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Abstract

We prove uniqueness theorems in the class of piecewise Lipschitz continuous solutions of the Cauchy-Problem for 2×2 conservation laws and improve the results of DiPerna [Di] in this class.

Keywords: uniqueness, conservation laws, 2×2 system

Subject Classification: 35L65, 35A05, 35L67, 35L60

1. Introduction.

We consider strictly hyperbolic systems of two conservation laws in one dimension:

(1.1.a) $$\partial_t U + \partial_x f(U) = 0, \qquad (x,t) \in R \times R^+;$$

(1.1.b) $$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{und} \quad f(U) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix}$$

with the intitial data

(1.2) $$U(x,0) = U_0(x),$$

where $R = (-\infty, +\infty)$ und $R^+ = [0, +\infty)$.

Here the condition of *strict hyperbolicity* means that the Jacobian $\bigtriangledown f$ of f has two real and distinct eigenvalues:

$$\sigma_1(U) < \sigma_2(U)$$

and the function f is in $C^3(\bar{\Omega})$ for some open set Ω in R^2.

The system (1.1) is *genuinely nonlinear*, if for each i with $i = 1, 2$

$$r_i \cdot \bigtriangledown \sigma_i \neq 0,$$

where r_i denotes the right eigenvector of $\bigtriangledown f$ corresponding to σ_i.

A bounded integrable function $U(\cdot,\cdot)$ is called a *weak solution* of the Cauchy-Problem: (1.1) and (1.2), if U satisfies (1.1) and (1.2) in the sense of distributions, i.e.,

$$\iint_{R\times R^+} (U\partial_t\phi + f(U)\partial_x\phi)\, dx\, dt + \int_R U_0(x)\phi(x,0)\, dx = 0$$

for any smooth function ϕ with compact support in $t \geq 0$.

It is well known that uniqueness is lost in the class of weak solutions. Thus we consider uniqueness in the following subclasses of the weak solutions. $|\cdot|$ denotes the Euclidean norm in R^2.

Definition 1.1 A weak solution $U(\cdot,\cdot)$ is called a solution in the *class 1*, if it satisfies the following conditions:

(1.4) U is bounded in the sense of Lebesque's measure, i.e.,

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \big|U(x,t)\big| < +\infty.$$

(1.5) U has a finite number of curves of discontinuity in each compact set in $R \times R^+$.

(1.6) Each curve of discontinuity is smooth.

(1.7) U is Lipschitz continuous in the region between any two curves of discontinuity.

(1.8) Across any discontinuity $x = \gamma(t)$ the weak solution U satisfies *Lax's entropy condition:*

(1.8.a)
$$\sigma_1(U^+) < \gamma' < \sigma_1(U^-)$$

or

(1.8.b)
$$\sigma_2(U^+) < \gamma' < \sigma_2(U^-),$$

where $U^\pm = U(x \pm 0, t)$ and $\gamma' = \frac{d\gamma(t)}{dt}$.

Remark The condition (1.5) refers to the count of the number of discontinuities. We consider the discontinuities after the intersection of two discontinuities as new discontinuities.

Definition 1.2 A weak solution $U(\cdot,\cdot)$ is called a solution in the *class 2*, if it satisfies the conditions: (1.4)-(1.7), and

(1.9) For any curve of the discontituity $x = \gamma(t)$ there is a number $\sigma > 0$ independent of t, such that across $x = \gamma(t)$ the weak solution U satisfies *the strong enropy condition:*

$$\begin{cases} \sigma_1(U^+) < \gamma'; \\ 0 < \sigma < \sigma_1(U^-) - \gamma' \end{cases}$$

or

$$\begin{cases} \gamma' < \sigma_2(U^-); \\ 0 < \sigma < \gamma' - \sigma_2(U^+). \end{cases}$$

We show in this paper the following two theorems.

Theorem 1.3 : Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} |U(x,t)| < C;$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} |\bar{U}(x,t)| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta;$$

$$|\bar{U}^{+} - \bar{U}^{-}| < \theta,$$

then $\bar{U} = U$ in $R \times R^{+}$.

