Some complementary estimates in the Dead Core problem

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Abstract

This article is concerned with some estimates in the problem $\Delta u = c^2 u^p$ in Ω, $u = 1$ on $\partial\Omega$. The quantities of interest here are the critical value of c for which a dead core Ω_0 exists, the location of Ω_0 and the effectiveness factor.

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1 Introduction

This article is concerned with the problem

(1.1)
$$
\begin{cases}
\Delta u & = & c^2 u^p & \text{in } \Omega \subset \mathbb{R}^N , \\
u & = & 1 & \text{on } \partial\Omega ,
\end{cases}
$$

with $p \in (0,1)$. This precise form of the nonlinearity was chosen for simplicity only and in fact one can replace u^p by a more general function $f(u)$ satisfying

$$
f(0) = 0, \ f'(s) \geq 0 \ \text{ and } \ \int_0^1 \frac{ds}{f(s)} < \infty .
$$

It was shown in [3] that for sufficiently large c a "dead core" Ω_0 develops in Ω, i.e. a region where $u \equiv 0$. Problem (1.1) and various generalizations have been studied since by many different authors. There is no attempt made here of giving a survey of the literature on this problem and only the papers directly related to the results of this paper will be cited.

The quantities of interest for which estimates will be derived are:

- the critical value c_0 of c above which a dead core will exist,

- the location of the dead core,

- the "effectiveness factor" $\eta := \dfrac{\displaystyle\int_\Omega u^p \, dx}{|\Omega|}$.

The estimates to be derived complement the corresponding ones given in [3].

2 Estimates derived from optimal supersolutions

In the following an essential assumption made is that the mean curvature of $\partial\Omega$ be nonnegative everywhere. Most of the estimates will in general no longer hold if this assumption is dropped, as counterexamples show. Hence, this is not merely a purely technical assumption.

We now construct an optimal supersolution by combining the one-dimensional version of (1.1) with a suitable linear problem defined on Ω. This idea is essentially contained in [5] in a different form. It was extended then to more general elliptic problems in $\mathbb{R}^2$ in [6], to problems on a two-dimensional manifold in [9] and to the case of nonlinear boundary conditions in [10].

Let now $X(s)$ be the solution of

(2.1)
$$\begin{cases} X''(s) = c^2 f(X) \text{ in } (0, s_0) \\ X'(0) = 0, \ X(s_0) = 1 \ , \end{cases}$$

where at the moment the precise form of $f(X)$ is not yet relevant.

As a first choice of a linear problem consider the "torsion problem", i.e.

(2.2)
$$\begin{cases} \Delta \psi + 1 = 0 \text{ in } \Omega \\ \psi = 0 \text{ in } \partial \Omega \ . \end{cases}$$

One then constructs a supersolution $\overline{u}(x)$ to (1.1) having the same level lines as the torsion function by setting

(2.3)
$$\overline{u}(x) = X(s(x)), \ \ x \in \Omega \ ,$$

where

(2.4)
$$s(x) = \sqrt{2(\psi_m - \psi(x))}, \ \ \psi_m = \max_\Omega \psi \ .$$

The choice of $s(x)$ is suggested by the one-dimensional version of (2.2) when $s(x) = x$. In problem (2.1) we thus choose $s_0 = \sqrt{2 \, \psi_m}$.

The main result from which the estimates follow can be stated as

Theorem 1 *Assume that the mean curvature of $\partial \Omega$ is nonnegative everywhere and $f(0) \geq 0$, $f'(s) \geq 0$ for $s \geq 0$. Then*

$$\begin{aligned} \overline{u}(x) &= X(s(x)) \text{ is a supersolution, i.e.} \\ \Delta \overline{u} &\leq c^2 f(\overline{u}) \text{ in } \Omega \\ \overline{u} &= 1 \text{ on } \partial \Omega \ . \end{aligned}$$

Proof: Calculate first

$$\begin{aligned} \nabla s &= -\frac{\nabla \psi}{s} \ , \\ \Delta s &= -\frac{\Delta \psi}{s} - \frac{|\nabla \psi|^2}{s^3} = \frac{1}{s} - \frac{|\nabla \psi|^2}{s^3} \ , \end{aligned}$$

and then

$$\Delta \overline{u} = X' \cdot \Delta s + X'' \cdot |\nabla s|^2 = \frac{1}{s} X'\Big(1 - \frac{|\nabla \psi|^2}{s^2}\Big) + X'' \cdot \frac{|\nabla \psi|^2}{s^2} \ ,$$

from where one finds

$$\Delta \overline{u} - c^2 f(\overline{u}) = \Big(\frac{1}{s} X' - c^2 f(X)\Big)\Big\{1 - \frac{|\nabla \psi|^2}{s^2}\Big\} \ .$$

