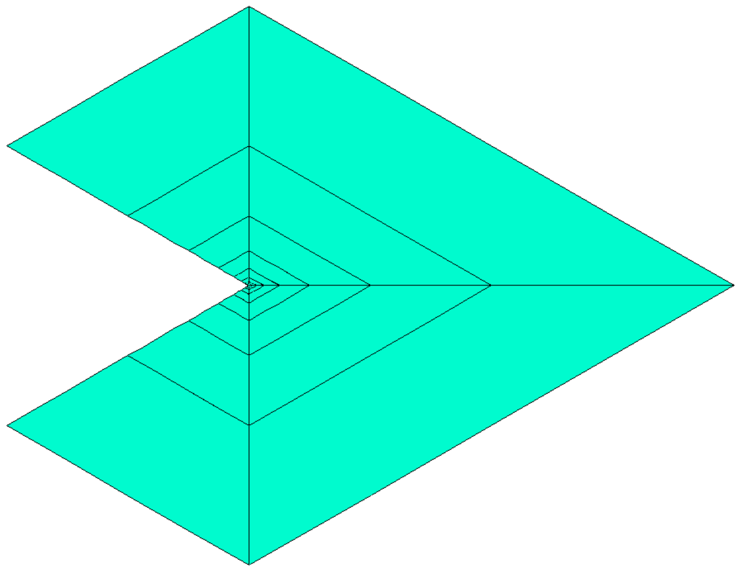

Figure 6: L-shaped domain with geometric mesh, grading factor 0.5

Figure 7: Errors $\|u^1 - u^1_{FE}\|_{H^1(\Omega)}$ and $\|p - p_{FE}\|_{L^2_0(\Omega)}$ versus $N^{1/3}$ for various grading factors s

6 Conclusions

In this paper, we have investigated the design and convergence properties of hp spaces for problems in fluid flow. For the mixed method for **Stokes Flow**, the following results follow from this paper and from [3], [4], [29]:

(1) For **parallelograms**, the combination $\mathcal{Q}^{k+1}$ (velocity) - $\mathcal{P}^k$ (pressure) is stable and optimal in h and k. The combination $\mathcal{Q}^{k+1} - \mathcal{Q}^{k-1}$ is stable in h, and has an inf-sup constant of $Ck^{-1/2}$ in k. It is non-optimal in h for the velocities. An optimized version of this element can be obtained, however - see Section 3.2 and [29].

(2) For **triangles**, the combination $\mathcal{P}^{k+1}$ (velocity) - $\mathcal{P}^{k-1}$ (pressure) is stable in h, but non-optimal in h in the velocity (an optimized version is given in Section 3.2). The stability constant is no worse than Ck^{-3} in k, but computational results in [6] indicate a behavior of Ck^{-1} for practical choices of k. If $\mathcal{P}^{k+1}$ (velocity) - $\mathcal{P}^k$ (pressure) spaces are used with **continuous** pressures, then the stability estimate is improved to $Ck^{-2}/\sqrt{\log k}$.

(3) Due to the above stability results for parallelograms and triangles, the hp version over geometric meshes leads to **exponential** convergence in both the velocity and the pressures.

For the three-field linearized limit model of **Non-Newtonian Flow**, we obtain, in addition:

(4) On **parallelograms**, the combination

$$Q^{k+3} \text{ (continuous stress) - } Q^{k+1} \text{ (velocity) - } Q^{k-1} \text{ or } \mathcal{P}^k \text{ (pressure)}$$

is stable in h. In terms of the degree, the stability constants are no worse than $\alpha(N) = C$ for the pressure space $\mathcal{P}^k$ and $\alpha(N) = Ck^{-1}$ for the pressure space $\mathcal{Q}^{k-1}$ and $\beta(N) = Ck^{-1}$ in either case (see equations (4.12) - (4.13)).

For **triangles**, the combination

$$\mathcal{P}^{k+3} \text{ (continuous stress) - } \mathcal{P}^{k+1} \text{ (velocity) - } \mathcal{P}^{k-1} \text{ (pressure)}$$

is stable in h, and the discrete inf-sup constants in k satisfy $\alpha(N) \geq Ck^{-6}$, $\beta(N) \geq Ck^{-2}$. It is expected that the actual behavior will be better.

Computational results in [30] confirm that it is necessary to choose higher degrees for the continuous stress than the velocity.

(5) For **discontinuous stresses** we get the combination

$$\mathcal{Q}^{k+1} \text{ (stresses) - } \mathcal{Q}^{k+1} \text{ (velocity) - } \mathcal{P}^k \text{ (pressures)}$$

on parallelograms, which is **stable** in **both** h **and** k. The only non-optimality present in this element is that for smooth solutions, even though one uses degree $k+1$ polynomials for $\boldsymbol{\sigma}$, we would only get $O(h^{k+1})$ and not $O(h^{k+2})$ convergence in $\boldsymbol{\sigma}$.

