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Stability analysis for the method of transport

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Abstract

In this paper, we provide analytical stability estimates for the method of transport. We first prove stability of the second-order method of transport applied to the linear advection equation with constant coefficient in one dimension by using the von Neumann method and with the positive operator technique. In a second step, we extend the proof to the linear advection equation with variable coefficient. Finally, we investigate and compare the existing multidimensional schemes from van Leer, Colella, and LeVeque for the linear advection equation with constant coefficients.

Keywords: Multidimensional scheme, method of transport, advection equation, stability, von Neumann method

Subject Classification: AMS(MOS) subject classifications (1991): 35L10, 35B35, 65M12

1. Introduction

Many numerical methods for computing multidimensional flow were developed during the last decades. Usually, such methods were constructed by applying one-dimensional schemes along all coordinate axes. Recently, truly multidimensional schemes for solving hyperbolic problems were constructed, examples of such schemes are van Leer's multidimensional differences scheme [16], Colella's CTU scheme [4], LeVeque's scheme [11], Roe's upwinding scheme [5], or Fey's method of transport [6], [7]. Once a new method is developed, the code has to be validated. This can be done either numerically or analytically. Numerical validation means to carry out computational experiments and comparing the results with those of physical experiments or with numerical results of other codes that are already well established. For the method of transport, this is made in [7], [12], and [13].

Analytical validation consists of stability or convergence proofs. A common technique to prove non-linear convergence is to first linearise the scheme and to prove stability for this linearisation [1], [2]. For that reason, stability analysis can be seen as a first step towards a convergence proof. Since linearised schemes have variable coefficients, it is necessary to derive stability estimates for linear, variable coefficients problems.

In this paper, we provide analytical stability estimates for the method of transport. A first stability analysis was provided by Fey and Schroll [8] for the first-order scheme applied to non-linear scalar equations. To make the proofs readable, we first prove stability of the second-order method of transport applied to the linear advection equation in one space dimension with constant coefficient. This is done by means of the von Neumann method. The von Neumann method essentially consists of proving that the symbol of the scheme is absolutely bounded by one. For treating variable coefficients problems, the von Neumann method has to be generalised. One technique to do this is given by the positive operator technique described by Zhu et al. in [17] which is applied in Sections 2.2 and 2.3. Analogies between this method and the von Neumann analysis are worked out.

In the two-dimensional case, we start with proving stability for the first-order method of transport with constant coefficients. Unfortunately, Zhu's technique for analysing stability of variable coefficients problems cannot be generalised to two dimensions. For the second-order scheme in two dimensions, the symbol becomes quite untractable and boundedness can no longer be proved analytically. However, comprehensive computational experiments show that the symbol does not exceed one.

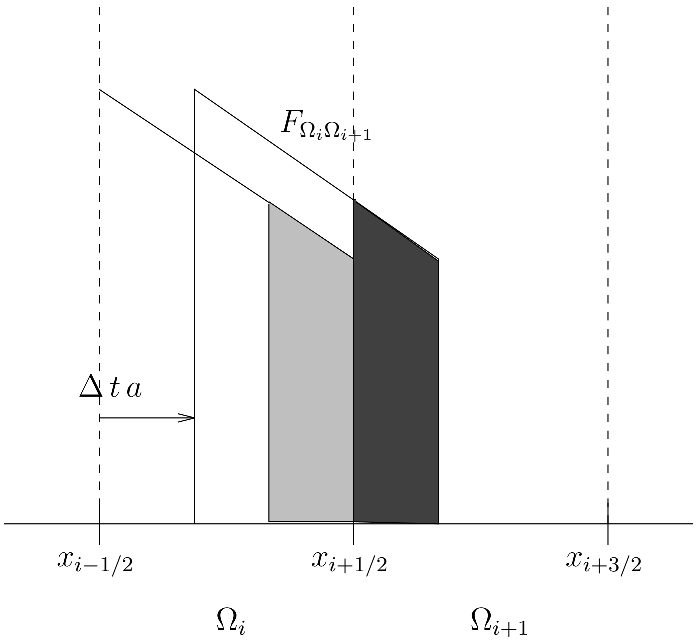

FIGURE 1. Flux $F_{\Omega_i\Omega_{i+1}}$ for $a > 0$.

In Section 4, the von Neumann analysis is carried out for the first-order method of transport in three dimensions.

2. One-dimensional second-order scheme

The one-dimensional advection equation with constant coefficient is given by

$$u_t + a\,u_x = 0$$

The method of transport is described by

$$u_i^{n+1} = u_i^n - \frac{1}{|\Omega_i|} \sum_{j\in\{i-1,i+1\}} (F_{\Omega_i\Omega_j} - F_{\Omega_j\Omega_i}),$$

Cell Ω_i is the interval $[x_{i-1/2}, x_{i+1/2}]$, $|\Omega_i|$ its length and Ω_{i-1}, Ω_{i+1} its neighbouring cells. For $a > 0$, the flow $F_{\Omega_i\Omega_{i+1}}$ from cell Ω_i into Ω_{i+1}, cf. Figure 1, is described by

$$F_{\Omega_i\Omega_{i+1}} = \int_{x_{i+1/2}}^{x_{i+1/2}+\Delta\,t\,a} u(x - \Delta\,t\,a)\,dx = \int_{x_{i+1/2}-\Delta\,t\,a}^{x_{i+1/2}} u(x)\,dx,$$

and for $a \leq 0$, $F_{\Omega_i\Omega_{i+1}}$ is identical to zero.

To get second-order accuracy for the fluxes, the transported quantity u has to be reconstructed linearly and the fluxes become

$$F_{\Omega_i \Omega_{i+1}} = \begin{cases} \Delta t \, (a \, u_i + \dfrac{\Delta x}{2}(1-\sigma) \, a \, (D \, u)_i) & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma = \dfrac{a \, \Delta t}{\Delta x}$ and $(D \, u)$ is some difference operator approximating the derivative u_x.

