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Mixed *hp*-FEM on anisotropic meshes

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Abstract

Mixed *hp*-FEM for incompressible fluid flow on anisotropic meshes are analyzed. A discrete inf-sup condition is proved with a constant independent of the meshwidth and the aspect ratio. For each polynomial degree $k \geq 2$, velocity-pressure subspace pairs are presented which are stable on quadrilateral mesh-patches, independently of the element aspect ratio implying in particular divergence stability on the so-called Shishkin-meshes. Moreover, the inf-sup constant is shown to depend on the spectral order k like $k^{-1/2}$ for quadrilateral meshes and like k^{-3} for meshes containing triangles. New consistency results for spectral elements on anisotropic meshes are also proved.

1 Introduction

Many problems in fluid mechanics exhibit, as is well known (see, e.g., [1], [9] or [10]), *boundary layers*. These are flowfields with rapid variation normal to the boundary and smooth behaviour tangentially to it. The efficient numerical approximation of boundary layers requires therefore anisotropic meshes (e.g. so called "viscous meshes") which involve cells of arbitrary high aspect ratio. For any FE discretization of viscous, incompressible flow with velocity spaces that are not discretely divergence-free stability amounts, as is well known (e.g. [5, 7, 15]), to the satisfaction of a discrete inf-sup condition by the velocity and pressure spaces. For many pairs of velocity/pressure spaces, this condition has been established (see [19, 5] and the references there for h-version FEM; and [21, 20, 17, 4] and the references there for p-version/spectral FEM). Nevertheless, all presently available techniques for establishing the divergence stability of velocity/pressure space pairs seem to require the *shape regularity* of the elements in some form. This precludes, of course, anisotropic boundary layer meshes as described above. Recently, some attention has been turned to this issue and it has been proved in [3] that a certain nonconforming element (the $\widetilde{Q}_1 \times P_0$ element) is, on axiparallel meshes, indeed divergence stable independent of the element aspect ratio. The result, however, seems to be limited to such meshes and elements of degree 1, as in numerical experiments divergence stability was lost on affinely mapped, anisotropic meshes that are not aligned with the coordinate axes.
To present an affine family of conforming velocity/pressure space pairs of any polynomial degree resp. spectral order k that are divergence stable on possibly mapped, stretched grids independently of the meshwidth h and of the element aspect ratio σ is the purpose of the present paper. In fact, the considered family are the "$\mathcal{P}_N \times \mathcal{P}_{N-2}$" elements already discussed in [4]. We prove that the inf-sup constant for such pairs is independent of h and σ and is bounded from below by $Ck^{-1/2}$ (we admit anisotropic polynomial degrees in different directions, thereby allowing for general anisotropic hp-refinements).
The plan of this paper is as follows. In Section 2, we briefly formulate the Stokes problem, collect some standard a priori estimates for mixed Finite Elements and we introduce the notion of boundary layer meshes. In Section 3, patchwise stability results are proved. Some numerical estimates of inf-sup constants on patches confirm our results. Section 4 states our main result whereas Section 5 illustrates the use of boundary layer meshes to approximate viscous boundary layers. Some new anisotropic hp-FEM/spectral element approximation results are also presented.
The usual notation is used in this paper: For a polygonal domain $D \subseteq \mathbb{R}^2$ or an interval $D = (a, b)$ we denote by $H^k(D)$ the Sobolev spaces of integer orders $k \geq 0$ equipped with the usual norms $\|\cdot\|_{k,D}$ and seminorms $|\cdot|_{k,D}$, $H^0(D) = L^2(D)$, $H^1_0(D) = \{u \in H^1(D) : \text{trace(u)} = 0 \text{ on } \partial D\}$, $L^2_0(D) = \{p \in L^2(D) : (p, 1)_D = 0\}$ where $(\cdot, \cdot)_D$ denotes the $L^2(D)$ inner product. For $s \geq 0$ nonintegral, the Sobolev spaces $H^s(D)$ with norm $\|\cdot\|_{s,D}$ are defined as usually via the K-method of interpolation (see, e.g., [22] or [11]). We will deal additionally with the Sobolev space $H^{1/2}_{00}(I)$, $I = (a, b)$ an interval, which are also defined by the K-method of interpolation [22], i.e.

$$H^{1/2}_{00}(I) = \left(L^2(I), H^1_0(I)\right)_{1/2,2}.$$

The interpolation norm on this space is denoted by $\|\cdot\|_{1/2,00,I}$. This norm is equivalent to the following norm which we denote also by $\|\cdot\|_{1/2,00,I}$ (see [11, Section I.11.5])

$$\|u\|_{1/2,00,I} = \left(\|u\|_{1/2,I}^2 + \left\| \rho^{-\frac{1}{2}} u \right\|_{0,I}^2 \right)^{\frac{1}{2}}$$

where $\rho(x) = (x-a)(b-x)$. Let now $I = (-1,1)$ and $D = I \times I$. In Section 5 we deal for $m, n \in \mathbb{N}$ with the tensor product spaces $H^{m,n}(D) := H^m(I, H^n(I))$ as defined e.g. in [11, Chapter I] or [4, Remarque I.1.18]. We write also u_x and u_y for the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial x}$. The set of all polynomials of total degree $\leq k$ on $D \subseteq \mathbb{R}^2$ will be denoted by $\mathcal{P}_k(D)$, the set of all polynomials of degree $\leq r$ in the first variable and of degree $\leq s$ in the second by $\mathcal{Q}_{r,s}(D)$. We write shortly $\mathcal{Q}_k(D)$ for $\mathcal{Q}_{k,k}(D)$. If $I = (a,b)$ is again an interval we define $\mathcal{P}_k(I)$ as the set of polynomials on I of degree $\leq k$. In the following we denote by C generic constants independent of the meshwidth, the polynomial degree and the element aspect ratio, but not necessarily identical at different places.

2 Problem Formulation

2.1 Stokes Problem

In a bounded, polygonal domain $\Omega \subset \mathbb{R}^2$ we consider the Stokes boundary value problem for incompressible fluid flow: Find a velocity field $\overline{u} \in [H_0^1(\Omega)]^2$ and a pressure $p \in L_0^2(\Omega)$ such that

$$\nu \left(\nabla \overline{u}, \nabla \overline{v}\right)_\Omega - \left(p, \nabla \cdot \overline{v}\right)_\Omega = \left(\overline{f}, \overline{v}\right)_\Omega, \qquad (1)$$

$$\left(q, \nabla \cdot \overline{u}\right)_\Omega = 0 \qquad (2)$$

for all $(\overline{v}, q) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$. Hereby, $\nu > 0$ is the kinematic viscosity which is related to the Reynolds number Re of the flow by $\nu = 1/Re$. It is well-known (see, e.g., [7]) that for every $\overline{f} \in [L^2(\Omega)]^2$ there exists a unique weak solution $(\overline{u}, p)$ of (1), (2) due to the continuous *inf-sup condition*

$$\inf_{0 \neq p \in L_0^2(\Omega)} \sup_{0 \neq \overline{v} \in \left[H_0^1(\Omega)\right]^2} \frac{(\nabla \cdot \overline{v}, p)_\Omega}{|\overline{v}|_{1,\Omega} \|p\|_{0,\Omega}} \geq C(\Omega) > 0. \qquad (3)$$

A conforming FE-discretization of (1), (2) is obtained in the usual way: Given finite dimensional subspaces $\overline{V}_N \subset [H_0^1(\Omega)]^2$ and $M_N \subset L_0^2(\Omega)$, find $(\overline{u}_N, p_N) \in \overline{V}_N \times M_N$ such that (1), (2) holds for any $(\overline{v}, q) \in \overline{V}_N \times M_N$. A family $\{\overline{V}_N \times M_N\}_N$ is $\gamma(N)$-stable, if the following *discrete inf-sup condition* holds

$$\inf_{0 \neq p \in M_N} \sup_{0 \neq \overline{v} \in \overline{V}_N} \frac{(\nabla \cdot \overline{v}, p)_\Omega}{|\overline{v}|_{1,\Omega} \|p\|_{0,\Omega}} \geq \gamma(N) > 0. \qquad (4)$$

If $\gamma(N)$ in (4) does not depend on N, we say that the family $\{\overline{V}_N \times M_N\}_N$ is stable. If a familiy is $\gamma(N)$-stable, the rate of convergence of the FE approximations $\{(\overline{u}_N, p_N)\}_N$ of

$(\overline{u}, p)$ is determined by that of the best approximations of $(\overline{u}, p)$ in $\left\{\overline{V}_N \times M_N\right\}_N$, i.e. we have the error estimates [5]

$$\begin{aligned} \|\overline{u} - \overline{u}_N\|_{1,\Omega} &\leq C\gamma^{-1}(N) \inf_{\overline{v} \in \overline{V}_N} \|\overline{u} - \overline{v}\|_{1,\Omega} + C \inf_{q \in M_N} \|p - q\|_{0,\Omega}\,, \qquad (5) \\ \|p - p_N\|_{0,\Omega} &\leq C\gamma^{-2}(N) \inf_{\overline{v} \in \overline{V}_N} \|\overline{u} - \overline{v}\|_{1,\Omega} + C\gamma^{-1}(N) \inf_{q \in M_N} \|p - q\|_{0,\Omega} \qquad (6) \end{aligned}$$

with C independent of N. The following lemma due to Fortin (see, e.g., [5] or [15]) is a useful tool for establishing the discrete inf-sup condition.

Lemma 2.1. *Assume that the continuous inf-sup condition (3) holds with a constant* $C_1 = C_1(\Omega) > 0$. *Assume further that there exists a projector* $\overline{\Pi}_N : [H^1_0(\Omega)]^2 \to \overline{V}_N$ *such that for every* $\overline{v} \in [H^1_0(\Omega)]^2$

$$\begin{aligned} (\nabla \cdot \overline{v}, p)_\Omega &= \left(\nabla \cdot \overline{\Pi}_N \overline{v}, p\right)_\Omega = 0 \qquad \forall p \in M_N, \\ \left|\overline{\Pi}_N \overline{v}\right|_{1,\Omega} &\leq C_N \left|\overline{v}\right|_{1,\Omega}. \end{aligned}$$

Then the discrete inf-sup condition (4) holds with $\gamma(N) = C_1/C_N$.

2.2 Finite Element Spaces

We define the pairs $\overline{V}_N \times M_N$ to be analyzed below.

2.2.1 Meshes

A partition $\mathcal{T}$ of Ω into quadrilateral and/or triangle elements $\{K\}$ is called a *regular* mesh on Ω if the intersection of any two elements K and $K' \in \mathcal{T}$ is either empty, a vertex or an entire side. It is κ*-regular* if

$$0 < \kappa \leq \max_{K \in \mathcal{T}} \frac{h_K}{\rho_K} \leq \kappa^{-1} < \infty \qquad (7)$$

where $h_K = \text{diam}(K)$ and where

$$\rho_K = \sup\left\{2r : B_r \subseteq \overline{K}\right\}$$

is the diameter of the largest circle B_r inscribed into K. $\mathcal{T}$ is an *affine mesh* if each $K \in \mathcal{T}$ is affine equivalent to a reference element $\hat{K}$ which is either the square $\hat{Q} = (-1, 1)^2$ or the triangle $\hat{T} = \{(x, y) : 0 < x < 1, 0 < y < x\}$, i.e.

$$K = F_K(\hat{K}), \quad F_K(\cdot) \text{ affine}.$$

Definition 2.2. *Consider a (coarse)* κ*-regular affine mesh* $\mathcal{T}_m$ *on* Ω *which is split into three parts* $\mathcal{T}^{BL}_m$, $\mathcal{T}^C_m$ *and* $\mathcal{T}^I_m$*; the boundary layer patches, the corner patches and the interior patches, respectively, where* $\mathcal{T}^{BL}_m$ *consists only of parallelograms. The (finer) regular affine mesh* $\mathcal{T}$ *is called a* boundary layer mesh *with* macro-element mesh $\mathcal{T}_m$ *if it is obtained from* $\mathcal{T}_m$ *by the following refinements:*

- *Interior patches* $K \in \mathcal{T}^I_m$ *and corner patches* $K \in \mathcal{T}^C_m$ *are partitioned into* κ'*-regular subtriangles and/or -quadrilaterals for some* $\kappa' \geq \kappa/2$.

- *Boundary layer patches $K \in \mathcal{T}_m^{BL}$ are anisotropically refined as follows: Let $\mathcal{T}_x^K$ be a mesh on $(-1,1)$ (depending on K) given by a partition $\{K_x\}$ of $(-1,1)$ into subintervals. Consider the corresponding product mesh on $\hat{Q} = (-1,1)^2$*

$$\hat{\mathcal{T}}^K = \left\{ \hat{K} : \hat{K} = K_x \times (-1,1), \quad K_x \in \mathcal{T}_x^K \right\}$$

(cf. Figure 3.3). Then K is partitioned into $\left\{ F_K(\hat{K}) \right\}_{\hat{K} \in \hat{\mathcal{T}}^K}$.

Some or all quadrilateral elements $K \in \mathcal{T}_m$ are therefore partitioned into "strips". Typically, the partition $\mathcal{T}_m$ of Ω into the macro-elements will be as in Figure 2.1 and the boundary layer mesh is then obtained by anisotropic subdivision of the boundary layer patches. In Figure 2.2 we show boundary layer meshes near corners. The macro-element

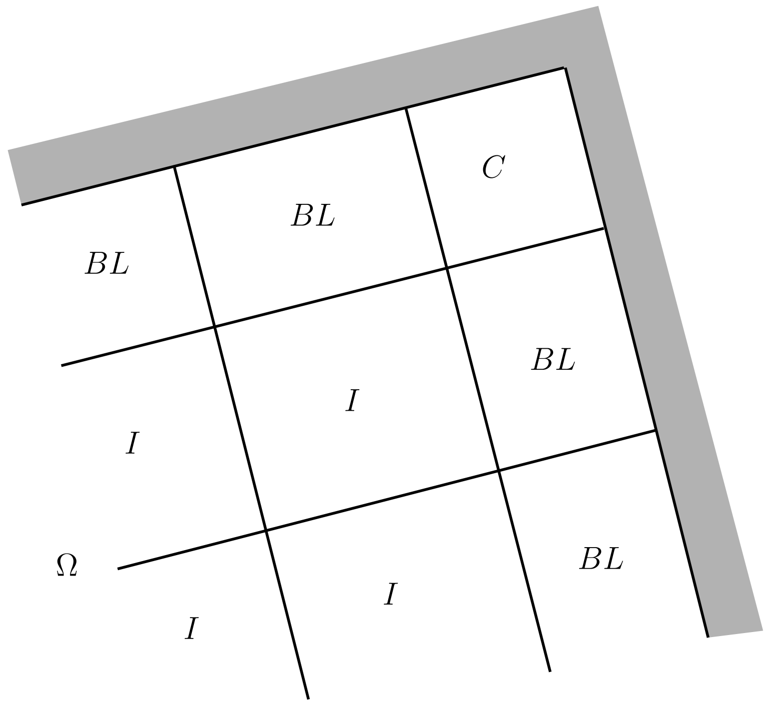

Figure 2.1: Partition of Ω into boundary layer, corner and interior patches. The abbreviations BL, C, I indicate the sets $\mathcal{T}_m^{BL}$, $\mathcal{T}_m^{C}$ and $\mathcal{T}_m^{I}$, respectively.

partitions are indicated by bold lines. Note the geometric mesh refinement in the corner patches towards O, which is required to resolve corner singularities. The refinements in $\mathcal{T}_m^{BL}$ towards the boundary require corresponding refinements towards P_1 and P_2 in the corner patch to ensure the κ'-regularity of the corner meshes.