If, instead, we take $\mathcal{Q}^{k-1}$ for pressures, then the stability in h is preserved, but the inf-sup conditions become $\alpha(N) \geq Ck^{-1}$, $\beta(N) \geq \beta > 0$. This element is less optimal in h for both $\boldsymbol{\sigma}$ *and* $\boldsymbol{u}$. Finally, for triangles, we may take

$$\mathcal{P}^k \text{ (stresses) - } \mathcal{P}^{k+1} \text{ (velocity) - } \mathcal{P}^{k-2} \text{ (pressures)} ,$$

for which we can prove $\alpha(N) \geq Ck^{-6}$, $\beta(N) \geq \beta > 0$.

(6) A **stable** and **optimal** choice (in **both** h and k) with **continuous** stresses is obtained by using the modified EVSS method and taking (on parallelograms)

$$\mathcal{Q}^{k'} \text{ (stresses) - } \mathcal{Q}^{k+1} \text{ (velocity) - } \mathcal{P}^k \text{ (pressures)} .$$

In comparison with the stable element in (5) above, although a smaller space is used for $\boldsymbol{\sigma}$, we now have an extra (stress) unknown $\boldsymbol{d}$, and we get non-symmetric matrices.

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References

[1] I. Babuška and B.Q. Guo. Regularity of the solution of elliptic problems with piecewise analytic data. Part I and II, *SIAM J. Math. Anal.*, **19**, (1988), 172-203 and **20**, (1989), 763-781.

[2] C. Bernardi and Y. Maday: Approximations spectrales de problèmes aux limites elliptiques. Springer Verlag, Paris, (1992).

[3] C. Bernardi and Y. Maday. Preprint, 1997.

[4] E. Boillat. On the right inverse of the divergence operator in spaces of continuous, piecewise polynomials. Preprint 1995, Dept. of Mathematics, EPFL, in press in Math. Meth. & Mod. Appl. Sci., (1998).

[5] F. Brezzi and M. Fortin. Mixed and Hybrid Finite Element Methods. Springer Verlag, Heidelberg, (1991).

[6] L.K. Chilton. Locking-free mixed *hp* finite element methods for linear and geometrically non-linear elasticity. Ph.D. Dissertation, Department of Mathematics and Statistics, UMBC, Baltimore, MD, (1997).

[7] L.K. Chilton and M. Suri. On the selection of a locking-free *hp* element for elasticity problems. *Int. J. Num. Meth. Engg.*, **40**, (1997), 2045-2062.

[8] M.J. Crochet and J.M. Marchal. A new mixed finite element for calculating viscoelastic flows. *J. Non-Newtonian Fluid. Mech.*, **26**, (1987), 77-114.

[9] L. Demkowicz, K. Gerdes, C. Schwab, A. Bajer and T. Walsh. HP90: A general & flexible *hp*-FE code in Fortran 90. In preparation.

[10] M. Fortin and R. Pierre. On the convergence of the mixed method of Crochet and Marchal for viscoelastic flows. *CMAME*, **73** (1989), 341-350.

[11] M. Fortin and R. Guenette. A new mixed finite element method for computing viscoelastic flows. JNNFM, **60**, (1995), 27-52.

[12] M. Fortin, R. Guenette and R. Pierre. Numerical analysis of the modified EVSS method. Preprint Dept. Math., Univ. Laval, Quebec, (1996).

[13] V. Girault and P.A. Raviart. Finite Elements for Navier Stokes Equations. Springer-Verlag (1986).

[14] B. Gui and I. Babuška: The *h*-, *p*- and *hp*-FEM. *Num. Math.*, **49**, (1985), 577-683.

[15] B.Q. Guo and I. Babuška. The *hp*-version of the FEM I: The basic approximation results, II: Further Results and Applications. *Comp. Mech.*, **1** (1986), 21–41 and 203–220 .

[16] B.Q. Guo and I. Babuška. On the regularity of elasticity problems with piecewise analytic data. *Adv. Appl. Math.*, **14**, (1993), 307-347.

[17] B. Khomami, K.K. Talwar and H.K. Ganpule. A comparative study of higher- and lower-order finite element techniques for computation of viscoelastic flows. *J. Rheol.*, **38** (2) (1994), 255–289.

[18] B. Khomami. Personal communication. (1996).

[19] V.A. Kozlov, V.G. Maz'ya and C. Schwab. On singularities of solutions to the Dirichlet Problem of hydrodynamics near the vertex of a cone. *J. reine angew. Math.*, **456** (1994), 65-97.

[20] G.G. Lipscomb, R. Keunings and M.M. Demkovicz. Implication of boundary singularities in complex geometries. *J. Non-Newt. Fluid Mech.*, **24** (1987), 85–96.

[21] V.G. Maz'ya and B.A. Plamenevskii. The first boundary value problem for the classical equations of mathematical physics on piecewise smooth domains, parts I and II. *Z. Anal. Anw.*, **2** (1983), 335–359 and 523–551.

[22] R.A. Nicolaides. Existence, uniqueness and approximation for generalized saddle point problems. *SIAM J. Numer. Anal.*, **19**, (1982), 349-357.