2.1. **Von Neumann method.** The von Neumann method offers a simple way of assessing the stability properties of linear schemes with constant coefficients when the boundary conditions are assumed to be periodic. The method is based on the Fourier decomposition of the numerical scheme, from which a stability estimate follows. More information about the method can be found in [9].
If a is assumed to be positive and if the operator $(D \, u)_i = (D_0 \, u)_i = \dfrac{u_{i+1} - u_{i-1}}{2 \, \Delta x}$ denotes the second-order centred differences, the scheme reads

$$(1) \quad u_i^{n+1} = u_i^n - \sigma \left[\, (u_i^n - u_{i-1}^n) + \frac{1}{4}(1-\sigma) \, (u_{i+1}^n - u_i^n - u_{i-1}^n + u_{i-2}^n) \, \right].$$

By replacing u_i^n by the exponential and writing the symbol Λ of the scheme as the quotient of u_i^{n+1} divided by u_i^n

$$\Lambda = 1 - \sigma[(1 - e^{-i\gamma}) + \frac{1}{4}(1-\sigma)(e^{i\gamma} - 1 - e^{-i\gamma} + e^{-i2\gamma})]$$

and the square of its absolute value
(2)

$$\begin{aligned} \Lambda \cdot \overline{\Lambda} &= 1 + \frac{1}{8}(2\,\sigma^4 - 4\,\sigma^3 + 14\,\sigma^2 - 12\,\sigma) + \frac{1}{8}\cos(\gamma)\,(-\sigma^4 + 2\,\sigma^3 - 17\,\sigma^2 + 16\,\sigma) \\ &\quad + \frac{1}{8}\cos(2\,\gamma)\,(-2\sigma^4 + 4\,\sigma^3 + 2\,\sigma^2 - 4\,\sigma) + \frac{1}{8}\cos(3\,\gamma)\,(\sigma^4 - 2\,\sigma^3 + \sigma^2) \\ &= 1 + \frac{1}{2}\,\sigma\,(\sigma - 1)\,(-\sigma^2 + \sigma + 3) \\ &\quad + (\cos^2(\gamma) - 2\,\cos(\gamma) + 1)\,(\sigma^4 - 2\,\sigma^3 + 2\,\sigma^2 - \sigma) \\ &\quad + \frac{1}{2}(\cos^3(\gamma) - 3\,\cos^2(\gamma) + 3\,\cos(\gamma) - 1)\,(\sigma^4 - 2\,\sigma^3 + \sigma^2) \\ &= 1 + \frac{1}{2}\,\sigma\,(\sigma - 1)\,(-\sigma^2 + \sigma + 3) + (\cos(\gamma) - 1)^2\,\sigma\,(\sigma - 1)\,(\sigma^2 - \sigma + 1) \\ &\quad + \frac{1}{2}(\cos(\gamma) - 1)^3\,\sigma^2\,(\sigma - 1)^2. \end{aligned}$$

If the CFL number σ is restricted to the interval $[0, 1]$, it follows that that $|\Lambda| \le 1$ for all γ and hence the one-dimensional scheme is stable. It is interesting to notice that the scheme is also stable for a CFL number of 2, if the derivatives are approximated by first-order backward differences and is stable for a CFL number of 1, if the derivatives are approximated by first-order forward differences. However, the second-order scheme for which the derivatives are approximated by second-order forward differences is no longer stable. The explanation is given by looking at the stencil of the scheme, in fact too much information is taken from the downwind side of the characteristic. Similar investigations were done by Childs and Morton for the one-dimensional ECG schemes in [3].

2.2. **Positive operator technique for constant coefficients.** Next we are interested to analyse stability of the advection equation with variable coefficients. In [14] it was proved that under certain constraints for linear, non-constant coefficient problems a local von Neumann analysis provides a necessary condition for stability. The analysis was carried out by freezing the coefficients at their value at a certain point and then applying the von Neumann method. This provides a local stability estimate. In order to get a sufficient condition for stability, additional restrictions on the amplitude of the symbol have to be made. Kreiss (1964) has found that a scheme is stable, if the amplification matrix is hermitian, uniformly bounded, and Lipschitz continuous in x, and if the scheme is dissipative of order $2\,r$ and accurate of order $(2\,r-1)$, for some integer r. However, since the scheme treated here is second order accurate, the method of Kreiss cannot be applied.
The goal of this section is to prove stability by applying the positive operator technique, described by Zhu et al. in [17]. We have to show that the stability inequality

$$||u^{n+1}||_2^{\Delta\,x} \le (1 + \Delta\,x)\,||u^n||_2^{\Delta\,x},$$

where $||\cdot||_2^{\Delta\,x}$ is the energy norm given by

$$||u||_2^{\Delta\,x} = \sqrt{\Delta\,x \sum_i u_i^2}.$$

To introduce the technique, we consider the method of transport for the advection equation with constant coefficient. To this end we use a

matrix formulation of the scheme. The discrete norm of (1) is given by
(3)

$$
\begin{aligned}
(||u^{n+1}||_2^{\Delta x})^2 &= \Delta x \sum_i \left(u_i^n - \sigma\,(u_i^n - u_{i-1}^n) - \frac{\sigma}{4}(1-\sigma)\,(u_{i+1}^n - u_i^n - u_{i-1}^n + u_{i-2}^n)\right)^2 \\
&= (||u^n||_2^{\Delta x})^2 + \Delta x \sum_i \bar{u}_I^T\,\mathbf{A}\,\bar{u}_I,
\end{aligned}
$$

where $\bar{u}_I = (u_{i+1}, u_i, u_{i-1}, u_{i-2})^T$ and the amplification matrix $\mathbf{A}$ is defined as