Remark 2.3. It would be natural to use tensor products of geometrically refined boundary layer patches in the corners of the domain. It seems that one can not prove divergence stability of such meshes with the methods used in this paper. However, by techniques for meshes with hanging nodes it is possible to obtain divergence stability on such mesh-patches with an inf-sup constant only depending on the geometrical grading factor. This will be presented in [14]. Note that we use here only one quadrilateral corner patch. But it is of course also possible to choose the boundary layer patches perpendicular to $\partial\Omega$ and to introduce several triangular/quadrilateral corner patches.

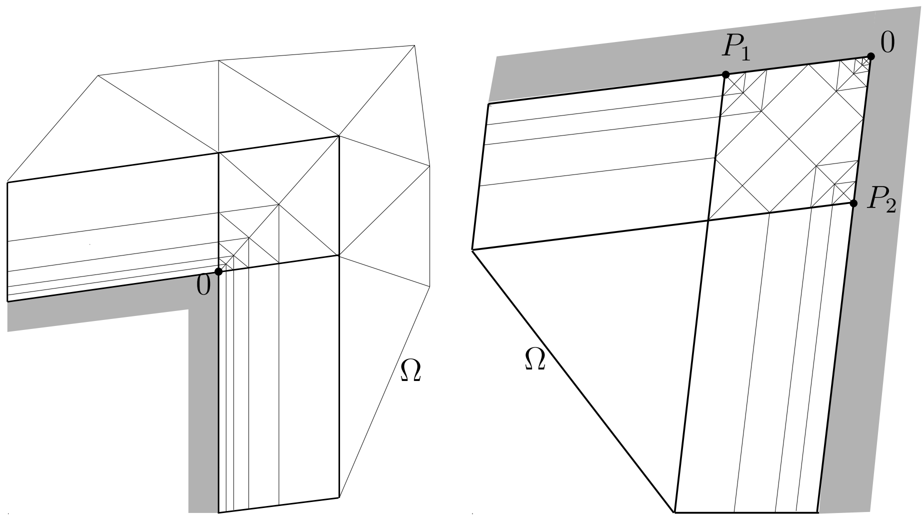

Figure 2.2: Boundary layer mesh patches near reentrant or convex corner.

2.2.2 Polynomial degree distribution

Let $\mathcal{T}$ be a regular affine mesh on Ω. With each quadrilateral element $K \in \mathcal{T}$ we associate two polynomial degrees r_K and s_K and set, if K is a triangle, $r_K = s_K =: k_K$. We combine all polynomial degrees in degree vectors

$$\underline{r} = \{r_K : K \in \mathcal{T}\}, \qquad \underline{s} = \{s_K : K \in \mathcal{T}\}.$$

We denote by $V^{r,s}(\hat{K})$ and $M^{r,s}(\hat{K})$ generic velocity and pressure spaces on the reference element $\hat{K}$ (specific examples will follow shortly) and define

$$S^{\underline{r},\underline{s},1}(\Omega,\mathcal{T}) = \left\{u \in H^1(\Omega) : u|_K \circ F_K \in V^{r_K,s_K}(\hat{K}), K \in \mathcal{T}\right\}, \tag{8}$$

$$S^{\underline{r},\underline{s},0}(\Omega,\mathcal{T}) = \left\{q \in L^2(\Omega) : u|_K \circ F_K \in M^{r_K,s_K}(\hat{K}), K \in \mathcal{T}\right\}. \tag{9}$$

If the two polynomial degrees are the same (i.e. $r_K = s_K =: k_K\ \forall K \in \mathcal{T}$) we use the shorthand notation $S^{\underline{k},1}(\Omega,\mathcal{T})$ and $S^{\underline{k},0}(\Omega,\mathcal{T})$, and if one or both of the degrees are constant (e.g. $s_K =: s\ \forall K \in \mathcal{T}$) we write analogously $S^{\underline{r},s,0}(\Omega,\mathcal{T})$ etc. We define further

$$S_0^{\underline{r},\underline{s},1}(\Omega,\mathcal{T}) = S^{\underline{r},\underline{s},1}(\Omega,\mathcal{T}) \cap H_0^1(\Omega), \qquad S_0^{\underline{r},\underline{s},0}(\Omega,\mathcal{T}) = S^{\underline{r},\underline{s},0}(\Omega,\mathcal{T}) \cap L_0^2(\Omega). \tag{10}$$

We will analyze the following specific subspaces: For $\hat{K} = \hat{Q}$ and $r, s \geq 2$ we choose in (8), (9)

$$V^{r,s}(\hat{Q}) = \mathcal{Q}_{r,s}(\hat{Q}), \qquad M^{r,s}(\hat{Q}) = \mathcal{Q}_{r-2,s-2}(\hat{Q}). \tag{11}$$

If especially $r = s =: k$ we see that the above spaces become

$$V^k(\hat{Q}) := V^{k,k}(\hat{Q}) = \mathcal{Q}_k(\hat{Q}), \qquad M^k(\hat{Q}) := M^{k,k}(\hat{Q}) = \mathcal{Q}_{k-2}(\hat{Q}). \tag{12}$$

If $\hat{K} = \hat{T}$ and $k \geq 2$ we set in (8), (9) $r_K = s_K = k_K$ and

$$V^k(\hat{T}) = \mathcal{P}_k(\hat{T}), \qquad M^k(\hat{T}) = \mathcal{P}_{k-2}(\hat{T}). \tag{13}$$

We note that in (11) - (13) the indices for the pressure spaces are shifted. Our purpose in the following sections is to establish the discrete inf-sup condition (4) for the above pairs with an inf-sup constant $\gamma(N) > 0$ independent of the aspect ratio of the elements in the boundary layer patches. (Though, γ will depend weakly on the polynomial degrees.) As usual, we will do this by first verifying local, i.e. patchwise inf-sup conditions; this is done in the following section. The local stability results are then used in Section 4 along with the fact that the discrete pressures are discontinuous to obtain the main results.

3 Local stability results

3.1 Stability on the reference triangle and square

Theorem 3.1. *Let $\hat{K} = \hat{Q}$, $k \geq 2$ and let the generic velocity and pressure spaces be given by (12). Then there exists a constant $C > 0$ independent of k such that*

$$\inf_{0 \neq p \in M_0^k(\hat{Q})} \sup_{0 \neq \overline{v} \in \left[V_0^k(\hat{Q})\right]^2} \frac{(\nabla \cdot \overline{v}, p)_{\hat{Q}}}{|\overline{v}|_{1,\hat{Q}} \, \|p\|_{0,\hat{Q}}} \geq C k^{-\frac{1}{2}} \tag{14}$$

where $V_0^k(\hat{Q}) := V^k(\hat{Q}) \cap H_0^1(\hat{Q})$, $M_0^k(\hat{Q}) := M^k(\hat{Q}) \cap L_0^2(\hat{Q})$. If $\hat{K} = \hat{T}$, $k \geq 2$ and the generic velocity and pressure spaces are given by (13), then there holds

$$\inf_{0 \neq p \in M_0^k(\hat{T})} \sup_{0 \neq \overline{v} \in \left[V_0^k(\hat{T})\right]^2} \frac{(\nabla \cdot \overline{v}, p)_{\hat{T}}}{|\overline{v}|_{1,\hat{T}} \, \|p\|_{0,\hat{T}}} \geq C k^{-3} \tag{15}$$

with C independent of k.

Proof. (14) is proved in [20] and (15) in [17]. □

Remark 3.2. While (14) is known to be optimal, (15) is likely suboptimal.

Remark 3.3. The meshes we use in the corner and interior patches of the macro-partition $\mathcal{T}_m$ are κ'-regular. Applying standard arguments (see, e.g., [7, Section II.1.4] or [15, 20]) we get from Theorem 3.1 divergence stability on these patches with inf-sup constants independent of the meshwidth used in the patches but depending on κ' and depending on the polynomial degree k as in (14) and (15).

3.2 Stability on boundary layer patches

In this section we consider a boundary layer patch on $\Omega = (-1, 1)^2$. Therefore, let $\mathcal{T}_x$ be an *arbitrary* one dimensional mesh on $I = (-1, 1)$, given by a partition of I into subintervals $\{K_x\}$. We associate with $\mathcal{T}_x$ the affine product mesh

$$\mathcal{T} = \{K : K = K_x \times I, \quad K_x \in \mathcal{T}_x\},$$

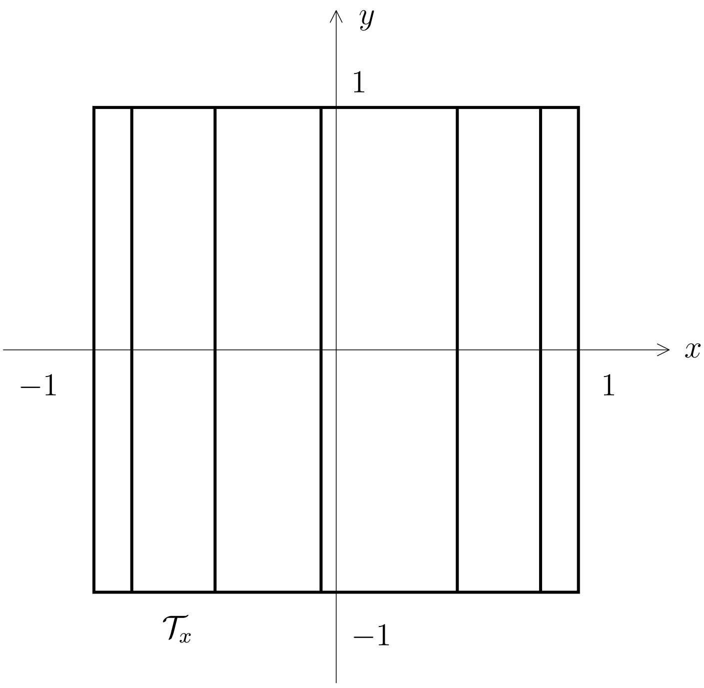

Figure 3.3: Anisotropic product mesh on $(-1,1)^2$.

as shown in Figure 3.3.

We want to prove the discrete divergence stability (4) for the pairings

$$\left\{\left[S_0^{\underline{r},s,1}(\Omega,\mathcal{T})\right]^2 \times S_0^{\underline{r},s,0}(\Omega,\mathcal{T})\right\}_{\underline{r},s}$$

in (8), (9) with an inf-sup constant independent of the mesh $\mathcal{T}_x$. Moreover, the dependence on $\underline{r}$ and s will be given explicitly. To this end, we use on the reference square the notations in Figure 3.4 and set

$$H(\hat{Q}) = \left\{v \in H^1(\hat{Q}) : v|_{\Gamma_1} = 0 = v|_{\Gamma_3}\right\}.$$

Note that for $v \in H(\hat{Q})$ we have $v|_{\Gamma_2} \in H_{00}^{1/2}(\Gamma_2)$ and $v|_{\Gamma_4} \in H_{00}^{1/2}(\Gamma_4)$. Further, we introduce a projection operator $\Pi_{r,s}^{\hat{Q}}$ on $H(\hat{Q})$ similar to the one in [21].

Definition 3.4. *For $v \in H(\hat{Q})$ and $r, s \geq 2$, $\Pi_{r,s}^{\hat{Q}} v$ is defined as the unique function in $\mathcal{Q}_{r,s}(\hat{Q}) \cap H(\hat{Q})$ satisfying the following equations:*

$$\left(\Pi_{r,s}^{\hat{Q}} v\right)(N_i) = 0, \qquad i = 1, \dots, 4 \tag{16}$$

$$\int_{\Gamma_i} \left(\Pi_{r,s}^{\hat{Q}} v\right)(s) q(s) ds = \int_{\Gamma_i} v(s) q(s) ds, \qquad \forall q \in \mathcal{P}_{r-2}(\Gamma_i), \quad i = 1, 3 \tag{17}$$

$$\int_{\Gamma_i} \left(\Pi_{r,s}^{\hat{Q}} v\right)(s) q(s) ds = \int_{\Gamma_i} v(s) q(s) ds, \qquad \forall q \in \mathcal{P}_{s-2}(\Gamma_i), \quad i = 2, 4 \tag{18}$$

$$\int_{\hat{Q}} \left(\Pi_{r,s}^{\hat{Q}} v\right)(\overline{x}) q(\overline{x}) d\overline{x} = \int_{\hat{Q}} v(\overline{x}) q(\overline{x}) d\overline{x}, \qquad \forall q \in \mathcal{Q}_{r-2,s-2}(\hat{Q}) \tag{19}$$

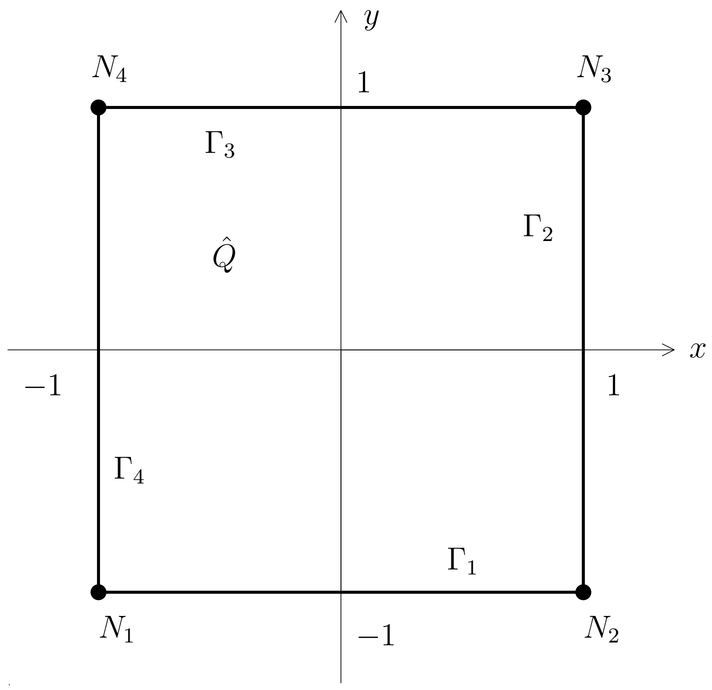

Figure 3.4: Reference square $\hat{Q}$ and its notations.

