An Accuracy Barrier for Stable Three-Time-Level Difference Schemes for Hyperbolic Equations

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Abstract

We consider three-time-level difference schemes for the linear constant coefficient advection equation $u_t = cu_x$. In 1985 it was conjectured that the barrier to the local order p of schemes which are stable is given by

$$p \le 2\min\{R, S\}.$$

Here R and S denote the number of downwind and upwind points, respectively, in the difference stencil with respect to the characteristic of the differential equation through the update point. Here we prove the conjecture for a class of explicit and implicit schemes of maximal accuracy. In order to prove this result, the existing theory on order stars has to be generalized to the extent where it is applicable to an order star on the Riemann surface of the algebraic function associated with a difference scheme. Proof of the conjecture for all schemes relies on an additional conjecture about the geometry of the order star.

Keywords: scalar advection equation, difference scheme, accuracy, stability, order star, algebraic function, Riemann surface

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1   Introduction

Suppose we have a difference scheme for an initial-boundary value problem for a system of hyperbolic partial differential equations. A **global difference scheme** for the solution of this problem generally consists of an **interior scheme** and a **boundary scheme**. By the Lax-Richtmyer Equivalence Theorem these difference schemes result in solutions which are convergent to the exact solution only when they are consistent with the initial-boundary value problem and are stable. Consistency is the minimal requirement that the order of accuracy p is one for interior and boundary schemes. Their stability in the global framework was investigated by Kreiss [18], [19] and in the influential paper by Gustafsson, Kreiss and Sundström [4]. In the latter paper the following necessary condition was given for the global scheme to be stable, namely that the corresponding interior scheme has to be stable in the Von Neumann sense when applied to the pure Cauchy problem for the scalar advection equation. Goldberg and Tadmor [3], [2] gave more practical sufficient conditions for stability of global schemes. These conditions entail, among others, that the boundary scheme also has to be stable in the same sense as mentioned above for the interior scheme.

From these results it can be concluded that accurate and stable difference schemes for the scalar advection equation are of fundamental importance in the construction of useful global schemes. For this reason we consider a Cauchy problem for the scalar advection equation

$$(1) \qquad \frac{\partial}{\partial t}\, u(t,x) \;=\; c\, \frac{\partial}{\partial x}\, u(t,x),\; x \in \mathbb{R},\; t \geq 0\,,$$

$$u(0,x) \;=\; u_0(x)\;\text{ given}\,,$$

and a class of multistep $((k+1)$-time-level) difference schemes of the form

$$(2) \qquad \sum_{i=0}^{k}\; \sum_{j=-r_i}^{s_i}\; a_{ij}\, u_{n+i,m+j} = 0$$

which are used to determine an approximate solution of (1). The coefficients a_{ij} are responsible for the two above-mentioned features, namely the **accuracy** and **stability** of the scheme. In general the requirement of stability imposes a bound on the order of a scheme. This paper focuses on this barrier to the order imposed by the requirement of stability for schemes of type (2).

One-step schemes ($k = 1$) were extensively studied in [6, 7, 9, 10, 15, 22] and results for multistep schemes were given in [11, 13, 17, 23]. In [11, 15] it was conjectured that the order barrier for stable multi-time-level schemes (2) should be

$$(3) \qquad\qquad p \leq 2\min\{R, S\}\,.$$

Here, for counting purposes, we let the "zero line" be the characteristic through the point on the new time level for which one solves. Then R denotes the number of downwind

points and S the number of upwind points of a given scheme with respect to this zero line. This means that a stable scheme of order p needs to have on each side of the characteristic at least $\lceil \frac{p}{2} \rceil$ points in the stencil. (Here $\lceil \alpha \rceil$ denotes the smallest integer which is not smaller than α). If $p = 1$, this conjecture reduces to the Courant-Friedrichs-Lewy condition. The bound (3) therefore has the quality of being an extension of the Courant-Friedrichs-Lewy condition, [1].

The bound (3) was proved in [15] for two-time-level schemes. In [17] it was partially proved for a small subclass of explicit three-time-level schemes. In [11], [13] many examples in support of (3) were given for multi-time-level schemes. In [12] the first lower bound in (3) for the $(k+1)$-level case was given by actually showing stability of schemes with long and slender stencils (only one step in space). Such schemes may be useful as high-order boundary schemes. In Sections 3-8 of this paper we generalize the results in [17] to some maximal order explicit and implicit three-time level schemes with a property to which we refer as convexity (see Section 2). The results for all other schemes follow from a conjecture presented in Section 6.

The analysis in this paper is based on the order star technique, which was introduced in [24] and treated extensively in [5], [8]. The ideas treated there, however, to be applicable to our analysis, have to be generalized for order stars on the Riemann surface of an algebraic function. This algebraic function is treated in Section 4. An additional complication is that our order stars are defined with respect to the comparison function z^μ. This comparison function was first used in [22], [6]. The analysis in Sections 3-8 is a continuation of the work in [16], [17], viz. a study of the order stars on a two-sheeted Riemann surface. Since z^μ is multiple-valued with a logarithmic singularity at $z = 0$, extreme care has to be taken with the integration path used for the application of the argument principle. Notwithstanding these complications the order stars basically retain the elegant features which make them so useful in the sense that they allow a simple geometrical interpretation of the relationship between accuracy and stability.

For explicit schemes the order of the logarithmic singularity at $z = 0$ determines the maximum multiplicity of components of the order star. The various possible geometric configurations and the corresponding multiplicities of these geometries are investigated in Section 6. Except for a small subclass of schemes the derived bounds on order of the schemes do not lead to a proof of (3). This leads to the introduction of a conjecture, based on our observations, in which we propose that certain geometric configurations are not possible. The incorporation of the branch points into the analysis, in Section 7, leads to a proof of this conjecture for some schemes of maximal order.

For implicit schemes the poles of the algebraic function also play an important role. The geometry of components containing poles is investigated in Section 8. Section 9 combines the results of the previous sections to provide the proof of (3).

We believe that this paper indicates the direction which the generalization of (3) to the $(k+1)$-time-level case will take. Since we restrict ourselves to schemes which can be considered convex, i.e. with an increasing stencil, we only work with poles of the

algebraic function. In order to allow convexity for negative time as well, i.e. convexity in the reverse time direction, equivalently, concave schemes, the zeros of the algebraic function also have to be taken into account. Note that by excluding concave schemes we exclude the possibility of a branch point of the algebraic function added to the logarithmic singularity at $z = 0$.

In a parallel investigation in [14] concerning three-time-level schemes for the wave equation we build on work started in [20], [21]. In that case the symmetry-properties of the schemes lead to a considerable simplification of the order star theory as treated in Sections 5- 9. By also taking into account the role of the zeros of all of the polynomials defining the algebraic function the class of schemes can be treated there without imposing a restriction such as convexity. Furthermore, because of symmetry there is no possibility of a branch point at $z = 0$.

2   Order, stability and normalization of schemes

We consider three-time-level difference schemes of the form

$$(4) \qquad \sum_{j=-r_2}^{s_2} a_{2j}\, u_{n+2,m+j} + \sum_{j=-r_1}^{s_1} a_{1j}\, u_{n+1,m+j} + \sum_{j=-r_0}^{s_0} a_{0j}\, u_{n,m+j} = 0$$

$$n = 0, 1, 2, ..., \quad m = 0, \pm 1, \pm 2, ...$$

The step sizes in the time and space variables are denoted by Δt and Δx, resp., while $\mu = \frac{c\Delta t}{\Delta x}$ denotes the Courant number which is assumed to be fixed. The coefficients a_{ij} are real and depend in general on μ, $a_{ij} = a_{ij}(\mu)$. Further r_i, $s_i \in \mathbb{Z}$ with $r_2 \geq 0$, $s_2 \geq 0$ and $-r_i \leq s_i$, $i = 0,1$ and $a_{i,-r_i} \neq 0$, $a_{i,s_i} \neq 0$ for $i = 0,1,2$. A value u_{nm} is meant to be an approximation of $u(n\Delta t, m\Delta x)$. If $r_2 = s_2 = 0$ a scheme (4) is said to be **explicit.** Otherwise it is called **implicit.**

We shall concentrate on schemes with a stencil satisfying

$$(5) \qquad \begin{cases} 0 \leq r_0 - r_1 \leq r_1 - r_2 \\ 0 \leq s_0 - s_1 \leq s_1 - s_2 \,. \end{cases}$$

A scheme with a stencil satisfying (5) will be referred to as a **convex scheme** with an increasing stencil. From a computational point of view these schemes seem to yield the most interesting stencils.

A Fourier Transform enables us to associate with (4) on time level $n + i$ a function

$$(6) \qquad a_i(z) = \sum_{j=-r_i}^{s_i} a_{ij}\, z^j, \qquad i = 0, 1, 2$$
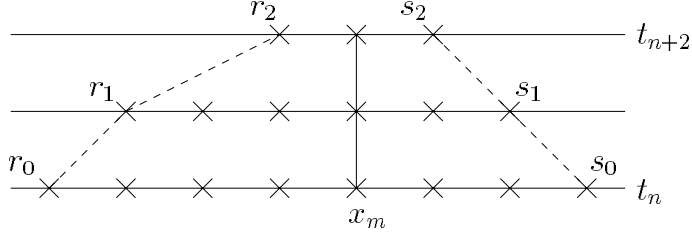
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Figure 1: Convex stencil

and to introduce the **characteristic function**

$$\Phi(z, w) = a_2(z)\, w^2 + a_1(z)w + a_0(z) \,,$$

which is assumed to be irreducible (see [13]).

In order to be able to solve (4) for the values on the new time level in the implicit case, we impose the necessary and sufficient condition

$$a_2(z) \neq 0 \quad \text{for } |z| = 1 \,.$$

We also require our schemes to satisfy the following **normalization condition** (see [9, 13]):

$$(7) \qquad \begin{cases} r_2 &= \text{number of zeros of } a_2(z) \text{ with } |z| < 1 \\ s_2 &= \text{number of zeros of } a_2(z) \text{ with } |z| > 1 \,. \end{cases}$$

A scheme (4) is said to be **stable** if

$$(8) \qquad \begin{aligned} \Phi(z, w) &= 0 \\ |z| &= 1 \end{aligned} \Biggr\} \implies \begin{cases} |w| \leq 1 \text{ and if } |w| = 1, \\ \text{then } w \text{ is a simple root} \,. \end{cases}$$

A scheme (4) has **error order** p if for any smooth solution $u(t, x)$ of (1) we have

$$\sum_{i=0}^{2} \sum_{j=-r_i}^{s_i} a_{ij}\, u(t + i\Delta t,\ x + j\Delta x) \;=\; C\, \frac{\partial^{p+1}}{\partial x^{p+1}} u(t, x)(\Delta x)^{p+1} + O((\Delta x)^{p+2})$$

$$\text{if } \Delta x \to 0 \text{ and } \mu = \text{constant} \,.$$

Since we are interested only in schemes with positive order, we assume that

$$\Phi(1, 1) = \sum_{i=0}^{2} \sum_{j=-r_i}^{s_i} a_{ij} = 0 \,.$$

The next result expresses the order of a scheme as a property of the solution w of $\Phi(z, w) = 0.$

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Proposition 2.1 (Equivalent Order Conditions, [13, 23]) *Let a scheme (4) with characteristic function $\Phi(z, w)$ and Courant number μ be stable and satisfy $\Phi(1, 1) = 0$. Then the following three conditions are equivalent.*

a) *The scheme has order p.*

b) *$\Phi(z, z^\mu) = O((z - 1)^{p+1})$ as $z \to 1$.*

c) *The algebraic function w given by $\Phi(z, w(z)) \equiv 0$ has exactly one branch w_1 which is analytic in a neighbourhood of $z = 1$ and satisfies*

$$z^\mu - w_1(z) = O((z - 1)^{p+1}) \text{ as } z \to 1 .$$ □

The next theorem gives the highest possible order that a scheme can have if stability is ignored. We introduce the index set of the difference stencil

$$I = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 0 \le i \le 2, -r_i \le j \le s_i\} .$$

A scheme (4) is said to be **regular** if a characteristic through any given stencil point does not pass through any other point of the difference stencil (see [13]).

Proposition 2.2 (Regular Stencil, [13]) *Let a scheme (4) have a regular difference stencil with index set I. Then the highest possible order that the scheme can have is*

$$p = |I| - 2 ,$$

where $|I|$ denotes the number of indices in I. □

3 Main result: bound on order of stable schemes

Suppose we have a convex scheme (4) which is also regular. The characteristic through the point (t_{n+2}, x_m) (of the normalized scheme (4)) will be taken to be our "zero line". Then it is possible to interpret the order bound of stable schemes in a simple geometrical way. This entails that the number of stencil points on each side of this zero line has to be "balanced" in order to give the highest order. If R denotes the total number of downwind points and S the total number of upwind points with respect to this zero line, then the order p of a stable scheme satisfies

$$p \le 2 \min\{R, S\} .$$

This result can be related to the indices r_i, s_i of (4) in the following way: Define

$$R_1 = \begin{cases} 0 & \text{if } \mu < -r_1 \\ \lfloor r_1 + \mu \rfloor + 1 & \text{if } -r_1 < \mu < s_1 , \\ r_1 + s_1 + 1 & \text{if } \mu > s_1 \end{cases} \quad S_1 = r_1 + s_1 + 1 - R_1$$

$$R_0 = \begin{cases} 0 & \text{if } 2\mu < -r_0 \\ \lfloor r_0 + 2\mu \rfloor + 1 & \text{if } -r_0 < 2\mu < s_0 , \\ r_0 + s_0 + 1 & \text{if } 2\mu > s_0 \end{cases} \quad S_0 = r_0 + s_0 + 1 - R_0 ,$$

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where $\lfloor \alpha \rfloor$ denotes the largest integer not exceeding α, and

(9)
$$R = R_0 + R_1 + r_2, \quad S = S_0 + S_1 + s_2 .$$

Then our main result can be formulated as follows.

Theorem 3.1 (Maximal Order of Stable Convex Schemes) *Let a convex scheme (4) with an increasing stencil be normalized and have a fixed Courant number μ satisfying $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, then the order p of the scheme is bounded by*

$$p \leq 2 \min\{R, S\} .$$

$\square$

Remark 3.2   *a) In [17] the bound (3) was proved for a small subclass of explicit schemes of type (4). In this paper we generalize it, making use of a conjecture introduced in Section 6, for the class of (explicit and implicit) schemes of type (4) which are convex and have an increasing stencil.*

   b) In Section 7 we provide a partial proof of (3). In particular, for the maximal order schemes, $p = |I| - 2$,

$$p \leq \begin{cases} 2R & -\frac{1}{2} < \mu < 0 \\ 2S & 0 < \mu < \frac{1}{2}. \end{cases}$$

   c) The result (3) can be extended to $|\mu| > \frac{1}{2}$ by making use of the following transformation. Assume that a stable scheme is represented by $\Phi(z, w)$, where $w(z)$ approximates z^μ in a neighbourhood of the point $z = 1, w = 1$. Then we consider the scheme represented by the characteristic function

$$\widetilde{\Phi}(z, u) = z^2 \, \Phi(z, \tfrac{u}{z}) .$$

Since $u = zw$, the new scheme is stable and approximates $z^{\widetilde{\mu}} = z^{\mu+1}$ with the same order as the original scheme. The stencil undergoes the following transformations:

$$\widetilde{r}_1 = r_1 - 1, \quad \widetilde{s}_1 = s_1 + 1, \quad \widetilde{r}_0 = r_0 - 2, \quad \widetilde{s}_0 = s_0 + 2 .$$

4   Properties of the algebraic function w

The algebraic function w, satisfying $\Phi(z, w(z)) \equiv 0$, is multiple-valued, consisting in general for a given z of two values $w_1(z)$ and $w_2(z)$. Associated with this algebraic function is the Riemann surface M,

$$M = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : \ \Phi(z, w) = 0\} ,$$

which consists of two sheets, one above the other, and interacts at a finite number of branch points z_i (where $w_1(z_i) = w_2(z_i)$). It is known that M is a closed, connected set and that on M the function w is single-valued and, except for a finite number of singular points, also analytic.

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Remark 4.1 (Branch points of w)   *a) The branch points of w (except those that can occur at 0 and ∞) occur at points z_i where $a_1^2(z_i) - 4a_2(z_i)\,a_0(z_i) = 0$. Since the coefficients of this polynomial equation are real, the branch points are either real or they occur in complex conjugate pairs. Branch cuts along which the two sheets of M are connected can therefore always be taken to fall on the real axis, or to be perpendicular and symmetric with respect to the real axis or to occur in conjugate pairs.*

*b) If a scheme is stable, the corresponding algebraic function cannot have a branch point at $z = 1$ (see (8)). The sheet of M on which the point $z = 1$, $w = 1$ occurs, is called the **principal sheet**. Since the Riemann surface is connected, this notion is basically a local property in a neighbourhood of $z = 1$. We make the convention that the principal sheet will refer to that part of M which can be connected to $z = 1$, $w = 1$ without crossing a branch cut. The remaining part of M will be called the **secondary sheet**.*

Remark 4.2 (Poles of w) *The function w has a pole at every point where $a_2(z) = 0$. In view of the normalization condition (8) there will therefore be in total r_2 poles of w with $|z| < 1$ away from $z = 0$ on the two sheets of M. Moreover, if $\max\{r_0, r_1, r_2\} > 0$, then w can have a pole at $z = 0$ on one or both sheets of M (see Proposition 4.4).*

Remark 4.3 (Zeros of w) *The finite points where w has zeros coincide with the points where the function $v(z) = \frac{1}{w(z)}$ has poles. Similarly to the preceding remark, we can conclude that these points occur where $a_0(z)$ has zeros and occasionally also at $z = 0$.*

The expansion of the function w at $z = 0$ will be very important in subsequent considerations. This is determined by means of Newton's polygons and is given in the next proposition.