$$
\mathbf{A} = \frac{1}{16}\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12} & A_{22} & A_{23} & A_{24} \\ A_{13} & A_{23} & A_{33} & A_{34} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix},
$$

with

$$
\begin{aligned}
A_{11} &= A_{14} = A_{44} = \sigma^4 - 2\,\sigma^3 + \sigma^2 \\
A_{12} &= A_{24} = -\sigma^4 - 2\,\sigma^3 + 7\,\sigma^2 - 4\,\sigma \\
A_{13} &= A_{34} = -\sigma^4 + 6\,\sigma^3 - 5\,\sigma^2 \\
A_{22} &= \sigma^4 + 6\,\sigma^3 + \sigma^2 - 24\,\sigma \\
A_{23} &= \sigma^4 - 2\,\sigma^3 - 19\,\sigma^2 + 20\,\sigma \\
A_{33} &= \sigma^4 - 10\,\sigma^3 + 25\,\sigma^2.
\end{aligned}
$$

The trace and the sum of the entries of the first, second, and third subdiagonals are given by

$$
\begin{aligned}
D_0 &= \frac{1}{8}(2\,\sigma^4 - 4\,\sigma^3 + 14\,\sigma^2 - 12\,\sigma) \\
D_1 &= \frac{1}{8}(-\sigma^4 + 2\,\sigma^3 - 17\sigma^2 + 16\,\sigma) \\
D_2 &= \frac{1}{8}(-2\sigma^4 + 4\,\sigma^3 + 2\,\sigma^2 - 4\,\sigma) \\
D_3 &= \frac{1}{8}(\sigma^4 - 2\,\sigma^3 + \sigma^2).
\end{aligned}
$$

We recognise here the coefficients of 1, $\cos(\gamma)$, $\cos(2\,\gamma)$, and $\cos(3\,\gamma)$ in the von Neumann stability analysis (2). By analogy with trigonometric equalities used before, we can decompose the matrix $\mathbf{A}$ into the sum of three matrices $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ with

$$
\mathbf{B} = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 \\ B_{12} & B_{22} & B_{23} & B_{24} \\ B_{13} & B_{23} & B_{33} & B_{34} \\ 0 & B_{24} & B_{34} & B_{44} \end{pmatrix}.
$$

$$\begin{aligned}
B_{11} &= 2\,\sigma^4 - 4\,\sigma^3 + 2\,\sigma^2\\
B_{12} &= -4\,\sigma^4 + 4\,\sigma^3 + 4\,\sigma^2 - 4\,\sigma\\
B_{13} &= -B_{24} = 2\,\sigma^4 - 2\,\sigma^2\\
B_{22} &= 6\,\sigma^4 - 4\,\sigma^3 + 2\,\sigma^2 - 20\,\sigma\\
B_{23} &= -12\,\sigma^2 + 12\,\sigma\\
B_{33} &= -6\,\sigma^4 + 4\,\sigma^3 + 2\,\sigma^2 + 16\,\sigma\\
B_{34} &= 4\,\sigma^4 - 4\,\sigma^3 + 8\,\sigma^2 - 8\,\sigma\\
B_{44} &= -2\,\sigma^4 + 4\,\sigma^3 - 6\,\sigma^2 + 4\,\sigma.
\end{aligned}$$

Obviously $\mathbf{B}$ is a pseudo-null matrix, which means that the trace and the sum of the entries of each subdiagonal are zero. The matrices $\mathbf{C}$ and $\mathbf{D}$ are given by

$$\mathbf{C} = -\frac{1}{16}\,\sigma^2\,(1-\sigma)^2 \begin{pmatrix} 1\\ -3\\ 3\\ -1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \end{pmatrix} = -\frac{1}{16}\,\sigma^2\,(1-\sigma)^2 \begin{pmatrix} 1 & -3 & 3 & -1\\ -3 & 9 & -9 & 3\\ 3 & -9 & 9 & -3\\ -1 & 3 & -3 & 1 \end{pmatrix}$$

and

$$\mathbf{D} = \frac{1}{4}\sigma(\sigma-1)(\sigma^2-\sigma+1) \begin{pmatrix} 0\\ 1\\ -2\\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix} = \frac{1}{4}\sigma(\sigma-1)(\sigma^2-\sigma+1) \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & -2 & 1\\ 0 & -2 & 4 & -2\\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 & -3 & 3 & -1 \end{pmatrix}^T$ defining $\mathbf{C}$ consists of the coefficients of

$$(\cos(\gamma) - 1)^3 = \cos(\gamma)^3 - 3\,\cos(\gamma)^2 + 3\,\cos(\gamma) - 1$$

and the vector $\begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}^T$ defining $\mathbf{D}$ of the coefficients of

$$(\cos(\gamma) - 1)^2 = \cos(\gamma)^2 - 2\,\cos(\gamma) + 1.$$

For CFL numbers $\sigma \in [0, 1]$, $\mathbf{C}$ and $\mathbf{D}$ are both negative semi-definite matrices. Hence, (3) becomes

$$\begin{aligned}
(||u^{n+1}||_2^{\Delta x})^2 &= (||u^n||_2^{\Delta x})^2 + \Delta x \sum_i \bar{u}_I^T\ [\mathbf{B} + \mathbf{C} + \mathbf{D}]\ \bar{u}_I\\
&\le (||u^n||_2^{\Delta x})^2 + \Delta x \sum_i \bar{u}_I^T\,\mathbf{B}\,\bar{u}_I.
\end{aligned} \tag{4}$$