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It was shown in [4] that under our assumption on $\partial\Omega$ one has

$$|\nabla\psi|^2 \leq 2(\psi_m - \psi(x)) \, ,$$

which means that the term $\{\}$ is nonnegative. It remains to check the sign of the other factor. To this end consider

$$g(s) = X' - c^2 s f(X)$$

which satisfies $g(0) = 0$ and

$$g'(s) = X'(s) - c^2 f(X(s)) - c^2 \frac{df}{dX} \cdot X'(s) = -c^2 \frac{df}{dX} \cdot X'(s) \leq 0 \, ,$$

since $\frac{df}{dX} \geq 0$ and $X' \geq 0$.

Therefore $g(s) \leq 0$ for $s \geq 0$ so that $\overline{u}(x)$ satisfies the required differential inequality. If we select $s_0 = \sqrt{2\,\psi_m}$ in (2.1) the boundary condition is satisfied as well.

For the particular choice $f(u) = u^p$, $0 \leq p < 1$ the usual properties of sub- or supersolutions still hold even if $f'(0)$ becomes unbounded (see [11]).

As a first application of Theorem 1 we note

Corollary 1.1 *The critical value satisfies*

(2.5)
$$c_0^2 \leq \frac{p+1}{(1-p)^2 \psi_m} \, ,$$

with equality if Ω degenerates to an infinite slab of width $2\sqrt{2\psi_m}$.

Proof: For $f(u) = u^p$ the value c_0 in the one dimensional problem(2.1) is given by

(2.6)
$$c_0^2 = \frac{2(p+1)}{(1-p)^2} \cdot \frac{1}{s_0^2} \, .$$

Since $\overline{u}(x) = X(s(x))$ is a supersolution inequality (2.6) follows immediately.

Remarks:

a) In order to make inequality (2.5) more explicit we need a lower bound for ψ_m. It was shown in [4] that

$$\psi_m \geq \frac{A^2}{2L^2} \quad (A = |\Omega|, \ L = |\partial\Omega|)$$

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with equality for a slab (taking appropriate limits) under our assumption on $\partial\Omega$. For a strictly convex plane domain it follows from inequality (3) of Webb [12] that even

$$\psi_m \geq \frac{A^2}{L^2} \cdot \frac{L - k_0\,A}{2L - 3k_0\,A}\ , \quad \text{(curvature } k \geq k_0 > 0\text{)}$$

with equality for a circle or an infinite strip.

b) It was shown in [3] that

$$(2.7) \qquad c_0^2 \geq \frac{2(p+1)}{(p-1)^2 \cdot \rho^2}\ , \quad \rho = \text{radius of largest ball in } \Omega\ ,$$

again with equality for a slab. Hence (2.5) is the optimal counterpart to (2.7) since $\psi_m \leq \frac{\rho^2}{2}$ as noted in [4].

There is also information on the location and size of Ω_0 contained in Theorem 1, which may be stated as

Corollary 1.2 *The dead core Ω_0 contains the set*

$$\left\{ x \in \Omega \,|\, \psi(x) \geq d(p,c)\left[\sqrt{2\,\psi_m} - \frac{1}{2}\,d(p,c)\right] \right\}\ ,$$

where $d(p,c) = \dfrac{\sqrt{2(p+1)}}{(1-p)c}$.

Proof: In the one-dimensional problem (2.1) with $f(X) = X^p$ one can easily calculate the dead core as the interval $(0, \sigma(p,c))$ where

$$(2.8) \qquad\qquad\qquad \sigma(p,c) = s_0 - d(p,c)\ .$$

The level set $\sqrt{2(\psi_m - \psi(x))} = s_0 - d(p,c) = \sqrt{2\,\psi_m} - d(p,c)$ must be contained in Ω_0 since $X(s(x))$ is a supersolution. This implies the statement of Corollary 1.2.

Remarks:

a) It was shown in [3] that the dead core is contained in the set

$$\{x \in \Omega \,|\, \text{dist}(x, \partial\Omega) \geq d(p,c)\}\ .$$

b) The torsion function $\psi(x)$ is only known explicitly in some special cases (e.g. ellipse, equilateral triangle). If $\psi(x)$ is not known explicitly Corollary 1.2 is still useful if one makes use of the monotonic behavior of $\psi(x)$ with respect to the domain: if $\widetilde{\Omega} \subset \Omega$ then the corresponding solutions satisfy $\widetilde{\psi}(x) \leq \psi(x)$ for any $x \in \widetilde{\Omega}$.