Proposition 4.4 (Expansion at $z = 0$) *Let*

$$\Phi(z, w) = \left(a_{2,-r_2} z^{-r_2} + \,...\, + a_{2,s_2} z^{s_2}\right) w^2 + \left(a_{1,-r_1} z^{-r_1} + \,...\, + a_{1,s_1} z^{s_1}\right) w$$
$$+ \left(a_{0,-r_0} z^{-r_0} + \,...\, + a_{0,s_0} z^{s_0}\right)$$

be the characteristic function of a convex scheme (4) with an increasing stencil. Then the algebraic function w satisfying $\Phi(z, w) = 0$ does not have a branch point at $z = 0$ and has the following expansions at $z = 0$:

a) if $r_1 - r_2 > r_0 - r_1$, then

$$\begin{aligned} w_1(z) &= z^{-(r_1 - r_2)}(c_0 + c_1 z + c_2 z^2 + \,...),\\ w_2(z) &= z^{-(r_0 - r_1)}(d_0 + d_1 z + d_2 z^2 + \,...). \end{aligned}$$

where one has $c_0 = -\dfrac{a_{1,-r_1}}{a_{2,-r_2}}$ and $d_0 = -\dfrac{a_{0,-r_0}}{a_{1,-r_1}}$.

b) if $r_1 - r_2 = r_0 - r_1$, then

$$
\begin{aligned}
w_{1,2}(z) &= z^{-(r_1-r_2)}(-a_1(z)z^{r_1} \pm \sqrt{D})/(2a_2(z)z^{r_2}) \\
D(z) &= (a_1^2(z) - 4a_2(z)\,a_0(z))z^{2r_1} \\
&= d + O(z)
\end{aligned}
$$

where $d = a_{1,-r_1}^2 - 4a_{2,-r_2}\,a_{0,-r_0}$.

$\square$

Remark 4.5 *From Proposition 4.4 we observe that $z = 0$ is not a branch point of w if $r_1 - r_2 > r_0 - r_1$. If $2r_1 = r_0 + r_2$ and $d \neq 0$ then again $z = 0$ is not a branch point. If $d = 0$ then $z = 0$ can be a branch point. In the following we shall restrict ourselves to schemes where $z = 0$ is not a branch point and then one has the two expansions*

$$
\begin{aligned}
w_1(z) &= z^{-(r_1-r_2)}(c_0 + c_1 z + c_2 z^2 + \ldots) \\
w_2(z) &= z^{-(r_0-r_1)}(d_0 + d_1 z + d_2 z^2 + \ldots)\,.
\end{aligned}
$$

5   Order Stars

An order star is defined on the Riemann surface M of the algebraic function w in the following way. Define the function φ by

$$
\varphi(z, w) = z^{-\mu}w, \quad (z, w) \in M
$$

and the **order star** Ω by

$$
\Omega = \{(z, w) \in M : \ |\varphi(z, w)| > 1\}\,.
$$

Because of the factor $z^{-\mu}$ the function φ is multiple-valued on M. However, the order star Ω, being defined by means of the modulus of φ, is again well defined on M. Ω^c denotes the complement of Ω, i.e. $\Omega^c = M\backslash\Omega$. Because the coefficients a_{ij} are real, it is easily verified that Ω is symmetric with respect to the real axis.

The order and stability of a scheme, which were interpreted in Section 2 as properties of the function w associated with the scheme, can now be expressed as properties of the corresponding order star. We give without proof those properties which are standard results in investigations involving order stars (e.g. in [24], [8]).

Lemma 5.1 (Stability) *If a scheme is stable, then*

$$
\Omega \cap \{(z, w) \in M : \ |z| = 1\} = \emptyset\,.
$$

$\square$

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Lemma 5.2 (Order) *A scheme (4) has order p if and only if at the point $z = 1$ on the principal sheet of M the order star consists of $p + 1$ sectors of angle $\frac{\pi}{p+1}$, separated by $p + 1$ sectors of Ω^c, each with the same angle.* $\qquad\qquad\qquad\square$

A subset A (with boundary ∂A) of Ω is said to be an **Ω-component** if $\partial A \subset \partial \Omega$ and A is connected. Ω^c-components are defined similarly. An Ω-component is said to be of **multiplicity** m if it contains m Ω-sectors at $z = 1$ on the principal sheet. Similarly for Ω^c-components.

Note that the curve on M which has the projection $|z| = 1$ in the z-plane separates M into two well defined subsets. We call the portion on M with $|z| < 1$ the unit disk Δ. The set with $|z| > 1$ is called the outside of the unit disk. Since we only deal with stable schemes, we deduce from Lemma 5.1 that there is a clear distinction between the portion of the order star inside and the portion outside the unit disk. In Section 7 we shall see that it is sufficient to consider the portion of Ω inside Δ. Hence our subsequent investigation will be restricted to Ω-components occurring inside Δ. Such components have the property that they are bounded, where a component Ω_1 is said to be **bounded** if $\sup\limits_{(z,w)\in\Omega_1} |z| < \infty$.

In order to emphasize important features, our pictures of Ω-components will not always be the exact geometrical embeddings of M into $\mathbb{R}^3$. They will, however, display the basic connectivity relations and cuts and elucidate the important properties of both macroscopic and microscopic scale.

According to Remark 4.5 we restrict ourselves to schemes where there is no branch point of w at $z = 0$. Thus there are two values w_1^0 and w_2^0 with $\Phi(0, w_i^0) = 0$, i.e. there are two zero points $(0, w_1^0)$ and $(0, w_2^0)$ on M. Depending on the values of the indices r_i we know from Proposition 4.4 that there can be a pole of w at one or both of the zero points. Further, because of the factor $z^{-\mu}$ occurring in φ, this function in general no longer has an integer-valued leading exponent of z at the zero points.

We know from Remark 4.2 that there are r_2 poles of w away from $z = 0$ inside Δ (if our scheme is normalized). Since the expansion of $z^{-\mu}$ from any point $z_0 \neq 0$ has the form

$$z^{-\mu} = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ..., \quad |z - z_0| < |z_0| \; ,$$

there will be poles of φ of exactly the same orders at the points with these z-values on one of the sheets of M.

The influence of poles and negative leading exponents of φ at $z = 0$ on the multiplicity of the components in which they occur, is studied by means of the argument principle with respect to the function φ (see [24]). In this regard we have to note that the factor $z^{-\mu}$ of φ introduces onto the Riemann surface M a new structure in the sense that it defines on M another Riemann surface which in general has infinitely many sheets. In

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order to define $z^{-\mu}$ uniquely on M, we have to introduce branch cuts from the zero points $(0, w_i^0)$ to the points at infinity (∞, w_i^∞) on the two sheets of M. We indicate these cuts by L_i and first discuss the rules for making them.

Rule 1 for cuts L_i: The branch cuts L_i have to be such that their projections onto the z-plane are either identical, or "enclose" a "sector" of $\mathbb{C}$ which does not contain a branch cut of M.

If we adhere to Rule 1, then $z^{-\mu}$ is defined uniquely on M (see [17], Section 5.1), even if the cuts L_i are allowed to cross a branch cut of M. In the present context, however, we can always avoid this. To this extent we introduce the following rule.

Rule 2 for cuts L_i. The cuts L_i have to be such that each cut occurs only on one sheet of M, i.e. L_1 between $(0, w_1^0)$ and (∞, w_1^∞) on the principal sheet and L_2 between $(0, w_2^0)$ and (∞, w_2^∞) on the secondary sheet.

We can always adhere to Rules 1 and 2 by choosing the branch cuts L_i to go along two radial lines which have the same projection onto the z-plane and for whom this projection onto the z-plane does not pass through the point $z = 1$. In the sequel this will be our convention for making the cuts L_i, with the exception of a few situations where the deviation from this convention will be pointed out explicitly.

6 The Role of the Zero Points on Multiplicity

We start by restricting ourselves to the order stars of explicit schemes. Since then $r_2 = 0$ there are no poles away from $z = 0$ inside Δ and there has to be at least one of the zero points $(0, w_1^0), (0, w_2^0)$ inside every bounded Ω-component (see [17], Lemma 5.1).

Our investigation of the relationship between the multiplicity of a component and the total order of poles/singularities of φ that it contains begins with a very simple type of Ω-component, Ω_1 (say), which occurs only on the principal sheet of M and which contains no branch points of w. This type of component was treated in [17], but the proof is repeated here since it displays the approach to the application of the argument principle under the prevailing circumstances.

Proposition 6.1 (Multiplicity) *Let Ω_1 be such that the principal branch can be defined as a single-valued function on the projection of Ω_1 onto the z-plane. Assume φ has a leading exponent of $-\alpha$ at $z = 0$. Then the multiplicity m of Ω_1 satisfies*

$$m \le \lfloor \alpha \rfloor + 1 \, .$$

Proof: If Ω_1 is of multiplicity m, there are $m - 1$ Ω^c-components emerging from (1,1) to the "inside" of Ω_1. With the cut L_1 made according to our convention, we construct a
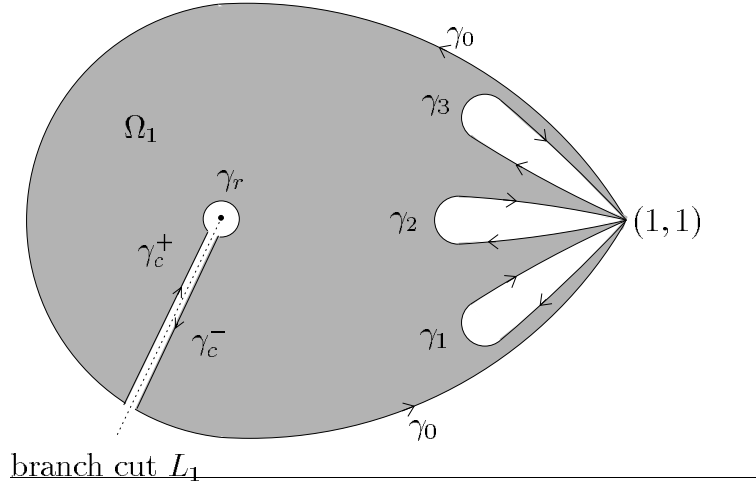
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Figure 2: Component Ω_1 illustrating the integration path

closed curve γ and will compute $\frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z,w)}{\varphi(z,w)} \, dz$. The curve γ is composed of the following curves (see Fig. 2):

γ_0:   positively oriented (w.r.t. $z = 0$) "outward" boundary of Ω_1. According to [24] (proof of Proposition 4) the argument of φ decreases along γ_0.

γ_c^+, γ_c^-:   two sides of a Jordan curve which connects the "outward" boundary of Ω_1 with a circle around $z = 0$. According to [17] (Lemma 4.4) the contributions to the integral along γ_c^+ and γ_c^- cancel out.

γ_r: circular curve with small radius r, traversed clockwise. According to [17] (Lemma 4.3) the contribution of this curve to the integral is α.

$\gamma_1, ..., \gamma_{m-1}$:   boundary of Ω_1 along $m - 1$ Ω^c-components emerging from (1,1) to the "inside" of Ω_1. Again the argument of φ decreases along each path γ_i and since φ is single-valued in $\Omega_1 \backslash L_1$, every time the boundary γ returns to (1,1) the argument has decreased by at least 2π.

Then

$$\gamma = \gamma_0 + \gamma_c^+ + \gamma_r + \gamma_c^- + (\gamma_1 + \gamma_2 + ... + \gamma_{m-1}) \, .$$

By application of the argument principle and since there are assumed to be no zeros or

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poles of φ inside γ, we have

$$
\begin{aligned}
0 &= \frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z,w)}{\varphi(z,w)}\,dz \\[2mm]
&= \underbrace{\frac{1}{2\pi i}\int_{\gamma_0}}_{<0} + \underbrace{\frac{1}{2\pi i}\left(\int_{\gamma_c^+} + \int_{\gamma_c^-}\right)}_{=0} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_r}}_{=\alpha} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_1 + \ldots + \gamma_{m-1}}}_{\leq -(m-1)} .
\end{aligned}
$$

We introduce the notation γ_0^E to indicate the curve "γ_0 extended" to include the first three terms in the above expression, viz.

$$
\int_{\gamma_0^E} = \int_{\gamma_0} + \int_{\gamma_c^+} + \int_{\gamma_r} + \int_{\gamma_c^-} .
$$

Then it follows that

$$
\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'(z,w)}{\varphi(z,w)}\,dz \leq \lfloor \alpha \rfloor
$$

and the complete expression above leads to

$$
m \leq \lfloor \alpha \rfloor + 1 . \qquad \square
$$

In the sequel, when we use the notation γ^E (curve "γ extended", such as in the above proof), the direction in which the curve γ_r around a zero point is traversed will depend on the direction in which the boundary curve γ encounters the corresponding branch cut L_i. If γ encounters the branch cuts L_i on both sheets of M, then the same procedure is followed with every encounter.

Clearly the component being treated in Proposition 6.1 basically involves only one sheet of M, although nothing prevents it in general from crossing a branch cut of M from one sheet to the other. We shall refer to a component of this kind as a **non-binary component.** With two sheets of M available there also exist components which essentially involve both sheets of M in a very specific manner and which will be called binary.

Definition 6.2 (Binary Component) *An Ω-component Ω_1 with one zero point is said to be* **binary** *if*

(a) *both sheets of M are essentially involved in Ω_1 in the sense that a branch cut of M on the negative real axis allows the component to extend to both sheets;*

(b) *the "outward", (positively oriented with respect to Ω_1), boundary curve(s) enclosing the zero point inside Ω_1 do(es) so (when starting from the point (1,1)) after having crossed the branch cut on the negative real axis;*

(c) *the zero point not inside Ω_1 is also "basically involved" in Ω_1 in the sense that it is enclosed by (an) "inner" boundary curve(s) of Ω_1.*
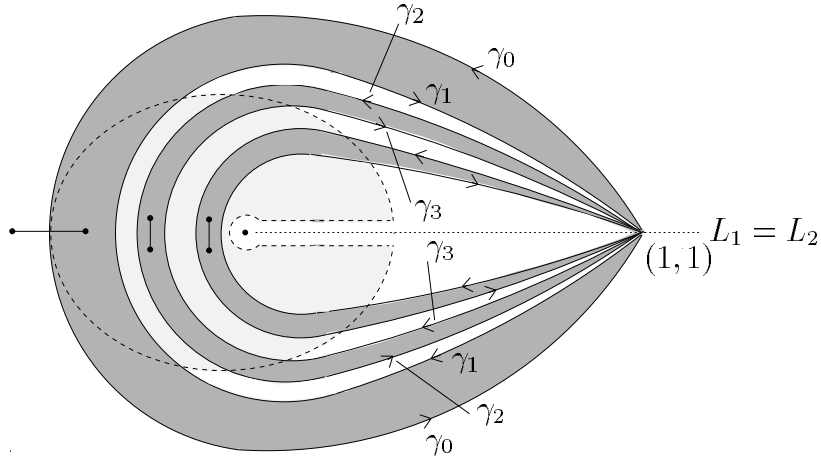
Figure 3: Symmetric binary component and cuts L_i going through $z = 1$

We basically get two types of binary components, the symmetric and the unsymmetric ones, and we first treat them separately, starting with a symmetric binary component.

Lemma 6.3 (Symmetric Binary) *Let Ω_1 be a symmetric binary component containing one zero point $(0, w_2^0)$ (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Let δ_i denote the non-integer part of α_i, i.e.*

$$\delta_i = \alpha_i - \lfloor \alpha_i \rfloor, \;\; i = 1, 2 \;.$$

Then the multiplicity m of Ω_1 satisfies

(10) $$m \le 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor \;.$$

Proof: The proof is conducted in two different ways depending on the way in which the branch cuts L_i are chosen. The first version highlights the binary character of the component, while the second leads to the bound (10).

Version 1: We first deviate from our convention of making the cuts L_i by choosing them such that their projection onto the z-plane is the positive real axis (see Fig. 3). With the integration along γ_0^E we have the situation that, on the principal sheet, we have gone once around the zero point $(0, w_1^0)$ without crossing L_1. Hence we end up with γ_0^E at a point where $z^{-\mu}$ and therefore also φ has a value which differs from the value with which it started at the point (1,1). We then have to return along γ_1 to our starting point at (1,1). Only then has the complete boundary of a component with respect to the function φ been traversed. Hence, in terms of counting the multiplicity of Ω_1 at (1,1), we can regard the point where γ_0^E went over into γ_1 as a point away from (1,1). When we apply the argument principle as we did in Proposition 6.1, this only accounts for the sectors of Ω_1 "on one side" of the cut L_1. Disregarding the sectors in the lower halfplane $Im\, z < 0$
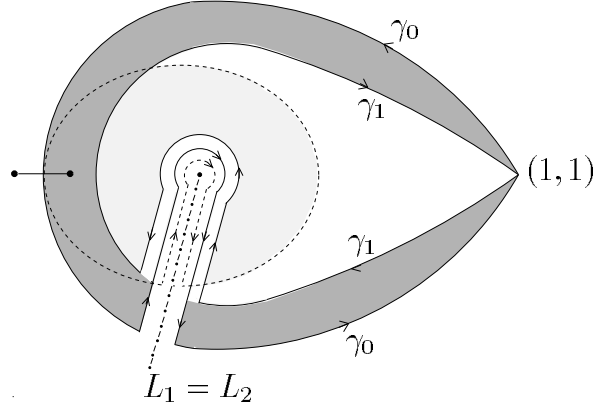
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Figure 4: Symmetric binary component and cuts L_i made according to convention

we have a component with $m/2$ sectors in the upper halfplane which is with respect to these $m/2$ sectors a non-binary component. Hence, we have

$$\frac{1}{2\pi i} \int_{\gamma_0^E + \gamma_1} \frac{\varphi'(z,w)}{\varphi(z,w)}\, dz \leq \lfloor \alpha_2 \rfloor$$

and similar to the proof of Proposition 6.1 we find

(11) $$0 \leq \lfloor \alpha_2 \rfloor - \left(\tfrac{m}{2} - 1\right)$$

which implies that

$$m \leq 2\lfloor \alpha_2 \rfloor + 2\,.$$

The binary part is seen from the fact that an integration which gives a drop of the argument by 2π, e.g. along $\gamma_2 + \gamma_3$, contributes here two sectors while in a non-binary component it contributes only one sector.