In fact, (16) and (17) are redundant, since $v \in H(\hat{Q})$ vanishes on $\Gamma_1 \cup \Gamma_3$. The heart of our analysis are anisotropic norm estimates of the operator $\Pi^{\hat{Q}}_{r,s}$ which are presented in the subsequent Theorem. The proof of this Theorem, however, is postponed to the next subsection.

Theorem 3.5. *Let $r, s \geq 2$. Then there exist constants $C > 0$ independent of r and s such that*

$$\left\| (\Pi^{\hat{Q}}_{r,s} v)_x \right\|^2_{0,\hat{Q}} \leq C s \left\| v_x \right\|^2_{0,\hat{Q}} \tag{20}$$

$$\left\| (\Pi^{\hat{Q}}_{r,s} v)_y \right\|^2_{0,\hat{Q}} \leq C r \left\| v_y \right\|^2_{0,\hat{Q}} + C \frac{s^2}{r} \left(\left\| v \right\|^2_{1/2,00,\Gamma_2} + \left\| v \right\|^2_{1/2,00,\Gamma_4} \right) \tag{21}$$

for all $v \in H(\hat{Q})$.

The trace operator $t : H(\hat{Q}) \to H^{1/2}_{00}(\Gamma_2) \times H^{1/2}_{00}(\Gamma_4)$ is continuous (cf. [8, Theorem 1.5.2.3]). Hence, there exists a constant $C > 0$ just depending on the reference square $\hat{Q}$ such that

$$\left\| v \right\|^2_{1/2,00,\Gamma_2} + \left\| v \right\|^2_{1/2,00,\Gamma_4} \leq C \left\| v \right\|^2_{1,\hat{Q}} \leq C \left| v \right|^2_{1,\hat{Q}} ,$$

where we used also the Poincaré inequality. We conclude with (20) and (21) that

$$\left\| (\Pi^{\hat{Q}}_{r,s} v)_y \right\|^2_{0,\hat{Q}} \leq C r \left\| v_y \right\|^2_{0,\hat{Q}} + C \frac{s^2}{r} \left| v \right|^2_{1,\hat{Q}} . \tag{22}$$

Using the projection operator $\Pi^{\hat{Q}}_{r,s}$ on the reference square we define on $K \in \mathcal{T}$

$$\Pi^K_{r,s} : H(K) \to \mathcal{Q}_{r,s}(K) \cap H(K) \quad \text{by} \quad \Pi^K_{r,s} v = \left[\Pi^{\hat{Q}}_{r,s} (v \circ F_K) \right] \circ F_K^{-1}.$$

Here $H(K) = \left\{ v \in H^1(K) : v \circ F_K \in H(\hat{Q}) \right\}$.
The estimates of Theorem 3.5 can be anisotropically scaled as follows.

Corollary 3.6. *Let $K \in \mathcal{T}$ and $r, s \geq 2$. Then there exists a constant $C > 0$ independent of r, s and the element K such that*

$$\left|\Pi^K_{r,s} v\right|^2_{1,K} \leq C \left(\max(r,s) + \frac{s^2}{r} \right) |v|^2_{1,K} \qquad \forall v \in H(K). \tag{23}$$

Proof. K is of the form $(x_1, x_2) \times (-1, 1)$, where $-1 \leq x_1 < x_2 \leq 1$. Hence, the affine transformation $F_K : \hat{Q} \to K$ is given by

$$x = h\hat{x} + \frac{1}{2}(x_1 + x_2), \qquad y = \hat{y},$$

where we denote by $\hat{x}$ and $\hat{y}$ the coordinates on $\hat{Q}$. Further, $h = \frac{1}{2}(x_2 - x_1)$. Scaling gives

$$\begin{aligned}
\|(v \circ F_K)_{\hat{x}}\|^2_{0,\hat{Q}} &= h \|v_x\|^2_{0,K}, \qquad \left\|(\Pi^{\hat{Q}}_{r,s}[v \circ F_K])_{\hat{x}}\right\|^2_{0,\hat{Q}} = h \left\|(\Pi^K_{r,s} v)_x\right\|^2_{0,K} \\
\|(v \circ F_K)_{\hat{y}}\|^2_{0,\hat{Q}} &= \frac{1}{h} \|v_y\|^2_{0,K}, \qquad \left\|(\Pi^{\hat{Q}}_{r,s}[v \circ F_K])_{\hat{y}}\right\|^2_{0,\hat{Q}} = \frac{1}{h} \left\|(\Pi^K_{r,s} v)_y\right\|^2_{0,K}.
\end{aligned}$$

Since further $|v \circ F_K|^2_{1,\hat{Q}} \leq \frac{1}{h} |v|^2_{1,K}$, the inequalities (20) and (22) yield for $K \in \mathcal{T}$

$$\begin{aligned}
h \left\|(\Pi^K_{r,s} v)_x\right\|^2_{0,k} &\leq hCs \|v_x\|^2_{0,K} \\
\frac{1}{h} \left\|(\Pi^K_{r,s} v)_y\right\|^2_{0,k} &\leq \frac{1}{h} Cr \|v_y\|^2_{0,K} + \frac{1}{h} C \frac{s^2}{r} |v|^2_{1,K},
\end{aligned}$$

from which (23) follows. ☐

If $\overline{v} = (v_1, v_2) \in [H(K)]^2$ we set $\overline{\Pi}^K_{r,s} \overline{v} = \left(\Pi^K_{r,s} v_1, \Pi^K_{r,s} v_2 \right)$. Another very important property of $\Pi^K_{r,s}$ in order to prove divergence stability is the following one.

Proposition 3.7. *Let $K \in \mathcal{T}$, $\overline{v} \in [H(K)]^2$ and $r, s \geq 2$. Then*

$$(\nabla \cdot \overline{v}, p)_K = \left(\nabla \cdot \overline{\Pi}^K_{r,s} \overline{v}, p \right)_K \qquad \forall p \in \mathcal{Q}_{r-2,s-2}(K). \tag{24}$$

Proof. By the formula of Green we see that for $p \in \mathcal{Q}_{r-2,s-2}(K)$

$$(\nabla \cdot \overline{v}, p)_K = -(\overline{v}, \nabla p)_K + (\overline{v} \cdot \overline{n}, p)_{\partial K},$$

$\overline{n}$ being the unit exterior normal to ∂K. Since $\nabla p \in [\mathcal{Q}_{r-2,s-2}(K)]^2$ and the mapping F_K is affine, we get from Definition 3.4

$$(\overline{v}, \nabla p)_K = \left(\overline{\Pi}^K_{r,s} \overline{v}, \nabla p \right)_K, \qquad (\overline{v} \cdot \overline{n}, p)_{\partial K} = \left(\overline{\Pi}^K_{r,s} \overline{v} \cdot \overline{n}, p \right)_{\partial K}.$$

Integrating by parts again, the claim follows. ☐

Let now a polynomial degree distribution

$$\underline{r} = \{r_{K_x} : r_{K_x} \geq 2, K_x \in \mathcal{T}_x\}$$

on the one dimensional mesh $\mathcal{T}_x$ be given. For $s \geq 2$ we define the global projector

$$\overline{\Pi}_{\underline{r},s} : \left[H_0^1(\Omega)\right]^2 \to \left[S_0^{\underline{r},s,1}(\Omega, \mathcal{T})\right]^2 \tag{25}$$

elementwise via

$$\left(\overline{\Pi}_{\underline{r},s}\overline{v}\right)|_K = \overline{\Pi}^K_{r_{K_x},s}\left(\overline{v}|_K\right), \qquad K = K_x \times I.$$

Here we use the generic velocity and pressure spaces as defined in (11). Due to (16)-(18) continuity across the element boundaries is ensured. Hence, $\overline{\Pi}_{\underline{r},s}$ is well-defined. We state our main result in this section, a local inf-sup condition on the anisotropically refined boundary layer patch $(\Omega, \mathcal{T})$.

Theorem 3.8. *Let $\Omega = (-1,1)^2$ and let $\mathcal{T}_x$ be an* arbitrary *mesh on $I = (-1,1)$. Let $\underline{r} = \{r_{K_x} : r_{K_x} \geq 2, K_x \in \mathcal{T}_x\}$ be a polynomial degree distribution on $\mathcal{T}_x$. Let $\mathcal{T}$ be the product mesh $\mathcal{T}_x \times I$ and $s \geq 2$. Let the generic velocity and pressure spaces be given by (11). Then there exists a constant $C > 0$, independent of $\underline{r}$, s and $\mathcal{T}_x$, such that*

$$\left|\overline{\Pi}_{\underline{r},s}\overline{v}\right|^2_{1,\Omega} \leq C\left\{\max_{K_x\in\mathcal{T}_x}\left(\max(r_{K_x}, s)\right) + \frac{s^2}{\min\limits_{K_x\in\mathcal{T}_x} r_{K_x}}\right\}|\overline{v}|^2_{1,\Omega} \qquad \forall \overline{v} \in \left[H_0^1(\Omega)\right]^2. \tag{26}$$

Further, for $\overline{v} \in \left[H_0^1(\Omega)\right]^2$ we have

$$(\nabla\cdot\overline{v}, p)_\Omega = \left(\nabla\cdot\overline{\Pi}_{\underline{r},s}\overline{v}, p\right)_\Omega \qquad \forall p \in S_0^{\underline{r},s,0}(\Omega,\mathcal{T}). \tag{27}$$

If we set

$$M_N = S_0^{\underline{r},s,0}(\Omega,\mathcal{T}), \qquad \overline{V}_N = \left[S_0^{\underline{r},s,1}(\Omega,\mathcal{T})\right]^2$$

there holds the divergence stability

$$\inf_{0\neq p\in M_N}\sup_{0\neq\overline{v}\in\overline{V}_N}\frac{(\nabla\cdot\overline{v},p)_\Omega}{|\overline{v}|_{1,\Omega}\,\|p\|_{0,\Omega}} \geq C\left\{\max_{K_x\in\mathcal{T}_x}\left(\max(r_{K_x}, s)\right) + \frac{s^2}{\min\limits_{K_x\in\mathcal{T}_x} r_{K_x}}\right\}^{-\frac{1}{2}} \tag{28}$$

with $C > 0$ independent of $\underline{r}$, s and $\mathcal{T}_x$.

Proof. These statements are direct consequences of Lemma 2.1, Proposition 3.7 and Corollary 3.6. □

If in particular $r_K = r = s =: k$ for all $K \in \mathcal{T}$ we are able to establish the divergence stability for the elements $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ on $(\Omega, \mathcal{T})$ with an inf-sup constant independent of the mesh $\mathcal{T}_x$. In that case we write $\overline{\Pi}_k$ instead of $\overline{\Pi}_{\underline{r},s}$.

Corollary 3.9. *Let* $\Omega = (-1,1)^2$ *and let* $\mathcal{T}_x$ *be* any *mesh on* $I = (-1,1)$*. Let* $\mathcal{T}$ *be the product mesh* $\mathcal{T}_x \times I$ *and* $k \geq 2$*. The velocity and pressure spaces are given by (12). Then there exists a constant* $C > 0$*, independent of* k *and* $\mathcal{T}_x$*, such that*

$$\left|\overline{\Pi}_k \overline{v}\right|^2_{1,\Omega} \leq Ck \left|\overline{v}\right|^2_{1,\Omega} \qquad \forall \overline{v} \in \left[H^1_0(\Omega)\right]^2. \tag{29}$$

Further, for $\overline{v} \in \left[H^1_0(\Omega)\right]^2$ *we have*

$$(\nabla \cdot \overline{v}, p)_\Omega = \left(\nabla \cdot \overline{\Pi}_k \overline{v}, p\right)_\Omega \qquad \forall p \in S^{k,0}_0(\Omega, \mathcal{T}). \tag{30}$$

If we set

$$M_N = S^{k,0}_0(\Omega, \mathcal{T}), \qquad \overline{V}_N = \left[S^{k,1}_0(\Omega, \mathcal{T})\right]^2$$

there holds the divergence stability

$$\inf_{0 \neq p \in M_N} \sup_{0 \neq \overline{v} \in \overline{V}_N} \frac{(\nabla \cdot \overline{v}, p)_\Omega}{|\overline{v}|_{1,\Omega} \|p\|_{0,\Omega}} \geq Ck^{-\frac{1}{2}} \tag{31}$$

with $C > 0$ *independent of* k *and* $\mathcal{T}_x$*.*

Remark 3.10. We emphasize that the constant in (31) is independent of $\mathcal{T}_x$. This means that we can use for the mesh $\mathcal{T}_x$ rectangles of arbitrary high aspect ratio.

Remark 3.11. The divergence stability (31) implies immediately the same result if we set in (12)

$$V^k(\hat{Q}) = \mathcal{Q}_k(\hat{Q}), \qquad M^k(\hat{Q}) = \mathcal{P}_{k-2}(\hat{Q})$$

since in that case the pressure space has been reduced. Argueing analogously we see that (31) holds also in the case of continuous pressure spaces.

Remark 3.12. The inf-sup constant $\gamma(N)$ in (78) is completely independent of the one-dimensional meshes $\mathcal{T}_x$ used in the construction of the anisotropic refinements in the boundary layer patches in Definition 2.2. Though, one has to observe that the refinements in the boundary layer patches must be such that the κ'-regularity in the corner patches is satisfied. If the polynomial degree k is fixed and $\mathcal{T}_x$ is chosen to be a Shishkin mesh (see, e.g., [13]), Theorem 3.5 implies that the pairs $\overline{V}_N \times M_N$ of subspaces are divergence stable on two dimensional Shishkin meshes.