We can rewrite the last term as

$$\begin{aligned}\Delta x \sum_i \bar{u}_I^T \mathbf{B}\, \bar{u}_I &= \Delta x \sum_i (B_{11}\, u_{i+1}^2 + B_{22}\, u_i^2 + B_{33}\, u_{i-1}^2 + B_{44}\, u_{i-2}^2 + 2\, B_{12}\, u_{i+1}\, u_i \\ &\quad + 2\, B_{23}\, u_i\, u_{i-1} + 2\, B_{34}\, u_{i-1}\, u_{i-2} + 2\, B_{13}\, u_{i+1}\, u_i + 2\, B_{24}\, u_i\, u_{i-2}) \\ &= \Delta x \sum_i (B_{11} + B_{22} + B_{33} + B_{44})\, u_i^2 \\ &\quad + 2\, \Delta x \sum_i (B_{12} + B_{23} + B_{34})\, u_i\, u_{i-1} + 2\, \Delta x \sum_i (B_{13} + B_{24})\, u_i\, u_{i-2}\end{aligned}$$

Since $\mathbf{B}$ is pseudo-null, $\Delta x \sum_i \bar{u}_I^T \mathbf{B}\, \bar{u}_I = 0$ and (4) is given by

$$(||u^{n+1}||_2^{\Delta x})^2 \leq (||u^n||_2^{\Delta x})^2.$$

We have found the same results as with the von Neumann method, i.e. the scheme is stable in one dimension for CFL numbers between 0 and 1. The advantage of this method is that it can easily be extended to linear equations with variable coefficients.

2.3. **Positive operator technique for variable coefficients.** The linear advection equation with variable coefficients in conservative form is given by

$$u_t + (a(x)\, u)_x = 0.$$

For future investigations, we assume $a(x)$ to be differentiable. Note that $a(x)$ is therefore Lipschitz continuous and its Lipschitz constant is given by L_a. The second-order fluxes are given by

$$F_{\Omega_i \Omega_{i+1}} = \begin{cases} \Delta\, t(a_i u_i + \dfrac{\Delta\, x}{2}(1 - \lambda a_i)\mathrm{Rec}_i & \text{if } \mathrm{vel}_{\Omega_i\Omega_{i+1}} > 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\mathrm{Rec}_i = (a_i\, (Du)_i + u_i\, (Da)_i)),$$

$$\mathrm{vel}_{\Omega_i\Omega_{i+1}} = a_i + \frac{\Delta\, x}{2}(1 - \lambda\, a_i)\, (D\, a)_i,$$

and $\lambda = \dfrac{\Delta\, t}{\Delta\, x}$. The method of transport is described by

$$\begin{aligned} u_i^{n+1} &= u_i^n - \lambda\, [(a_i^n\, u_i^n - a_{i-1}\, u_{i-1}^n) \\ &\quad + \frac{\Delta\, x}{2}(1 - \lambda\, a_i^n)(\, a_i^n\, (D\, u)_i^n + u_i^n\, (D\, a)_i^n) \\ &\quad - \frac{\Delta\, x}{2}(1 - \lambda\, a_{i-1}^n)\, (a_{i-1}^n\, (D\, u)_{i-1}^n + u_{i-1}^n\, (D\, a)_{i-1}^n)]. \end{aligned} \tag{5}$$

or by rearranging the terms

(6)

$$
\begin{aligned}
u_i^{n+1} &= u_i^n - \lambda\,[a_i^n\,u_i^n + \frac{\Delta\,x}{2}(1-\lambda\,a_i^n)\,a_i^n\,(D\,u)_i^n] \\
&\quad +\lambda\,[a_{i-1}^n\,u_{i-1}^n + \frac{\Delta\,x}{2}(1-\lambda\,a_{i-1}^n)\,a_{i-1}^n(D\,u)_{i-1}^n] \\
&\quad -\lambda\frac{\Delta\,x}{2}(1-\lambda\,a_i^n)\,u_i^n(D\,a)_i^n + \lambda\frac{\Delta\,x}{2}(1-\lambda\,a_{i-1}^n)u_{i-1}^n(D\,a)_{i-1}^n].
\end{aligned}
$$

We define

$$R_i = \lambda\,[a_i^n\,u_i^n + \frac{\Delta\,x}{2}(1-\lambda\,a_i^n)\,a_i^n\,(D\,u)_i^n]$$

and

$$Q_i = \lambda\frac{\Delta\,x}{2}(1-\lambda\,a_i^n)\,u_i^n(D\,a)_i^n,$$

so that we can rewrite (5) as

$$u_i^{n+1} = u_i^n - R_i + R_{i-1} - Q_i + Q_{i-1}. \tag{7}$$

(8)

$$
\begin{aligned}
(||u^{n+1}||_2^{\Delta\,x})^2 =& \Delta\,x\sum_i(u_i^n,u_i^n) \\
&+\Delta\,x\sum_i[-2(u_i^n,R_i)+2(u_i^n,R_{i-1})-2(R_i,R_{i-1})+(R_i,R_i)+(R_{i-1},R_{i-1})] \\
&+\Delta\,x\sum_i[-2\,(u_i^n,Q_i)+2\,(u_i^n,Q_{i-1})+2\,(R_i,Q_i)-2\,(R_i,Q_{i-1}) \\
&-2\,(R_{i-1},Q_i)+2\,(R_{i-1},Q_{i-1})+(Q_i,Q_i)+(Q_{i-1},Q_{i-1})] \\
=& (||u^n||_2^{\Delta\,x})^2 + T_1 + T_2.
\end{aligned}
$$

The term R_{i-1} can be separated into a part similar to the constant coefficients case and an additional part arising from the variation of the coefficients. To this end, we replace a_{i-1} by $a_i + \Delta\,x\,a_\xi$, where $a_\xi = a_x(\xi)$, with $\xi \in [x_{i-1}, x_i]$.