Version 2: While illuminating in version 1 the binary nature of Ω_1, the basic involvement of both zero points in the integration process is not expressed in (11). Further we deviated there from our convention of choosing the cuts L_i.

By making the cuts according to convention (see Fig. 4) we now obtain

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'(z,w)}{\varphi(z,w)}\, dz \leq \lfloor \alpha_1 + \alpha_2 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'(z,w)}{\varphi(z,w)}\, dz \leq \lfloor -\alpha_1 \rfloor\,.$$

For an integration along γ_2^E and γ_3^E we obtain

$$\frac{1}{2\pi i} \int_{\gamma_2^E} \frac{\varphi'(z,w)}{\varphi(z,w)}\, dz + \frac{1}{2\pi i} \int_{\gamma_3^E} \frac{\varphi'(z,w)}{\varphi(z,w)}\, dz \leq \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor = -1\,.$$
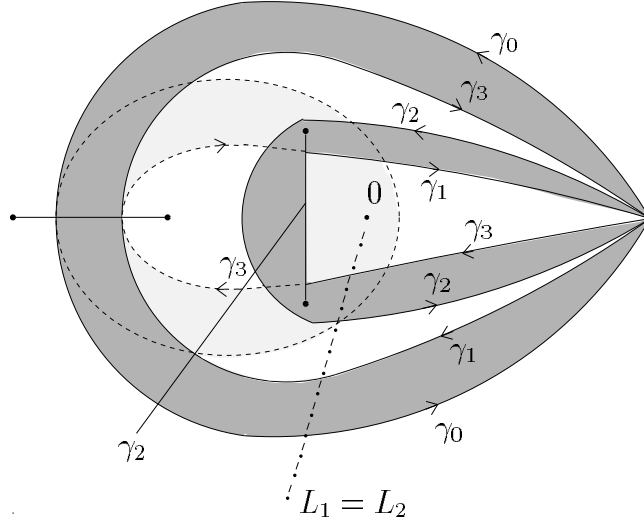
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Figure 5: Symmetric binary component with three "inner" boundary curves

Hence similar to our proof version 1 there is a drop of the argument by 2π when integrating along $\gamma_2^E + \gamma_3^E$. As in version 1 we assume that $m/2 - 1$ pairs of curves like $\gamma_2 + \gamma_3$ are present. Hence the argument principle leads to

$$(12) \qquad\qquad 0 \le \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor - \left(\tfrac{m}{2} - 1 \right) .$$

Hence,
$$m \le 2\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor) + \lfloor -\alpha_1 \rfloor\} + 2$$

which implies that
$$m \le 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor ,$$

where we have made use of the fact that

$$\lfloor \alpha \rfloor + \lfloor -\alpha \rfloor = -1 \ \text{ if } \ \alpha \notin \mathbb{Z} . \qquad\qquad \square$$

Remark 6.4 *a) Instead of a binary component having only one "inner" boundary curve, the "inner" boundary can also consist of more than one curve such as in Fig. 5.*

If the integration process is done such as in version 2 of the proof of Lemma 6.3, we obtain

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \le \lfloor \alpha_1 + \alpha_2 \rfloor, \ \ \frac{1}{2\pi i} \int_{\gamma_1^E} \le \lfloor -\alpha_1 \rfloor, \ \ \frac{1}{2\pi i} \int_{\gamma_2^E} \le \lfloor \alpha_1 \rfloor, \ \ \frac{1}{2\pi i} \int_{\gamma_3^E} \le \lfloor -\alpha_1 \rfloor ,$$

leading to the inequality

$$0 \le \{\lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor\} - \frac{(m-4)}{2} ,$$

with m again bounded by
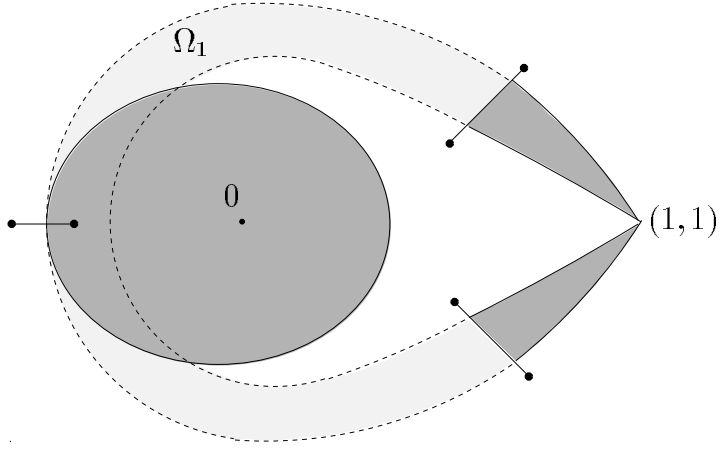$$m \le 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor .$$

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Figure 6: Binary component containing the zero point on the principal sheet

b) A similar result to that in Lemma 6.3, one only has to replace α_2 in (10) by α_1, is obtained if we assume that Ω_1 contains the zero point $(0, w_1^0)$ and "encloses" $(0, w_2^0)$ such as in Fig. 6.

We now treat an unsymmetric binary component.

Lemma 6.5 (Unsymmetric Binary) *Let Ω_1 be an unsymmetric binary component containing one zero point $(0, w_2^0)$ (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$. Then the multiplicity m of Ω_1 satisfies*

$$m \le 2\lfloor \alpha_2 \rfloor + 1 .$$

Proof: The proof is again conducted in two ways as a result of two different choices of the branch cuts L_i.

Version 1: L_1 and L_2 are chosen such that their projection onto the z-plane passes through $z = 1$. Since their choice in the form of straight lines would lead to a transgression of Rule 2, we choose them such as in Fig. 7. Then Ω_1 is unsymmetric with respect to L_1. In a manner similar to that in the version 1 proof of Lemma 6.3, we first treat the sectors of Ω_1 "on one side" of L_1 and assume that there is a total of m_1 of them. We start integrating along γ_0^E to end "on the same side" of L_1 from which we started. However, to complete the boundary of a component, we carry on along γ_1 to end up at a point at which φ is different to the initial value. Integration along γ_2 enables us to return to our starting point, where the complete boundary has also been traversed. The contributions of the integrals now are

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'(z,w)}{\varphi(z,w)} \, dz \le \lfloor \alpha_2 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_1+\gamma_2} \frac{\varphi'(z,w)}{\varphi(z,w)} \, dz \le -1 .$$

Application of the argument principle leads to

(13) $$0 \le \lfloor \alpha_2 \rfloor - 1 - (m_1 - 2) .$$

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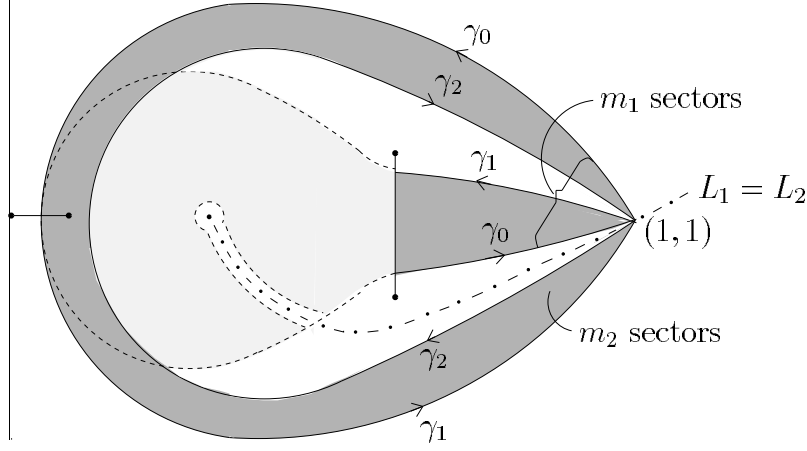
Figure 7: Unsymmetric binary component and cuts L_i going through $z = 1$

Observing that in the optimal case $m_1 = (m-1)/2 + 1$ we obtain the result. We can argue also as follows.

The process is now repeated for the portion of Ω_1, "on the other side" of L_1, where we assume a total of m_2 sectors. The integration process is started along γ_2 to end up at a point A (say) where φ is different to the initial value. So we carry on along γ_0^E and return to A again, and then along γ_1 to return to our starting point. Then we obtain

$$\frac{1}{2\pi i} \int_{\gamma_2 + \gamma_0^E + \gamma_1} \frac{\varphi'(z,w)}{\varphi(z,w)} \, dz \leq \lfloor \alpha_2 - 1 \rfloor \ ,$$

where the -1 accounts for the fact that we returned to A, on which occasion the argument must have decreased by at least 2π. Application of the argument principle leads to

$$(14) \qquad\qquad\qquad 0 \leq \lfloor \alpha_2 \rfloor - 1 \ - \ (m_2 - 1) \ .$$

By combining (13)and (14) we obtain the following bound on the total number $m = m_1 + m_2$ of sectors of Ω_1:

$$m \leq 2\lfloor \alpha_2 \rfloor + 1 \ .$$

Version 2: By choosing the branch cuts L_i according to our convention (see Fig. 8), we obtain the following for the integrals:

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor \alpha_2 \rfloor, \ \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor \alpha_1 \rfloor, \ \frac{1}{2\pi i} \int_{\gamma_2^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor -\alpha_1 \rfloor \ .$$

Application of the argument principle leads to

$$0 \leq \{ \lfloor \alpha_2 \rfloor + \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor \} - \frac{(m-3)}{2} \ ,$$
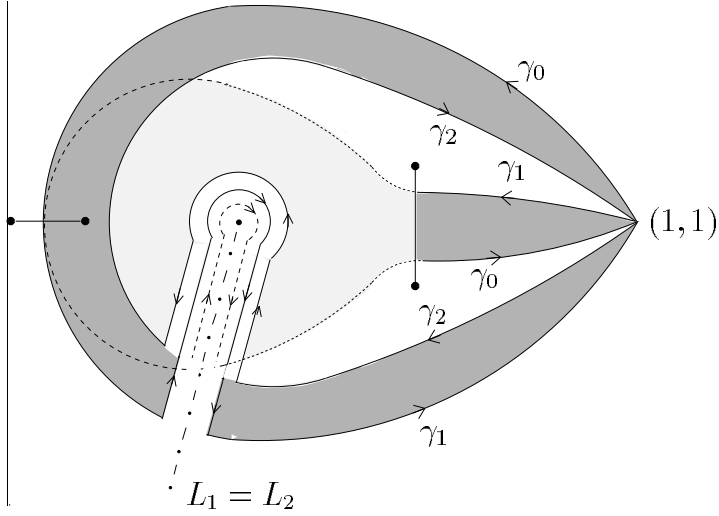
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Figure 8: Unsymmetric binary component and cuts L_i made according to convention

where the factor 2 accounts for the binary nature of the component and 3 is subtracted from m because γ_0, γ_1 and γ_2 contribute 3 sectors. Hence, we again obtain

$$m \leq 2\lfloor \alpha_2 \rfloor + 1 \ .$$

$\square$

In view of Lemmas 6.3 and 6.5 we have to conclude that the bound (10) is in general too sharp if the non-integer parts of α_1 and α_2 satisfy $0 < \delta_1 + \delta_2 < 1$. A combination of Lemmas 6.3 and 6.5 leads to the following general result for binary components.

Proposition 6.6 (Binary) *Let Ω_1 be a binary component containing one zero point. Assume that φ has a leading exponent of $-\alpha_1$ at the zero point inside Ω_1 and $-\alpha_2$ at the other zero point. Let δ_1 and δ_2 be the non-integer parts of α_1 and α_s, respectively, i.e.*

$$\delta_1 = \alpha_1 - \lfloor \alpha_1 \rfloor, \quad \delta_2 = \alpha_2 - \lfloor \alpha_2 \rfloor \ .$$

Then the multiplicity m of Ω_1 satisfies

$$m \leq 2\lfloor \alpha_1 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\} \ .$$

$\square$

Remark 6.7 (Efficiency)   *a) In view of the factor 2 accompanying $\lfloor \alpha \rfloor$ in the bound for binary components, we declare that the zero point inside a binary component has a **higher efficiency** than the zero point inside a non-binary component. Further, we say that the zero point contributes to the multiplicity of the component in a binary manner.*
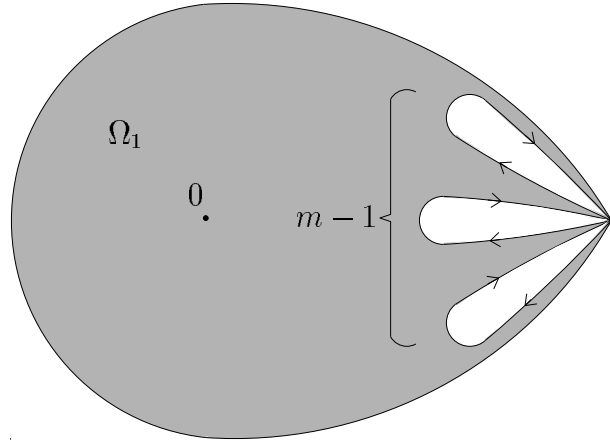
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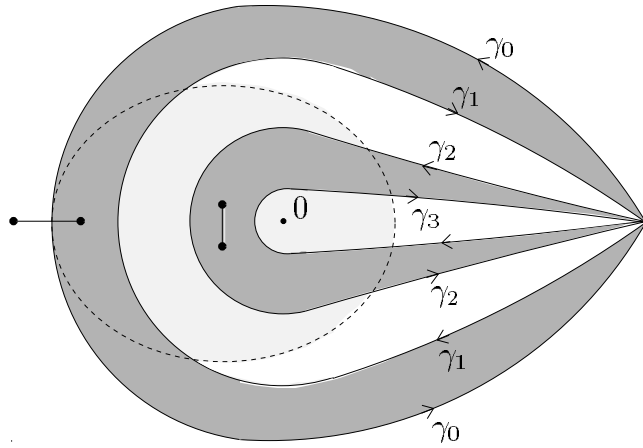Figure 9: Non-binary component with $m-1$ Ω^c-components

Figure 10: Binary component with one "binary loop"

b) In Proposition 6.1 the total multiplicity m of the non-binary component Ω_1 is achieved by $m-1$ Ω^c-components "to the inside" of Ω_1 such as in Fig. 9. In the case of binary components the maximum multiplicity is achieved by means of curves which circle around the zero points such as γ_2 and γ_3 in Fig. 10 and which are connected to the "basic component" by means of branch cuts. These extra curves constitute what we in [17] referred to as "binary loops" ([17], Lemma 5.10).

c) At this stage an important question is whether there exists under the present circumstances a type of component with one zero point where the zero point has a higher efficiency than in the case of binary components. Relevant considerations in this regard are the following:

(i) both sheets of M are basically involved in a binary component;

(ii) both zero points are utilized in a very specific manner;

(iii) when applying the argument principle (in version 1 of the proofs of Lemmas 6.3 and 6.5) the branch cut L_1 through (1,1) has only two sides from where "separate components" w.r.t. φ can contribute to the total multiplicity.

In view of these considerations we conclude that the efficiency of a binary component is the highest which can be achieved on a Riemann surface with 2 sheets and no pole outside $z = 0$.

Since two zero points are available inside Δ, the question arises as to which combination of two components, each with one zero point, leads to the highest possible total multiplicity at (1,1) inside Δ. The combination of two binary components would seem the obvious answer to this question. However, such a combination is impossible, as we prove in the next proposition.

Proposition 6.8 *There can be at most one binary component containing one zero point inside the unit disk Δ.*

Proof: Let Ω_1 be a binary component inside Δ and say $(0, w_2^0)$ is contained in Ω_1, while $(0, w_1^0)$, which does not belong to Ω_1, is enclosed by (an) "inner" boundary curve(s) of Ω_1 (see Definition 6.2). Suppose $(0, w_1^0)$ belongs to a second Ω-component Ω_2 (say). For Ω_2 to be binary, it has to have (an) "outward" boundary curve(s) going through a branch cut on the negative real axis and then enclosing $(0, w_1^0)$. However, this is impossible since $(0, w_1^0)$ is already enclosed by (an) "inner" boundary curve(s) of Ω_1. Hence, Ω_2 cannot be binary.
□

A binary component Ω_1 can be combined with a non-binary component Ω_2 (say), as is illustrated in Fig. 11. For such a combination the following theorem follows as a consequence of Propositions 6.1 and 6.6.

Theorem 6.9 (Binary plus non-binary) *Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M and suppose $(0, w_1^0)$ belongs to a non-binary component and $(0, w_2^0)$ to a binary component. Then the highest total multiplicity m that these two components can contribute at (1,1) is given by*

$$(15) \qquad m \leq (\lfloor \alpha_1 \rfloor + 1) + (2\lfloor \alpha_2 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\}),$$

where $\delta_i = \alpha_i - \lfloor \alpha_i \rfloor$, $i = 1, 2$.
□

Remark 6.10 *The bound (15) is a sharper bound than (5.20) in [17], p. 29, Proposition 5.15, in the sense that the factor 2 does not occur with $\lfloor \alpha_1 \rfloor$. This portrays in (15) the exact non-binary character of one of the components of the combination.*
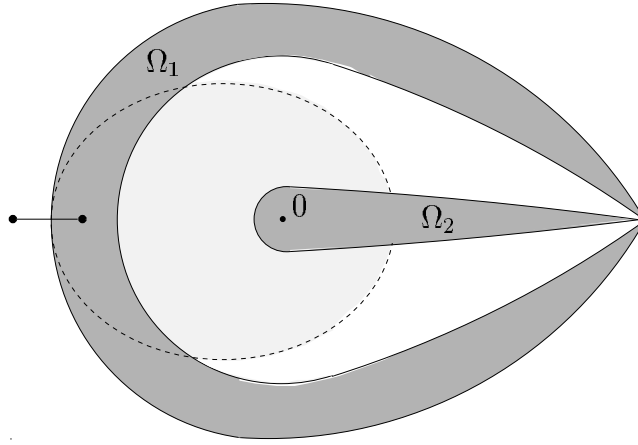
Figure 11: Binary component Ω_1 combined with non-binary component Ω_2

6.1 Components containing two zero points

An obvious way of obtaining a component with two zero points is to connect two components with one zero point each by means of a branch cut of M. We shall show that our technique to prove bounds for the multiplicity in such a situation will give a bound which is larger than what one would obtain by ignoring the connecting cut and applying the results of the previous section to each component separately. Since we have not found any example which shows that this larger bound is sharp we conjecture that the smaller bound (15) is correct in all cases.

