Symplectic Integrators for Hill's Lunar Problem

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Abstract

Hill's lunar problem is one of the simplest, yet realistic, gravitational three-body problems. It models, e.g., the motion of the Earth's moon or the motion of Saturn's coorbital satellites. It still has a great importance since it is being used as a reference problem in certain more accurate perturbation models of the moon's motion. In regularized coordinates Hill's lunar problem may be described by a polynomial Hamiltonian of degree 6.

Here we will use an additive splitting of this Hamiltonian in order to implement symplectic composition integrators for Hill's lunar problem. Experiments indicate that these integrators are very accurate for orbits consisting of many revolutions. E.g., in integrations of quasiperiodic or homoclinic orbits the Hamiltonian remains nearly constant until the orbit escapes. A comparison with high-order Taylor series methods shows that these methods are considerably faster and therefore quite competitive, although the Hamiltonian seems to grow linearly with time.

1. Introduction

Symplectic integrators are often seen as a major advance in the numerical treatment of Hamiltonian systems. These integrators have the property that the map corresponding to one integration step is a symplectic map, i.e. it shares important properties (e.g. conservation of areas) with the flow defined by the differential equation. A difficulty of symplectic integrators is the fact that symplecticity is lost when the step size is changed. Furthermore, many of the well known symplectic integrators are implicit and therefore rather slow.

For comprehensive articles on symplectic integrators the reader is referred to, e.g., review papers by Sanz-Serna [8] or Yoshida [17]. A comprehensive and modern account of the entire field of numerical treatment of ordinary differential equations is contained in the volumes by Hairer, Nørsett, and Wanner [1,2].

The purpose of this paper is to use a particular class of *explicit* symplectic integrators, namely the composition methods [5,17] and a particular problem of celestial mechanics, namely Hill's lunar problem [3], for demonstrating advantages and disadvantages of symplectic integration.

Composition methods are chosen in order to minimize the overhead of the method. Therefore explicit methods are preferred. An example of Hamiltonians permitting explicit methods are the *separable* Hamiltonians $H(q,p) = V(q) + T(p)$, where $q \in \mathbb{R}^n$ are the coordinates and p are the conjugated momenta. The concept of the composition methods, dating back as far as 1959 [13,7], is a generalization of separability and is applicable to differential equations of the form

$$(1) \qquad\qquad x_1' = A(x) + B(x), \quad x \in \mathbb{R}^N, \quad ' \equiv \frac{d}{ds}$$

as long as the individual flows corresponding to

$$(2) \qquad\qquad x_1' = A(x_1), \quad x_2' = B(x_2)$$

are known explicitly.

Hill's lunar problem, as one of the simplest gravitational problems, is chosen for several reasons:

   i) It is a problem of practical relevance in lunar theories, coorbital motion etc.

  ii) The complexity of the family of orbits leaves no doubt that the problem is non-integrable. The rigorous proof of this conjecture, however, is still missing.

 iii) Hill's lunar problem has orbits with arbitrarily many revolutions about the origin [10]. The use of regularized coordinates permits a constant step size in the numerical integration of these orbits. Furthermore, the regularized Hamiltonian permits an additive splitting with explicit partial flows, thus allowing for the direct application of composition methods.

In the next section the regularized differential equations of Hill's problem as well as the partial flows will be summarized, and details about the individual symplectic maps will be given. Then, we describe the orbits considered, together with the performance of the symplectic integrators. Finally, the results are compared with a commercially available explicit integrator (ODE45 of MATLAB) and with high-order Taylor series integrators. These latter integrators turn out to be highly competitive for high-precision computations.

2. Hill's Lunar Problem Regularized

In this section we summarize part of the article [16] with the goal of collecting the relations defining the individual flows corresponding to Equ. (2). Hill's lunar equations describe the motion of the moon under the influence of the earth and an infinitely remote sun on a circular orbit. In a uniformly rotating geocentric frame of reference, where x is the tangential coordinate, the equations of motion are [3,10]

$$\ddot{x} - 2\dot{y} - 3x + xr^{-3} = 0, \quad r = \sqrt{x^2 + y^2}$$
$$\ddot{y} + 2\dot{x} \qquad + yr^{-3} = 0,$$
(3)

where dots denote derivatives with respect to time t. The equations of motion (3) imply the existence of the Jacobi integral

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{3}{2}x^2 - \frac{1}{r} = h,$$
(4)

where h is the Jacobi constant.