3.3 Proof of Theorem 3.5

We introduce in this subsection a projector slightly different from $\Pi^{\hat{Q}}_{r,s}$ in Definition 3.4, namely the following one:

Definition 3.13. *For* $r, s \geq 2$ *and* $v \in H^{1+\epsilon}(\hat{Q})$ *(*$\epsilon > 0$*),* $I_{r,s}v$ *is the unique function in* $\mathcal{Q}_{r,s}(\hat{Q})$ *satisfying the following* $(r+1)(s+1)$ *equations:*

$$(I_{r,s}v)(N_i) = v(N_i), \qquad i = 1, \ldots, 4 \tag{32}$$

$$\int_{\Gamma_i} (I_{r,s}v)(s) q(s) ds = \int_{\Gamma_i} v(s) q(s) ds, \qquad \forall q \in \mathcal{P}_{r-2}(\Gamma_i), \quad i = 1, 3 \tag{33}$$

$$\int_{\Gamma_i} (I_{r,s}v)(s) q(s) ds = \int_{\Gamma_i} v(s) q(s) ds, \qquad \forall q \in \mathcal{P}_{s-2}(\Gamma_i), \quad i = 2, 4 \tag{34}$$

$$\int_{\hat{Q}} (I_{r,s}v)(\overline{x}) q(\overline{x}) d\overline{x} = \int_{\hat{Q}} v(\overline{x}) q(\overline{x}) d\overline{x}, \qquad \forall q \in \mathcal{Q}_{r-2,s-2}(\hat{Q}) \tag{35}$$

Let $\mathcal{I}(\hat{Q})$ be the set of all polynomials on the reference square $\hat{Q} = (-1,1)^2$ and define

$$\mathcal{I}_0(\hat{Q}) = \left\{ v \in \mathcal{I}(\hat{Q}) : v(N_i) = 0,\ i = 1, \ldots, 4 \right\}.$$

We want to analyze $I_{r,s}$ defined on $\mathcal{I}(\hat{Q})$ following [20] and [21]. This is sufficient since we will prove Theorem 3.5 by a density argument. Observe also that

$$I_{r,s}|_{\mathcal{I}(\hat{Q}) \cap H(\hat{Q})} = \Pi_{r,s}^{\hat{Q}}|_{\mathcal{I}(\hat{Q}) \cap H(\hat{Q})}.$$

First, we establish some properties of the space $\mathcal{I}(\hat{Q})$.

3.3.1 The space $\mathcal{I}(\hat{Q})$

Let $\{L_i(x)\}_{i \geq 0}$ be the set of Legendre polynomials of degree $i \geq 0$ on $I = (-1,1)$. Define further $L_{-1} = L_{-2} \equiv 0$. For $i \geq 0$ set $\gamma_i = \frac{1}{2i+1}$ and $\gamma_{-1} = 1$. Note that

$$\int_I L_i(x) L_j(x) dx = 2\gamma_i \delta_{ij}, \qquad i \geq 0, j \geq 0. \tag{36}$$

Define for $i \in \mathbb{N}_0$

$$U_i(x) = \gamma_{i-1} \left(L_i(x) - L_{i-2}(x) \right), \qquad x \in I. \tag{37}$$

The following properties are well-known:

$$\begin{array}{l} U_0(x) = 1, \qquad U_1(x) = x, \\ U_i(x) = \int_{-1}^{x} L_{i-1}(x) dx \qquad i > 1, \\ U_i(\pm 1) = 0 \qquad i > 1, \\ U_i'(x) = L_{i-1}(x) \qquad i \geq 0. \end{array} \tag{38}$$

Since $\{U_i\}_{i=0}^k$ is a basis of $\mathcal{P}_k(I)$ and $\{U_i(x)U_j(y)\}_{0 \leq i \leq r, 1 \leq j \leq s}$ of $\mathcal{Q}_{r,s}(\hat{Q})$, each $v \in \mathcal{I}(\hat{Q})$ can uniquely be written in the form

$$v(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} U_i(x) U_j(y), \tag{39}$$

where only finitely many terms are nonzero. Note that for $v \in \mathcal{I}_0(\hat{Q})$ we have in (39)

$$a_{00} = a_{01} = a_{10} = a_{11} = 0. \tag{40}$$

For $v \in \mathcal{Q}_{r,s}(\hat{Q})$ we may write

$$v(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{s} b_{ij} U_i(x) U_j(y).$$

Lemma 3.14. *Let $v \in \mathcal{I}(\hat{Q})$ be given in the form (39). Then we have*

$$\|v_x\|_{0,\hat{Q}}^2 = 4\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\gamma_{i-1}\gamma_j\left(\gamma_{j-1}a_{ij}-\gamma_{j+1}a_{i,j+2}\right)^2 \tag{41}$$

$$\|v_y\|_{0,\hat{Q}}^2 = 4\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\gamma_i\gamma_{j-1}\left(\gamma_{i-1}a_{ij}-\gamma_{i+1}a_{i+2,j}\right)^2 \tag{42}$$

$$|v|_{1,\Gamma_1}^2 + |v|_{1,\Gamma_3}^2 = 4\sum_{i=1}^{\infty}\gamma_{i-1}\left(a_{i0}^2+a_{i1}^2\right) \tag{43}$$

$$|v|_{1,\Gamma_2}^2 + |v|_{1,\Gamma_4}^2 = 4\sum_{j=1}^{\infty}\gamma_{j-1}\left(a_{0j}^2+a_{1j}^2\right) \tag{44}$$

$$\|v\|_{0,\Gamma_1}^2 + \|v\|_{0,\Gamma_3}^2 = 4\sum_{i=0}^{\infty}\sum_{j=0}^{1}\gamma_i\left(a_{ij}\gamma_{i-1}-a_{i+2,j}\gamma_{i+1}\right)^2 \tag{45}$$

$$\|v\|_{0,\Gamma_2}^2 + \|v\|_{0,\Gamma_4}^2 = 4\sum_{i=0}^{1}\sum_{j=0}^{\infty}\gamma_j\left(a_{ij}\gamma_{j-1}-a_{i,j+2}\gamma_{j+1}\right)^2 \tag{46}$$

Proof. (41), (42), (43) and (44) are proven in [21] and [20]. Nevertheless, we present the proofs here for the sake of completeness. By changing the roles of x and y it suffices to show only (41), (43) and (45).
Proof of (41): For v in the from (39) we have with (38)

$$v_x = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}a_{ij}L_{i-1}(x)U_j(y). \tag{47}$$

Hence using (36),

$$\begin{aligned}
\|v_x\|_{0,\hat{Q}}^2 &= \sum_{i=1}^{\infty}2\gamma_{i-1}\int_{-1}^{1}\left(\sum_{j=0}^{\infty}a_{ij}U_j(y)\right)^2 dy\\
&= \sum_{i=1}^{\infty}2\gamma_{i-1}\left\{\underbrace{\sum_{j=0}^{\infty}a_{ij}^2\int_{-1}^{1}U_j(y)^2dy}_{=:A_i}+\underbrace{2\sum_{k<j}a_{ik}a_{ij}\int_{-1}^{1}U_k(y)U_j(y)dy}_{=:B_i}.\right\}
\end{aligned}$$

First we bound the terms $\{A_i\}$. Due to (37) we have

$$\begin{aligned}
A_i &= 2\sum_{j=0}^{\infty}a_{ij}^2\gamma_{j-1}^2\int_{-1}^{1}\left(L_j^2-2L_jL_{j-2}+L_{j-2}^2\right)\\
&= \sum_{j=0}^{\infty}2a_{ij}^2\gamma_{j-1}^2\gamma_j+\sum_{j=2}^{\infty}2a_{ij}^2\gamma_{j-1}^2\gamma_{j-2} = 2\sum_{j=0}^{\infty}\left(a_{ij}^2\gamma_{j-1}^2\gamma_j+a_{i,j+2}^2\gamma_{j+1}^2\gamma_j\right).
\end{aligned} \tag{48}$$

Similarly,

$$
\begin{aligned}
B_i &= \sum_{k<j} a_{ik} a_{ij} \gamma_{k-1} \gamma_{j-1} \int_{-1}^{1} \left(L_k - L_{k-2}\right) \left(L_j - L_{j-2}\right) dy \\
&= 2 \sum_{k=0}^{\infty} a_{ik} a_{i,k+2} \gamma_{k-1} \gamma_{k+1} \left(-2\gamma_k\right) = -4 \sum_{k=0}^{\infty} a_{ik} a_{i,k+2} \gamma_{k-1} \gamma_k \gamma_{k+1}.
\end{aligned} \tag{49}
$$

(48) and (49) imply

$$
\|v_x\|_{0,\hat{Q}}^2 = \sum_{i=1}^{\infty} 2\gamma_{i-1} \left\{ \sum_{j=0}^{\infty} 2a_{ij}^2 \gamma_{j-1}^2 \gamma_j - 4a_{ij} a_{i,j+2} \gamma_{j-1} \gamma_j \gamma_{j+1} + a_{i,j+2}^2 \gamma_{j+1}^2 \gamma_j \right\},
$$

which is (41).
Proof of (43): Inserting $y = \pm 1$ in (47) and using (38) we get

$$
v_x(x, \pm 1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_{i-1}(x) U_j(\pm 1) = \sum_{i=0}^{\infty} \left(a_{i0} \pm a_{i1}\right) L_{i-1}(x).
$$

Hence

$$
\int_{-1}^{1} v_x(x, \pm 1)^2 dx = \int_{-1}^{1} \left(\sum_{i=0}^{\infty} \left(a_{i0} \pm a_{i1}\right) L_{i-1}(x) \right)^2 dx = 2 \sum_{i=1}^{\infty} \left(a_{i0} \pm a_{i1}\right)^2 \gamma_{i-1}.
$$

We obtain

$$
\|v_x\|_{0,\Gamma_1}^2 + \|v_y\|_{0,\Gamma_3}^2 = 4 \sum_{i=1}^{\infty} \gamma_{i-1} \left(a_{i0}^2 + a_{i1}^2\right).
$$

This is (43).
Proof of (45): Due to (39) and (38) we have

$$
\begin{aligned}
\int_{-1}^{1} v(x, \pm 1)^2 dx &= \int_{-1}^{1} \left(\sum_{i,j=0}^{\infty} a_{ij} U_i(x) U_j(\pm 1) \right)^2 dx \\
&= \int_{-1}^{1} \left(\sum_{i=0}^{\infty} (a_{i0} \pm a_{i1}) U_i(x) \right)^2 dx \\
&= \sum_{i=0}^{\infty} (a_{i0} \pm a_{i1})^2 \int_{-1}^{1} U_i^2(x) dx \\
&+ 2 \sum_{k<l} (a_{k0} \pm a_{k1})(a_{l0} \pm a_{l1}) \int_{-1}^{1} U_k(x) U_l(x) dx
\end{aligned}
$$

such that

$$
\|v\|_{0,\Gamma_1}^2 + \|v\|_{0,\Gamma_3}^2 = 2 \sum_{j=0}^{1} \left\{ \underbrace{\sum_{i=0}^{\infty} a_{ij}^2 \int_{-1}^{1} U_i^2(x) dx}_{=:A_j} + \underbrace{2 \sum_{k<l} a_{kj} a_{lj} \int_{-1}^{1} U_k(x) U_l(x) dx}_{=:B_j} \right\}.
$$

Using (36) and (37) we get for the terms $\{A_j\}$

$$
\begin{aligned}
A_j &= \sum_{i=0}^{\infty} a_{ij}^2 \gamma_{i-1}^2 \int_{-1}^{1} \{L_i(x) - L_{i-2}(x)\}^2 \, dx \\
&= 2 \sum_{i=0}^{\infty} a_{ij}^2 \gamma_{i-1}^2 \gamma_i + 2 \sum_{i=2}^{\infty} a_{ij}^2 \gamma_{i-1}^2 \gamma_{i-2} \\
&= 2 \sum_{i=0}^{\infty} \left\{ a_{ij}^2 \gamma_{i-1}^2 \gamma_i + a_{i+2,j}^2 \gamma_{i+1}^2 \gamma_i \right\}.
\end{aligned}
$$

The second term treated analogously results in

$$
\begin{aligned}
B_j &= 2 \sum_{k<l} a_{kj} a_{lj} \gamma_{k-1} \gamma_{l-1} \int_{-1}^{1} \left(L_k(x) - L_{k-2}(x)\right)\left(L_l(x) - L_{l-2}(x)\right) dx \\
&= -4 \sum_{k=0}^{\infty} a_{kj} a_{k+2,j} \gamma_{k-1} \gamma_k \gamma_{k+1}
\end{aligned}
$$

Adding together, we get

$$
A_j + B_j = 2 \sum_{i=0}^{\infty} \gamma_i \left(a_{ij} \gamma_{i-1} - a_{i+2,j} \gamma_{i+1}\right)^2. \tag{50}
$$

The identity (50) implies (45). □

3.3.2 Norm estimates

Proposition 3.15. *We write $v \in \mathcal{I}(\hat{Q})$ in the form (39). Then*

$$
I_{r,s} v(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{s} a_{ij} U_i(x) U_j(y).
$$

Proof. Let $I_{r,s} v$ be given in the form

$$
I_{r,s} v(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{s} b_{ij} U_i(x) U_j(y).
$$

Because of (32) and (38) we obtain easily

$$
a_{ij} = b_{ij}, \qquad 0 \le i,j \le 1. \tag{51}
$$

Next, we consider the sides $y = \pm 1$: Let $2 \le n \le r$. Since $L'_{n-1} \in \mathcal{P}_{r-2}(I)$ we have from (33) that

$$
\int_{-1}^{1} I_{r,s} v(x, \pm 1) L'_{n-1}(x) dx = \int_{-1}^{1} v(x, \pm 1) L'_{n-1}(x) dx
$$

which can be written as

$$
\sum_{i=0}^{r} (b_{i0} \pm b_{i1}) \int_{-1}^{1} U_i(x) L'_{n-1}(x) dx = \sum_{i=0}^{\infty} (a_{i0} \pm a_{i1}) \int_{-1}^{1} U_i(x) L'_{n-1}(x) dx.
$$

Upon integrating by parts and observing (51) this simplifies to

$$\sum_{i=2}^{r}(b_{i0}\pm b_{i1})\int_{-1}^{1}L_{i-1}(x)L_{n-1}(x)dx=\sum_{i=2}^{\infty}(a_{i0}\pm a_{i1})\int_{-1}^{1}L_{i-1}(x)L_{n-1}(x)dx.$$

From the last equation we get

$$2\gamma_{n-1}\left(b_{n0}\pm b_{n1}\right)=2\gamma_{n-1}\left(a_{n0}\pm a_{n1}\right).$$