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As a third consequence of Theorem 1 one has

Corollary 1.3 *For given value of ψ_m the effectiveness factor η is a minimum for the slab of width $2\sqrt{2\,\psi_m}$. In particular, if $c \geq c_0$ one has*

(2.9)
$$\eta \geq \frac{1}{c\sqrt{(p+1)\psi_m}} \; .$$

Proof: Since $\overline{u} = X(s(x))$ is a supersolution which satisfies the boundary condition one has

$$\frac{\partial u}{\partial n} \geq \frac{\partial \overline{u}}{\partial n} \quad \text{on } \partial\Omega$$

which implies

$$c^2|\Omega|\,\eta \geq -\oint_{\partial\Omega} \frac{X'(s_0)}{s_0} \, \frac{\partial\psi}{\partial n} \, d\sigma = \frac{X'(s_0)}{s_0} \, |\Omega| \, , \quad (d\sigma = \text{element of } \partial\Omega)$$

that is

$$\eta \geq \frac{X'(s_0)}{c^2 \cdot s_0} = \text{effectiveness factor for slab of width } 2 \cdot s_0.$$

If $c \geq c_0$ one gets (see Aris [1], p. 146)

$$\eta = \sqrt{\frac{2}{c^2(p+1) \cdot s_0}} = \frac{1}{c\sqrt{(p+1)\psi_m}} \, ,$$

which completes the proof.

Remarks:

a) For $c < c_0$ the value of η in the one-dimensional case is determined from the relation (see [1], p. 148)

$$c \cdot s_0 = \sqrt{\frac{2}{1+p}} \, (1 - u_0^{p+1}) \cdot F\Big(1, \, \frac{p}{p+1} \, ; \frac{3}{2} \, ; 1 - u_0^{p+1}\Big) =: h(u_0, p)$$

and then,

$$\eta = \frac{1}{F\Big(1, \, \dfrac{1}{p+1} \, ; \dfrac{3}{2} \, ; 1 - u_0^{p+1}\Big)}$$

there $F(a, b; c; z)$ is the hypergeometric function and $u_0 = u(0) = \text{minimum}$ value. The function $h(u_0, p)$ is monotonically decreasing in u_0 for any $p \in (0, 1)$. One can also prove that η is a decreasing function of s_0, so that an upper bound for ψ_m is needed in Corollary 1.3. A number of upper bounds for ψ_m are known (see e.g. [2], [4], [8]). One has e.g. $\psi_m \leq \frac{\rho^2}{2}$ under our assumptions on $\partial\Omega$.

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b) It follows from Lemma 3.1 of [3] that for $c \geq c_0$

$$(2.10) \qquad \eta \leq \sqrt{\frac{2}{p+1}} \cdot \frac{L}{cA}$$

and the equality sign holds also in the limit if Ω degenerates into a slab.

A second choice of a linear problem is the fixed membrane problem on Ω i.e.

$$(2.11) \qquad \begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \ . \end{cases}$$

In this case we replace $s(x)$ as defined in (2.4) by

$$(2.12) \qquad t(x) = \frac{1}{\sqrt{\lambda_1}} \ \cos^{-1}\left(\frac{\varphi(x)}{\varphi_m}\right) ,$$

where λ_1 = first eigenvalue with associated eigenfunction $\varphi(x)$ and $\varphi_m = \max\limits_{\Omega} \varphi$. Also $s_0 = \frac{\pi}{2\sqrt{\lambda_1}}$ is now the length of the interval.

The analogue of Theorem 1 is now

Theorem 2 *Assume that the mean curvature of $\partial\Omega$ is nonnegative everywhere and $f(0) \geq 0$, $f'(s) \geq 0$ for $s \geq 0$. Then*

$$\begin{aligned} \overline{u}(x) &= X(t(x)) & \text{is a supersolution, i.e.} \\ \Delta\overline{u} &\leq c^2 f(\overline{u}(x)) & \text{in } \Omega \\ \overline{u} &= 1 & \text{on } \partial\Omega \end{aligned}$$

Proof: A straightforward calculation gives now

$$\Delta\overline{u} - c^2 f(\overline{u}) = \left(\lambda_1 - \frac{|\nabla\varphi|^2}{\varphi_m^2 - \varphi^2}\right)\left\{X' \cdot \cot(\sqrt{\lambda_1}\, t) \sqrt{\lambda_1} - c^2 f(X(t))\right\} .$$

By a result of Payne & Stakgold [7] one has $\lambda_1 \geq \dfrac{|\nabla\varphi|^2}{\varphi_m^2 - \varphi^2}$ if the mean curvature of $\partial\Omega$ is nonnegative.