For a general system we have

Theorem 1.4 : For every weak solution U in the class 2 there exists a number $\theta > 0$ depending on f und U with the following property.

If $\bar{U}$ is a solution in the class 1 with

$$\max_{(x,t)\in R\times R^{+}} \left|U(x,t) - \bar{U}(x,t)\right| < \theta,$$

then $\bar{U} = U$ in $R \times R^{+}$.

DiPerna [1] has shown

Theorem D : Suppose that the system (1.1) is genuinely nonlinear and admits an additional conservation law:

$$\eta(U)_t + q(U)_x = 0. \tag{1.10}$$

For every $\tilde{U} \in R^2$ there exists a number $\theta > 0$ depending on f und $\tilde{U}$ with the following property.

If U is a solution in the class 1 and $\bar{U}$ an another weak solution with

$$\eta(U)_t + q(U)_x \leq 0$$

in the sense of distribution and U and $\bar{U}$ satisfy

$$\max_{(x,t)\in R\times R^{+}} |(U - \tilde{U})(x,t)| \leq \theta$$

and

$$\max_{(x,t)\in R\times R^{+}} |(\bar{U} - \tilde{U})(x,t)| \leq \theta,$$

then $\bar{U} = U$ in $R \times R^{+}$.

Here *DiPerna* requires: (i) Since $\tilde{U}$ is a fix vector in R^2, $U(x,t)$ and $\bar{U}(x,t)$ are almost constant. They have to stay in a ball with the sufficiently small radius θ und the center $\tilde{U}$. (ii) The distance between U and $\bar{U}$ is sufficient small.

We do not require the assumption of existence of an additional conservation law (1.10). In the theorem 1.3 both solutions can have a large variation and there is no restriction for the distance between U and $\bar{U}$. In the theorem 1.4 both solutions can have an arbitrary large, but bounded variation and the system must not be genuinely nonlinear.

In our view the technique applied here to show the identity of two weak solutions is more important than the results. With this technique we can consider furthermore the rarefaction waves and the $n \times n$ systems. This will be confirmed in our next papers.

For more information on uniqueness we would like to refer to the papers: [Liu], [FX], [DG] and the references cited there.

2. Localization.

We show in this section that it is sufficient to prove the identity of two weak solutions in a small trapezoid, in order to prove the identity in the halfplane: $(x,t) \in R \times R^{+}$. In this trapezoid the discontinuities can be easily described.

Let $M > 0$ be a given constant,

$$K(a,b,c,d) := \left\{ (x,t) \in R \times [0,\ T] \middle| \begin{array}{c} M(t-a)+b \leq x \leq -M(t-a)+d; \\ a \leq t \leq c \end{array} \right\}$$

with $0 \leq a < c, b < d$ and $M(c-a) < \frac{d-b}{2}$ a trapezoid and U and $\bar{U}$ two weak solutions. Then we introduce a class of trapezoides depending on f, U and $\bar{U}$.

Definition 2.1 : A trapezoid $K(a,b,c,d)$ is called a trapezoid in the *class* $T(M)$, if $K(a,b,c,d)$ satisfies the following conditions.

(2.1) The weak solutions U and $\bar{U}$ are almost everywhere identical in the ground line:

$$\Big\{ (x,t) \in K(a,b,c,d) \Big| b \leq x \leq d, t = a \Big\}.$$

(2.2) There is no or a finite number of discontinuities that run from one point in the ground line and don't intersect with the side lines:

$$x = M(t-a)+b \ \text{ and } \ x = -M(t-a)+d, \quad a < t \leq c.$$

They don't intersect each other in the trapezoid $K(a,b,c,d)$ either and each of them intersects with the upper boundary

$$\Big\{ (x,t) \in K(a,b,c,d) \Big| b + Mc \leq x \leq d - Mc, t = c \Big\}$$

at only one point.

Definition 2.2 : A trapezoid $K(a,b,c,d)$ in the class $T(M)$ has the *property* T_0, if for each $(x,a) \in K(a,b,c,d)$ with $b < x < d$ there is $\epsilon > 0$ with $b \le x - \epsilon < x + \epsilon \le d$, so that U and $\bar{U}$ are almost everywhere identical in the trapezoid $K(a, x-\epsilon, a+\epsilon, x+\epsilon) \subset K(a,b,c,d)$.