So we conclude that

$$a_{n0}=b_{n0},\qquad a_{n1}=b_{n1},\qquad 2\leq n\leq r. \tag{52}$$

The sides $x=\pm 1$ give analogously

$$a_{0n}=b_{0n},\qquad a_{1n}=b_{1n},\qquad 2\leq n\leq s. \tag{53}$$

Finally, let $2\leq m\leq r$, $2\leq l\leq s$ and set

$$w=U_m''(x)U_l''(y)\in\mathcal{Q}_{r-s,s-2}(\hat{Q}).$$

By (35)

$$\int_{\hat{Q}}I_{r,s}v(\overline{x})w(\overline{x})d\overline{x}=\int_{\hat{Q}}v(\overline{x})w(\overline{x})d\overline{x}.$$

This is

$$\sum_{i=0}^{r}\sum_{j=0}^{s}b_{ij}\int_{-1}^{1}U_iU_m''dx\int_{-1}^{1}U_jU_l''dy=\sum_{i,j=0}^{\infty}a_{ij}\int_{-1}^{1}U_iU_m''dx\int_{-1}^{1}U_jU_l''dy.$$

Again upon integration by parts and using (51)-(53) this reduces to

$$\sum_{i=2}^{r}\sum_{j=2}^{s}b_{ij}\int_{-1}^{1}L_{i-1}L_{m-1}dx\int_{-1}^{1}L_{j-1}L_{l-1}dy=\sum_{i=2}^{r}\sum_{j=2}^{s}a_{ij}\int_{-1}^{1}L_{i-1}L_{m-1}dx\int_{-1}^{1}L_{j-1}L_{l-1}dy$$

which gives

$$4b_{ml}\gamma_{m-1}\gamma_{l-1}=4a_{ml}\gamma_{m-1}\gamma_{l-1}.$$

Hence,

$$a_{ml}=b_{ml},\qquad 2\leq m\leq r,\quad 2\leq l\leq s. \tag{54}$$

(51)-(54) prove the proposition. □

Proposition 3.16. *Let $r,s\geq 2$ and $v\in\mathcal{I}(\hat{Q})$ be given in the form (39). Then there exist constants $C>0$ independent of r, s and v such that*

$$\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 \leq Cs\|v_x\|_{0,\hat{Q}}^2+C\sum_{i=1}^{r}\gamma_{i-1}\gamma_{s-1}\left(a_{i0}^2+a_{i1}^2\right) \tag{55}$$

$$\|(I_{r,s}v)_y\|_{0,\hat{Q}}^2 \leq Cr\|v_y\|_{0,\hat{Q}}^2+C\sum_{j=1}^{s}\gamma_{j-1}\gamma_{r-1}\left(a_{0j}^2+a_{1j}^2\right). \tag{56}$$

Proof. By symmetry we must only show (55). Applying Proposition 3.15 we have

$$I_{r,s}v(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{s} a_{ij}U_i(x)U_j(y)$$

and get with Lemma 3.14

$$\begin{aligned}
\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 &= \sum_{i=1}^{r}\sum_{j=0}^{s-2} 4\gamma_{i-1}\gamma_j\left(\gamma_{j-1}a_{ij} - \gamma_{j+1}a_{i,j+2}\right)^2 \\
&+ \sum_{i=1}^{r}\left\{4\gamma_{i-1}\gamma_{s-1}\left(\gamma_{s-2}a_{i,s-1}\right)^2 + 4\gamma_{i-1}\gamma_s\left(\gamma_{s-1}a_{is}\right)^2\right\} \\
&\le \|v_x\|_{0,\hat{Q}}^2 + \underbrace{\sum_{i=1}^{r} 4\gamma_{i-1}\gamma_{s-1}\left(\gamma_{s-2}a_{i,s-1}\right)^2}_{=:A} + \underbrace{\sum_{i=1}^{r} 4\gamma_{i-1}\gamma_s\left(\gamma_{s-1}a_{is}\right)^2}_{=:B}.
\end{aligned}$$

We next bound the terms A and B.
Let first s be odd, $s \ge 3$: Then for each $i = 1, \ldots, r$ we use the telescoping series

$$\gamma_{s-2}a_{i,s-1} = -\sum_{m=0}^{\frac{s-3}{2}}\left(\gamma_{2m-1}a_{i,2m} - \gamma_{2m+1}a_{i,2m+2}\right) + a_{i0}$$

and get by squaring and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\left(\gamma_{s-2}a_{i,s-1}\right)^2 &\le 2\left\{\sum_{m=0}^{\frac{s-3}{2}}\left(\gamma_{2m-1}a_{i,2m} - \gamma_{2m+1}a_{i,2m+2}\right)\right\}^2 + 2a_{i0}^2 \\
&\le 2\left(\frac{s-3}{2}+1\right)\sum_{m=0}^{\frac{s-3}{2}}\left(\gamma_{2m-1}a_{i,2m} - \gamma_{2m+1}a_{i,2m+2}\right)^2 + 2a_{i0}^2.
\end{aligned}$$

Hence, because $\gamma_{s-1} \le \gamma_{2m}$ for $m = 0, \ldots, \frac{s-3}{2}$ there holds

$$\begin{aligned}
A &= \sum_{i=1}^{r} 4\gamma_{i-1}\gamma_{s-1}\left(\gamma_{s-2}a_{i,s-1}\right)^2 \\
&\le 2\left(\frac{s-3}{2}+1\right)\sum_{i=1}^{r}4\gamma_{i-1}\sum_{m=0}^{\frac{s-3}{2}}\gamma_{2m}\left(\gamma_{2m-1}a_{i,2m} - \gamma_{2m+1}a_{i,2m+2}\right)^2 + 8\sum_{i=1}^{r}\gamma_{i-1}\gamma_{s-1}a_{i0}^2 \\
&\le (s-1)\,\|v_x\|_{0,\hat{Q}}^2 + 8\sum_{i=1}^{r}\gamma_{i-1}\gamma_{s-1}a_{i0}^2. \qquad (57)
\end{aligned}$$

Here we used once more Lemma 3.14. Treating the second term B analogously by writing

$$\gamma_{s-1}a_{is} = -\sum_{m=0}^{\frac{s-3}{2}}\left(\gamma_{2m}a_{i,2m+1} - \gamma_{2m+2}a_{i,2m+3}\right) + a_{i1}$$

we get

$$B \leq (s-1) \|v_x\|_{0,\hat{Q}}^2 + 8 \sum_{i=1}^{r} \gamma_{i-1}\gamma_s a_{i1}^2 \leq (s-1) \|v_x\|_{0,\hat{Q}}^2 + 8 \sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} a_{i1}^2. \tag{58}$$

Together (57) and (58) imply that

$$\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 \leq (2s-1) \|v_x\|_{0,\hat{Q}}^2 + 8 \sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} \left(a_{i0}^2 + a_{i1}^2\right) \tag{59}$$

which is (55).
Let now $s \geq 2$ be even: In that case we get similarly the bounds

$$\begin{aligned} A &= \sum_{i=1}^{r} 4\gamma_{i-1}\gamma_{s-1} a_{i1}^2, \qquad s = 2 \\ A &\leq (s-2) \|v_x\|_{0,\hat{Q}}^2 + 8 \sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} a_{i1}^2, \qquad s \geq 4 \\ B &\leq s \|v_x\|_{0,\hat{Q}}^2 + 8 \sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} a_{i0}^2, \qquad s \geq 2 \end{aligned}$$

such that (59) holds also. □

Proposition 3.17. *Let $v \in \mathcal{I}(\hat{Q})$ be given in the form (39) and $r, s \geq 2$. Then there exist constants $C > 0$ independent of r, s and v such that*

$$\sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} \left(a_{i0}^2 + a_{i1}^2\right) \leq C\frac{1}{s} \left(|v|_{1,\Gamma_1}^2 + |v|_{1,\Gamma_3}^2\right) \tag{60}$$

$$\sum_{j=1}^{s} \gamma_{j-1}\gamma_{r-1} \left(a_{0j}^2 + a_{1j}^2\right) \leq C\frac{1}{r} \left(|v|_{1,\Gamma_2}^2 + |v|_{1,\Gamma_4}^2\right). \tag{61}$$

Let $v \in \mathcal{I}_0(\hat{Q})$ be given in the form (39) and $r, s \geq 2$. Then there exist constants $C > 0$ independent of r, s and v such that

$$\sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} \left(a_{i0}^2 + a_{i1}^2\right) \leq C\frac{r^4}{s} \left(\|v\|_{0,\Gamma_1}^2 + \|v\|_{0,\Gamma_3}^2\right) \tag{62}$$

$$\sum_{j=1}^{s} \gamma_{j-1}\gamma_{r-1} \left(a_{0j}^2 + a_{1j}^2\right) \leq C\frac{s^4}{r} \left(\|v\|_{0,\Gamma_2}^2 + \|v\|_{0,\Gamma_4}^2\right) \tag{63}$$

$$\sum_{i=1}^{r} \gamma_{i-1}\gamma_{s-1} \left(a_{i0}^2 + a_{i1}^2\right) \leq C\frac{r^2}{s} \left(\|v\|_{1/2,00,\Gamma_1}^2 + \|v\|_{1/2,00,\Gamma_3}^2\right) \tag{64}$$

$$\sum_{j=1}^{s} \gamma_{j-1}\gamma_{r-1} \left(a_{0j}^2 + a_{1j}^2\right) \leq C\frac{s^2}{r} \left(\|v\|_{1/2,00,\Gamma_2}^2 + \|v\|_{1/2,00,\Gamma_4}^2\right). \tag{65}$$

Proof. Again, we remark that it suffices by symmetry to show only (60), (62) and (64).
Proof of (60): By Lemma 3.14 we have

$$4\sum_{i=1}^{\infty}\gamma_{i-1}\left(a_{i0}^2+a_{i1}^2\right)=|v|_{1,\Gamma_1}^2+|v|_{1,\Gamma_3}^2.$$

It follows that

$$\sum_{i=1}^{r}\gamma_{i-1}\gamma_{s-1}\left(a_{i0}^2+a_{i1}^2\right)\le\frac{\gamma_{s-1}}{4}\left\{|v|_{1,\Gamma_1}^2+|v|_{1,\Gamma_3}^2\right\}=\frac{1}{8s-4}\left\{|v|_{1,\Gamma_1}^2+|v|_{1,\Gamma_3}^2\right\}.$$

This is (60).
Proof of (62): Let now $v\in\mathcal{I}_0(\Omega)$. We write for $j=0$ or $j=1$

$$I_j=\sum_{i=1}^{r}\gamma_{i-1}\gamma_{s-1}a_{ij}^2.$$

We want to bound I_j. Since $\gamma_r\le\gamma_{i-1}$ for all $i=1,\ldots,r$, there holds

$$\gamma_r^2 I_j\le\gamma_{s-1}\gamma_r\sum_{i=1}^{r}\gamma_{i-1}^2a_{ij}^2\le\gamma_{s-1}\gamma_r r\max_{i=1}^{r}\left\{\gamma_{i-1}^2a_{ij}^2\right\}. \tag{66}$$

There exist indices $i(j)\in\{1,\ldots,r\}$, $j=0,1$, such that

$$\gamma_{i(j)-1}^2a_{i(j),j}^2=\max_{i=1}^{r}\left\{\gamma_{i-1}^2a_{ij}^2\right\}=:m_j. \tag{67}$$

Case 1: Assume that $m_j>0$ for $j=0$ and $j=1$. Since also (40) holds, we have that $i(j)\ge 2$.
Let $i(j)$ be odd, $i(j)\ge 3$: Using a telescoping series we can write

$$\gamma_{i(j)-1}a_{i(j),j}=-\sum_{m=0}^{\frac{i(j)-3}{2}}\left(\gamma_{2m}a_{2m+1,j}-\gamma_{2m+2}a_{2m+3,j}\right),$$

where we note that due to (40) $a_{1j}=0$. Applying the Cauchy-Schwarz inequality as before we can estimate

$$\begin{aligned}\left(\gamma_{i(j)-1}a_{i(j),j}\right)^2 &\le \frac{i(j)-1}{2}\sum_{m=0}^{\frac{i(j)-3}{2}}\left(\gamma_{2m}a_{2m+1,j}-\gamma_{2m+2}a_{2m+3,j}\right)^2\\ &\le \frac{r-1}{2}\sum_{m=0}^{\frac{r-3}{2}}\left(\gamma_{2m}a_{2m+1,j}-\gamma_{2m+2}a_{2m+3,j}\right)^2.\end{aligned}$$

Putting this into (66) there results

$$\begin{aligned}\gamma_r^2 I_j &\le \gamma_{s-1}\gamma_r r\frac{r-1}{2}\sum_{m=0}^{\frac{r-3}{2}}\left(\gamma_{2m}a_{2m+1,j}-\gamma_{2m+2}a_{2m+3,j}\right)^2\\ &\le \gamma_{s-1}r\frac{r-1}{2}\sum_{m=0}^{\frac{r-3}{2}}\gamma_{2m+1}\left(\gamma_{2m}a_{2m+1,j}-\gamma_{2m+2}a_{2m+3,j}\right)^2\end{aligned}$$

(because $\gamma_r \leq \gamma_{2m+1}$ for $m = 0, \ldots, \frac{r-3}{2}$). Together with Lemma 3.14

$$\begin{aligned} I_0 + I_1 &\leq \frac{\gamma_{s-1}}{\gamma_r^2} r \frac{r-1}{2} \sum_{j=0}^{1} \sum_{m=0}^{\frac{r-3}{2}} \gamma_{2m+1} \left(\gamma_{2m} a_{2m+1,j} - \gamma_{2m+2} a_{2m+3,j}\right)^2 \\ &\leq \frac{\gamma_{s-1}}{\gamma_r^2} \frac{r^2}{8} \left\{ \|v\|_{0,\Gamma_1}^2 + \|v\|_{0,\Gamma_3}^2 \right\}. \end{aligned} \tag{68}$$

This is (62).

Let $i(j)$ be even, $i(j) \geq 2$: Using the telescoping series

$$\gamma_{i(j)-1} a_{i(j),j} = - \sum_{m=0}^{\frac{i(j)-2}{2}} \left(\gamma_{2m-1} a_{2m,j} - \gamma_{2m+1} a_{2m+2,j}\right)$$

we see that (68) holds also.

Case 2: If $m_1 = 0$ or $m_2 = 0$ in (67), then (68) holds trivially which finishes the proof of (62).