Following Levi-Civita's regularization [4] procedure for removing the collision singularity at $x = y = 0$, we construct a Hamiltonian equivalent to Equs (3). By using the coordinates and conjugated momenta

$$q_1 = x, \quad q_2 = y, \quad p_1 = \dot{q}_1 - q_2, \quad p_2 = \dot{q}_2 + q_1$$

Hill's lunar problem is represented by the Hamiltonian

$$H(q, p) = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 - q_1^2 + \frac{1}{2}q_2^2 - \frac{1}{r}, \quad r = \sqrt{q_1^2 + q_2^2}.$$
(5)

Levi-Civita's regularization consists of introducing a new independent variable s (instead of time t) and a new Hamiltonian K according to

$$dt = r\,ds, \quad K = r(H - h),$$
(6)

where the constant h is the fixed energy $H(q, p) = h$ of the orbit under consideration, and h agrees with the definition (4) of the Jacobi constant. Furthermore, new coordinates u_1, u_2 and new momenta v_1, v_2 are introduced according to the canonical transformation

$$q_1 + iq_2 = (u_1 + iu_2)^2, \quad p_1 + ip_2 = \frac{v_1 + iv_2}{2(u_1 - iu_2)},$$
(7)

where complex notation has been used. In terms of the new coordinates and momenta the new Hamiltonian becomes

(8) $$K(u,v) = K_1(u,v) + K_2(u)$$

where

(9) $$K_1(u,v) = \frac{1}{8}\left(v_1^2 + v_2^2\right) - (u_1^2 + u_2^2)^2\left[\frac{1}{2}\left(u_1 v_2 - u_2 v_1\right) + h\right] - 1$$

(10) $$K_2(u) = (u_1^2 + u_2^2)(-u_1^4 + 4u_1^2\,u_2^2 - u_2^4)\ ,$$

and the equations of motion (in vector notation) are

(11) $$\frac{du}{ds} = \frac{\partial K}{\partial v}\ ,\quad \frac{dv}{ds} = -\frac{\partial K}{\partial u}\ ,\quad \frac{dt}{ds} = u_1^2 + u_2^2\ .$$

The purpose of the splitting (8) is to write the system (11) in the manner of Equ. (1). It turns out that both the Hamiltonian systems derived from K_1 and K_2 are integrable; therefore the splitting (8) leads to a family of explicit symplectic integrators.

The Hamiltonian system derived from $K_2(u)$ is trivially integrable; with the initial conditions

(12) $$u(0) = u_0,\quad v(0) = v_0,\quad t(0) = t_0\,, u_0,\,v_0 \in \mathbb{C}$$

we obtain

(13) $$\begin{aligned} u(s) &= u_0 \\ v(s) &= v_0 - s\,\frac{\partial K_2}{\partial u}\,(u_0),\ \frac{\partial K_2}{\partial u} = \begin{pmatrix} 6u_1(-u_1^4 + 2u_1^2\,u_2^2 + u_2^4) \\ 6u_2(u_1^4 + 2u_1^2\,u_2^2 - u_2^4) \end{pmatrix} \\ t(s) &= t_0\ . \end{aligned}$$

The Hamiltonian system due to $K_1(u,v)$ is related to the regularized Kepler motion in rotating coordinates. To demonstrate its integrability we write the system in complex notation as

(14) $$\begin{aligned} \frac{du}{ds} &= \frac{v}{4} - \frac{u\overline{u}}{2}\,iu \\ \frac{dv}{ds} &= u(2h + Im\,\overline{u}v) - \frac{u\overline{u}}{2}\,iv \\ \frac{dt}{ds} &= u\overline{u} \end{aligned}$$

and introduce new coordinates $U = e^{it/2}u$, $V = e^{it/2}v$ satisfying the initial conditions