Conjecture 6.11 *Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M. Then the multiplicity m of the $\Omega-$component Ω_1 containing both zero points satisfies*

$$ m \leq \lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\} $$

Clearly, if Ω_1 in the conjecture can be separated into two components then the conjecture is proved.

We can prove this conjecture for a class of schemes of maximal order, $p = |I| - 2$. This proof, however, relies on first assuming the converse and then obtaining a contradiction. Hence, for now, it is appropriate to assume the converse and examine its implications. With this in mind, we introduce another definition for a component containing two zero points.

Definition 6.12 (Double Binary Component) *An Ω-component Ω_1, which contains both zero points is said to be **double-binary** if both of the zero points contribute to the multiplicity of the component in a binary manner.*
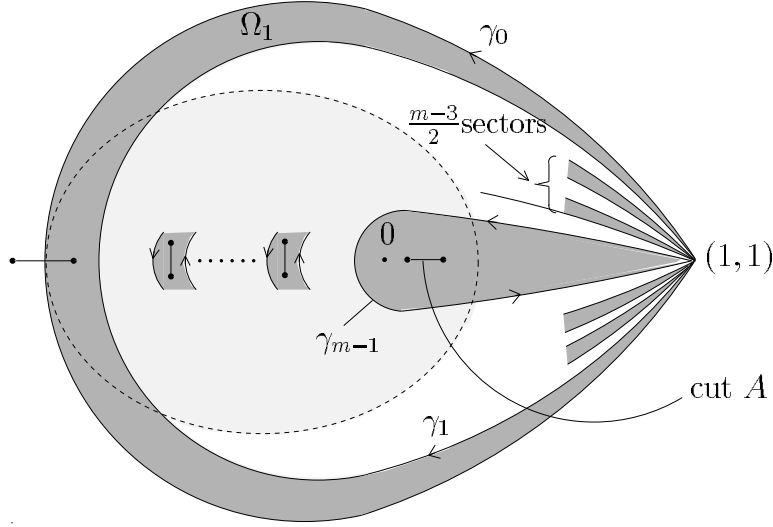
Figure 12: Symmetric Double-Binary Component

As in the situation of a single component containing just one zero point we basically get two types of double-binary components, those which are symmetric and those which are unsymmetric.

We first consider Figure 12 which depicts a modification of the symmetric binary component illustrated in Figs. 3, 4, in the sense that the "inner" boundary curve is moved inward so as to include the second zero point, and hence give a component containing both zero points.

Lemma 6.13 (Symmetric Double Binary Component) *Let Ω_1 be a symmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$ with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Then the multiplicity m of Ω_1 satisfies*

$$m \leq 2[\alpha_1] + 2[\alpha_2] + 1 + 2[\delta_1 + \delta_2].$$

Proof: Clearly if cut A would not be present one could apply Theorem 6.9 and we would have Conjecture 6.11. But, since the cut A can't be removed we apply the argument principle to the whole component Ω_1. Hence

$$
\begin{aligned}
0 &= \frac{1}{2\pi i} \int \frac{\gamma'}{\gamma} \, dz \\
&= \frac{1}{2\pi i} \int_{\gamma_0^E} + \frac{1}{2\pi i} \int_{\gamma_1^E} + \frac{1}{2\pi i} \sum_{j=1}^{\frac{m-3}{2}} \left(\int_{\gamma_{2j}^E} + \int_{\gamma_{2j+1}^E} \right) + \frac{1}{2\pi i} \int_{\gamma_{m-1}^E} \\
&\leq \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \tfrac{m-3}{2} \, (-1) + \lfloor \alpha_1 \rfloor
\end{aligned}
$$

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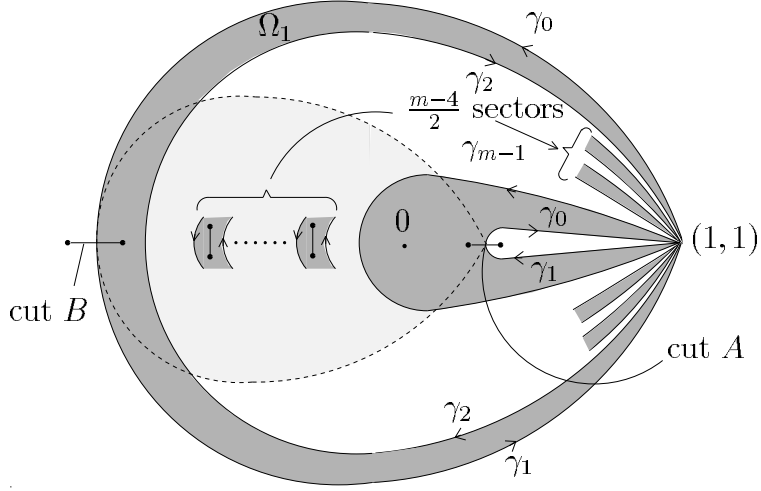
Figure 13: Unsymmetric Double Binary

and this gives the bound

$$(16) \qquad m \leq 2\lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor .$$

$\square$

Clearly, if $\lfloor \alpha_1 \rfloor > 0$ then the bound (16) is not as sharp as the conjecture bound. While in this example the component could have been separated by removing cut A such a separation is not as evident in the following example depicted in Fig. 13. A comparison with Figs. 7 and 8 reveals that Fig. 13 is obtained from an unsymmetric binary component by replacing the cut perpendicular to the real axis by cut A, on the real axis. Therefore a component of this kind will be called unsymmetric double-binary.

Lemma 6.14 (Unsymmetric Double Binary) *Let Ω_1 be an unsymmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Then the multiplicity m of Ω_1 satisfies*

$$m \leq 2(\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1).$$

Proof: As usual we apply the argument principle to the whole component Ω_1. Hence

$$
\begin{aligned}
0 &= \frac{1}{2\pi i} \int \frac{\gamma'}{\gamma} \, dz \\
&= \frac{1}{2\pi i} \int_{\gamma_0^E} + \frac{1}{2\pi i} \int_{\gamma_1^E} + \frac{1}{2\pi i} \int_{\gamma_2^E} + \frac{1}{2\pi i} \sum_{j=1}^{\frac{m-4}{2}} \left(\int_{\gamma_{2j+1}^E} + \int_{\gamma_{2j+2}^E} \right) + \frac{1}{2\pi i} \int_{\gamma_{m-1}^E} \\
&\leq \lfloor \alpha_2 \rfloor + \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor + \tfrac{m-4}{2} \left(\lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor \right) + \lfloor \alpha_1 \rfloor
\end{aligned}
$$

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and this gives the bound

$$(17) \qquad m \leq 2\lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + 1 \ .$$

$\square$

Again, if $\lfloor \alpha_1 \rfloor > 0$ then the bound (17) is not as sharp as the conjectured bound. As in the previous example the bound is wrong by the factor 2 in the term containing $\lfloor \alpha_1 \rfloor$.

We observe that the results suggested by Lemmas 6.13 and 6.14 lead to the following general result for a double binary component.

Theorem 6.15 (Double Binary) *Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to one double-binary component, Ω_1. Then the highest multiplicity, M, that this component can contribute at $(1,1)$ is given by*

$$m \leq 2(\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor) + 1 + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\}$$

$\square$

7 The role of the branch cuts on multiplicity

Close inspection of the results derived until now will reveal that these results actually rely on the occurrence of branch cuts to connect binary loops to the portion of the component containing one or both zero points. If there are insufficient branch cuts, or equivalently insufficient branch points, these multiplicities will not be achievable. It therefore becomes appropriate to reformulate the earlier results in terms of the minimum number of branch points used by a component in order to achieve a certain multiplicity. The implication is that we now consider components which we will call **suboptimal**; for example, for a binary component, Ω_1, we allow for the possibility that there are insufficient branch points for the zero point to be completely binary, and hence that there are also sectors of Ω_1 at $z = 1$ which contribute only a factor 1 rather than a factor 2 to the multiplicity.

To standardize our approach we will adopt the following notation:

$$
\begin{aligned}
K &:= \text{ number of branch points utilized by the component} \\
m_1 &:= \text{ number of sectors at } z = 1 \text{ due to binary loops} \\
m_2 &:= \text{ number of sectors at } z = 1 \text{ due to non-binary loops.}
\end{aligned}
$$

We also make the assumption, without loss of generality, $\lfloor \alpha_2 \rfloor \geq \lfloor \alpha_1 \rfloor$. Furthermore, to ease comparison with the previous results, we will refer to the components as classified as in the earlier sections. In each case we will employ the "Version 2" type proofs since these give the tighter bounds.
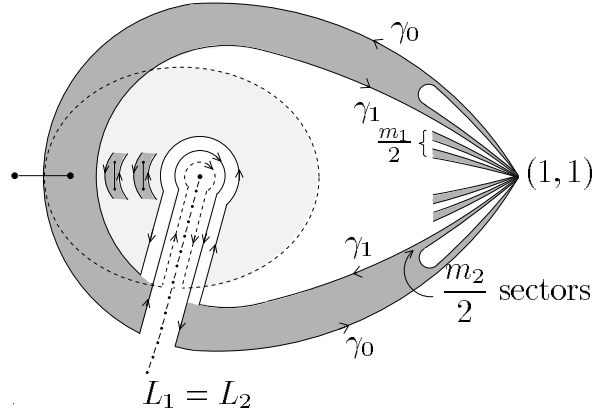
24

Figure 14: Suboptimal Binary

Lemma 7.1 (Suboptimal Symmetric Binary Component, (cf. Lemma 6.3)) *Let Ω_1 be a symmetric binary component containing one zero point $(0, w_2^0)$, (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies*

$$m \leq \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + \min\{\lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor, \lfloor \frac{K+1}{2} \rfloor\}.$$

Proof: The proof follows as for Lemma 6.3 but note now that the number of binary loops is limited by the number of branch points K. Hence, consider Fig. 4 modified as indicated in Fig. 14 and apply the argument principle. Each of the $\frac{m_1}{2}$ binary loops contributes -1 to the argument but two sectors at $z = 1$. The m_2 non-binary loops also contribute -1 to the argument decrease but one sector at $z = 1$. Hence, the total number of sectors at $z = 1$ is

$$m = m_1 + m_2 + 2.$$

Further, by the argument principle

$$0 \leq \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor - \frac{m_1}{2} - m_2.$$

Therefore

$$\begin{aligned} m = m_1 + m_2 + 2 &\leq \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \frac{m_1}{2} + 2 \\ &= \lfloor \alpha_2 \rfloor - 1 + \lfloor \delta_1 + \delta_2 \rfloor + 2 + \frac{m_1}{2}. \end{aligned}$$

Now each binary loop uses at least one branch cut to connect that loop to Ω_1. Also one branch point is required to make the component binary. Therefore
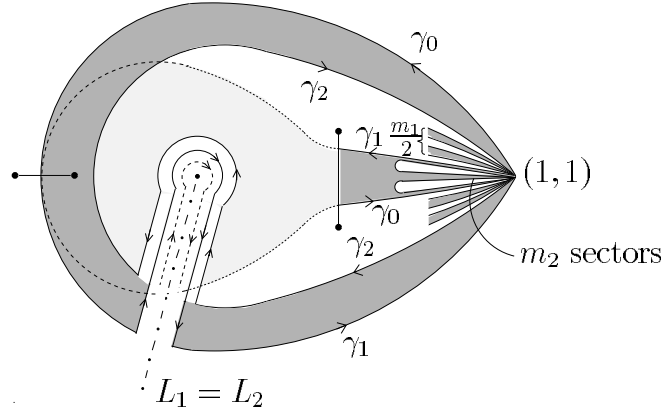
$$m_1 \leq K - 1,$$

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Figure 15: Unsymmetric Binary Suboptimal

and

$$\frac{m_1}{2} \leq \lfloor \frac{K-1}{2} \rfloor.$$

Thus

$$m \leq \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + 1 + \lfloor \frac{K-1}{2} \rfloor$$

and by Lemma 6.3 the result follows. □

Corollary 7.2 *The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.1, for which the maximum multiplicity, as indicated by Lemma 6.3, is obtained, is given by*

$$\lfloor \frac{K_{\min}+1}{2} \rfloor = \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor.$$

Proof: This follows immediately by observing that when $m_2 = 0$, $K \geq m-1$, and hence $K_{\min} = m_{\max} - 1$, where $m_{\max} = 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor$. □

Consider now the modification of Fig. 8 to include the possibility of non-binary loops at $z = 1$. Application of the argument principle to this modified, suboptimal component, illustrated in Fig. 15, as in Lemma 7.1 immediately leads to the following lemma, and its corollary.

Lemma 7.3 (Suboptimal Unsymmetric Binary Component, (cf. Lemma 6.5))
Let Ω_1 be an unsymmetric binary component containing one zero point $(0, w_2^0)$, (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Further, suppose that Ω_1 contains at most K branch points of $w(z,\mu)$. Then the multiplicity m of Ω_1 satisfies

$$m \leq \lfloor \alpha_2 \rfloor + \min\{\lfloor \alpha_2 \rfloor + 1, \lfloor \frac{K+1}{2} \rfloor\}.$$

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Corollary 7.4 *The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.3, for which the maximum multiplicity, as indicated by Lemma 6.5, is obtained, is given by*

$$\lfloor \frac{K_{\min} + 1}{2} \rfloor = \lfloor \alpha_2 \rfloor + 1.$$

□

These suboptimal binary components can be combined with a non-binary component in exactly the same way as a binary component is combined with a non-binary component in Theorem 6.9:

Theorem 7.5 (Suboptimal Binary plus Non-binary) *Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose $(0, w_1^0)$ belongs to a nonbinary component while $(0, w_2^0)$ belongs to a binary component containing K branch points. Then the highest total multiplicity m that these two components can contribute at (1,1) is given by*

$$m \le (\lfloor \alpha_1 \rfloor + 1) + (\lfloor \alpha_2 \rfloor + \min\{\lfloor \frac{K+1}{2} \rfloor + \lfloor \delta_1 + \delta_2 \rfloor, \lfloor \alpha_2 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\})$$

□

Theorem 7.5 implies that there is actually a combination of two components which yields a multiplicity between that which would be indicated by i) both components of non-binary type and ii) one component optimal binary and the other non-binary. In the same way double-binary components can also be suboptimal with a multiplicity greater than that indicated by Theorem 6.9 but less than that indicated by Theorem 6.15. Such components are again limited in multiplicity by the number of branch points they contain.

Lemma 7.6 (Suboptimal Symmetric Double Binary, (cf. Lemma 6.13)) *Let Ω_1 be a symmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies*

$$m \le \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + \min\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + \lfloor \delta_1 + \delta_2 \rfloor, \lfloor \frac{K+1}{2} \rfloor\}.$$

Proof: The argument principle is applied to Fig. 16, which is a modification of Fig. 12 to allow for the existence of m_2 non-binary loops of Ω_1 at $z = 1$, and $\frac{m_1}{2}$ binary loops of Ω_1 at $z = 1$. In this case
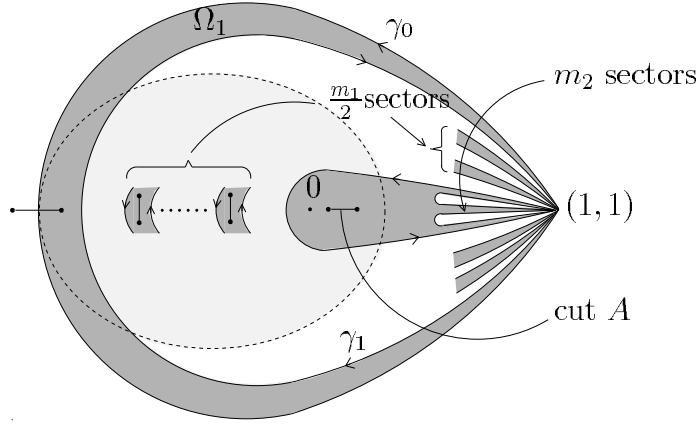
$$m = m_2 + m_1 + 3,$$

Figure 16: Suboptimal Symmetric Double Binary Component

and by the argument principle,

$$0 \leq \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \lfloor \alpha_1 \rfloor - m_2 - \frac{m_1}{2}.$$

Therefore

$$m \leq \lfloor \alpha_1 + \alpha_2 \rfloor - 1 + 3 + \frac{m_1}{2},$$

where m_1 is limited by the total number of branch points K,

$$m_1 \leq K - 3.$$

Therefore,

$$m \leq \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + \lfloor \frac{K+1}{2} \rfloor,$$

and the result follows in combination with Lemma 6.13. $\qquad \square$

Again there is a minimum number of branch points for which an optimal double-binary component can be obtained.