Lemma 2.3 : Suppose that U and $\bar{U}$ are two weak solutions satisfying the conditions (1.4)–(1.7).

If there is a number $M > 0$, so that any trapezoid in $T(M)$ has the property T_0, then $U = \bar{U}$ in $R \times R^+$.

Proof: Suppose that $U \ne \bar{U}$ in $R \times R^+$. Let $W := U - \bar{U}$ and

$$D(0,b,c,d) := \iint_{K(0,b,c,d)} |W|^2 \, dx \, dt.$$

Then there are b_0, c_0 and d_0, so that

$$D(0, b_0, c_0, d_0) > 0.$$

Let

$$t_0 := \sup\{t \in [0, c_0] \mid D(0, b_0, t, d_0) = 0\}.$$

It follows that

$$\begin{aligned} & t_0 < c_0; \\ (2.3) \qquad & D(0, b_0, t, d_0) > 0, \quad t \in (t_0, c_0); \\ & D(0, b_0, t_0, d_0) = 0 \end{aligned}$$

and U and $\bar{U}$ are identical in the segment

$$\Big\{(x,t) \in K(a,b,c,d) \Big| b_0 + Mt_0 \le x \le d_0 - Mt_0, t = t_0\Big\}.$$

Since U and $\bar{U}$ satisfy the conditions (1.4)–(1.7), it is obvious in the geometric views, that there is in the class $T(M)$ a finite number of trapezoides $\{K(t_0, b_i, t_0+\varepsilon, d_i)\}_{i=1}^n$ with $\epsilon > 0$, so that

$$\cup_{i=1}^n K(t_0, b_i, t_0+\varepsilon, d_i) = K(t_0, b_0, t_0+\varepsilon, d_0).$$

It follows from the property T_0 and compactness that

$$D(t_0, b_0, t_0+\epsilon, d_0) = 0.$$

Thus

$$\begin{aligned} D(0, b_0, t_0+\varepsilon, d_0) &= D(0, b_0, t_0, d_0) + D(t_0, b_0, t_0+\varepsilon, d_0) \\ &= 0. \end{aligned}$$

This is a contradiction to (2.3).∎

Without loss of generality we assume that the trapezoid

$$K := \big\{(x,t) \;\big|\; Mt - 1 \le x \le -Mt + 1, 0 \le t \le c\big\} \tag{2.4}$$

is in the class $T(M)$ and any discontinuities in K start from (0,0). Then we shall show the identity of U and $\bar{U}$ in K.

With the help of the entropy condition we can reduce the number of discontinuities in the trapezoid K.

Lemma 2.4 : Suppose that U and $\bar{U}$ satisfy the conditions (1.4)–(1.7). Then U (resp. $\bar{U}$) has in K at most two discontinuities. One of them $\alpha(\cdot)$ (resp. $\bar{\alpha}(\cdot)$) satisfies (1.8.a) and the another $\beta(\cdot)$ (resp. $\bar{\beta}(\cdot)$) satisfies (1.8.b).

Lemma 2.5 : Suppose that U and $\bar{U}$ satisfy the conditions (1.4)–(1.7). Then the discontinuities satisfy

$$\alpha(t) < \beta(t)$$
$$\bar{\alpha}(t) < \bar{\beta}(t)$$

for any $t \in (0, c)$.

The proof is trivial and we omit it.

3. Hyperbolicity and Symmetrizer.

Lemma 3.1 : For every $C > 0$ there exists $\theta > 0$ depending on C und f, so that for each V with $|V| \le C/2$ there is a neighborhood of V

$$\Omega_\theta(V) = \big\{U \in R^2 \big| |U - V| \le \theta, |U| \le C\big\},$$

in which there is a smooth, regular and real matrix L with the following properties:

(3.1) It holds that

$$\bigtriangledown f = L^{-1} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} L \quad \text{for all } U \in \Omega_\theta(V).$$

(3.2) Let $A := L^\tau L$. There is a positive constant depending on f and C with

$$< W, A(U)W > \geq const < W, W >$$

for all $U \in \Omega_\theta(V)$ and all $W \in R^2$, where the notation $< \cdot, \cdot >$ is the *scalar product* in R^2.