(15) $$U(0) = e^{it_0/2}u_0,\quad V(0) = e^{it_0/2}v_0\ .$$

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The system (14) is transformed into

$$(16) \qquad \frac{dU}{ds} = \frac{V}{4} \,, \quad \frac{dV}{ds} = (2h + U_1 V_2 - U_2 V_1)\, U, \quad \frac{dt}{ds} = U\overline{U} \,,$$

a system which has the angular momentum $Im(\overline{U}V) = U_1 V_2 - U_2 V_1$ as a first integral. Therefore the first two equations of (16) define a harmonic oscillator with frequency $\frac{\omega}{2}$, where

$$(17) \qquad \omega := \sqrt{-2h - Im(\overline{u}_0 v_0)} \,,$$

and in view of the initial conditions (15) their solution is

$$\begin{aligned} U(s) &= e^{i \frac{t_0}{2}} \left[u_0 \cos\left(\frac{\omega}{2}\, s\right) + \frac{u_0}{2\omega} \sin\left(\frac{\omega}{2}\, s\right) \right] \\ V(s) &= e^{i \frac{t_0}{2}} \left[v_0 \cos\left(\frac{\omega}{2}\, s\right) - 2\omega\, u_0 \sin\left(\frac{\omega}{2}\, s\right) \right] . \end{aligned}$$

The final step is to determine $t(s)$ by integrating the third relation of (16).

We now collect the equations given in [16] defining the flow due to K_1. Since the frequency ω in (17) may be 0 or imaginary we write the map in terms of K. Stumpff's "c functions" [12,11]

$$(18) \qquad \begin{aligned} c_0(z) &= \cos(\sqrt{z}) & c_1(z) &= \frac{\sin(\sqrt{z})}{\sqrt{z}} \\ c_2(z) &= \frac{1 - \cos(\sqrt{z})}{z}, & c_3(z) &= \frac{\sqrt{z} - \sin(\sqrt{z})}{z^{3/2}} \,. \end{aligned}$$

Methods for accurately evaluating the Stumpff functions will be discussed in the appendix. Let $x_0 = (u_0, v_0, t_0)$ contain the initial point in phase space (complex notation for u, v) and the initial time. Then the time $t(s)$ and the position $(u(s), v(s))$ are given by

$$(19) \qquad \begin{aligned} t(s) &= t_0 + u_0 \overline{u}_0\, \frac{s}{2} \left(1 + c_1(\omega^2 s^2)\right) + Re(\overline{u}_0 v_0)\, \frac{s^2}{2}\, c_2(\omega^2 s^2) \\ &\quad + v_0 \overline{v}_0\, \frac{s^3}{8}\, c_3(\omega^2 s^2) \\ u(s) &= e^{-\frac{i}{2}\,(t(s) - t_0)} \left[u_0\, c_0\left(\frac{\omega^2 s^2}{4}\right) + v_0\, \frac{s}{4}\, c_1\left(\frac{\omega^2 s^2}{4}\right) \right] \\ v(s) &= e^{-\frac{i}{2}\,(t(s) - t_0)} \left[- u_0\, \omega^2 s\, c_1\left(\frac{\omega^2 s^2}{4}\right) + v_0\, c_0\left(\frac{\omega^2 s^2}{4}\right) \right] . \end{aligned}$$

3. Composition Methods

Consider the differential equation (1), and denote the individual flows induced by the vector fields A, B by $\exp_A(s)$, $\exp_B(s)$, respectively, i.e. the solution $x(s)$ of the initial value problem $x' = A(x)$, $x(0) = x_0$ is written as

$$x(s) = \exp_A(s)(x_0) \,.$$

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Composition of flows is written as juxtaposition, e.g.

$$x(s_1 + s_2) = \exp_A(s_2) \; \exp_A(s_1) (x_0) \,.$$

A composition algorithm approximates the flow induced by the vector field $A + B$ by a repeated composition of the flows due to A and B. A popular and simple algorithm is the *leapfrog* method which uses 2 compositions and is of error order $p = 2$:

$$(20) \qquad \exp_{A+B}(s) = \exp_A \left(\frac{s}{2}\right) \; \exp_B(s) \; \exp_A \left(\frac{s}{2}\right) + O(s^{p+1}) \,.$$