Corollary 7.7 *The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described by Lemma 7.6, for which the maximum multiplicity, as indicated by Lemma 6.13, is obtained, is given by*

$$\lfloor \frac{K_{\min} + 1}{2} \rfloor = \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + 1.$$

$\qquad \square$

The unsymmetric double binary component, as illustrated by Fig. 13, is also modified by the assumption of non-binary sectors of Ω_1 at $z = 1$, see Fig. 17.
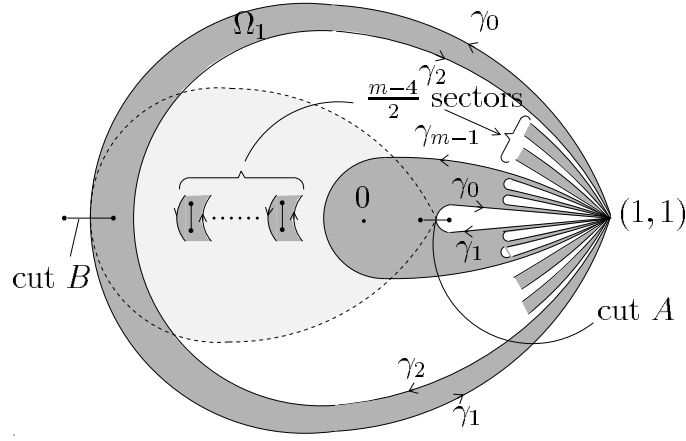
28

Figure 17: Suboptimal Unsymmetric Double Binary Component

Lemma 7.8 (Suboptimal Unsymmetric Double Binary, (cf. Lemma 6.14)) *Let Ω_1 be an unsymmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies*

$$m \leq \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + \min\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1, \lfloor \frac{K+1}{2} \rfloor\}.$$

Corollary 7.9 *The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.8, for which the maximum multiplicity, as indicated by Lemma 6.14, is obtained, is given by*

$$\lfloor \frac{K_{\min} + 1}{2} \rfloor = \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1.$$

□

We note that other geometries, for example as illustrated in Fig. 27, in the appendix A, may lead to less efficient use of the branch points, (assuming $0 < |\mu| < \frac{1}{2}$), and hence we conclude, when both zero points are contained within one component, and both exhibiting binary behavior, that the following theorem can be deduced from Lemmas 7.6 and 7.8.

Theorem 7.10 (Suboptimal Double Binary) *Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose that both zero points belong to a double-binary component Ω_1 which also*

contains K branch points. Then the highest total multiplicity, m, that this component can contribute at (1,1) is given by

$$m \leq \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \max\{\min\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 2, \lfloor \frac{K+1}{2} \rfloor + 1\} ,$$

$$\min\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + 2\lfloor \delta_1 + \delta_2 \rfloor, \lfloor \frac{K+1}{2} \rfloor + \lfloor \delta_1 + \delta_2 \rfloor\}\},$$

□

7.1 Proof of Conjecture 6.11 for Maximum Order Schemes

Here we derive bounds on the order, p, of explicit schemes, under the assumption that the double-binary components described in Sections 6.1 and 7, exist. But we note that their multiplicities depend very intimately on the number of branch points contained inside the unit circle. These components cannot use branch cuts completely outside the unit circle since then stability would be violated. We also know, by Lemma 5.2, that the total number, m, of sectors at $z = 1$, both from inside and from outside the unit circle, satisfies $p + 1 = m$. Let m_I and m_O be the number of sectors of Ω at $z = 1$ from inside, and outside the unit circle, respectively. Then $m = m_I + m_O$, and by Lemma 5.2,

$$m_O - 1 \leq m_I \leq m_O + 1.$$

Therefore, if considered independently

(18)
$$p \leq \max\{2m_I, 2m_O\}.$$

But, since m_O can be seen to be limited by the number of branch points of $w(z, \mu)$ outside the unit circle, the bound on m_O actually depends on the bound on m_I, via the number of branch points, K_I, inside the unit circle, limiting the number of branch points, K_O, outside the unit circle, because for convex schemes

$$2(r_1 + s_1) = K_I + K_O.$$

We will show that for the schemes of maximal order, $p = |I| - 2$, an argument in which both m_I and m_O are determined is required to derive a tight bound on p. But first we have to consider how to obtain m_O.

Instead of repeating the analyses of the earlier sections we apply a symmetry argument for the Ω-sectors outside Δ. For generality, we consider here implicit schemes. The functions a_i defined in (6) will here be written in the form $a_i(z, \mu)$ to emphasize the μ-dependence of the coefficients a_{ij}. Suppose we have a stable scheme (4) and that its stencil is regular for a certain value of μ, with R downwind and S upwind stencil points according to (9). Let ℓ denote the number of Ω-sectors of the corresponding order star Ω emerging from the point (1,1) outside the unit disk Δ. Then consider the reversed scheme

(19)
$$\sum_{j=-s_2}^{r_2} a_{2,-j}(-\mu) \, u_{n+2,m+j} + \sum_{j=-s_1}^{r_1} a_{1,-j}(-\mu) u_{n+1,m+j} +$$
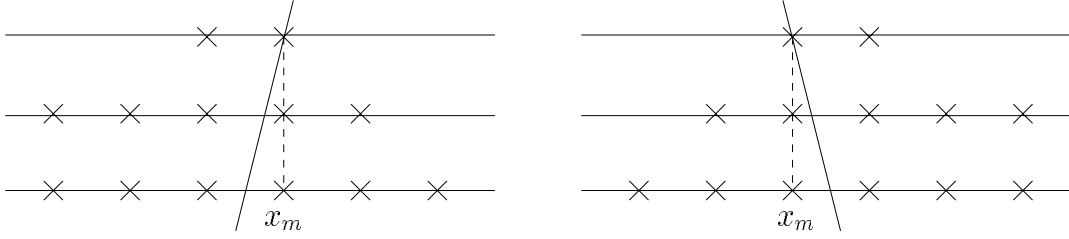$$\sum_{j=-s_0}^{r_0} a_{0,-j}(-\mu) u_{n,m+j} = 0 \ .$$

Figure 18: Stencil of Reversed Scheme

This scheme can be thought of as being obtained by a transformation of the space variable x into $-x$. Hence the stencil is reflected about the line $x = x_m$.

The new scheme has $R^* = S$ downwind and $S^* = R$ upwind stencil points with respect to the characteristic $\mu^* = -\mu$. The characteristic function Φ^* of the reversed scheme is obtained from the characteristic function Φ by using the transformation

$$z \to \frac{1}{z} \text{ and } \mu \to -\mu \;.$$

Hence
$$
\begin{aligned}
\Phi^*(z, w^*, \mu^*) &= \Phi(\tfrac{1}{z}, w^*, -\mu) \\
&= a_2(\tfrac{1}{z}, -\mu)(w^*)^2 + a_1(\tfrac{1}{z}, -\mu)w^* + a_0(\tfrac{1}{z}, -\mu) \;. \\
&= a_2^*(z, \mu^*)w^{*2} + a_1^*(z, \mu^*)w^* + a_0^*(z, \mu^*).
\end{aligned}
$$

Therefore the algebraic function w^* satisfies

(20)
$$w^*(z, \mu) = w(\tfrac{1}{z}, -\mu) \;.$$

From this relationship it follows that the reversed scheme is stable and of order p if and only if the original scheme is stable and of order p. The order star Ω^* of the reversed scheme (19) is related to Ω of (4) in the sense that the portion of Ω outside the unit disk Δ is mapped to the inside of Δ and vice versa by the mapping $z \to \frac{1}{z}$.

Therefore, in order to determine m_O for a given value of μ, we should map $z \to \frac{1}{z}$ and investigate the portion of Ω inside Δ for $\mu^* = -\mu$. For explicit schemes the corresponding leading exponents of φ will be $-\beta_1$ and $-\beta_2$ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively, where $\beta_1 = s_0 - s_1 + \mu^*$, $\beta_2 = s_1 - s_2 + \mu^*$. Effectively, this maps the pole at infinity to zero, so that the effects of the infinity points can be examined via the effects of the zero points for μ^*. Note here that the form of β_1 and β_2 explicitly assumes $s_1 - s_2 \geq s_0 - s_1$, see Proposition 4.4 and Remark 4.5. Hence the argument we adopt enforces convexity on both sides of the characteristic line.

To complete the methodology we also need to be able to count the number of branch points of $w^*(z, \mu^*)$ inside the unit circle for μ^*. But, by (20) this is just the number of branch points of $\omega(\frac{1}{z}, -\mu^*)$, which we have already denoted by K_O.

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Lemma 7.11 (Conjecture 6.11 for explicit maximal order schemes) *Suppose we have an explicit stable scheme of type (4) of maximal order, $p = |I| - 2$, with a convex increasing stencil, with a fixed Courant number μ, $-\frac{1}{2} < \mu < 0$. Assume that the algebraic function w of $\Phi(z, w) = 0$ has no branch point at $z = 0$ and that φ has leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M. Then the multiplicity m of the $\Omega-$component Ω_1 containing both zero points satisfies*

$$m \le \lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\}.$$

Proof: Note that this is a statement of Conjecture 6.11 for the schemes of maximal order, $p_{opt} = |I| - 2$. Hence what we seek to prove is that for these schemes there cannot exist components of double-binary or suboptimal binary type. In particular we show that if these components exist the order is necessarily less that p_{opt}.

From Proposition 4.4 and Remark 4.5 we have the following expansions of $\varphi(z, w_i(z))$ at $z = 0$:

$$
\begin{aligned}
\varphi(z, w_1(z)) &= z^{-r_1 - \mu}(b_0 + b_1 z + b_2 z^2 + \ldots), \\
\varphi(z, w_2(z)) &= z^{-(r_0 - r_1) - \mu}(c_0 + c_1 z + c_2 z^2 + \ldots).
\end{aligned}
$$

It should be noted here that we have no means of associating a certain expansion with the zero point on a specific sheet. Hence the expansions will be associated with the zero points in the way which leads to the highest possible multiplicity.

Equivalently, for $\mu^* = -\mu$ we have the expansions of $\varphi(z, w_i^*(z))$ at $z = 0$

$$
\begin{aligned}
\varphi(z, w_1^*(z)) &= z^{-s_1 - \mu^*}(b_0^* + b_1^* z + b_2^* z^2 + \ldots), \\
\varphi(z, w_2^*(z)) &= z^{-(s_0 - s_1) - \mu^*}(c_0^* + c_1^* z + c_2^* z^* + \ldots),
\end{aligned}
$$

obtained via the transformation $z = \frac{1}{z}$. To avoid confusion we denote the exponents associated with μ by

$$
\begin{aligned}
\alpha_1 &= r_0 - r_1 + \mu \\
\alpha_2 &= r_1 + \mu,
\end{aligned}
$$
(21)

and those associated with μ^* by

$$
\begin{aligned}
\beta_1 &= s_0 - s_1 + \mu^*, \\
\beta_2 &= s_1 + \mu^*.
\end{aligned}
$$
(22)

By convexity, $\lfloor \alpha_2 \rfloor \ge \lfloor \alpha_1 \rfloor$ and $\lfloor \beta_2 \rfloor \ge \lfloor \beta_1 \rfloor$. Furthermore, we explicitly assume $\lfloor \alpha_1 \rfloor > 0$. Otherwise the zero point at $(0, w_1^0)$ does not lie inside Ω and the argument is considerably simplified. Similarly, assume $\lfloor \alpha_2 \rfloor > \lfloor \alpha_1 \rfloor$.

Clearly $-\frac{1}{2} < \mu < 0$ implies $\lfloor \delta_1 + \delta_2 \rfloor = \lfloor 2\delta_1 \rfloor = \lfloor 2(1 + \mu) \rfloor = 1$.

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(i) By Theorems 7.5 and 7.10 we see that the optimal configuration is dependent on the number of branch points utilised by the components. In particular define K_L and K_U to be the minimum number of branch points for which an optimal binary-non-binary configuration, and a double binary configuration is possible, respectively. Then

$$(23) \qquad m_I \leq \begin{cases} \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 2 + \lfloor \frac{K_I+1}{2} \rfloor, & K_I < K_L \\ \lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + 2, & K_I = K_L \\ \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + \lfloor \frac{K_I+1}{2} \rfloor, & K_L < K_I < K_U \\ 2(\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1) + 1, & K_I \geq K_U. \end{cases}$$

Note that the bounds in (23) imply that the binary components are of symmetric type. Hence by corollaries 7.2 and 7.7

$$(24) \qquad \lfloor \frac{K_L + 1}{2} \rfloor = \lfloor \alpha_2 \rfloor + 1 \text{ and } \lfloor \frac{K_U + 1}{2} \rfloor = \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 2.$$

Substituting for $\lfloor \alpha_1 \rfloor$ and $\lfloor \alpha_2 \rfloor$ in (23) and (24) we obtain

$$(25) \qquad m_I \leq \begin{cases} r_0 + \lfloor \frac{K_I+1}{2} \rfloor, & \lfloor \frac{K_I+1}{2} \rfloor < r_1 \\ r_0 + r_1, & \lfloor \frac{K_I+1}{2} \rfloor = r_1 \\ r_0 - 1 + \lfloor \frac{K_I+1}{2} \rfloor, & r_1 < \lfloor \frac{K_I+1}{2} \rfloor < r_0 \\ 2r_0 - 1, & \lfloor \frac{K_I+1}{2} \rfloor \geq r_0. \end{cases}$$

(ii) Now we consider the outside of the unit circle and apply an equivalent argument with respect to β_1, β_2 and $\mu^* = -\mu$. In this case in the bound for m_O we use $\lfloor \delta_1 + \delta_2 \rfloor = \lfloor 2\delta_1 \rfloor = \lfloor 2\mu \rfloor = 0$. Thus by Theorems 7.5 and 7.10

$$(26) \qquad m_O \leq \begin{cases} \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1 + \lfloor \frac{K_O+1}{2} \rfloor, & K_O < K_L \\ \lfloor \beta_1 \rfloor + 1 + 2\lfloor \beta_2 \rfloor + 1, & K_O = K_L \\ \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + \lfloor \frac{K_O+1}{2} \rfloor + 1, & K_L < K_O < K_U \\ 2(\lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1), & K_O \geq K_U. \end{cases}$$

These components are unsymmetric and therefore, by corollaries 7.4 and 7.9,

$$(27) \qquad \lfloor \frac{K_L + 1}{2} \rfloor = \lfloor \beta_2 \rfloor + 1 \text{ and } \lfloor \frac{K_U + 1}{2} \rfloor = \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1.$$

As in (i), substitution of values for $\lfloor \beta_1 \rfloor$ and $\lfloor \beta_2 \rfloor$ in (26) and (27) leads to

$$(28) \qquad m_O \leq \begin{cases} s_0 + 1 + \lfloor \frac{K_O+1}{2} \rfloor, & \lfloor \frac{K_O+1}{2} \rfloor < s_1 + 1 \\ s_0 + s_1 + 2, & \lfloor \frac{K_O+1}{2} \rfloor = s_1 + 1 \\ s_0 + \lfloor \frac{K_O+1}{2} \rfloor + 1, & s_1 + 1 < \lfloor \frac{K_O+1}{2} \rfloor < s_0 + 1 \\ 2(s_0 + 1), & \lfloor \frac{K_O+1}{2} \rfloor \geq s_0 + 1. \end{cases}$$

(iii) We now combine the results from inside and outside Δ.

First, observe that if K_I is even, so is K_O, and because the total number of branch points is $2(r_1 + s_1)$,

$$\lfloor \frac{K_I + 1}{2} \rfloor + \lfloor \frac{K_O + 1}{2} \rfloor = r_1 + s_1,$$

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whereas, if K_I and K_O are odd,

$$\lfloor \frac{K_I + 1}{2} \rfloor + \lfloor \frac{K_O + 1}{2} \rfloor = r_1 + s_1 + 1.$$

Therefore bounds on K_I imply bounds on K_O, and vice versa. Hence, only certain combinations of components inside and outside Δ are possible.

In particular, suppose that inside Δ there is a double-binary or suboptimal double binary configuration. Then by (25) $\lfloor \frac{K_I+1}{2} \rfloor > r_1$ and

$$\lfloor \frac{K_O + 1}{2} \rfloor < s_1 + \begin{cases} 0 & K_I \text{ even} \\ 1 & K_I \text{ odd.} \end{cases}$$

Therefore outside Δ there can be at most a suboptimal binary configuration and

$$\begin{aligned} m_I + m_O &\leq r_0 - 1 + s_0 + 1 + \lfloor \frac{K_O+1}{2} \rfloor + \lfloor \frac{K_I+1}{2} \rfloor \\ &= r_0 + s_0 + r_1 + s_1 + \begin{cases} 0 & K_I \text{ even.} \\ 1 & K_I \text{ odd} \end{cases} \end{aligned}$$

Hence

$$m_I + m_O \leq |I| - 3 + \begin{cases} 0 & K_I \text{ even} \\ 1 & K_I \text{ odd.} \end{cases}$$

By Lemma 5.2, therefore

$$p \leq |I| - 3,$$

and (15) follows for $-\frac{1}{2} < \mu < 0$. □

We could now repeat the arguments for $0 < \mu < \frac{1}{2}$. But, by the discussion before Lemma 7.11, it is sufficient to examine the order star outside Δ for $-\frac{1}{2} < \mu < 0$. Hence again we would like to show that the bounds on m_O, (28), for double binary components, lead to a contradiction. For these components $\lfloor \frac{K_O+1}{2} \rfloor \geq s_1 + 1$, and hence $\lfloor \frac{K_I+1}{2} \rfloor \leq r_1$. Thus by (25)

$$m_I \leq r_0 + \lfloor \frac{K_I + 1}{2} \rfloor,$$

and

$$\begin{aligned} m_O + m_I &\leq r_0 + s_0 + \lfloor \frac{K_I+1}{2} \rfloor + \lfloor \frac{K_O+1}{2} \rfloor + 1 \\ &\leq r_0 + s_0 + r_1 + s_1 + 1 + \begin{cases} 0 & K_O \text{ even} \\ 1 & K_O \text{ odd.} \end{cases} \end{aligned}$$

This time

$$m_O + m_I \leq |I| - 2 + \begin{cases} 0 & K_O \text{ even} \\ 1 & K_O \text{ odd,} \end{cases}$$

and we do not obtain the required contradiction, unless it can be demonstrated that K_O is even.