$$
\begin{aligned}
R_{i-1} &= \lambda\,[a_{i-1}^n\,u_{i-1}^n + \frac{\Delta\,x}{2}(1-\lambda\,a_{i-1}^n)\,a_{i-1}^n\,(D\,u)_{i-1}^n] \\
&= \lambda\,[(a_i^n - \Delta\,x\,a_\xi)\,u_{i-1}^n + \frac{\Delta\,x}{2}(1-\lambda\,(a_i^n-\Delta\,x\,a_\xi))(a_i^n-\Delta\,x\,a_\xi)\,(D\,u)_{i-1}^n] \\
&= \lambda\,[a_i^n\,u_{i-1}^n + \frac{\Delta\,x}{2}(1-\lambda\,a_i^n)\,a_i^n\,(D\,u)_{i-1}^n] \\
&\quad -\lambda[\Delta\,x\,a_\xi\,u_{i-1}^n - \frac{\Delta\,x^2}{2}a_\xi(2\,\lambda\,a_i - \lambda\,\Delta\,x\,a_\xi - 1)(D\,u)_{i-1}^n] \\
&= R_{i-1}^c + R_{i-1}^v.
\end{aligned}
$$

In the previous section, we have investigated the norm of T_1 for the part R^c_{i-1} with $D = D_0$, so that we can rewrite T_1 as

$$T_1 = \Delta\, x \sum_i \bar{u}_I^T\, [\mathbf{B} + \mathbf{C} + \mathbf{D}]\; \bar{u}_I + \Delta\, x \sum_i [2\,(u_i^n, R^v_{i-1}) - 2\,(R_i, R^v_{i-1}) + (R^v_{i-1}, R^v_{i-1})],$$

where $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are defined as before, but depend now on the x-variable, $\mathbf{B} = \mathbf{B}_i$, $\mathbf{C} = \mathbf{C}_i$, and $\mathbf{D} = \mathbf{D}_i$, with $\sigma = \sigma_i$. $\mathbf{C}$ and $\mathbf{D}$ are negative semi-definite if the CFL numbers are between 0 and 1 and $\mathbf{B}$ is a pseudo-null matrix. The second term can be written with the help of some matrix $\mathbf{E}$, defined by

$$\mathbf{E} = \frac{\Delta\, x}{4} \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & E_{34} \\ E_{14} & E_{24} & E_{34} & E_{44} \end{pmatrix},$$

where

$$E_{ij} = a_\xi\, f_j(a_\xi, a_i, \Delta\, x, \lambda).$$

T_1 can be approximated by

$$T_1 \leq \Delta\, x \sum_i \bar{u}_I^T\, \mathbf{B}\, \bar{u}_I + \Delta\, x \sum_i \bar{u}_I^T\, \mathbf{E}\, \bar{u}_I.$$

The matrix $\mathbf{B}$ has the same form as before, but now the components are functions of σ_i. Therefore the first term becomes

$$\begin{aligned} \Delta\, x \sum_i \bar{u}_I^T\, \mathbf{B}\, \bar{u}_I \;&= \Delta\, x \sum_i ((B_i)_{11}\, u_{i+1}^2 + (B_i)_{22}\, u_i^2 + (B_i)_{33}\, u_{i-1}^2 + (B_i)_{44}\, u_{i-2}^2 \\ &\quad +2\,(B_i)_{12}\, u_{i+1}\, u_i + 2\,(B_i)_{23}\, u_i\, u_{i-1} + 2\,(B_i)_{34}\, u_{i-1}\, u_{i-2} \\ &\quad +2\,(B_i)_{13}\, u_{i+1}\, u_{i-1} + 2\,(B_i)_{24}\, u_i\, u_{i-2}) \\ &= \Delta\, x \sum_i ((B_{i-1})_{11} + (B_i)_{22} + (B_{i+1})_{33} + (B_{i+2})_{44})\, u_i^2 \\ &\quad +2 \sum_i ((B_{i-1})_{12} + (B_i)_{23} + (B_{i+1})_{34})\, u_i\, u_{i-1} \\ &\quad +2 \sum_i ((B_{i-1})_{13} + (B_i)_{24})\, u_i\, u_{i-2} \end{aligned}$$

By using the Lipschitz continuity of $a(x)$ and the pseudo-null property of $\mathbf{B}$, we can approximate the following term as

$$\begin{aligned}\sum_i |(B_{i-1})_{11} + (B_i)_{22} + (B_{i+1})_{33} + (B_{i+2})_{44}| \\ &= \sum_i |(B_{i-1})_{11} - (B_{i+2})_{11} + (B_i)_{22} - (B_{i+2})_{22} + (B_{i+1})_{33} - (B_{i+2})_{33}| \\ &\leq \sum_i |(B_{i-1})_{11} - (B_{i+2})_{11}| + \sum_i |(B_i)_{22} - (B_{i+2})_{22}| + \sum_i |(B_{i+1})_{33} - (B_{i+2})_{33}| \\ &< L_0 \, \Delta\, x\end{aligned}$$

and similarly for

$$\sum_i |(B_{i-1})_{12} + (B_i)_{23} + (B_{i+1})_{34}| < L_1 \, \Delta\, x$$

and

$$\sum_i |(B_{i-1})_{13} - (B_i)_{24}| < L_2 \, \Delta\, x.$$

So that $\Delta\, x \sum_i \bar{u}_I^T \, \mathbf{B} \, \bar{u}_I$ becomes

$$\Delta\, x \sum_i \bar{u}_I^T \, \mathbf{B} \, \bar{u}_I < L_0 \, \Delta\, x (||u^n||_2^{\Delta\, x})^2 + 2\, L_1 \, \Delta\, x \sum_i (u_i, u_{i-1}) + 2\, L_2 \, \Delta\, x \sum_i (u_i, u_{i-2}),$$

and finally by using the Cauchy-Schwarz inequality, we get

$$\Delta\, x \sum_i \bar{u}_I^T \, \mathbf{B} \, \bar{u}_I < (L_0 + 2\, L_1 + 2\, L_2) \Delta\, x (||u^n||_2^{\Delta\, x})^2 = C \, \Delta\, x \, (||u^n||_2^{\Delta\, x})^2.$$