The general *symmetric* composition method of order p with $2m$ compositions is of the form

$$(21) \qquad \begin{aligned} \exp_{A+B}(s) \;&=\; \exp_A(a_1 s) \; \exp_B(b_1 s) \; \exp_A(a_2 s) \ldots \exp_B(b_{1+[m/2]} s) \ldots \\ &\ldots \exp_A(a_2 s) \; \exp_B(b_1 s) \; \exp_A(a_1 s) + O(s^{p+1}) \end{aligned}$$

if m is odd. If m is even the factor in the middle is $\exp_A(a_{1+[m/2]} s)$. Symmetric methods are preferred since they are automatically reversible (i.e. backwards integration of an orbit always ends at the starting point).

In the case of Hamiltonian systems the individual maps used in the composition (21) are all symplectic; therefore every composition method is a symplectic integrator.

The theory of composition methods [5] is based on Lie algebras and uses the Baker-Campbell-Hausdorff (BCH) formula describing the composition of flows,

$$(22) \; \exp_A \; \exp_B = \exp_C, \quad \text{where} \quad C = A + B + \frac{1}{2} \left[A, B\right] + \frac{[A, [A, B]] + [B, [B, A]]}{12} + \ldots$$

and

$$[A, B] := A \cdot \nabla B - B \cdot \nabla A$$

is the commutator bracket of the vector fields A, B.

In order to find sets of coefficients $a_1, b_1, a_2, \ldots$ for a given error order p the BCH expansion of the composition must be equated with the Lie series expansion of the exact solution up to order p. This results in a system of algebraic equations for $a_1, b_1, a_2, \ldots$ in analogy to the determining equations for the classical Runge-Kutta coefficients. Therefore the resulting methods are referred to as (compositional) Runge-Kutta methods.

In the following we restrict ourselves to vector field splittings satisfying

$$(24) \qquad\qquad [B, [B, [B, A]]] = 0$$

since, in fact, this condition is met by the splitting (8). This relation considerably simplifies the conditions on the coefficients $a_1, b_1, a_2, \ldots$. Methods of this class are referred to as Runge-Kutta-Nyström (RKN) methods since the separable Hamiltonians

$H(q, p) = V(q) + T(p)$ are in this class, and if $T(p) = \frac{1}{2} \sum p_j^2$ the system is in the class considered by Nyström.

In the experiments of the next section we use a convenient Nyström method of order $p = 4$ with $2m = 8$ compositions, referred to as RKN-4 and defined by the coefficients [5]

$$(25) \qquad a_1 = \frac{1}{2} - \sqrt{\frac{7}{72}} \,, \quad b_1 = 1, \quad a_2 = \sqrt{\frac{7}{72}} - \frac{1}{3} \,, \quad b_2 = -\frac{1}{2} \,, \quad a_3 = \frac{2}{3}$$

and a more accurate method of order $p = 6$ with $2m = 14$ compositions, referred to as RKN-6 and defined by

$$(26) \qquad \begin{aligned} a_1 &= 1.01308797891717472981, & b_1 &= 0.00016600692650009894, \\ a_2 &= 1.18742957373254270702, & b_2 &= -0.37962421426377360608, \\ a_3 &= -0.01833585209646059034, & b_3 &= 0.68913741185181063674, \\ a_4 &= 0.34399425728109261313, & b_4 &= 0.38064159097092574080. \end{aligned}$$

MacLachlan [5] lists methods of orders up to $p = 8$.

4. Results and Conclusions

In this section we report about a few experiments carried out with homoclinic orbits in Hill's lunar problem [10]. The family of orbits considered may be uniquely characterized by its Jacobi constant h and the property that

$$(27) \qquad \lim_{t \to -\infty} x(t) =: c \,,$$

exists. These orbits are referred to as nonoscillating orbits [10], and we have $h = -0.375 \, c^2$. For certain values of h the corresponding nonoscillating orbit chaotically revolves about the origin hundreds of times before escaping to infinity.