Note that at no point did we explicitly impose stability in the above proof, because K_I and K_O are simply the number of branch points used by Ω from inside and outside Δ, respectively. But if a component inside Δ utilised a cut outside Δ then this would in fact

require that the stability condition, Lemma 5.1, is violated. Hence K_I and K_O do actually refer to the number of branch points inside and outside Δ, respectively, and stability is required.

8 Implicit schemes: the role of poles away from $z = 0$ on multiplicity

8.1 Components containing zero points and poles

Assume we have inside a binary component poles of a total multiplicity p. When applying the argument principle $-p$ is added on the left side of the equation, e.g. (11) or (12). This leads to a bound for m with an additional term $2p$. If the component is nonbinary then this additional term is clearly only p. Hence, having poles in a nonbinary component is less efficient than having one in a binary component. However, we shall show that we then can get a contribution $3p$ if the poles have multiplicity 1. To be able to do this the component is not allowed to contain a zeropoint. Such components are treated in the next section.

8.2 Components containing only poles away from $z = 0$

A class of components which still deserves special attention is the class of components containing poles away from $z = 0$ but no zero points. We now study the relationship between such poles and the multiplicities of the components in which they occur. We start off by considering three examples of components, each containing a single pole P of multiplicity p. It will become clear that, rather than the specific location of the pole P, it is the way in which the zero points are "excluded" from the components which leads to changes in the efficiency of P.

Example 8.1   *The pole P occurs on the positive real axis (for simplicity we locate it on the principal sheet in Fig. 19), or at any other location away from the real axis inside Δ. In this case the zero points do not have any effect on the component Ω_1, since the cuts L_i can be chosen such that they do not interact with Ω_1 in any way. Then Ω_1 corresponds to the type of component treated in [24], where it was shown that the multiplicity m of Ω_1 is bounded by*
$$m \leq p \, .$$

Clearly, the factor 1 in front of p is due to the fact that the component Ω_1 only involves one "sheet of the Riemann surface". We shall see that one obtains a factor 2 if one involves
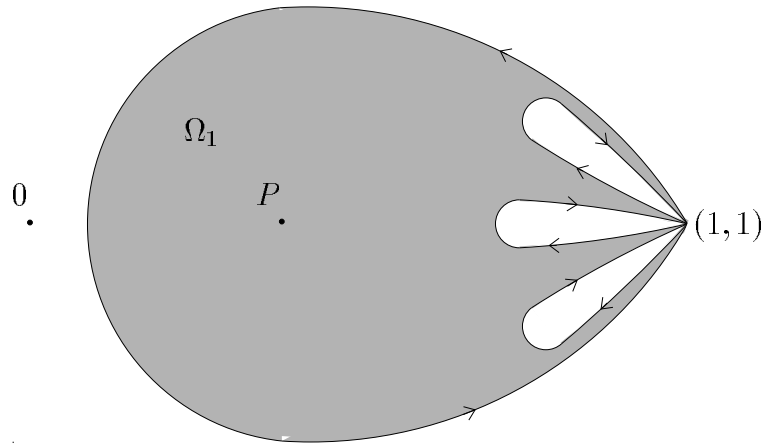
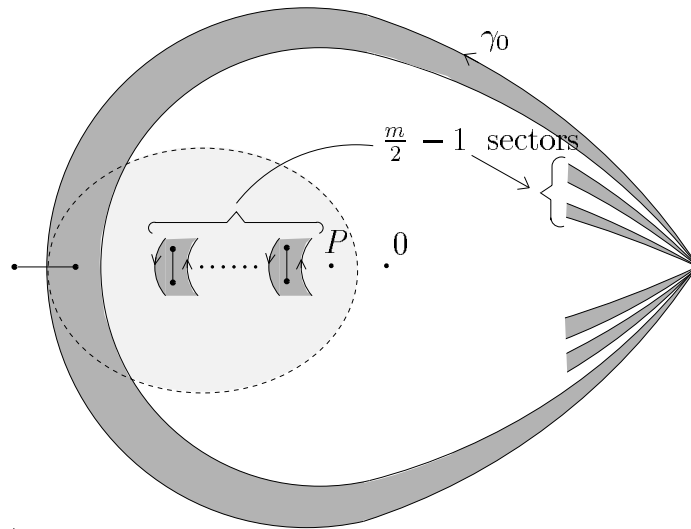Figure 19: Component with pole P on positive real axis on principal sheet of M

Figure 20: "Binary" component with pole P on negative real axis and one zeropoint encircled
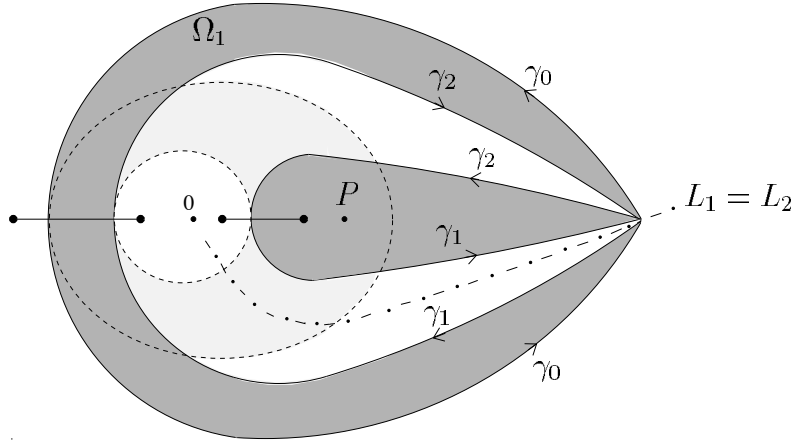
Figure 21: Component with pole P and with both zero points encircled

both sheets in a similar way as with binary component having one zeropoint. For this we consider the following example.

Example 8.2   *Let us consider the components of Fig. 3, 4 modified in the way that γ_0 does not encircle the zero point on the secondary sheet. Applying the argument principle yields*

$$
\begin{aligned}
-p &= \frac{1}{2\pi i}\int_{\gamma_0^E} + \frac{1}{2\pi i}\int_{\gamma_1^E} + \frac{1}{2\pi i}\sum_{j=1}^{m/2-1}\int_{\gamma_{2j}^E + \gamma_{2j+1}} \\
&\leq \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor - (m/2 - 1)\,.
\end{aligned}
$$

Hence, $m \leq 2p$ as expected.

To see whether one obtains a factor larger than 2 in the term with p, let us consider the following example.

Example 8.3 *Let Ω_1 be such as in Fig. 21, with the pole P on the positive real axis on the secondary sheet of M. Both the zero points are deliberately excluded in the sense that they are encircled by "inner" boundary curves of Ω_1. The branch cuts L_i are chosen such that their projection onto the z-plane passes through $z = 1$. By first working "on one side" of L_1, we obtain the following for the integrals:*

$$
\frac{1}{2\pi i}\int_{\gamma_0^E + \gamma_1}\frac{\varphi'}{\varphi}\,dz \leq \lfloor \alpha_2 \rfloor, \quad \frac{1}{2\pi i}\int_{\gamma_2^E}\frac{\varphi'}{\varphi}\,dz \leq \lfloor -\alpha_2 \rfloor\,.
$$

By applying the argument principle with the pole P inside the component and m_1 sectors at $z = 1$, we obtain

$$
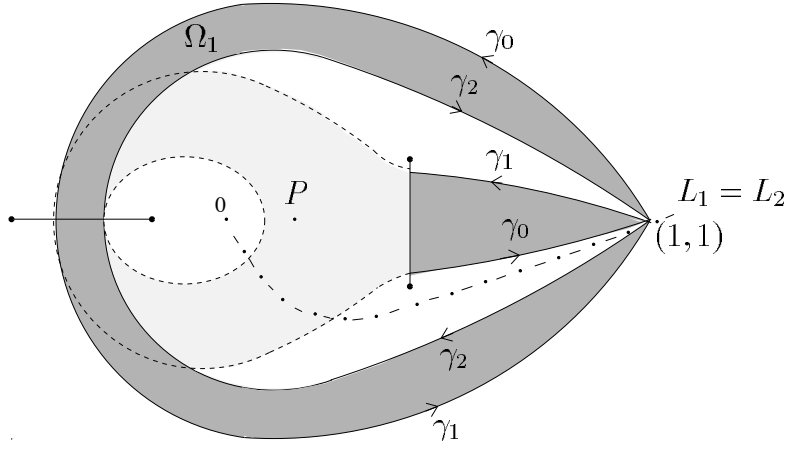-p \leq \lfloor \alpha_2 \rfloor + \lfloor -\alpha_2 \rfloor - (m_1 - 2)\,.
$$

Figure 22: Component with pole P and with both zero points encircled

When we treat the component "on the other side" of L_1, we obtain

$$\frac{1}{2\pi i} \int_{\gamma_2 + \gamma_1^E + \gamma_2^E} \frac{\varphi'}{\varphi}\, dz \le \lfloor -\alpha_2 + \alpha_2 - 1 \rfloor \ .$$

By applying the argument principle with the pole P inside the component and m_2 sectors at $z = 1$, we obtain

$$-p \le -1 - (m_2 - 1) \ .$$

A combination of these results leads to the following bound on the total multiplicity $m = m_1 + m_2$ of the component:

(29) $$m \le 2p + 1 \ .$$

Example 8.4   *Let Ω_1 be such as in Fig. 22, with the pole P again occurring on the positive real axis on the secondary sheet of M. Again the exclusion of the two zero points is achieved by means of an "inner" boundary curve (viz. γ_2) of Ω_1 which encircles them. By choosing the branch cuts L_i to go through $z = 1$ and by applying the argument principle in exactly the same way as in Example 8.3, we again obtain the bound $m \le 2p + 1$ occurring in (29).*

The question arises whether a pole P away from $z = 0$ can yield multiplicity higher than in Example 8.3 and 8.4. But since in these examples both the available zero points have been excluded in the "most efficient" way, we conclude that a pole away from $z = 0$ cannot yield a higher multiplicity than that expressed in the bound (29). We formulate this result in the following lemma.

Lemma 8.5 (Multiplicity of a single pole) *Let P be a pole of order p away from $z = 0$ inside an Ω-component Ω_1 from which the two zero points are excluded in the "most efficient" way. Then the multiplicity m of Ω_1 is bounded by*

$$m \leq 2p + 1 \; .$$

$\square$

We can deduce from Lemma 8.5 that the highest possible multiplicity of a component, Ω_1 relative to the order p of a pole P away from $z = 0$ inside it, is obtained if $p = 1$. Or, directly formulated: the most efficient poles away from $z = 0$ are simple poles. For such a simple pole we obtain the bound $m \leq 3$ on the multiplicity of the corresponding component. This entails that, if we have a normalized scheme which introduces into the corresponding order star poles of total order $p > 1$ inside Δ, then the highest possible contribution of these poles to the number of Ω-sectors inside Δ is obtained if these poles are simple and occur on the real axis. But then the complication occurs that the symmetry of components with respect to the real axis does not allow the simultaneous occurrence of two separate components, each with a multiplicity of 3. As before this problem is overcome by means of two components of multiplicity 3 each of which "join up" to form one component of multiplicity 6. In this new component the two separate simple poles still "act separately" as if they belong to separate components. This is illustrated in Fig. 23, where there are three simple poles, each leading to a contribution of 3 sectors to the total multiplicity inside Δ. The rightmost pole belongs to a separate component, while the other two have "joined up" to form a component of multiplicity 6. This situation allows itself to be generalized to the case where we have p simple poles inside Δ and can be formulated as follows.

Proposition 8.6 (Maximum multiplicity of poles) *Let the order star of a stable, normalized scheme have poles of total multiplicity p away from $z = 0$ inside the unit disk Δ. Then the highest possible contribution of these poles to the multiplicity m of components inside Δ is obtained if the poles are simple and real, leading to a multiplicity bounded by*

(30) $$m \leq 3p \; .$$

$\square$

Remark 8.7 *Instead of two components with simple poles "joining up" to form one component such as in Fig. 23, a component containing a simple pole can also "join up" with a component containing a zero point, such as in Fig. 24. There the component with the simple pole has "joined up" with a component of multiplicity one containing the zero point $(0, w_1^0)$ to form a component of multiplicity 4. (The zero point $(0, w_2^0)$ belongs to a separate binary component of multiplicity 2). It can be seen that the efficiency of the pole and of the zero point $(0, w_1^0)$ remain unchanged as if they occur in separate components.*

Remark 8.8 *Observe from Figures 21- 24 that in order for the poles in these components to each contribute a maximum multiplicity, of 3, branch cuts are required. In particular, for a single pole a minimum of 3 branch points inside Δ are utilised. But for components containing more than one pole the branch points are used more efficiently. For two poles*
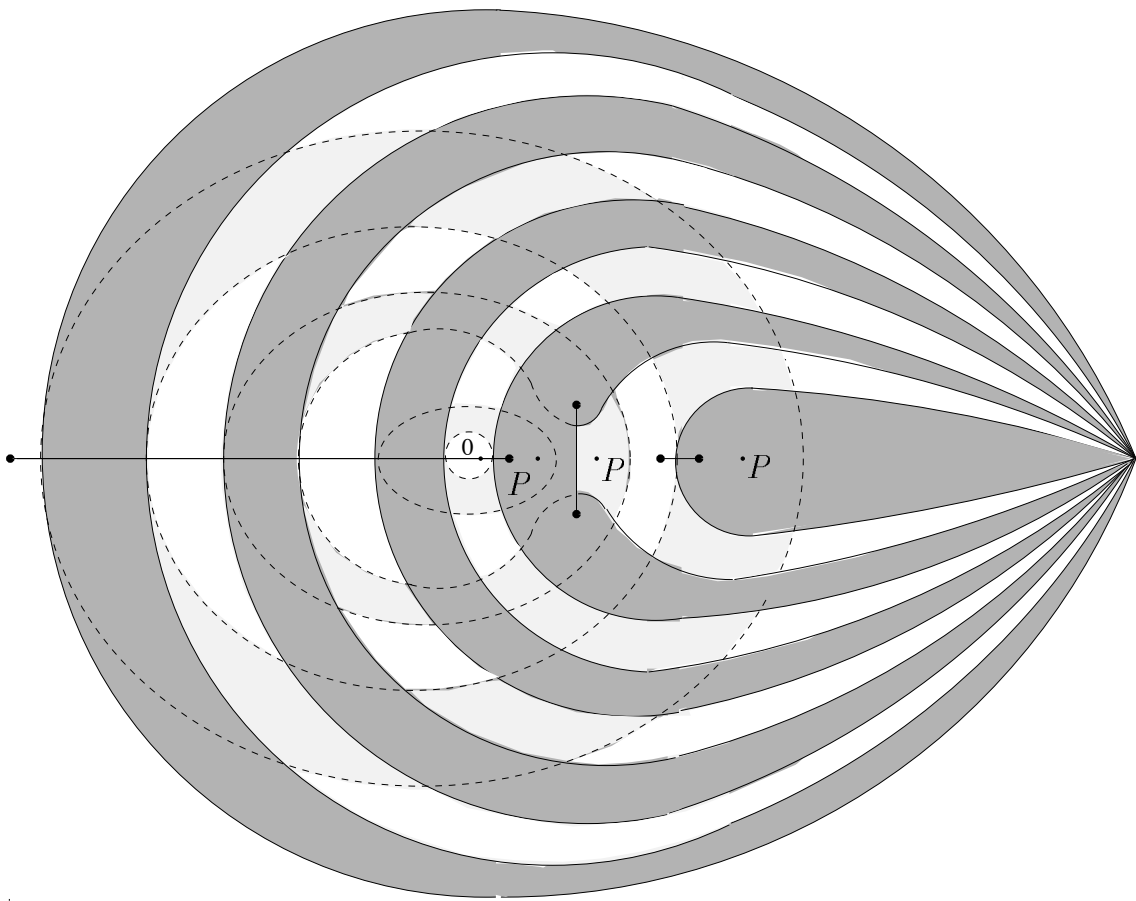
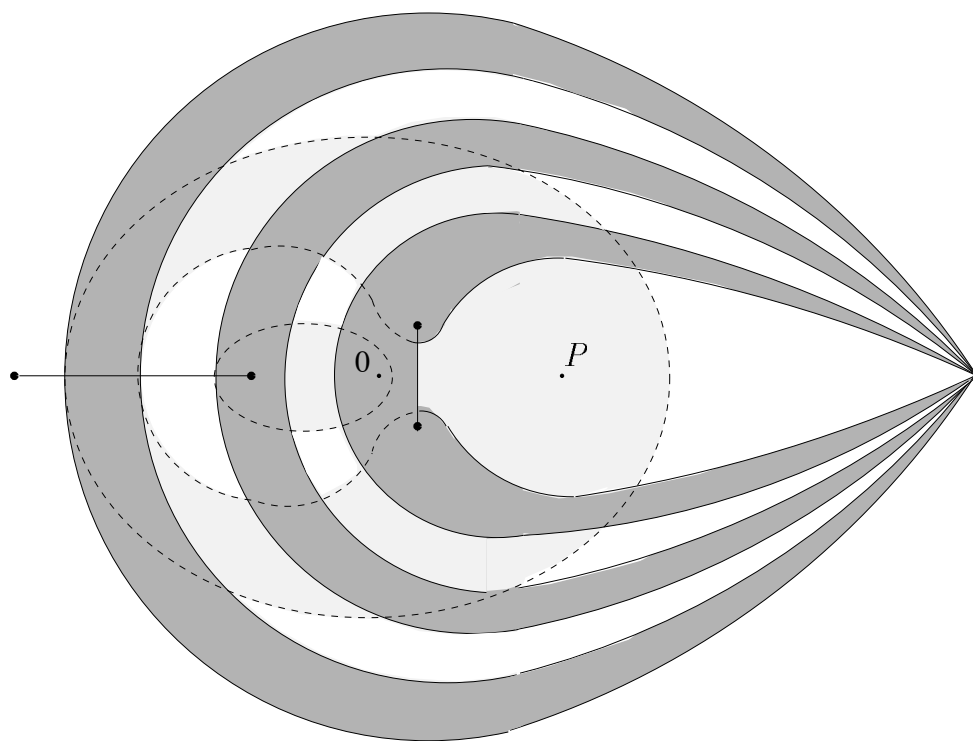Figure 23: Three simple poles, each leading to multiplicity 3, inside Ω-components