Proof of (64): This statement is proved by an interpolation argument. We therefore point out that for each $v \in \mathcal{I}_0(\Omega)$ we have $v|_{\Gamma_i} \in H_0^1(\Gamma_i)$ and $v|_{\Gamma_i} \in H_{00}^{1/2}(\Gamma_i)$. As already mentioned in the introduction the Sobolev space $H_{00}^{1/2}(\Gamma_i)$ is defined via the K-method of interpolation:

$$H_{00}^{1/2}(\Gamma_i) = \left(L^2(\Gamma_i), H_0^1(\Gamma_i)\right)_{1/2,2}, \qquad i = 1, \ldots, 4.$$

We consider the edges Γ_1 and Γ_3 and define the linear space

$$\mathcal{I}_0(\Gamma_1, \Gamma_3) = \{(v|_{\Gamma_1}, v|_{\Gamma_3}) : v \in \mathcal{I}_0(\Omega)\}.$$

Note that for $v \in \mathcal{I}(\Omega)$ given in the form (39) there holds

$$\begin{aligned} v|_{\Gamma_1}(x) &= v(x, +1) = \sum_{i=0}^{\infty} (a_{i0} + a_{i1}) U_i(x) \\ v|_{\Gamma_3}(x) &= v(x, -1) = \sum_{i=0}^{\infty} (a_{i0} + a_{i1}) U_i(x). \end{aligned}$$

Let l_γ^2 be the space of sequences $\{\lambda_i\}_{i=1}^r$ of length r equipped with the weighted norm

$$\|\{\lambda_i\}_{i=1}^r\|_\gamma^2 = \sum_{i=1}^{r} \gamma_{i-1} \gamma_{s-1} \lambda_i^2. \tag{69}$$

We define the operator $T : \mathcal{I}_0(\Gamma_1, \Gamma_3) \to l_\gamma^2 \times l_\gamma^2$ by

$$\left(\sum_{i=0}^{\infty} (a_{i0} + a_{i1}) U_i, \sum_{i=0}^{\infty} (a_{i0} - a_{i1}) U_i \right) \mapsto \left(\{a_{i0}\}_{i=1}^r, \{a_{i1}\}_{i=1}^r\right). \tag{70}$$

T is well-defined and linear. On the one hand we equip $\mathcal{I}_0(\Gamma_1, \Gamma_3)$ with the product norm induced by $L_2(\Gamma_1) \times L_2(\Gamma_3)$, on the other hand with the one induced by $H_0^1(\Gamma_1) \times H_0^1(\Gamma_3)$. We get two normed spaces which we denote by $(\mathcal{I}_0(\Gamma_1,\Gamma_3), L_2 \times L_2)$ and $(\mathcal{I}_0(\Gamma_1,\Gamma_3), H_0^1 \times H_0^1)$, respectively. By definition of these spaces and by (60), (62) we know that

$$T : (\mathcal{I}_0(\Gamma_1,\Gamma_3), L_2 \times L_2) \to l_\gamma^2 \times l_\gamma^2$$

and

$$T : \left(\mathcal{I}_0(\Gamma_1,\Gamma_3), H_0^1 \times H_0^1\right) \to l_\gamma^2 \times l_\gamma^2$$

are bounded linear operators whose squared norms are bounded by $C\frac{r^4}{s}$ and $C\frac{1}{s}$, respectively. Noting that the product space $H_{00}^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_3)$ equipped with the product norm and the interpolation space $(L^2(\Gamma_1) \times L^2(\Gamma_3), H_0^1(\Gamma_1) \times H_0^1(\Gamma_3))_{1/2,2}$ equipped with the interpolation norm are isomorphic, we can use interpolation theory for normed spaces. Hence, we conclude that the square of the norm of

$$T : \left(\mathcal{I}_0(\Gamma_1,\Gamma_3), H_{00}^{1/2} \times H_{00}^{1/2}\right) \to l_\gamma^2 \times l_\gamma^2$$

is bounded by $C\frac{r^2}{s}$. This is exactly (64). □

$I_{r,s}$ satisfies the following estimates.

Proposition 3.18. *Let $r, s \geq 2$. Then there exist constants $C > 0$ independent of r, s such that*

$$\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 \leq Cs\,\|v_x\|_{0,\hat{Q}}^2 + C\frac{1}{s}\left(|v|_{1,\Gamma_1}^2 + |v|_{1,\Gamma_3}^2\right) \tag{71}$$

$$\|(I_{r,s}v)_y\|_{0,\hat{Q}}^2 \leq Cr\,\|v_y\|_{0,\hat{Q}}^2 + C\frac{1}{r}\left(|v|_{1,\Gamma_1}^2 + |v|_{1,\Gamma_3}^2\right) \tag{72}$$

for all $v \in \mathcal{I}(\hat{Q})$. Additionally there exist constants $C > 0$ independent of r, s such that

$$\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 \leq Cs\,\|v_x\|_{0,\hat{Q}}^2 + C\frac{r^4}{s}\left(\|v\|_{0,\Gamma_1}^2 + \|v\|_{0,\Gamma_3}^2\right) \tag{73}$$

$$\|(I_{r,s}v)_y\|_{0,\hat{Q}}^2 \leq Cr\,\|v_y\|_{0,\hat{Q}}^2 + C\frac{s^4}{r}\left(\|v\|_{0,\Gamma_2}^2 + \|v\|_{0,\Gamma_4}^2\right) \tag{74}$$

as well as

$$\|(I_{r,s}v)_x\|_{0,\hat{Q}}^2 \leq Cs\,\|v_x\|_{0,\hat{Q}}^2 + C\frac{r^2}{s}\left(\|v\|_{1/2,00,\Gamma_1}^2 + \|v\|_{1/2,00,\Gamma_3}^2\right) \tag{75}$$

$$\|(I_{r,s}v)_y\|_{0,\hat{Q}}^2 \leq Cr\,\|v_y\|_{0,\hat{Q}}^2 + C\frac{s^2}{r}\left(\|v\|_{1/2,00,\Gamma_2}^2 + \|v\|_{1/2,00,\Gamma_4}^2\right) \tag{76}$$

for all $v \in \mathcal{I}_0(\hat{Q})$.

Proof. The statements of this Theorem are direct consequences of Propositions 3.16 and 3.17. □

Corollary 3.19. *Theorem 3.5 holds.*

Proof. We have

$$I_{r,s}|_{\mathcal{I}(\hat{Q})\cap H(\hat{Q})} = \Pi_{r,s}^{\hat{Q}}|_{\mathcal{I}(\hat{Q})\cap H(\hat{Q})}.$$

Since $\mathcal{I}(\hat{Q}) \cap H(\hat{Q})$ is dense in $H(\hat{Q})$ there exists a unique norm preserving extension of $\Pi_{r,s}^{\hat{Q}}|_{\mathcal{I}(\hat{Q})\cap H(\hat{Q})}$ to $H(\hat{Q})$ which coincides with the operator in Definition 3.4. Therefore, Theorem 3.5 follows from Proposition 3.18. □

3.4 Numerical estimates of inf-sup constants

We present some numerical estimates of inf-sup constants which confirm our stability results on the boundary layer patches. To this end we consider an anisotropic two-element mesh on the patch $\Omega = (-1,1)^2$ where the one-dimensional mesh $\mathcal{T}_x(\delta)$ is given by $\{(-1,\delta),(\delta,1)\}$, $\delta \in [0,1)$. The associated boundary layer mesh is as in Figure 3.3 given by $\mathcal{T}_\delta = \mathcal{T}_x(\delta) \times I$. We computed some inf-sup constants for $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ elements on $(\Omega, \mathcal{T}_\delta)$. By Corollary 3.9 there holds

$$\inf_{0\neq p\in S_0^{k,0}} \sup_{0\neq \overline{v}\in \left[S_0^{k,1}\right]^2} \frac{(\nabla \cdot \overline{v}, p)_\Omega}{|\overline{v}|_{1,\Omega} \, \|p\|_{0,\Omega}} \geq C k^{-\frac{1}{2}} \tag{77}$$

with C independent of δ. In Figure 3.5 we show inf-sup constants for $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ elements for different values of δ at some fixed polynomial degrees. By symmetry, only the range $\delta \in [0,1)$ is plotted. The graph is in agreement with (77); the inf-sup constant does not deteriorate as δ approaches one. Also, we remark that the values of the inf-sup constants $\gamma(N)$ are rather moderate. In Figure 3.6 we vary the polynomial degree k for several

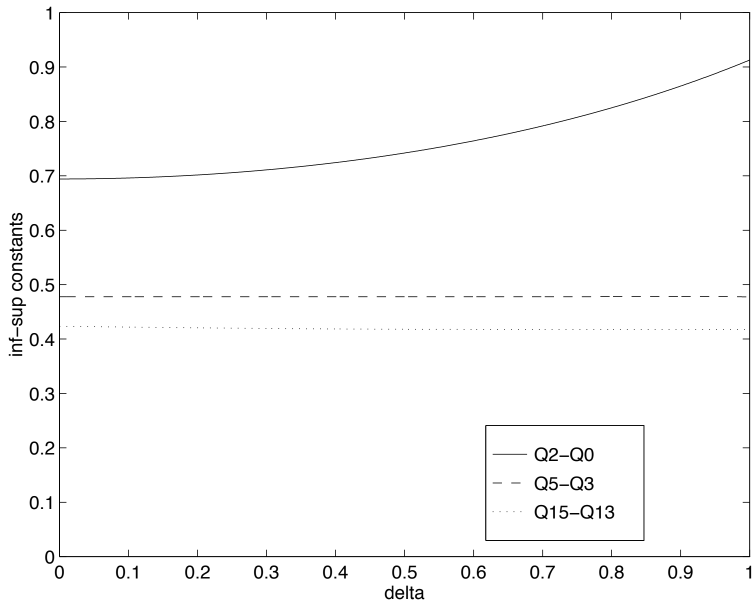

Figure 3.5: Inf-sup constants for $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ elements on the two-element mesh $\mathcal{T}_\delta$.

fixed $\delta \in [0,1)$. Both figures indicate that the $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ elements are indeed stable independently of δ and that this robustness increases with the spectral order k.

4 Stability on boundary layer meshes

We combine the local stability results in the previous section to a general stability result of hp-FEM on boundary layer meshes $\mathcal{T}$ as defined in Definition 2.2. Our main stability result is

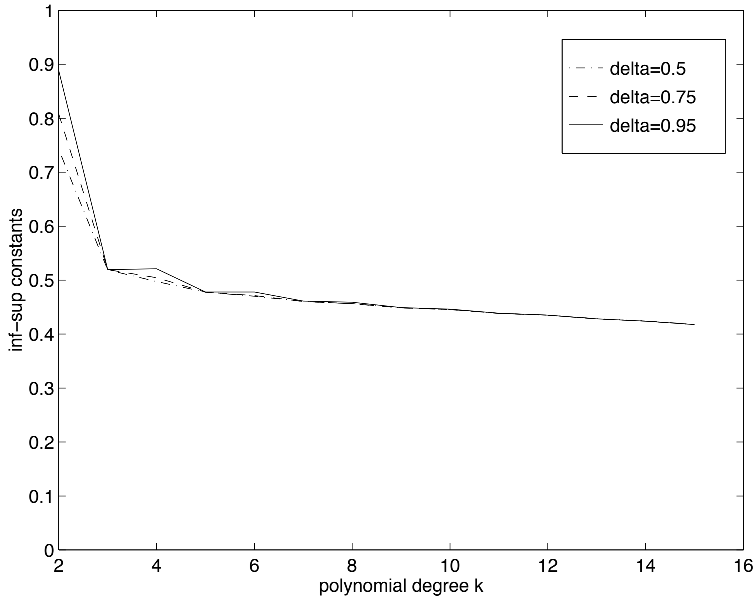

Figure 3.6: Inf-sup constants for various $[\mathcal{Q}_k]^2 \times \mathcal{Q}_{k-2}$ elements on $\mathcal{T}_\delta$.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygon and $\mathcal{T}$ be a boundary layer mesh with macro-element mesh $\mathcal{T}_m$ as in Definition 2.2. Assume that in each boundary layer patch $K \in \mathcal{T}_m^{BL}$ the polynomial degrees are identical (i.e. $r = s$) and constant. Denote the polynomial degree vector by $\underline{k} = \{k_K \geq 2 : K \in \mathcal{T}\}$ and let $|\underline{k}| = \max\{k_K : K \in \mathcal{T}\}$. For*

$$\overline{V}_N = \left[S_0^{\underline{k},1}(\Omega, \mathcal{T})\right]^2, \qquad M_N = S_0^{\underline{k},0}(\Omega, \mathcal{T})$$

with generic velocity and pressure spaces on $\hat{K}$ given by (12) and (13), respectively, there holds

$$\inf_{0 \neq p \in M_N} \sup_{0 \neq \overline{v} \in \overline{V}_N} \frac{(\nabla \cdot \overline{v}, p)_\Omega}{|\overline{v}|_{1,\Omega} \|p\|_{0,\Omega}} \geq \gamma(N) > 0 \tag{78}$$

with $\gamma(N) = C\,|\underline{k}|^{-\frac{1}{2}}$ if $\mathcal{T}$ does not contain triangles, $\gamma(N) = C\,|\underline{k}|^{-3}$ otherwise. Here C depends only on the shape regularity constants κ and κ' of $\mathcal{T}$ and on Ω. In particular, the constant C is independent of the aspect ratio of the elements in the boundary layer patches and of the polynomial degree vector $\underline{k}$.

Proof. To prove (78), we pick any $0 \neq p \in M_N$ and decompose it into $p = p^* + p_m$ where p_m is piecewise constant with vanishing mean-value on the macro-element mesh $\mathcal{T}_m$. By standard theory, there exists $\overline{v}_m \in \overline{V}_N$ such that

$$(\nabla \cdot \overline{v}_m, p_m)_{0,\Omega} \geq C\,\|p_m\|_{0,\Omega}^2, \qquad |\overline{v}_m|_{1,\Omega} \leq \|p_m\|_{0,\Omega}, \qquad C = C(\kappa, \Omega) > 0.$$

Next, consider $p^* \in M_N$. Since p^* is discontinuous and $p^* \in L_0^2(K)$ for every $K \in \mathcal{T}_m$, we can construct for every $p^*|_K$ a corresponding $\overline{v}_K^* \in \overline{V}_N \cap [H_0^1(K)]^2$ satisfying (78), according to Theorem 3.1, Remark 3.3 and Corollary 3.9. Putting $\overline{v}^* = \sum_{K \in \mathcal{T}_m} \overline{v}_K^*$, $\overline{v} = \overline{v}^* + \delta \overline{v}_m$ with an suitably choosen $\delta > 0$ will give (78) globally in the usual way (see, e.g, [7, Section II.1.4] or [15, 20]). □

Remark 4.2. We could select in the boundary layer patches also anisotropic polynomial degrees $\underline{r}$ and s as in Theorem 3.8; Theorem 4.1 would still be valid, with $\gamma(N)$ depending now on the lower bound in (28).