The comparisons of this section will be carried out using the nonoscillating orbit with the energy $h = -1.03895\,34169\,0923$. For convenience, the asymptotics of nonoscillating orbits [15] and a preliminary high-precision integration were used in order to generate the nearby point

$$(28) \qquad \begin{aligned} u_0 &= 1.14311\,78537\,8775 \;+\; 0.27028\,78925\,4599 \cdot i \\ v_0 &= -2.73213\,07672\,5326 \;-\; 1.06280\,27746\,4126 \cdot i \end{aligned}$$

on the orbit to be considered.

For the symplectic integrations regularized variables were used. The vector x of dependent variables is defined as $x = (Re\,u, Im\,u, Re\,v, Im\,v, t)$, and the maps $\exp_A(s)(x_0)$ and $\exp_B(s)(x_0)$ are given by Equs (19) and Equs (13), respectively.

Typical of many differential equations and of many numerical integration procedures is the sensitivity to changes in the initial data and to a badly chosen step size. This is very much so for the differential equation and integrators considered here. In general, the deviation between neighbouring orbits grows exponentially within time. However, a symplectic integrator is expected to reproduce the correct qualitative behaviour of an orbit in spite of too large a step.

We consider Hill's lunar problem in the splitting (8) with the initial data (28). The orbits produced by the integrator RKN-4 (Equ. (25)) with steps $\Delta s = 1/16$ and $\Delta s = 1/32$ are shown in Fig. 1.

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Fig. 1:   Integrator RKN-4

Step $\Delta s = 1/16$, escape near $s = 284$ (left)
Step $\Delta s = 1/32$, escape near $s = 783$ (right).

The correct orbit escapes near $s = 424$, see Fig. 2, 4.

None of the orbits in Fig. 1 is correct after a long time; nevertheless they show the same qualitative behaviour. This is the most obvious advantage of symplectic integration. Unfortunately, it is rather difficult to predict a good step size. With $\Delta s = 1/256$ the true orbit up to the true escape time near $s = 424$ would be obtained at least in drawing accuracy. We point out, however, that such a *quantitatively* correct symplectic long-term integration is rather expensive if a low-order method is used.

Better accuracy and faster integration is obtained with a higher-order integrator, e.g. with RKN-6 (Equ. (26)). Here we claim that $\Delta s = 1/64$ suffices to integrate the orbit reliably until it escapes near $s = 424$, see Fig. 2.

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Fig. 2:   Integrator RKN-6, escape near $s = 424$. The orbit is believed to be correct to drawing accuracy, except for the "pull back" of the escaping branch. This is an artefact of the regularized coordinates which would require the reduction of the step Δs inversely proportional to the distance from the origin.

However, it is not possible to verify this claim on the basis of the information given so far. Also, the attempt of "computing" a long orbit may not be meaningful due to its inherent sensitivity to small changes of the initial data or small errors. In our case the correctness of an orbit may be established with high confidence by recomputing it with a smaller step or with a different algorithm, see Fig. 4.

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Fig. 3: The regularized Hamiltonian K for the integration of Fig. 2 (RKN-6). During the escape the situation changes dramatically due to the constant (instead of decreasing) step size.

The value of the Hamiltonian (a constant in theory) on the computed orbit provides a convenient rough estimate of the accuracy that is being achieved. Typically the regularized Hamiltonian K ($= 0$ on the orbit) oscillates chaotically but very stably at a very small scale and remains bounded away from 0, see Fig. 3. Only at the onset of the escaping phase of the motion the symplectic integration breaks down, and the behaviour of K changes dramatically. From the asymptotics of hyperbolic escape there follows that the ideal step Δs in regularized time should be inversely proportional to the distance from the origin.

To widen the spectrum of integrators the same orbit was recomputed by means of the commercially available integrator ODE45 of MATLAB (an explicit non-symplectic Runge-Kutta method with step size control), using the smallest possible tolerance of $2.22 \cdot 10^{-14}$. The orbit of Fig. 2 was reproduced in every detail (except for the artefact). Therefore we believe this orbit to be correct to at least drawing accuracy. The execution time in MATLAB was roughly the same as for RKN-6 with $\Delta s = \frac{1}{64}$. It must be said, however, that time comparisons in MATLAB are to be handled with care.