(3.3) the Matrixes L and A are uniformly bounded in $U \in \Omega_\theta(V)$, i.e., there is a positive constant depending on f and C with

$$\max\{|L|, |A|\} \leq const$$

for all $U \in \Omega_\theta(V)$ and all $W \in R^2$.

(3.4) The derivatives of L and A uniformly bounded in $U \in \Omega_\theta(V)$, i.e., there is a positive constant depending on f and C with

$$\max\{|\partial_{u_1} L|, |\partial_{u_2} L|\} \leq const$$

and

$$\max\{|\partial_{u_1} A|, |\partial_{u_2} A|\} \leq const$$

for all $U \in \Omega_\theta(V)$ and all $W \in R^2$.

Note that (3.1) implies

$$A \bigtriangledown f = L^\tau \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} L$$

and

$$(A \bigtriangledown f)^\tau = A \bigtriangledown f.$$

The row vectors of L are the left eigenvectors of $\bigtriangledown f$. We can prove this lemma by directly calculating L. In order to show the properties (3.1)–(3.4) we must apply the strict hyperbolicity of $\bigtriangledown f$. We omit here the deteils of proof.

A matrix A is called a *symmtrizer of the Matrix* $\bigtriangledown f$, if A is positive definite and $(A \bigtriangledown f)^\tau = A \bigtriangledown f$ holds. With its help one can show the existence und the uniqueness of classical solution of conservation laws (1.1). We refer to [KL]. We apply in this paper the symmetrizer to show the uniqueness of solution with discontinuities and avoid the assumption of existence of the entropy pair (η, q) [Dip].

4. Local Estimation.

Let t_1, t_2 with $0 \leq t_1 < t_2 < c$

$$\begin{cases} \Gamma_l : x = \gamma_l(t), & t \in [t_1, t_2] \\ \Gamma_r : x = \gamma_r(t), & t \in [t_1, t_2] \end{cases}$$

be two smooth curves in K and

$$G := \{(x,t) \in K \mid \gamma_l(t) \leq x \leq \gamma_r(t),\ t_1 \leq t \leq t_2\}$$

be a subset of K.

In this section we assume, that there is a symmetrizer satisfying (3.1)–(3.4), there is no discontinuity in G and U and $\bar{U}$ satisfy the condition (1.7) and

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} |U(x,t)| < C$$

and

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} |\bar{U}(x,t)| < C.$$

We measure the distance between U and $\bar{U}$ in G by the integration

$$J(t) := \int_{\gamma_l(t)}^{\gamma_r(t)} < W, A(U)W > \phi\, dx, \quad t \in [t_1, t_2]$$

with a strictly positive smooth function ϕ. Let

$$\begin{aligned} I(\tau, c, \phi) := \iint_{G(\tau)} [&< W, A(U)W > (\partial_t \phi) \\ &+ < W, A(U) \bigtriangledown f(U) W > (\partial_x \phi) \\ &+ c < W, W > \phi]\, dx\, dt, \end{aligned}$$

where $\tau \in [t_1, t_2]$, $G(\tau) := \{(x,t) \in G | t_1 \leq t \leq \tau\}$, $W = \bar{U} - U$ and $c > 0$ is a real number. Then we have

Lemma 4.1 : There is $c > 0$ depending on f, U and $\bar{U}$, so that for any $\tau \in [t_1, t_2]$

$$J(\tau) \leq I(\tau, c, \phi) + J(t_1) - J(\Gamma_l, t_1, \tau) + J(\Gamma_r, t_1, \tau)$$

holds, where

$$J(\Gamma_r, t_1, \tau) := \int_{t_1}^{\tau} < W, A[\gamma_r'(t) \cdot E - \bigtriangledown f] W > \phi \Big|_{x=\gamma_r(t)} dt,$$

$$J(\Gamma_l, t_1, \tau) := \int_{t_1}^{\tau} < W, A[\gamma_l'(t) \cdot E - \bigtriangledown f] W > \phi \Big|_{x=\gamma_l(t)} dt$$

and E is the unit matrix.