[23] E.T. Olsen and J. Douglas Jr. Bounds on spectral condition numbers of matrices arising in the *p*-version of the FEM. *Numer. Math.*, **69**, (1995), 333-352.

[24] D. Rajagopalan, R.A. Brown and R.C. Armstrong. Finite Element Methods for calculation of steady viscoelastic flow using constitutive equations with a Newtonian viscosity. *J. Non-Newtonian Fluid Mech.*, **36**, (1990), 159-199.

[25] D. Schötzau: Doctoral Dissertation, ETH Zürich (to appear).

[26] D. Schötzau, C. Schwab and R. Stenberg. To appear.

[27] C. Schwab. Lecture Notes on *hp*- and spectral FEM. ETH Zürich, (1996).

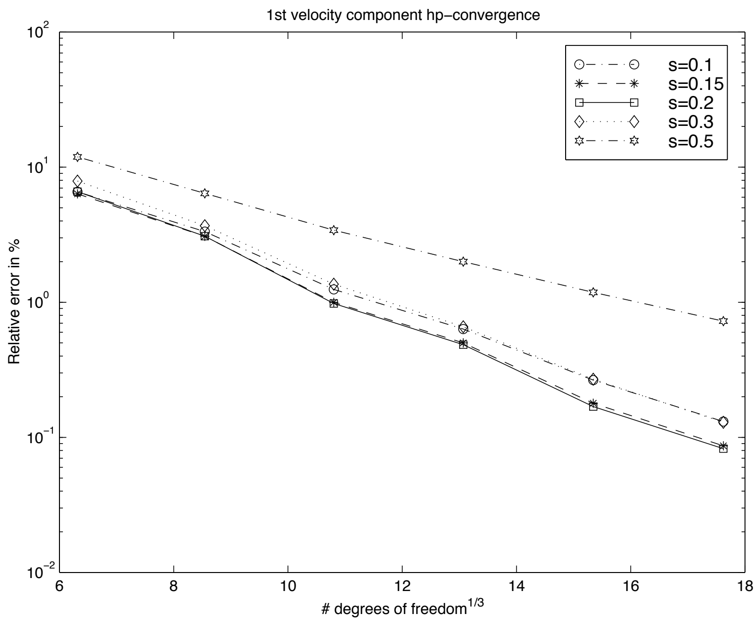
[28] R. Stenberg. Error Analysis of some FEM for the Stokes Problem. *Math. Comp.*, **54** (1990), 495-508.

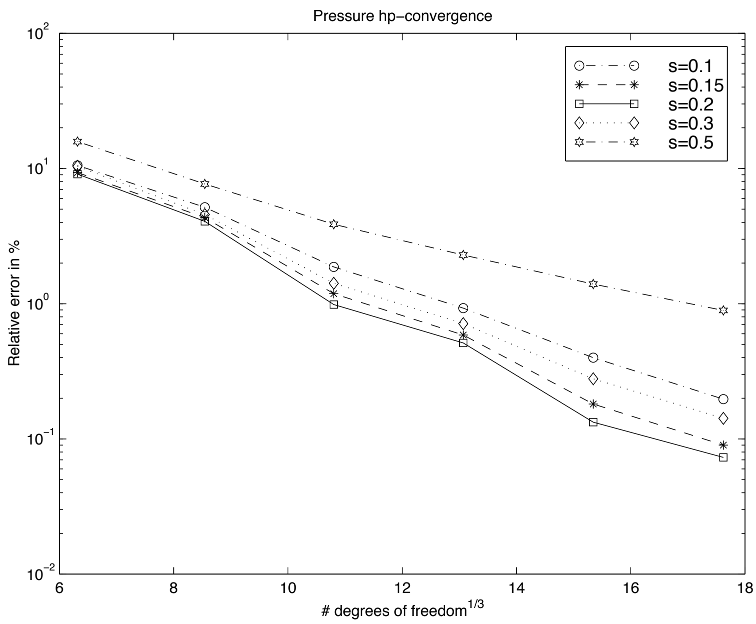
[29] R. Stenberg and M. Suri. *hp*-Mixed FEM for elasticity and Stokes flow. *Numer. Math.*, **72** (1996), 367-389.

[30] K.K. Talwar, H.K. Ganpule and B. Khomami. A note on selection of spaces in computation of viscoelastic flows using the *hp*-FEM. *J. Non-Newtonian Fluid Mech.*, **52**, (1994), 293-307.

[31] K.K. Talwar and B. Khomami. Accuracy and Convergence of the *p*- and *hp*-type FEM for the Stokes and Navier Stokes Equation. *AICHE Journal*, **38**, No. 1, (1992), 83-91.

[32] F. Yurun and M.J. Crochet. High-order finite element methods for steady viscoelastic flows. *J. Non-Newt. Fluid Mech.*, **57** (1995), 283-311.


1st velocity component hp−convergence
Relative error in %
# degrees of freedom^1/3
s=0.1
s=0.15
s=0.2
s=0.3
s=0.5


Pressure hp−convergence
Relative error in %
# degrees of freedom^1/3
s=0.1
s=0.15
s=0.2
s=0.3
s=0.5

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