Remark 4.3. As in Remark 3.11, Theorem 4.1 holds also if we use $[\mathcal{Q}_k]^2 \times \mathcal{P}_{k-2}$ quadrilateral elements instead of (12).

Remark 4.4. The subspace pairs $\overline{V}_N \times M_N$ are stable uniformly in h. If only h-stability at fixed degree $\underline{k}$ is desired, one can likely succeed with smaller spaces $\overline{V}_N$. For example, the serendipity element $[\mathcal{Q}_2']^2 \times \mathcal{P}_0$ is also stable on boundary layer patches independently of $\mathcal{T}_x$. The velocity space $\mathcal{Q}_2'$ can even be replaced by a certain, still smaller, space.

5 Boundary layer approximations

In this section we want to show how one can use boundary layer meshes to resolve efficiently boundary layers. We use the already existing boundary layer *hp*-approximation theory of [16] and present some new anisotropic *hp*-approximation results. As indicated in the Introduction, viscous boundary layers near smooth portions of the wall $\partial\Omega$ are, in the simplest case, solution components of the velocity field which have the form

$$u(\xi, \rho) = c(\xi) \exp(-\rho\sqrt{Re}), \tag{79}$$

Re being the (eventually very large) Reynolds number. They arise, as is well-known (see, e.g., [1], [9] or [10]), in laminar solutions of the Navier-Stokes equations. The other solution components are smooth in the sense that their derivatives are bounded independently of the Reynolds number. Above, (ξ, ρ) are the usual boundary-fitted coordinates in a tubular neighborhood of $\partial\Omega$, with ρ denoting the normal distance to the wall and the function $c(\xi)$ is smooth independently of Re. We remark that not resolving these boundary layers with appropriate meshes may pollute the entire numerical computation. Since a rigorous asymptotic expansion of laminar solutions of the Navier-Stokes equations near walls does not seem to be available, we confine ourselves here to the model solution (79) and the "reference" boundary layer patch which is given by a partition of the square $\Omega = (-1, 1)^2$ into high aspect ratio rectangles; that is we consider again on Ω the product mesh

$$\mathcal{T} = \{K : K = K_x \times I, K_x \in \mathcal{T}_x\}$$

where $I = (-1, 1)$ and $\mathcal{T}_x$ is an one dimensional mesh on I given by a partition of I into subintervals $\{K_x\}$ (cf. Figure 3.3). Remember that the refinements in the boundary layer patches are obtained by mapping affinely this reference situation.

5.1 Approximation of exponential boundary layers

Let now u be an exponential boundary layer function on Ω which we assume to be of the form

$$u(x, y) = c(y) \exp\left(-\frac{1-x}{d}\right) \qquad |x|, |y| < 1 \tag{80}$$

where $c(y)$ is smooth on I, $d = 1/\sqrt{Re}$ is a small parameter $\in (0, 1]$ that can approach zero. $1 - x$ is the (normal) distance to the boundary $\{x = 1\}$. We wish to approximate u

by a function $u_{\underline{r},s}$ belonging to the FE-space $S^{\underline{r},s,1}(\Omega,\mathcal{T})$. Based on (80) we look for $u_{\underline{r},s}$ in the form

$$u_{\underline{r},s}(x,y) = u_{\underline{r}}(x)u_s(y)$$

with $u_s \in \mathcal{P}_s(I)$, $u_{\underline{r}} \in S^{\underline{r},1}(I,\mathcal{T}_x)$. Here, the one dimensional FE-space $S^{\underline{r},1}(I,\mathcal{T}_x)$ is defined analogously to (8), i.e.

$$S^{\underline{r},1}(I,\mathcal{T}_x) = \left\{ u \in H^1(I) : u|_{K_x} \in \mathcal{P}_{r_{K_x}}(I), K_x \in \mathcal{T}_x \right\},$$

$\underline{r} = \{r_{K_x} \geq 1, K_x \in \mathcal{T}_x\}$ being as usual a polynomial degree distribution.

Lemma 5.1. *Let u be a boundary layer function of the form (80). Then we have*

$$\begin{aligned} & \|u - u_{\underline{r},s}\|_{0,\Omega}^2 + d^2 \left|u - u_{\underline{r},s}\right|_{1,\Omega}^2 \\ \leq\ & K\left\{ \|c\|_{1,I}^2 + \|c - u_s\|_{1,I}^2 \right\} \left\{ d^2 \left| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right|_{1,I}^2 + \left\| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right\|_{0,I}^2 \right\} + Kd\,\|c - u_s\|_{1,I}^2 \end{aligned}$$

for any $u_{\underline{r}} \in S^{\underline{r},1}(I,\mathcal{T}_x)$, $u_s \in \mathcal{P}_s(I)$. The constant $K > 0$ is independent of d, $\underline{r}$, s and c.

Proof. Writing $u - u_{\underline{r},s} = u - u_{\underline{r}}c + u_{\underline{r}}c - u_{\underline{r}}u_s$ we get by the triangle inequality

$$\begin{aligned} \|u - u_{\underline{r},s}\|_{0,\Omega}^2 &\leq K\left\{ \|c\|_{0,I}^2 \left\| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right\|_{0,I}^2 + \|u_{\underline{r}}\|_{0,I}^2 \|c - u_s\|_{0,I}^2 \right\} \\ &\leq K\left\{ \left(\|c\|_{1,I}^2 + \|c - u_s\|_{1,I}^2 \right) \left\| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right\|_{0,I}^2 + \left\| e^{-\frac{1-x}{d}} \right\|_{0,I}^2 \|c - u_s\|_{1,I}^2 \right\}. \end{aligned}$$

Similarly, there holds

$$\begin{aligned} & d^2 \left\| \frac{\partial}{\partial x}(u - u_{\underline{r},s}) \right\|_{0,\Omega}^2 \\ \leq\ & K\left\{ d^2 \left| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right|_{1,I}^2 \|c\|_{0,I}^2 + d^2 \left|u_{\underline{r}}\right|_{1,I}^2 \|c - u_s\|_{0,I}^2 \right\} \\ \leq\ & K\left\{ \left(\|c\|_{1,I}^2 + \|c - u_s\|_{1,I}^2 \right) d^2 \left| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right|_{1,I}^2 + d^2 \left| e^{-\frac{1-x}{d}} \right|_{1,I}^2 \|c - u_s\|_{1,I}^2 \right\} \end{aligned}$$

and

$$\begin{aligned} & d^2 \left\| \frac{\partial}{\partial y}(u - u_{\underline{r},s}) \right\|_{0,\Omega}^2 \\ \leq\ & K\left\{ d^2 \left\| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right\|_{0,I}^2 |c|_{1,I}^2 + d^2 \|u_{\underline{r}}\|_{0,I}^2 \left|c - u_s\right|_{1,I}^2 \right\} \\ \leq\ & K\left\{ \left(\|c\|_{1,I}^2 + \|c - u_s\|_{1,I}^2 \right) \left\| e^{-\frac{1-x}{d}} - u_{\underline{r}} \right\|_{0,I}^2 + \left\| e^{-\frac{1-x}{d}} \right\|_{0,I}^2 \|c - u_s\|_{1,I}^2 \right\}. \end{aligned}$$

Since $\left\| e^{-\frac{1-x}{d}} \right\|_{0,I}^2 \leq Kd$ and $\left| e^{-\frac{1-x}{d}} \right|_{1,I}^2 \leq Kd^{-1}$, the above estimates imply the statement of Lemma 5.1. □

Lemma 5.1 reduces the approximation of the boundary layer function (80) to two one-dimensional problems, namely the approximation of $e^{-\frac{1-x}{d}}$ in $S^{\underline{r},1}(I,\mathcal{T}_x)$ and the approximation of c in $\mathcal{P}_s(I)$. The later imposes no problems since c is smooth and can be achieved easily; we will not discuss this issue further. But for the x-direction we will show that for the particular solution (80) we can achieve, by proper choice of $\mathcal{T}_x$ and $\underline{r}$, exponential convergence independent of $d = 1/\sqrt{Re}$.

Remark 5.2. The above analysis is based on the assumed form (79) of the boundary layer. However, similar arguments can be applied to any other separable form $u(\xi,\rho) = c(\xi)U(\rho, Re)$; here $U(\rho, Re)$ can be, for example, a similarity solution of the boundary layer equations (see, e.g., the discussions in [1, Section 6], [9] or [10]). Once again, the mesh $\mathcal{T}_x$ and the degree distribution $\underline{r}$ can be choosen *a-priori* to achieve robust exponential convergence in the layer.

Several choices for the mesh $\mathcal{T}_x$ and the degree vector $\underline{r}$ are of course possible. We present here two of them, the first being the two-element mesh as in [16].

Theorem 5.3. *Let $I = (-1,1)$ and $u(x) = \exp\left(-\frac{1-x}{d}\right)$. Let further the one dimensional mesh $\mathcal{T}_x$ and polynomial degree vector $\underline{r}$ on $\mathcal{T}_x$ be such that for $r \geq 1$*

$$\begin{aligned} \underline{r} &= \{1, r\}, \qquad \mathcal{T}_x = \{(-1, 1-\lambda r d), (1-\lambda r d, 1)\} \qquad if\ \lambda r d < 2, \\ \underline{r} &= \{r\}, \qquad \mathcal{T}_x = \{(-1,1)\} \qquad otherwise \end{aligned}$$

where $0 < \lambda_0 \leq \lambda \leq \lambda_1 < \frac{4}{e}$ are independent of r and d. Then there exists $u_{\underline{r}} \in S^{\underline{r},1}(I,\mathcal{T}_x)$ such that $u_{\underline{r}}(\pm 1) = u(\pm 1)$ and

$$\|u - u_{\underline{r}}\|_{0,I} \leq C d^{\frac{1}{2}} \exp(-br), \qquad |u - u_{\underline{r}}|_{1,I} \leq C d^{-\frac{1}{2}} \exp(-br) \tag{81}$$

where $b > 0$ and $C > 0$ are constants independent of $\underline{r}$ and d but depend on λ_0 and λ_1.

Proof. As already mentioned, this can be found in [16]. □

Using two-element meshes in the boundary layer patches the construction of κ'-regular meshes in the corner patches becomes difficult. To circumvent this it may be more convenient to use for $\mathcal{T}_x$ a mesh which is geometrically refined towards $x = 1$. Therefore, we fix a grading factor $\sigma \in (0,1)$ and a number $L \in \mathbb{N}$ of refinements. A geometric mesh $\mathcal{T}_x = \{K_l\}_{l=0}^{L}$ is obtained by subdividing I into $L+1$ subintervals

$$K_0 = (1 - 2\sigma^L, 1), \qquad K_l = (1 - 2\sigma^{L-l}, 1 - 2\sigma^{L+1-l}),\ l = 1,\ldots,L.$$

Corollary 5.4. *Let $I = (-1,1)$ and $u(x) = \exp(-\frac{1-x}{d})$. Let $\mathcal{T}_x = \{K_l\}_{l=0}^{L}$ be a geometric mesh as defined above with grading factor $\sigma \in (0,1)$ and $L \in \mathbb{N}$ refinements such that the smallest element has width $O(d)$, i.e. let L be such that $2\sigma^L \leq C_1 d$ for some $C_1 > 0$. Let $0 < \lambda_1 < \frac{4}{e}$ and let the polynomial degree vector be constant, i.e. $\underline{r} = \{r\}_{l=0}^{L}$ for some $r \geq \frac{C_1}{\lambda_1}$. Then there exists $u_{\underline{r}} \in S^{r,1}(I,\mathcal{T}_x)$ such that $u_{\underline{r}}(\pm 1) = u(\pm 1)$ and*

$$\|u - u_{\underline{r}}\|_{0,I} \leq C d^{1/2} \exp(-br), \qquad |u - u_{\underline{r}}|_{1,I} \leq C d^{-1/2} \exp(-br) \tag{82}$$

where $b > 0$ and $C > 0$ are constants independent of $\underline{r}$ and d but depend on λ_1 and σ.

Proof. Let $r \geq 1$. Denote by $\mathcal{T}_{two}$ the two-element mesh with degree vector $\underline{r} = \{1, r\}$ introduced in Theorem 5.3. As in [12] it suffices to show that we can choose λ_0 and λ in Theorem 5.3 satisfying $0 < \lambda_0 \leq \lambda \leq \lambda_1 < \frac{4}{e}$ such that

$$S^{\underline{r},1}(I, \mathcal{T}_{two}) \subseteq S^{r,1}(I, \mathcal{T}_x). \tag{83}$$

(82) follows then from (81). If the polynomial degree r satisfies $\lambda_1 rd \geq 2$ we can choose $\lambda = \lambda_1$ and (83) holds. Let us therefore concentrate on the case $\lambda_1 rd < 2$. We observe that in this case (83) is valid if there exists $l \in \{1, \ldots, L\}$ such that $1 - 2\sigma^l = 1 - \lambda rd$ for $\lambda_0 \leq \lambda \leq \lambda_1$. Since $r \geq C_1/\lambda_1$, it is easy to see that this can be achieved for some $\lambda \in [\sigma\lambda_1, \lambda_1]$. □

Note that the number of layers in the geometric mesh depends (weakly) on d. Combining Lemma 5.1, Theorem 5.3 and Corollary 5.4 results in

Corollary 5.5. *Let $\mathcal{T}_x$ be a two-element mesh or a geometrically refined mesh with corresponding degree vector $\underline{r}$ as in Theorem 5.3 or Corollary 5.4, respectively. Let u be a boundary layer of the form (80). Then there exists $u_{\underline{r}} \in S^{\underline{r},1}(I, \mathcal{T}_x)$ such that*

$$\|u - u_{\underline{r},s}\|^2_{0,\Omega} + d^2 \, |u - u_{\underline{r},s}|^2_{1,\Omega} \leq K \left(\|c\|^2_{1,I} + \|c - u_s\|^2_{1,I} \right) \exp(-br) + Kd \, \|c - u_s\|^2_{1,I}$$

for any $u_s \in \mathcal{P}_s(I)$. Here, $K > 0$ and $b > 0$ are constants independent of $\underline{r}$, s, c and d but depend on λ_0, λ_1 or σ as in Theorem 5.3 or Corollary 5.4, respectively.

If in particular c is analytic in $[-1, 1]$ independent of d, robust exponential convergence results.