The term $\{\}$ is nonpositive as a similar reasoning as in the proof of Theorem 1 shows, now for $g(t) = X'\sqrt{\lambda_1}\, \cos(\sqrt{\lambda_1}\, t) - \sin(\sqrt{\lambda_1}\, t)\, f(X(t))$.

The counterparts of Corollaries 1.1 - 1.3 are now obvious:

Corollary 2.1 *The critical value c_0 satisfies*

$$(2.13) \qquad c_0^2 \leq \frac{8\lambda_1(p+1)}{(1-p)^2\pi^2} \ .$$

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Remarks: Since $\lambda_1 \geq \dfrac{\pi^2}{8\,\psi_m}$ as noted by Payne [5], (2.13) is weaker than (2.5).

The counterpart of Corollary 1.2 may still be useful in some cases (see e.g. Example 1). It now reads

Corollary 2.2 *The dead core is contained in the set*

$$\{x \in \Omega \,|\, \varphi(x) \geq \varphi_m \cdot \sin(\sqrt{\lambda_1}\, d(p,c)\} \ .$$

It is interesting to see however that inequality (2.9) can now be improved. In fact we have

Corollary 2.3 *For given value of λ_1 the effectiveness factor η is a minimum for the slab of width $\dfrac{\pi}{\sqrt{\lambda_1}}$. In particular if $c \geq c_0$ one has*

(2.14)
$$\eta \geq \frac{2}{c\pi}\,\sqrt{\frac{2\lambda_1}{p+1}} \ .$$
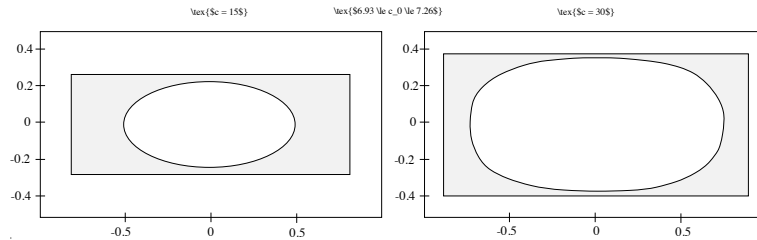
Examples

1. $\Omega =$ Rectangle of sides $a = 2$, $b = 1$. Inequalities (2.5) and (2.7) then yield

$$8 \leq \frac{(1-p)^2}{p+1}\, c_0^2 \leq 8.782 \ .$$

For $c \geq c_0$ (2.10) and (2.14) give

$$\sqrt{\frac{2}{p+1}} \cdot \frac{2.236}{c} \leq \eta \leq \sqrt{\frac{2}{p+1}} \cdot \frac{3}{c} \ .$$

For $p = \frac{1}{2}$ Corollary 1.2 and the estimate in [3] give the following pictures (the boundary of Ω_0 must lie in the shaded regions).

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2. Ω = Equilateral triangle of height 1. One has $\psi_m = \frac{1}{27}$ and $\lambda_1 = 4\pi^2$ and (2.5) and (2.7) now give

$$18 \leq \frac{(1-p)^2}{p+1} \, c_0^2 \leq 27 \; .$$

A better lower bound in some cases (for plane domains) is $c_0^2 \geq \dfrac{4\pi}{(1-p) \cdot A}$ as given in Corollary 3.1 of [3].
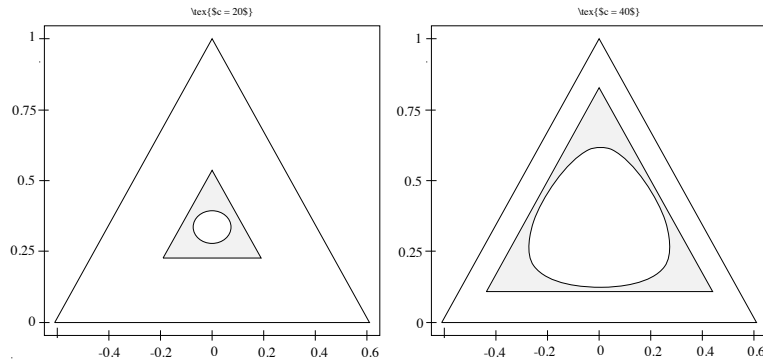
For $c \geq c_0$ (2.10) and (2.14) show that

$$\sqrt{\frac{2}{p+1}} \cdot \frac{4}{c} \leq \eta \leq \sqrt{\frac{2}{p+1}} \cdot \frac{6}{c} \; .$$

For $p = 0.5$ one has therefore

$$9.33 \leq c_0 \leq 12.73 \; ,$$

and one obtains the following pictures

The boundary of Ω_0 lies in the shaded region.

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