On the other hand, the regularized Hamiltonian K shows an almost perfect linear growth from 0 to $K = -1.1 \cdot 10^{-12}$ at $s = 424$. This fact puts a warning sign next to explicit Runge-Kutta integrators for long-term use! Symplectic integrators do not seem to have this problem, see Fig. 3.

To conclude, we discuss the surprising performance of the Taylor algorithm [1, 14] for this problem. A Taylor integrator, to be built for the individual differential equation under consideration, generates the Taylor series of the solution at the initial point and propagates the solution to the next point. The order and step size may be varied in a most flexible way. A simple strategy suggested by the author [14] is to sum the Taylor series up to a given error tolerance *tol* and keep the order between fixed bounds p_{min} and p_{max} by possibly halving or doubling the step.

The orbit obtained with $tol = 10^{-15}$, $p_{min} = 10$, $p_{max} = 20$ is shown in Fig. 4. Every detail of Fig. 2 is reproduced in this integration that runs about twice as fast as ODE45. Furthermore, the deviation of the Hamiltonian K is reduced by a factor of 50 and stays at the remarkably low level of $2 \cdot 10^{-14}$, see Fig. 5. Due to the slight variation in the truncation order of the Taylor series K varies irregularly. It also shows the tendency of linear growth, however. Therefore symplectic integration seems to be a possible way of keeping the energy constant.

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Fig. 4: Taylor integrator, $tol = 10^{-15}$, $10 \leq$ order ≤ 20, escape near $s = 424$. The orbit is correct at least to drawing accuracy.

Fig. 5: The regularized Hamiltonian K for the integration of Fig. 4 (Taylor).

5. Appendix: The Stumpff Functions

The family of entire functions

$$(29) \qquad c_n(z) := \sum_{k=0}^{\infty} \frac{(-z)^k}{(n+2k)!} , \quad n = 0, 1 \dots$$

was introduced by K. Stumpff [12] in his theory of uniform treatment of Kepler motion. In Equs (18) the first 4 Stumpff functions are written in terms of trigonometric functions. Equivalent representations in terms of hyperbolic functions may be used [11], e.g.

$$(30) \qquad c_0(z) = \cosh(\sqrt{-z}), \quad c_1(z) = \frac{\sinh(\sqrt{-z})}{\sqrt{-z}} , \quad c_2(z) = \frac{1 - \cosh(\sqrt{-z})}{z} .$$

Since $z = 0$ appears as a removable singularity, (18) or (30) may not be well suited for computing $c_n(z)$ if $|z| << 1$. The purpose of this section is to discuss methods (better than the Taylor series (29)) of evaluating $c_n(z)$ near the origin. In Section 2 the results are only needed for real values of z; however, they hold in the complex plane as well. Items i) through iv) below cover aspects of increasing complexity.

   i) $c_0(z)$ and $c_1(z)$ may be accurately evaluated by means of (18) and (30), together with $c_n(0) = 1/n!$, $(n = 0, 1, \dots)$, using the library functions since they usually provide constant relative accuracy, even near the origin.

   ii) The identity

$$(31) \qquad c_2(z) = \frac{1}{2} \left[c_1\left(\frac{z}{4}\right) \right]^2 ,$$

equivalent to the duplication formula of the cosine, directly reduces $c_2(z)$ to $c_1(\frac{z}{4})$ in a numerically stable way.

   iii) It is interesting to notice that (31) is the particular case $n = 1$ of the general quadrupling formula

$$2^n \, c_{n+1}(z) = c_1\left(\frac{z}{4}\right) c_n\left(\frac{z}{4}\right) + \sum_{k=1}^{[n/2]} \frac{c_{2k+1}(z/4)}{(n-2k)!} , \quad n = 0, 1, \dots .$$

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Unfortunately, for $n = 2$ this relation reads as

$$(32) \qquad 4c_3(z) = c_1\left(\frac{z}{4}\right) c_2\left(\frac{z}{4}\right) + c_3\left(\frac{z}{4}\right) ;$$

therefore it does not allow the direct evaluation of $c_3(z)$. However, by repeated application of (32) the argument z may be reduced to arbitrarily small values. For sufficiently small arguments a finite section of the Taylor series (29) or an appropriate Padé approximant may be used. This method was suggested by S. Mikkola [6].