5.2 Anisotropic *hp*-approximation on boundary layer patches

Since the NSE-equations are nonlinear, the boundary layers do not necessarily have the structure (79), but rather $u(\xi, \rho) = c(\xi) U(Re, \rho)$ where U is analytic and satisfies a certain nonlinear ordinary differential equation (e.g., [1, Section 6] or [10, Chapter 10]). In this case it is desirable to have general tensor product *hp*-/spectral approximation results on anisotropic patches available which will be derived here. We introduced in Section 3 the operator $I_{r,s}$ in order to prove divergence stability on boundary layer patches. However, by a tensor product argument the same projector can be used to obtain such results. They are of interest in their own right. Note that in our error estimates the dependence on the regularity of the functions and on the polynomial degrees is given explicitly. We start with a one dimensional approximation result.

Proposition 5.6. *Let $I = (-1, 1)$ and $u \in H^{m+1}(I)$ for some $m \geq 0$. Then there exist operators $T_r : H^1(I) \to \mathcal{P}_r(I)$ $(r \geq 1)$ such that $u(\pm 1) = T_r u(\pm 1)$ and*

$$\left\| u' - (T_r u)' \right\|^2_{0,I} \leq \frac{(r - \alpha_1)!}{(r + \alpha_1)!} \left\| u^{(\alpha_1 + 1)} \right\|^2_{0,I} \tag{84}$$

$$\|u - T_r u\|^2_{0,I} \leq \frac{(r - \alpha_2)!}{(r + \alpha_2)!} \frac{1}{r(r+1)} \left\| u^{(\alpha_2 + 1)} \right\|^2_{0,I} \tag{85}$$

for any integers $0 \leq \alpha_1, \alpha_2 \leq \min(r, m)$.

Proof. This is proved, for example, in [15]. □

Proposition 5.6 is proved by developing u' into the Legendre series $\sum_{i=0}^{\infty} b_i L_i(x)$ and then putting

$$(T_r u)'(x) := \sum_{i=0}^{r-1} b_i L_i(x), \qquad T_r u(x) := \int_{-1}^{x} (T_r u)'(t)dt + u(-1).$$

Therefore, if u is of the form $u(x) = \sum_{i=0}^{\infty} a_i U_i(x)$ where only finitely many a_i are non-vanishing (the polynomials U_i are defined in (37)) it is easy to see that

$$T_r u(x) = \sum_{i=0}^{r} a_i U_i(x). \tag{86}$$

For a function $u(x, y)$ we may use the above one dimensional operators T_r with respect to x or y. This will be indicated by the symbols $^{(x)}$ or $^{(y)}$, correspondingly. We introduce for $r, s \geq 1$ the tensor-product operator $T_{r,s}$ by

$$T_{r,s} := T_s^{(y)} \otimes T_r^{(x)}. \tag{87}$$

If $v \in \mathcal{I}(\hat{Q})$ is of the form (39), i.e.

$$v(x, y) = \sum_{i,j=0}^{\infty} a_{ij} U_i(x) U_j(y),$$

it is obvious from (86) that

$$(T_{r,s} v)(x, y) = \sum_{i=0}^{r} \sum_{j=0}^{s} a_{ij} U_i(x) U_j(y).$$

Because of the density of $\mathcal{I}(\hat{Q})$ in $H^{1+\epsilon}(\hat{Q})$ and the uniqueness of norm-preserving operator extensions $T_{r,s}$ is in fact nothing else than the projection operator $I_{r,s}$ already introduced in Definition 3.13 (cf. Proposition 3.15) and can therefore be defined on $H^{1+\epsilon}(\hat{Q})$. Note that $T_r^{(x)}$ and $T_s^{(y)}$ commute.

Proposition 5.7. *Let $u \in H^{m+1,n+1}(\hat{Q})$ for $m, n \geq 0$. Then $T_{r,s} : H^{1+\epsilon}(\hat{Q}) \to \mathcal{Q}_{r,s}(\hat{Q})$ $(r, s \geq 1)$ satifies $(T_{r,s} u)(N_i) = u(N_i)$, $i = 1, \dots, 4$, (33) and (34). Further*

$$\begin{aligned} \|u - T_{r,s} u\|_{0,\hat{Q}}^2 \leq C \Bigg\{ & \frac{(r-\alpha_1)!}{(r+\alpha_1)!} \frac{1}{r(r+1)} \left\| \frac{\partial^{\alpha_1+1} u}{\partial x^{\alpha_1+1}} \right\|_{0,\hat{Q}}^2 + \frac{(s-\beta_1)!}{(s+\beta_1)!} \frac{1}{s(s+1)} \left\| \frac{\partial^{\beta_1+1} u}{\partial y^{\beta_1+1}} \right\|_{0,\hat{Q}}^2 \\ & + \frac{1}{r(r+1)} \frac{1}{s(s+1)} \left[\frac{(r-\alpha_2)!}{(r+\alpha_2)!} \left\| \frac{\partial^{\alpha_2+2} u}{\partial y \partial x^{\alpha_2+1}} \right\|_{0,\hat{Q}}^2 + \frac{(s-\beta_2)!}{(s+\beta_2)!} \left\| \frac{\partial^{\beta_2+2} u}{\partial y^{\beta_2+1} \partial x} \right\|_{0,\hat{Q}}^2 \right] \Bigg\} \end{aligned}$$

and

$$\begin{aligned} |u - T_{r,s} u|_{1,\hat{Q}}^2 \leq \; & C \Bigg\{ \frac{(r-\alpha_1)!}{(r+\alpha_1)!} \left\| \frac{\partial^{\alpha_1+1} u}{\partial x^{\alpha_1+1}} \right\|_{0,\hat{Q}}^2 + \frac{(r-\alpha_2)!}{(r+\alpha_2)!} \frac{1}{r(r+1)} \left\| \frac{\partial^{\alpha_2+2} u}{\partial x^{\alpha_2+1} \partial y} \right\|_{0,\hat{Q}}^2 \\ & + \frac{(s-\beta_2)!}{(s+\beta_2)!} \left\| \frac{\partial^{\beta_2+1} u}{\partial y^{\beta_2+1}} \right\|_{0,\hat{Q}}^2 + \frac{(s-\beta_1)!}{(s+\beta_1)!} \frac{1}{s(s+1)} \left\| \frac{\partial^{\beta_1+2} u}{\partial y^{\beta_1+1} \partial x} \right\|_{0,\hat{Q}}^2 \Bigg\} \end{aligned}$$

for any $0 \le \alpha_1, \alpha_2 \le \min(r, m)$ *and* $0 \le \beta_1, \beta_2 \le \min(s, n)$*. Here,* $C > 0$ *is independent of* r*,* s*,* α_1*,* α_2*,* β_1 *and* β_2*.*

Proof. By density, it suffices to assume that $u \in C^\infty(\overline{\hat{Q}})$. Then

$$|u - T_{r,s}u|_{1,\hat{Q}} \le \underbrace{\left|u - T_r^{(x)}u\right|_{1,\hat{Q}}}_{=:E} + \underbrace{\left|u - T_s^{(y)}u\right|_{1,\hat{Q}}}_{=:F} + \underbrace{\left|\left(I - T_s^{(y)}\right) \otimes \left(I - T_r^{(x)}\right) u\right|_{1,\hat{Q}}}_{=:G}.$$

We bound E with Proposition 5.6 noting that $\frac{\partial}{\partial y}$ and $T_r^{(x)}$ commute.

$$\begin{aligned}
E^2 &= \left|\frac{\partial}{\partial x}\left(u - T_r^{(x)}u\right)\right|_{1,\hat{Q}}^2 + \left|\frac{\partial}{\partial y}\left(u - T_r^{(x)}u\right)\right|_{1,\hat{Q}}^2 \\
&= \int_{-1}^1 \left\|\frac{\partial}{\partial x}u(\cdot, y) - \frac{\partial}{\partial x}T_r^{(x)}u(\cdot, y)\right\|_{0,I}^2 dy + \int_{-1}^1 \left\|\frac{\partial}{\partial y}u(\cdot, y) - T_r^{(x)}\frac{\partial}{\partial y}u(\cdot, y)\right\|_{0,I}^2 dy \\
&\le \frac{(r-\alpha_1)!}{(r+\alpha_1)!}\int_{-1}^1 \left\|\frac{\partial^{\alpha_1+1}}{\partial x^{\alpha_1+1}}u(\cdot, y)\right\|_{0,I}^2 dy + \frac{(r-\alpha_2)!}{(r+\alpha_2)!}\frac{1}{r(r+1)}\int_{-1}^1 \left\|\frac{\partial^{\alpha_2+2}}{\partial x^{\alpha_2+1}\partial y}u(\cdot, y)\right\|_{0,I}^2 dy \\
&= \frac{(r-\alpha_1)!}{(r+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial x^{\alpha_1+1}}\right\|_{0,\hat{Q}}^2 + \frac{(r-\alpha_2)!}{(r+\alpha_2)!}\frac{1}{r(r+1)}\left\|\frac{\partial^{\alpha_2+2}u}{\partial x^{\alpha_2+1}\partial y}\right\|_{0,\hat{Q}}^2
\end{aligned}$$

for any $0 \le \alpha_1, \alpha_2 \le \min(r, m)$. The analogous estimates for term F yield

$$F^2 \le \frac{(s-\beta_1)!}{(s+\beta_1)!}\frac{1}{s(s+1)}\left\|\frac{\partial^{\beta_1+2}u}{\partial y^{\beta_1+1}\partial x}\right\|_{0,\hat{Q}}^2 + \frac{(s-\beta_2)!}{(s+\beta_2)!}\left\|\frac{\partial^{\beta_2+1}u}{\partial y^{\beta_2+1}}\right\|_{0,\hat{Q}}^2$$

where $0 \le \beta_1, \beta_2 \le \min(s, n)$. By writing $v = (I - T_r^{(x)})u$ we get for the term G in the same manner

$$G^2 = \left|v - T_s^{(y)}v\right|_{1,\hat{Q}}^2 \le \frac{(s-\beta_3)!}{(s+\beta_3)!}\frac{1}{s(s+1)}\left\|\frac{\partial^{\beta_3+2}v}{\partial y^{\beta_3+1}\partial x}\right\|_{0,\hat{Q}}^2 + \frac{(s-\beta_4)!}{(s+\beta_4)!}\left\|\frac{\partial^{\beta_4+1}v}{\partial y^{\beta_4+1}}\right\|_{0,\hat{Q}}^2.$$

Inserting u again in the last two terms above gives the estimates

$$\left\|\frac{\partial^{\beta_3+2}v}{\partial y^{\beta_3+1}\partial x}\right\|_{0,\hat{Q}}^2 \le \frac{(r-\alpha_3)!}{(r+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+\beta_3+2}u}{\partial x^{\alpha_3+1}\partial y^{\beta_3+1}}\right\|_{0,\hat{Q}}^2$$

and

$$\left\|\frac{\partial^{\beta_4+1}v}{\partial y^{\beta_4+1}}\right\|_{0,\hat{Q}}^2 \le \frac{(r-\alpha_4)!}{(r+\alpha_4)!}\frac{1}{r(r+1)}\left\|\frac{\partial^{\alpha_4+\beta_4+2}u}{\partial x^{\alpha_4+1}\partial y^{\beta_4+1}}\right\|_{0,\hat{Q}}^2.$$

Together this implies that

$$\begin{aligned}
G^2 &\le \frac{(s-\beta_3)!}{(s+\beta_3)!}\frac{1}{s(s+1)}\frac{(r-\alpha_3)!}{(r+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+\beta_3+2}u}{\partial x^{\alpha_3+1}\partial y^{\beta_3+1}}\right\|_{0,\hat{Q}}^2 \\
&+ \frac{(s-\beta_4)!}{(s+\beta_4)!}\frac{(r-\alpha_4)!}{(r+\alpha_4)!}\frac{1}{r(r+1)}\left\|\frac{\partial^{\alpha_4+\beta_4+2}u}{\partial x^{\alpha_4+1}\partial y^{\beta_4+1}}\right\|_{0,\hat{Q}}^2
\end{aligned}$$

for any $0 \leq \alpha_3, \alpha_4 \leq \min(r,m)$ and $0 \leq \beta_3, \beta_4 \leq \min(s,n)$. Choosing in particular $\alpha_3 = 0 = \beta_4$, $\alpha_2 = \alpha_4$ and $\beta_3 = \beta_1$ there results

$$
\begin{aligned}
|u - T_{r,s}u|^2_{1,\hat{Q}} \;\leq\; & C\left\{ \frac{(r-\alpha_1)!}{(r+\alpha_1)!} \left\| \frac{\partial^{\alpha_1+1}u}{\partial x^{\alpha_1+1}} \right\|^2_{0,\hat{Q}} + \frac{(r-\alpha_2)!}{(r+\alpha_2)!} \frac{1}{r(r+1)} \left\| \frac{\partial^{\alpha_2+2}u}{\partial x^{\alpha_2+1}\partial y} \right\|^2_{0,\hat{Q}} \right. \\
& \left. + \frac{(s-\beta_2)!}{(s+\beta_2)!} \left\| \frac{\partial^{\beta_2+1}u}{\partial y^{\beta_2+1}} \right\|^2_{0,\hat{Q}} + \frac{(s-\beta_1)!}{(s+\beta_1)!} \frac{1}{s(s+1)} \left\| \frac{\partial^{\beta_1+2}u}{\partial y^{\beta_1+1}\partial x} \right\|^2_{0,\hat{Q}} \right\}
\end{aligned}
$$

for $0 \leq \alpha_1, \alpha_2 \leq \min(r,m)$ and $0 \leq \beta_1, \beta_2 \leq \min(s,n)$. The constant C is independent of r, s, α_1, α_2, β_1 and β_2. The L^2-estimates are obtained in an analogous manner. □

Let now $(\Omega, \mathcal{T})$ be a boundary layer patch on $\Omega = (-1,1)^2$ with the mesh $\mathcal{T} = \mathcal{T}_x \times I$ as in Section 3. $\underline{r}$, s are as usual the polynomial degrees on $\mathcal{T}_x$ and I, correspondingly. Then the estimates of Proposition 5.7 can easily be scaled to high aspect ratio rectangles $K_x \times I \in \mathcal{T}$, since on the right hand side of the estimates in Proposition 5.7 there are only semi-norms. Continuity across the elements is also guaranteed. In this way one gets analogous approximation results for $S^{\underline{r},s,0}(\Omega, \mathcal{T})$ and $S^{\underline{r},s,1}(\Omega, \mathcal{T})$.

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96-05	R. Bodenmann, H.J. Schroll	Higher order discretisation of initial-boundary value problems for mixed systems