Figure 24: Component with a pole and one with a zero point which have "joined up" to form one component

again just 3 branch points in Δ are sufficient. As further poles are added to the component each requires an additional cut inside Δ. We therefore deduce the following corollary:

Corollary 8.9 *Let the order star of a stable, normalized scheme have n_p poles away from $z = 0$ inside a component Ω_1, inside Δ. Then the minimum number of branch points, $K_{\min}$, contained in Ω_1 such that the multiplicity of Ω_1 is given by $m = 3n_p$ satisfies*

$$K_{\min} = 2n_p - 1$$

$\square$

The question then arises as to whether Conjecture 6.11 can still be proved for the schemes of maximal order. This, however, turns out not to be so difficult. First let us consider both the order of the zero points and the number of branch points for the implicit schemes. When $r_2 > 0$, $s_2 > 0$ the exponents of φ in the expansion around $z = 0$ are given by

$$\begin{aligned}
\alpha_1 &= r_0 - r_1 + \mu \\
\alpha_2 &= r_1 - r_2 + \mu
\end{aligned}$$

and

$$\begin{aligned}
\beta_1 &= s_0 - s_1 + \mu^* \\
\beta_2 &= s_1 - s_2 + \mu^*,
\end{aligned}$$

as compared with (21) and (22), respectively, when $r_2 = s_2 = 0$. But, by convexity, we still have $\lfloor \alpha_2 \rfloor \geq \lfloor \alpha_1 \rfloor$ and $\lfloor \beta_2 \rfloor \geq \lfloor \beta_1 \rfloor$, even though $\lfloor \alpha_2 \rfloor$ and $\lfloor \beta_2 \rfloor$ are reduced by r_2 and s_2, respectively. Hence the number of branch points K utilised by the zero points is reduced by $2r_2 - 1$ if K is odd and $2r_2$ if K is even. But, by Corollary 8.9, the r_2 poles inside Δ need at least $2r_2 - 1$ branch points in order to contribute maximum multiplicity. Note further that for K even, it was demonstrated in Lemma 7.11 that $p = |I| - 2$ could not be achieved. Hence the branch points left unutilised by the reduction of $\lfloor \alpha_2 \rfloor$ and $\lfloor \beta_2 \rfloor$, when $r_2, s_2 > 0$, are immediately required to contribute maximum multiplicity from the poles. Furthermore, since the contribution due to the poles gives a factor 3, rather than 2, in front of n_p and $\lfloor \alpha_1 \rfloor$, respectively, we deduce that the optimal configuration uses branch points to maximise multiplicity due to the poles rather than due to the zero points. This leads us to conclude that Conjecture 6.11 is also valid for convex implicit schemes:

Lemma 8.10 (Conjecture 6.11 for Implicit Schemes of Maximal Order) *Suppose we have an implicit stable scheme of type (4) of maximal order, $p = |I| - 2$, with a convex increasing stencil, with a fixed Courant number μ, $-\frac{1}{2} < \mu < 0$. Assume that the algebraic function w of $\Phi(z, w) = 0$ has no branch point at $z = 0$ and that φ has leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two*

sheets of M. Then the multiplicity m of the Ω−component Ω_1 containing both zero points and no poles satisfies

$$m \le \lfloor \alpha_1 \rfloor + 1 + 2\lfloor \alpha_2 \rfloor + \max\{1, 2\lfloor \delta_1 + \delta_2 \rfloor\}.$$

$\square$

9 Proof of the main theorem

The proof of Theorem 3.1 is divided into stages. One part of it is proved in Lemma 9.1 and the other part in Lemma 9.2. The numbers R and S denote the number of downwind and upwind stencil points, respectively, with respect to the characteristic through (t_{n+2}, x_m) as defined in (9).

Lemma 9.1 (Maximal order with stability) *Suppose we have a convex, normalized scheme of type (4) with a fixed Courant number μ, $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, and Conjecture 6.11 is satisfied, then the order p of the scheme is bounded by*

(31) $$p \le 2R.$$

Proof: Since the scheme is stable, there will be a clear distinction between the portion of the corresponding order star Ω inside the unit disk Δ and the portion outside Δ. In this proof we restrict ourselves to the portion inside Δ.

From Proposition 4.4 we have the following expansions of $\varphi(z, w_i(z))$ at $z = 0$:

$$\varphi(z, w_1(z)) = z^{-(r_1 - r_2) - \mu}(b_0 + b_1 z + b_2 z^2 + \ldots),$$
$$\varphi(z, w_2(z)) = z^{-(r_0 - r_1) - \mu}(c_0 + c_1 z + c_2 z^2 + \ldots).$$

Again we have no means of associating a certain expansion with the zero point on a specific sheet. Hence the expansions will be associated with the zero points in the way which leads to the highest possible multiplicity.

In the remainder of the proof we have to work separately with the cases where $\mu < 0$ and where $\mu > 0$. Note also that where we assume Conjecture 6.11, Lemmas 7.11 and 8.10 give the result for $p = p_{opt}$, and $-\frac{1}{2} < \mu < 0$.

a) We first assume $-\frac{1}{2} < \mu < 0$. Then the following choices of the indices r_0, r_1, r_2 lead to different combinations of Ω-components inside Δ.

(i) $r_0 = r_1 = r_2 = 0$. Then also $R = 0$. According to the Courant-Friedrichs-Lewy condition the scheme cannot be convergent, i.e. it is impossible to have order $p \ge 1$ and stability simultaneously.

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(ii) $r_0 = r_1 = r_2 > 0$. There are r_2 poles away from $z = 0$ inside Δ, while both $\varphi(z, w_1(z))$ and $\varphi(z, w_2(z))$ have positive leading exponents of $-\alpha_i = -\mu$ at $z = 0$, implying that both zero points belong to Ω^c. We apply Proposition 8.6 to obtain

$$m \leq 3r_2 = r_0 + r_1 + r_2 = R \,.$$

(iii) $0 = r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -\mu > 0$, implying that $(0, w_1^0) \in \Omega^c$, and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that $(0, w_2^0) \in \Omega$. The highest possible multiplicity is obtained if $(0, w_2^0)$ belongs to a binary component, in which case we apply Proposition 6.6. If $r_2 > 0$, the poles away from $z = 0$ are again treated according to Proposition 8.6. Then we have

$$
\begin{aligned}
m &\leq 3r_2 + 2\lfloor r_1 - r_2 + \mu \rfloor + 2\lfloor 2 + 2\mu \rfloor \\
&= 3r_2 + 2(r_1 - r_2 - 1) + 2 \\
&= r_2 + 2r_1 = r_2 + r_1 + r_0 = R \,.
\end{aligned}
$$

(iv) $0 < r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. Since $\alpha_1 < \alpha_2$, the highest possible multiplicity is obtained by applying Conjecture 6.11, with $(0, w_1^0)$ inside a non-binary and $(0, w_2^0)$ inside a binary component. If $r_2 > 0$, the poles away from $z = 0$ are again treated according to Proposition 8.6. Then the total multiplicity m inside Δ is bounded by

$$
\begin{aligned}
m &\leq 3r_2 + \{2\lfloor r_1 - r_2 + \mu \rfloor + 2\lfloor 2 + 2\mu \rfloor\} + \{\lfloor r_0 - r_1 + \mu \rfloor + 1\} \\
&= 3r_2 + \{2(r_1 - r_2 - 1) + 2\} + \{(r_0 - r_1 - 1) + 1\} \\
&= r_0 + r_1 + r_2 = R \,.
\end{aligned}
$$

b) Assume $0 < \mu < \frac{1}{2}$. Then we have $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. Since $\alpha_1 \leq \alpha_2$, the highest possible multiplicity is obtained, assuming Conjecture 6.11, if α_1 is inside a non-binary and α_2 inside a binary component. If $r_2 > 0$, the poles away from $z = 0$ are treated according to Proposition 8.6. This leads to the bound

$$
\begin{aligned}
m &\leq 3r_2 + \{2\lfloor r_1 - r_2 + \mu \rfloor + 1\} + \{\lfloor r_0 - r_1 + \mu \rfloor + 1\} \\
&= 3r_2 + \{2(r_1 - r_2) + 1\} + \{(r_0 - r_1) + 1\} \\
&= r_0 + r_1 + r_2 + 2 = R \,.
\end{aligned}
$$

In all the foregoing cases we obtained

$$m \leq R \,.$$

The remainder of the proof makes use of Lemma 5.2 and hence Equation (18) to give for the order p of the scheme

$$p + 1 \leq m + (m + 1) \leq 2R + 1 \,,$$

which leads to $p \leq 2R$. $\qquad\square$

Concerning the upwind points of a difference stencil we now prove the following lemma.

Lemma 9.2 (Maximum order with stability) *Suppose we have a convex and normalized scheme of type (4) with a fixed Courant number μ satisfying $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, then the order p of the scheme is bounded by*

$$p \leq 2S .$$

Proof: Instead of repeating the argument of Lemma 9.1 for the Ω-sectors outside Δ, we apply the symmetry argument introduced in Section 7 to prove this result.

Hence, if (31) is proved for a value of μ for which the stencil is regular, we obtain by the mapping $z \to \frac{1}{z}$ and $\mu \to -\mu$ for $-\mu$ that

$$\ell \leq R^* = S .$$

By making use of Lemma 5.2 this result leads to

$$p \leq 2S .$$
$\qquad\square$

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Appendix A. Motivation of Conjecture 6.11

Here we return to the consideration of the multiplicities of double binary components. Observe that if cut B in Fig. 13 would not be present and the curves γ_0 and γ_1 would join together as in Fig. 25 we would still obtain bound (19). Note that Fig. 25 corresponds to Fig. 26 c) of [16]. Now let us deform the boundary of Ω_1 in Fig. 13. As long as the deformation is continuous this should not affect the argument principle if neither a zeropoint nor a zero crosses the boundary. Hence, the integrals along the curves given in Figs. 26-27 are the same since a branchpoint on a Riemann surface is not a special point with respect to integration. The connectivity of the curves in Fig. 26 b) can be interpreted in two ways, the one obtained from a continuous deformation of Fig. 26 a) and the one obtained from Fig. 27 by a continuous deformation. However, what does change in our way of applying the argument principle, when making this continuous deformation, is the pairing of the integrals. Hence, Fig. 26 a) leads to bound (17) while Fig. 27 leads to the bound

$$m \leq 2\lfloor \alpha_1 \rfloor + 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor .$$

Again the factor 2 in the term with $\lfloor \alpha_1 \rfloor$ is present but note now that this bound is sharper than that given by (17) and at the same time requires fewer branch cuts to obtain the multiplicity. However, the cut A has now a similar function as cut A in Fig. 12. If it would be removed Ω_1 would be separated into two disconnected Ω components, one which is binary and one which is non-binary. This would again give the bound of Theorem 6.9. We can actually remove the cut A in Figs. 12, 26 a), respectively, by deforming the boundary as indicated in Figs. 28. In Fig. 28 b) when binary loops are introduced the set Ω has been decomposed into two separate components, a binary and a nonbinary giving again the bound of Theorem 6.9. In a similar fashion one can move the cut A in Fig. 12 outside the order star and thus separate Ω_1 into two disconnected components, see Figs. 29 - 31.

The problem with these continuous deformations is that one has to be able to show that they can be performed while keeping the other conditions of the schemes unchanged
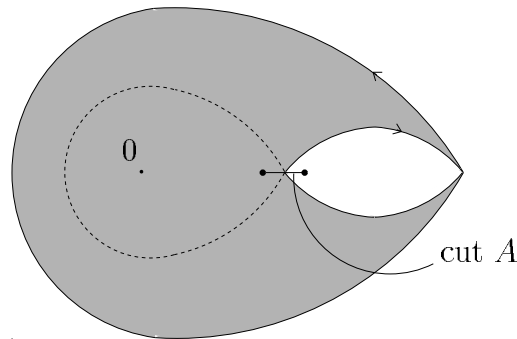

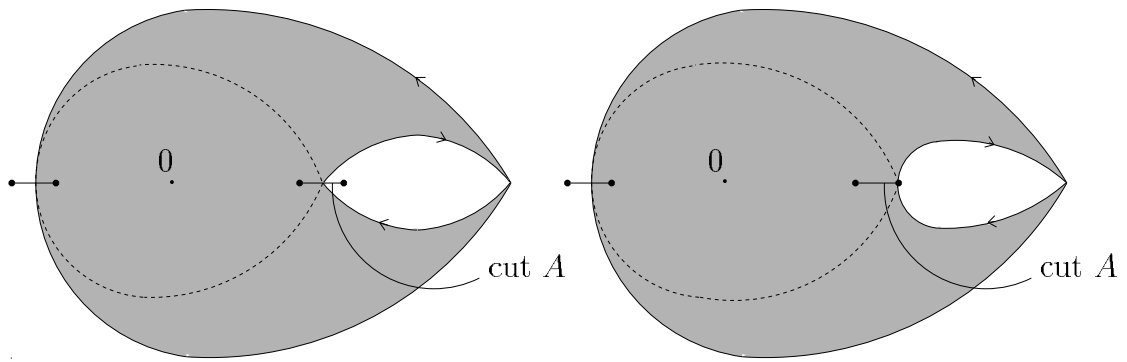Figure 25: Simplification of Unsymmetric Binary Component

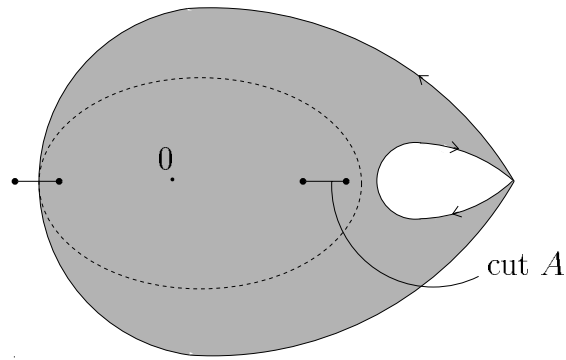Figure 26: a, b Deformation of the boundary curves of Unsymmetric Binary Component

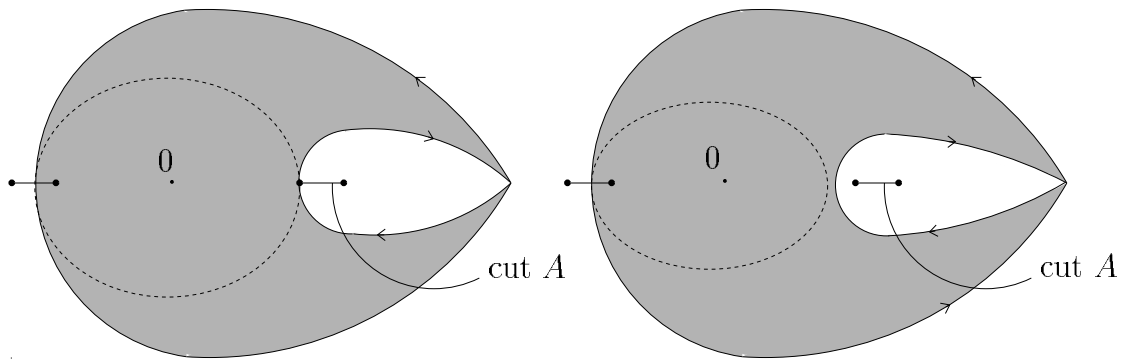Figure 27: Cut A moved inside the component

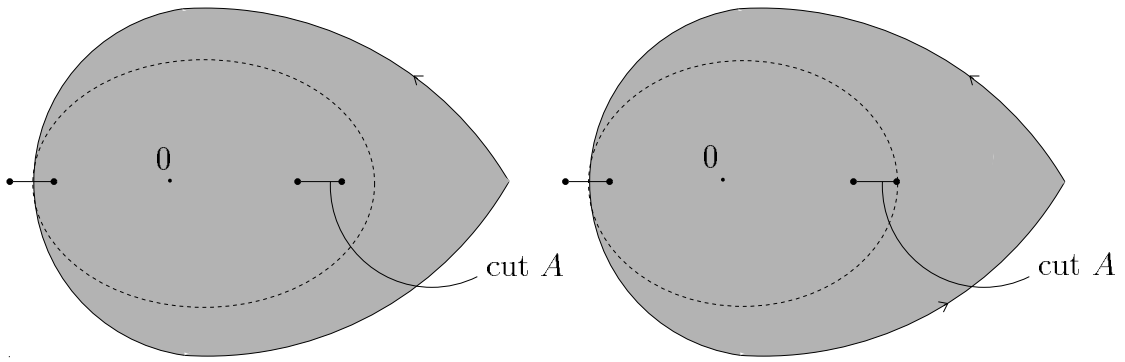Figure 28: a, b Removal of cut A from the component