For the second term $\Delta\, x \sum_i \bar{u}_I^T \, \mathbf{E} \, \bar{u}_I$, we use similar ideas and get

$$\begin{aligned}\Delta\, x \sum_i \bar{u}_I^T \, \mathbf{E} \, \bar{u}_I \leq \Delta\, x (&||E_{11}||_\infty^{\Delta\, x} + 2\, ||E_{12}||_\infty^{\Delta\, x} + 2\, ||E_{13}||_\infty^{\Delta\, x} + 2\, ||E_{14}||_\infty^{\Delta\, x} \\ &+ ||E_{22}||_\infty^{\Delta\, x} + 2\, ||E_{23}||_\infty^{\Delta\, x} + 2\, ||E_{24}||_\infty^{\Delta\, x} \\ &+ ||E_{33}||_\infty^{\Delta\, x} + 2\, ||E_{34}||_\infty^{\Delta\, x} + ||E_{44}||_\infty^{\Delta\, x}) (||u^n||_2^{\Delta\, x})^2,\end{aligned}$$

where $||E_{jk}||_\infty^{\Delta\, x} = \max_i \{ |E_{jk}(a_\xi, a_i, \Delta\, x, \lambda)| \}$. It follows that

$$T_1 \leq \Delta\, x (||u^n||_2^{\Delta\, x})^2.$$

Now we can write the norm of u^{n+1} as

$$\begin{aligned}(||u^{n+1}||_2^{\Delta\, x})^2 = (1 + \Delta\, x) \, (||u^n||_2^{\Delta\, x})^2 \\ &+ \Delta\, x \sum_i [-2\, (u_i^n, Q_i) + 2\, (u_i^n, Q_{i-1}) + 2\, (R_i, Q_i) \\ &-2\, (R_i, Q_{i-1}) - 2\, (R_{i-1}, Q_i) + 2\, (R_{i-1}, Q_{i-1}) \\ &-2\, (Q_i, Q_{i-1}) + (Q_i, Q_i) + (Q_{i-1}, Q_{i-1})].\end{aligned}$$

We have to investigate the other terms. First

$$\begin{aligned}-2\Delta\, x\sum_i (u_i^n, Q_i) &= -\lambda \Delta\, x^2 \textstyle\sum_i (u_i^n, (1-\lambda\, a_i^n)\, u_i^n\, (D\, a)_i) \\ &\leq -\lambda\, L_a (1+\lambda\, ||a||_\infty^{\Delta\, x})\, \Delta\, x\, (||u||_2^{\Delta\, x})^2 \\ &\leq C\, \Delta\, x\, (||u||_2^{\Delta\, x})^2,\end{aligned}$$

where $||a||_\infty^{\Delta\, x} = \max_i |a_i^n|$ and L_a is the Lipschitz constant of $a(x)$ such that $(D\, a) < L_a$ holds. Similarly,

$$2\, \Delta\, x \sum_i (u_i, Q_{i-1}) < C\, \Delta\, x\, (||u||_2^{\Delta\, x})^2.$$

For the terms in T_2, we get

$$\begin{aligned}2\, \Delta\, x \sum_i (R_i, Q_i) &= \lambda^2\, \Delta\, x^2 \sum_i (a_i^n\, u_i^n - \frac{\Delta\, x}{2} (1-\lambda\, a_i^n)\, a_i^n\, (D_0\, u)_i (1-\lambda\, a_i^n)\, u_i^n\, (D_0\, a)_i) \\ &\leq \lambda^2\, \Delta\, x ||a||_\infty^{\Delta\, x}\, (1+\lambda\, ||a||_\infty^{\Delta\, x})\, L_a\, ||u||_\infty^{\Delta\, x} \\ &\quad +\lambda^2\, \frac{\Delta\, x}{2} (1+\lambda\, ||a||_\infty^{\Delta\, x})^2\, ||a||_\infty^{\Delta\, x}\, L_a\, \Delta\, x \sum_i (\Delta\, x\, (D_0 u)_i, u_i^n) \\ &\leq C\, \Delta\, x\, (||u||_2^{\Delta\, x})^2,\end{aligned}$$

and similarly for $-2\Delta\, x \sum_i (R_i, Q_{i-1})$, $-2\Delta\, x \sum_i (R_{i-1}, Q_i)$, and $2\Delta\, x \sum_i (R_{i-1}, Q_{i-1})$. Finally,

$$\begin{aligned}\Delta\, x \sum_i (Q_i, Q_i) &= \lambda^2 \frac{\Delta\, x^3}{4} \sum_i ((1-\lambda\, a_i^n)\, u_i^n (D_0\, a)_i, (1-\lambda\, a_i^n)\, u_i^n\, (D_0\, a)_i) \\ &\leq \frac{\Delta\, x^2}{4} \lambda^2\, L_a^2 (1+\lambda\, ||a||_\infty^{\Delta\, x})^2\, (||u||_2^{\Delta\, x})^2 \\ &\leq C\, \Delta\, x^2\, (||u||_2^{\Delta\, x})^2,\end{aligned}$$

Estimates for the terms $-2\, \Delta\, x \sum_i (Q_i, Q_{i-1})$ and $\Delta\, x \sum_i (Q_{i-1}, Q_{i-1})$ are derived in the same way. Using the above estimates in (8), we get

$$(||u^{n+1}||_2^{\Delta\, x})^2 \leq (1+\Delta\, x)(||u^n||_2^{\Delta\, x})^2.$$

for CFL numbers between 0 and 1, this means that the scheme is stable. This proof can be adapted to other finite-difference operators, such as first-order backward differences or first-order forward differences. Here we have presented the proof for the centred differences, which is the most complicated of the three schemes, since it has the largest stencil. Moreover centred differences are always used in the limiting technique by flux selection developed in [13].