iv) An elegant algorithm for $c_3(z)$ is obtained via the triplication formula of the sine, which becomes

$$(33) \qquad c_3(z) = \frac{1}{9}\left[c_3\left(\frac{z}{9}\right) + \frac{4}{3} c_1\left(\frac{z}{9}\right)^3\right] .$$

This directly yields the infinite series

$$(34) \qquad c_3(z) = \frac{4}{3} \sum_{k=1}^{\infty} 9^{-k} c_1(9^{-k}z)^3 .$$

In order to evaluate this series efficiently the number N of terms necessary to meet a given error tolerance ϵ in the domain $|z| < r$ has to be decided in advance. Let the successive terms of the series (34) be denoted by

$$(35) \qquad a_k := 9^{-k} c_1(9^{-k}z)^3, \quad k = 1, 2, \ldots, N, \quad a := (a_1, \ldots, a_N)^T .$$

Once a_N is known, $a_{N-1}, a_{N-2}, \ldots a_1$ can be stably computed recursively with about 5 multiplications each. Since the series (34) or

$$(36) \qquad c_3(z) = \frac{4}{3} \sum_{k=1}^{N} a_k + R_N, \quad |R_N| < \epsilon$$

converges like a geometric series with convergence quotient $q = 9$ we follow the Neville-Aitken extrapolation procedure [9] (specialized to $q = 9$), and we consider the transformed sequence

$$(37) \qquad \tilde{a} = Ma$$

where M is the infinite matrix product

$$(38) \qquad M = \ldots \left(I + \frac{I - J}{80}\right)\left(I + \frac{I - J}{8}\right)(I - J)^{-1} \qquad \text{with } J = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} ,$$

and $I \in \mathbb{R}^{N \times N}$ is the unit matrix. Equ. (36) then becomes

$$(39) \qquad c_3(z) = \frac{4}{3} \tilde{a}_N + \tilde{R}_N .$$

The matrix M is handled by means of the function

$$(40) \qquad g(x) := \prod_{k=1}^{\infty} \left(1 + \frac{1+x}{9^k - 1}\right) ;$$

then we have

$$(41) \qquad M = g(-J)\,(I - J)^{-1} .$$

From the properties $g(-1) = 1$ and $g(9x) = (1 + x)\,g(x)$ we obtain the Taylor series

$$(42) \qquad g(x) = \gamma \sum_{k=0}^{\infty} \gamma_k\, x^k = \gamma\Big(1 + \frac{x}{8} + \frac{x^2}{8 \cdot 80} + \frac{x^3}{8 \cdot 80 \cdot 728} + \cdots\Big)$$

with

$$(43) \qquad \begin{aligned} &\gamma_0 = 1, \ \gamma_k = \prod_{\ell=1}^{k} (9^\ell - 1)^{-1}, \quad (k = 1, 2, \ldots) \\ &\gamma = \Big(\sum_{k=0}^{\infty} (-1)^k \gamma_k\Big)^{-1} = 1.14082\,27572\,64437\,28201\,66654\,27682 . \end{aligned}$$

Now the following final result is obtained from (37), (41), (42), (43):

$$(44) \quad \widetilde{a}_N = \gamma \sum_{k=0}^{N-1} \sigma_k\, a_{N-k} = \gamma\left(a_N + \frac{7}{8}\,a_{N-1} + \frac{561}{640}\,a_{N-2} + \ldots\right) \ \text{ with } \ \sigma_k = \sum_{\ell=0}^{k} (-1)^\ell \gamma_\ell ;$$

then $c_3(z)$ is given by (39). An analysis of the error $|\widetilde{R}_N|$ shows that, e.g., in the disk $|z| \le 5$ (for $|z| > 5$ (18) or (30) are acceptable) Equ. (44) yields about $N(N+2.5)/2$ correct decimal digits, hence for $N = 1, 2, 3, 4, 5, 6, 7, 8$ about $D = 1.75,\ 4.5,\ 8.25,\ 13,\ 18.75,\ 25.5,\ 33.25,\ 42$ correct digits are obtained, respectively.

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Acknowledgement

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