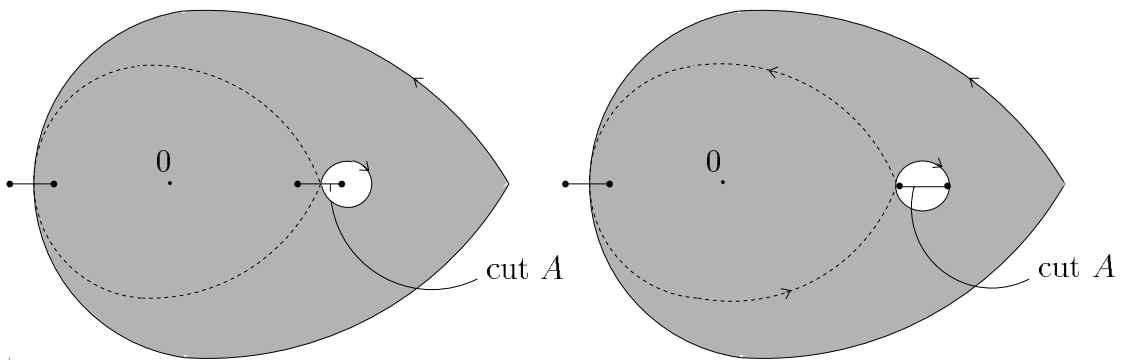Figure 29: a, b Cut A redundant
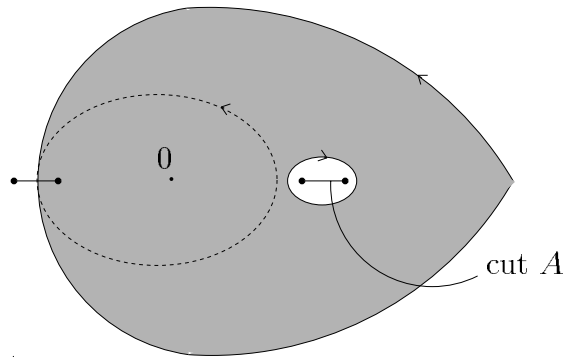
Figure 30: Cut A being moved outside the order star

Figure 31: Cut A outside the order star

Figs. 29 - 31   Transformation of γ_0 in Fig. 12 such as to move cut A outside
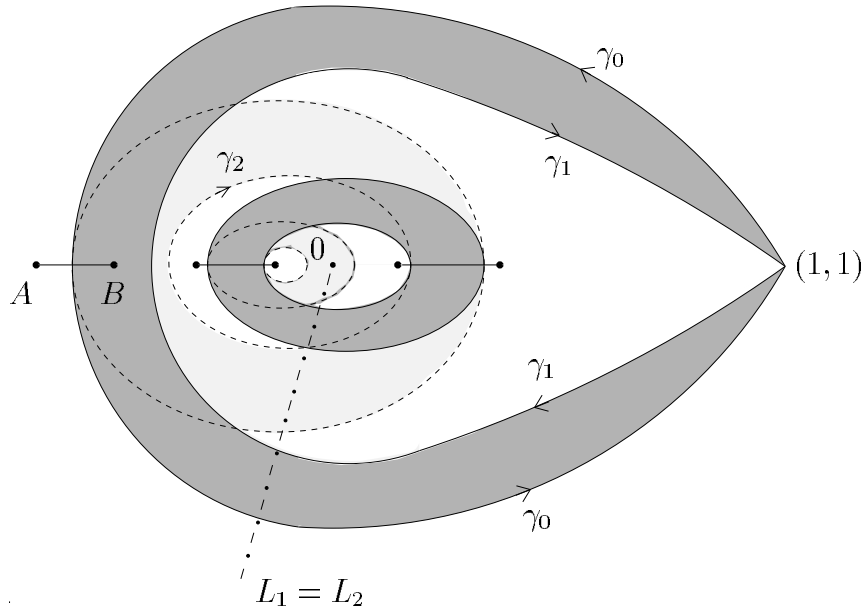the order star.

Figure 32: Symmetric binary component with boundary curves circling more than once around the zero points

such as stability, instability, real coefficients, error order, difference stencil at least on the downwind side. While such a proof seems to be technically extremely difficult the possible deformations suggested above support Conjecture 6.11.

We do observe, however, that for all double binary components there is one cut which is required to maintain the connection between the components. We have called this cut, cut A. To be effective cut A has to allow, in the application of the argument principle, that the multiplicity due to the zero point on one sheet is "carried" via the cut to the other sheet and then back to the original sheet by cuts inside binary loops. If the cut is excluded by integrating around it, in the application of the argument principle, the components are disconnected. But it appears it is still not possible to draw any conclusions about whether the multiplicity can be "carried" in this way because the integral around the cut amounts to an integration around a "hole" of the surface, which is not uniquely defined. We do feel that a further consideration of this approach will lead to a proof of Conjecture 6.11.

Appendix B. Components with one zero point which are basically different

The components we considered in the body of the paper have boundary curves which circle at most once around a zero point. The question arises as to what the effect will be if one or more boundary curves circle several times around the zero points.

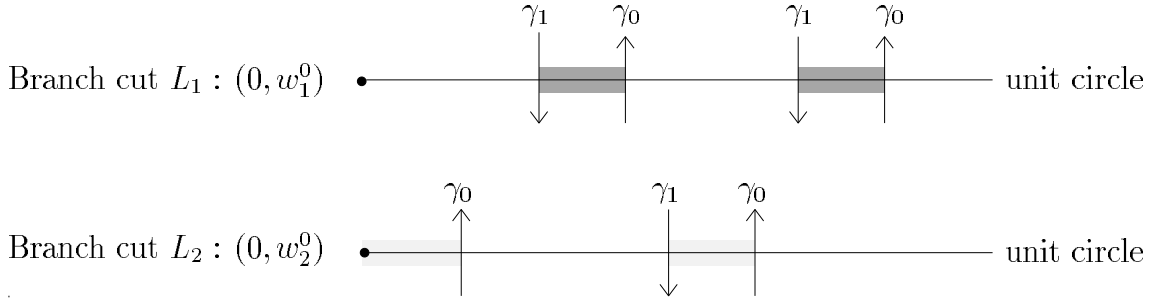Because a zero point has the highest possible efficiency if it occurs inside a binary

Figure 33: Boundary curves crossing the branch cuts

component, we initially consider binary components. The first one is a symmetric binary component depicted in Fig. 32.

By choosing the branch cuts L_i according to convention, the way in which the boundary curves cross the branch cuts can be represented schematically in Fig. 33.

Integration along the boundary curves leads to

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor 2\alpha_1 + 2\alpha_2 \rfloor \, ; \; \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor -2\alpha_1 - \alpha_2 \rfloor \, .$$

If we apply the argument principle and take into account the binary character of the component such as we did in Lemma 6.3, we obtain

(32) $$0 \leq 2\{\lfloor 2\alpha_1 + 2\alpha_2 \rfloor + \lfloor -2\alpha_1 - \alpha_2 \rfloor\} - (m-2) \, .$$

In order to simplify (32), let $\tilde{\alpha} = 2\alpha_1 + \alpha_2$, $\tilde{\delta} = \tilde{\alpha} - \lfloor \tilde{\alpha} \rfloor$ and $\delta_2 = \alpha_2 - \lfloor \alpha_2 \rfloor$ as before. Then we obtain

$$0 \leq \{\lfloor \tilde{\alpha} + \alpha_2 \rfloor + \lfloor -\tilde{\alpha} \rfloor\} - \frac{(m-2)}{2} \, ,$$

leading to the bound

$$m \leq 2\lfloor \alpha_2 \rfloor + 2\lfloor \tilde{\delta} + \delta_2 \rfloor$$

which is basically the same as in Lemma 6.3. The simplification in (32) is possible because the "outward" boundary curve γ_0 is "accompanied" by an "inner" boundary curve γ_1, ensuring that the component is well defined as it "switches" to and fro between the two sheets of M.

Remark *There are two variations to the component in Fig. 32 which deserve special attention.*

a) *Suppose we have a component with an "outward" boundary γ_0 which corresponds to that in Fig. 32, while the "inner" boundary curve γ_1 does not cross one of the branch cuts on the real axis, such as AB in Fig. 34. Then at least one other "inner" boundary curve γ_2 (say) is needed for the component to be well defined. Since the*

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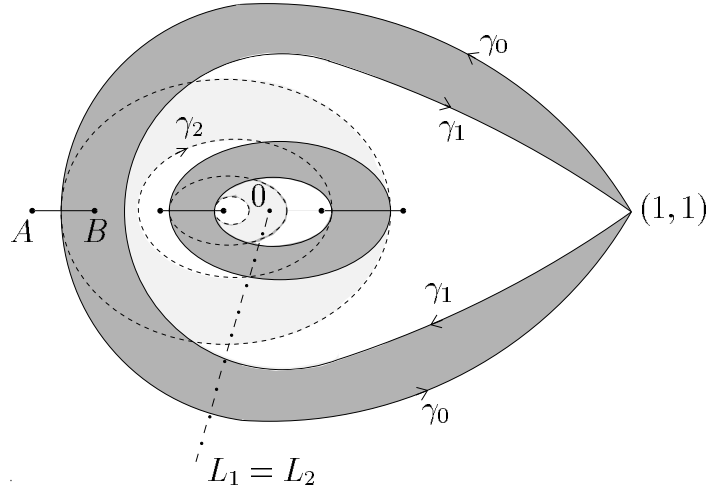Figure 34: Boundary curve γ_2 which is inefficient

curve γ_2 *does not pass through (1,1) it is not contributing to the multiplicity of the component. In [17] a curve of this kind is called an* **inefficient curve** *(see [17], Fig. 18). Integration along the boundary now leads to*

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor 2\alpha_1 + 2\alpha_2 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor -\alpha_1 \rfloor,$$

$$\frac{1}{2\pi i} \int_{\gamma_2^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor -\alpha_1 - \alpha_2 \rfloor \ .$$

Because $\lfloor -\alpha_1 \rfloor + \lfloor -\alpha_1 - \alpha_2 \rfloor \leq \lfloor -2\alpha_1 - \alpha_2 \rfloor$, *the inequality involving the argument principle can be simplified to (32), leading to the same bound as before. Since in general* $\lfloor \alpha \rfloor + \lfloor \beta \rfloor \leq \lfloor \alpha + \beta \rfloor$, *a simplification of this kind can be made wherever an inefficient curve "replaces" an efficient one. For this reason we will as far as possible make use of efficient curves.*

b) *Suppose we have a component with an "outward" boundary* γ_0 *which corresponds to that in Fig. 32, but with more than one efficient "inner" boundary curve such as* γ_1, γ_2 *and* γ_3 *in Fig. 35. Then a careful combination of the terms in the inequality involving the argument principle leads to an inequality similar to (32). (For this simplification it is convenient to make the assumption that the non-integer parts of* α_1 *and* α_2 *coincide, i.e. that* $\delta_1 = \delta_2$ *where* $\delta_i = \alpha_i - \lfloor \alpha_i \rfloor$, $i = 1, 2$. *From a practical point of view this assumption is not restrictive at all, since the factor* $z^{-\mu}$ *in* φ *is throughout responsible for the non-integer parts of* α_1 *and* α_2. *However, since this assumption is not needed in the rest of the discussion and since a component of the type in Fig. 35 is most unlikely to occur in any "real" order star, we do our analysis without making this assumption).*
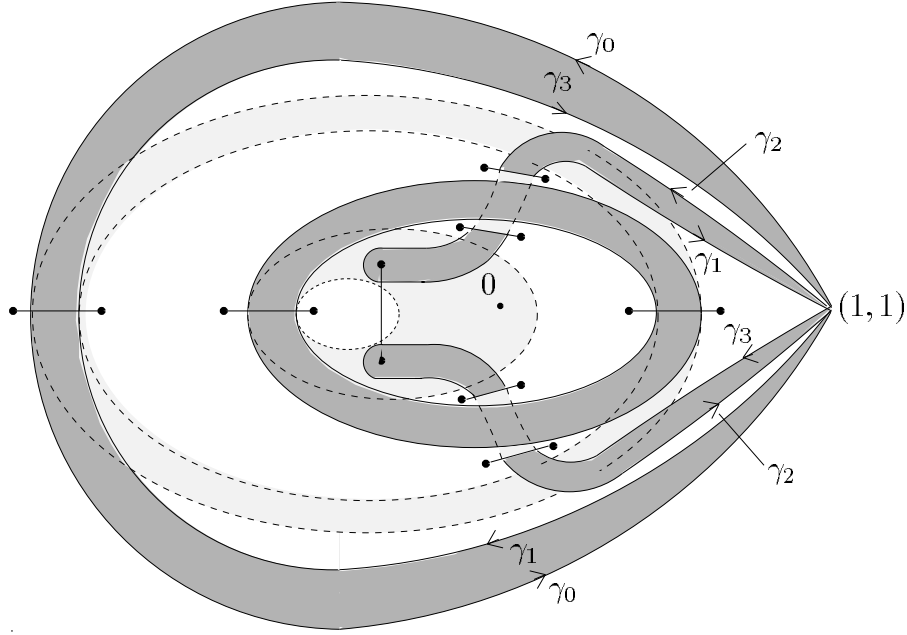
Figure 35: More than one efficient "inner" boundary curve circling more than once around the zero points

The situation in (32) can be generalized to the case where the two boundary curves "switch" several times from the one sheet to the other. It can again be represented schematically as in Fig. 36.

Integration along the boundary curves now leads to

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor k\alpha_1 + k\alpha_2 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'}{\varphi} \, dz \leq \lfloor -k\alpha_1 - (k-1)\alpha_2 \rfloor .$$

If we apply the argument principle and take into account the binary character of the component, we obtain

(33) $$0 \leq \{ \lfloor k\alpha_1 + k\alpha_2 \rfloor + \lfloor -k\alpha_1 - (k-1)\alpha_2 \rfloor \} - \frac{(m-2)}{2} .$$

The result (33) is simplified by taking $\tilde{\alpha} = k\alpha_1 + (k-1)\alpha_2$, $\tilde{\delta} = \tilde{\alpha} - \lfloor \tilde{\alpha} \rfloor$. Then we obtain as in (32)

$$m \leq 2\lfloor \alpha_2 \rfloor + 2\lfloor \tilde{\delta} + \delta_2 \rfloor .$$

We finally consider the case of an unsymmetric binary component such as the one depicted in Fig. 37. The presence of a branch cut such as AB now presents the complication that one more "inner" boundary curve γ_3 (say), is needed for the component to be well defined. With respect to the curve γ_3 we have the following possibilities:
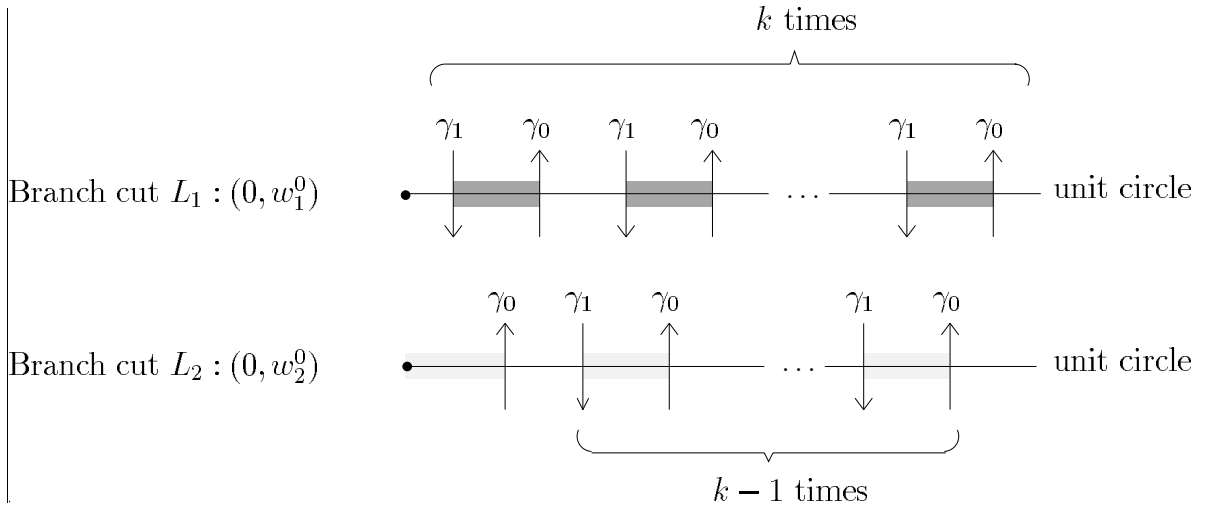
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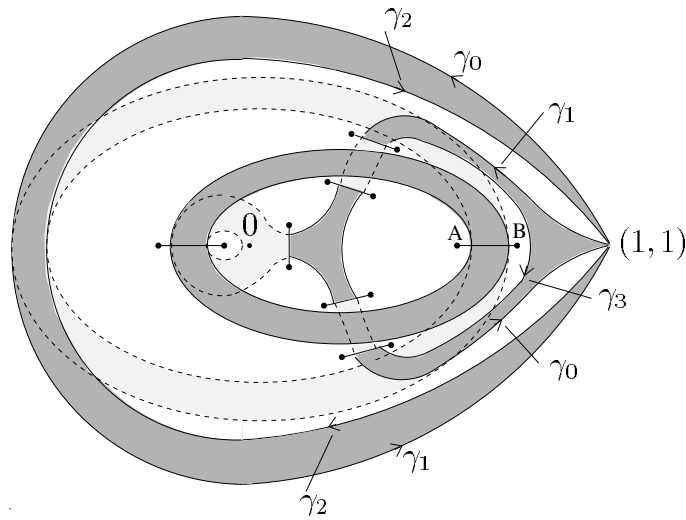Figure 36: Two boundary curves switching from one sheet to the other

Figure 37: Unsymmetric binary component with boundary curves circling more than once around the zero points

i) γ_3 is inefficient and Ω_1 is an unsymmetric binary component such as in Fig. 37;

ii) γ_3 passes through (1,1) to be efficient and Ω_1 becomes a symmetric binary component.

In both cases it can be shown that the bound in Proposition 6.6 remains valid.

In the foregoing examples we considered components with more than one "outward" and more than one "inner" boundary curve, being symmetric and unsymmetric with respect to the real axis and circling once or more than once around the zero point. In all these cases the bound on the multiplicity coincides with the result in Proposition 6.6. A similar analysis with respect to non-binary components leaves Proposition 6.1 unchanged.

It should also be noted that binary components such as those in Figs. 32, 34, 35, 37 cannot be combined with a non-binary component with odd multiplicity such as in Fig. 11.

In view of these considerations we have only worked with components with boundary curves which circle once around one or two zero points.