3. Two-dimensional first-order scheme

We consider now the scalar advection equation with constant coefficients in two dimensions

$$u_t + a\,u_x + b\,u_y = 0$$

and assume periodic boundary conditions on a closed interval.
The method of transport of first order gives the following numerical scheme

$$u_{i\,j}^{n+1} = u_{i\,j}^{n} - \sigma(1-\nu)(u_{i\,j}^{n} - u_{i-1\,j}^{n}) - \nu(1-\sigma)(u_{i\,j}^{n} - u_{i\,j-1}^{n}) - \nu\sigma(u_{i\,j}^{n} - u_{i-1\,j-1}^{n}),$$

for a and b positive. σ and ν are the CFL numbers, defined as

$$\sigma = a\frac{\Delta\,t}{\Delta\,x} \quad \text{and} \quad \nu = b\frac{\Delta\,t}{\Delta\,y}.$$

Van Leer presented this scheme in [16] and he has shown that this can be written in the form of a dimension splitting scheme

$$u_{i\,j}^{n+1} = (1 - \sigma\,D_{+x})\,(1 - \nu\,D_{+y})u_{i\,j}^{n}.$$

The symbol of this scheme is given by

$$\Lambda = 1 - \sigma(1-\nu)\,(1 - e^{-i\gamma}) - \nu(1-\sigma)\,(1 - e^{-i\delta}) - \nu\sigma(1 - e^{-i(\gamma+\delta)})$$

and we have to analyse

$$\begin{aligned}
\Lambda\cdot\overline{\Lambda} = 1 \;\; &+ 4\,\nu\,\sigma - 2\,\sigma - 2\,\nu + 2\,\nu^2 - 4\,\nu^2\,\sigma + 4\,\nu^2\,\sigma^2 - 4\,\nu\,\sigma^2 + 2\,\sigma^2 \\
&+ (\,4\,\nu^2\,\sigma + 4\,\nu\,\sigma^2 - 4\,\nu\,\sigma + 2\,\sigma - 2\,\sigma^2 - 4\,\nu^2\,\sigma^2\,)\,\cos(\gamma) \\
&+ (\,4\,\nu^2\,\sigma - 2\,\nu^2 + 4\,\nu\,\sigma^2 - 4\,\nu^2\,\sigma^2 + 2\,\nu - 4\,\nu\,\sigma\,)\,\cos(\delta) \\
&+ (\,4\,\nu^2\,\sigma^2 - 4\,\nu^2\,\sigma + 4\,\nu\,\sigma - 4\,\nu\,\sigma^2\,)\,\cos(\gamma)\,\cos(\delta) \\
= 1 \;\; &+ 2\,(\sigma - \sigma^2)(\cos(\gamma) - 1) + 2\,(\nu - \nu^2)(\cos(\delta) - 1) \\
&+ 4\,\sigma\nu(1-\sigma)(1-\nu)(1-\cos(\gamma))(1-\cos(\delta)) \\
= [1 \;&+ 2\,(\sigma - \sigma^2)\,(\cos(\gamma) - 1)]\,[1 + 2\,(\nu - \nu^2)\,(\cos(\delta) - 1)].
\end{aligned}$$

It is obvious that $|\Lambda| \leq 1$ for $0 \leq \sigma \leq 1$ and $0 \leq \nu \leq 1$. So that for all σ and ν between 0 and 1, the scheme is stable in two dimensions. The same result can be found in the paper of Fey and Schroll [8]. They also proved stability of the first-order method of transport for the two-dimensional Burgers' equation.

4. Two-dimensional second-order scheme

The method of transport of second order to solve the two-dimensional advection equation is of the following form

$$\begin{aligned}
u_{ij}^{n+1} = u_{ij}^{n} &- \sigma(1-\nu)\,[(u_{ij}^{n} - u_{i-1\,j}^{n}) - \frac{1}{2}(-(1-\sigma)\,(D_x\,u_{ij}^{n} - D_x\,u_{i-1\,j}^{n}) \\
&\quad +\nu(D_y\,u_{ij}^{n} - D_y\,u_{i-1\,j}^{n}))] \\
&- \nu\,(1-\sigma)\,[(u_{ij}^{n} - u_{i\,j-1}^{n}) - \frac{1}{2}(\sigma(D_x\,u_{ij}^{n} - D_x\,u_{i\,j-1}^{n}) \\
&\quad -(1-\nu)(D_y\,u_{ij}^{n} - D_y\,u_{i\,j-1}^{n}))] \\
&- \nu\sigma\,[(u_{ij}^{n} - u_{i-1\,j-1}^{n}) - \frac{1}{2}((1-\sigma)\,(D_x\,u_{ij}^{n} - D_x\,u_{i-1\,j-1}^{n}) \\
&\quad +(1-\nu)(D_y\,u_{ij}^{n} - D_y\,u_{i-1\,j-1}^{n}))].
\end{aligned}$$

If the operators D_x and D_y are approximated by centred differences, this scheme is identical to the second-order van Leer scheme [16] and the second-order corner transported upwind scheme (CTU) by Colella [4]. If the operators D_x and D_y are approximated by the first-order forward differences, the scheme is identical to LeVeque's $T^{2,2}$ scheme [11].

Van Leer, Colella, and LeVeque investigated stability of their scheme and all found that the condition

$$\max(|\sigma|, |\lambda|) \leq 1$$

guarantees stability. LeVeque gives some numerical results of his scheme.

The symbol of the scheme for backward differences is given by

$$\begin{aligned}
\Lambda = 1 + \frac{1}{2}\,\big(&-3\,\sigma - 3\,\nu + \sigma^2 + 8\,\sigma\,\nu + \nu^2 - 3\,\nu\,\sigma^2 - 3\,\sigma\,\nu^2 \\
&+ (-7\,\nu\,\sigma + 4\,\sigma - 2\,\sigma^2 + \sigma\,\nu^2 + 4\,\nu\,\sigma^2)\,\mathrm{e}^{-i\gamma} \\
&+ (-7\,\sigma\,\nu + 4\,\nu - 2\,\nu^2 + 4\,\sigma\,\nu^2 + \nu\,\sigma^2)\,\mathrm{e}^{-i\delta} \\
&+ (-\nu\,\sigma^2 + \nu\,\sigma - \sigma + \sigma^2)\,\mathrm{e}^{-2\,i\,\gamma} + (\sigma\,\nu - \nu\,\sigma^2)\,\mathrm{e}^{-i\,(2\,\gamma+\delta)} \\
&+ (-\nu + \sigma\,\nu + \nu^2 - \sigma\,\nu^2)\,\mathrm{e}^{-2\,i\,\delta} + (\sigma\,\nu - \sigma\,\nu^2)\,\mathrm{e}^{-i\,(\gamma+2\,\delta)} \\
&+ 2\,\sigma\,\nu\,\mathrm{e}^{-i\,(\gamma+\delta)}\big)\,.
\end{aligned}$$

Once again we are interested in $|\Lambda|$ which is now given by a huge algebraic term that can no longer be analysed analytically. However, computational tests supply evidence for this term to be less or equal to 1 for all γ and δ if $0 \leq \sigma,\, \nu \leq 1$. Unfortunately, the positive operator technique, which we used to prove stability of the one-dimensional advection equation with variable coefficients can not be extended to two-dimensions.

5. Three-dimensional first-order scheme

We consider the scalar advection equation with constant coefficients in three dimensions

$$u_t + a\,u_x + b\,u_y + c\,u_z = 0 \tag{9}$$

with periodic boundary conditions on a closed interval.
The method of transport of first order is given by the following numerical scheme

$$\begin{aligned} u^{n+1}_{i\,j\,k} = u^{n}_{i\,j\,k} &- \sigma\,(1-\nu)\,(1-\mu)\,(u^n_{i\,j\,k} - u^n_{i-1\,j\,k}) - \nu\,(1-\sigma)\,(1-\mu)\,(u^n_{i\,j\,k} - u^n_{i\,j-1\,k}) \\ &- \mu\,(1-\sigma)\,(1-\nu)\,(u^n_{i\,j\,k} - u^n_{i\,j\,k-1}) - \sigma\,\nu\,(1-\mu)\,(u^n_{i\,j\,k} - u^n_{i-1\,j-1\,k}) \\ &- \sigma\,\mu\,(1-\nu)\,(u^n_{i\,j\,k} - u^n_{i-1\,j\,k-1}) - \nu\,\mu\,(1-\sigma)\,(u^n_{i\,j\,k} - u^n_{i\,j-1\,k-1}) \\ &- \sigma\,\nu\,\mu\,(u^n_{i\,j\,k} - u^n_{i-1\,j-1\,k-1}), \end{aligned}$$

for a, b, and c positive. σ, ν, and μ are the CFL numbers, defined as

$$\sigma = a\frac{\Delta\,t}{\Delta\,x}, \quad \nu = b\frac{\Delta\,t}{\Delta\,y}, \quad \text{and} \quad \mu = c\frac{\Delta\,t}{\Delta\,z}.$$

The symbol of this scheme reads

$$\begin{aligned} \Lambda = 1 &- \sigma\,(1-\nu)\,(1-\mu)\,(1-e^{-i\gamma}) - \nu\,(1-\sigma)\,(1-\mu)\,(1-e^{-i\delta}) \\ &- \mu\,(1-\sigma)\,(1-\nu)\,(1-e^{-i\beta}) - \sigma\,\nu\,(1-\mu)\,(1-e^{(-i\gamma-i\delta)}) \\ &- \sigma\,\mu\,(1-\nu)\,(1-e^{(-i\gamma-i\beta)}) - \nu\,\mu\,(1-\sigma)\,(1-e^{(-i\delta-i\beta)}) \\ &- \nu\,\sigma\,\mu\,(1-e^{-i(\gamma+\delta+\beta)}). \end{aligned}$$

And we have to analyse $\Lambda\cdot\overline{\Lambda}$

$$\begin{aligned} \Lambda\cdot\overline{\Lambda} = {} & [1+2\,(\sigma-\sigma^2)\,(\cos(\gamma)-1)][1+2\,(\nu-\nu^2)\,(\cos(\delta)-1)] \\ & [1+2\,(\mu-\mu^2)\,(\cos(\beta)-1)]. \end{aligned}$$

It holds again that $|\Lambda| \le 1$, for $0\le\sigma\le 1$, $0\le\nu\le 1$, and $0\le\mu\le 1$. So that for all σ, ν and μ between 0 and 1, the scheme is stable in three dimensions.
The extension to second order can be realised in the same way as in two dimensions. The method of transport, where the derivatives are approximated with centred differences, is then identical to van Leer's scheme [16] and to the CTU scheme, cf. Saltzman [15]. If the derivatives are approximated by forward differences, it is identical to the $T^{2,2,2}$ scheme, cf. Langseth and LeVeque [10]. Notice, that these comparisons are valid only for the advection equation with constant coefficients.

6. Convergence

From the stability of a scheme, we are able to derive convergence, if the scheme satisfies the conditions of Lax's equivalence theorem [14].

Lax's Equivalence Theorem: *Given a properly posed initial-value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.*

For the advection equation with constant coefficients in one, two, and three dimensions, the second order method of transport therefore converges in the energy norm, defined in Section 2.2.
For the advection equation with variable coefficients in one dimension, the method of transport converges in the energy norm, if the coefficients are Lipschitz continuous.

7. Conclusion

In this paper, we have investigated stability for the method of transport applied to the advection equation. In the one-dimensional case, we proved stability estimates for both constant coefficient and variable coefficient problems. In the two- and three-dimensional cases, it was still possible to prove stability for constant coefficients problems. However sufficient stability condition for smooth variable coefficients problems are not yet found.

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