Proof: Let $U_n := \Psi_n * U$ and $\bar{U}_n := \bar{\Psi}_n * U$, where Ψ_n is a δ-function. U_n (resp. $\bar{U}_n$) is smooth, if U (resp. $\bar{U}$) is Lipschitz continuous. We estimate the integral:

$$\iint_{G_\tau} \partial_t[< W_n, A_n W_n > \phi] + \partial_x[< W_n, A_n(\bigtriangledown f)_n W_n > \phi]\, dx\, dt,$$

where $W_n = \bar{U}_n - U_n$, $A_n := A(U_n)$ and $(\bigtriangledown f)_n := f(U_n)$. By use of Green's theorem and the definition of weak solutions we can take apart this integration by the integrations along the boundaries of $G(\tau)$, i.e., by $J(\tau)$, $J(t_1)$, $J(\Gamma_l, t_1, \tau)$ and $J(\Gamma_r, t_1, \tau)$. The remaining term can be estimated by

$$\iint_{G(\tau)} c < W, W > \phi\, dx\, dt.$$

Then the conclusion follows, when $n \to \infty$.∎

Let

$$I_\triangle(\tau, c, \phi) := \iint_{G(\tau)} [< W, A(U)W > (\partial_t \phi) + < W, A(U)[f(\bar{U}) - f(U)] > (\partial_x \phi) + c < W, W > \phi]\, dx\, dt,$$

$$J_\triangle(\Gamma_r, t_1, \tau) := \int_{t_1}^{\tau} < W, A[\gamma_r'(t) \cdot W - (f(\bar{U}) - f(U))] > \phi \Big|_{x=\gamma_r(t)} dt$$

and

$$J_\triangle(\Gamma_l, t_1, \tau) := \int_{t_1}^{\tau} < W, A[\gamma_l'(t) \cdot W - (f(\bar{U}) - f(U))] > \phi \Big|_{x=\gamma_l(t)} dt.$$

Similarly we have

<u>**Lemma 4.2 :**</u> There is $c > 0$ depending on f, U and $\bar{U}$, so that for any $\tau \in [t_1, t_2]$

$$J(\tau) \leq I_\triangle(\tau, c, \phi) + J(t_1) - J_\triangle(\Gamma_l, t_1, \tau) + J_\triangle(\Gamma_r, t_1, \tau)$$

holds.

We omit here its proof.

<u>**Lemma 4.3 :**</u> Let c be the positive number determined in the Lemma 4.1 and the Lemma 4.2. Then there exists a positive constant M_1, so that for the testfunction

$$\phi_{\bar{c}}(x, t) := e^{-\frac{\bar{c}}{M_1} t}$$

with $\bar{c} > c$ the estimates:

$$I(\tau, \bar{c}, \phi_{\bar{c}}) \leq 0;$$

$$I_\triangle(\tau,\bar{c},\phi_{\bar{c}}) \le 0;$$

$$J(\tau) \le J(t_1) - J(\Gamma_l,t_1,\tau) + J(\Gamma_r,t_1,\tau)$$

and

$$J(\tau) \le J(t_1) - J_\triangle(\Gamma_l,t_1,\tau) + J_\triangle(\Gamma_r,t_1,\tau).$$

hold.

The conclusion follows from using $\phi_{\bar{c}}$ as a testfunction in the Lemma 4.1 and the Lemma 4.2.

5. Proof of the theorems.

At first we show the Theorem 1.3. According to the conclusions in the section 2 we shall show, that the trapezoid K determined in (2.4) has the property T_0. We set the constant M in the section 2 by

$$M := \max_{(x,t)\in R\times[0,T]} \Big\{ \sum_{i=1}^{2} |\sigma_i(U)| + |\sigma_i(\bar{U})| \Big\}.$$

Lemma 5.1 : For every point $(x,0)$ with $-1 \le x < 0$ or $0 < x \le 1$ there exists $\epsilon > 0$, so that $U = \bar{U}$ in the trapezoid $K(0,x-\epsilon,\epsilon,x+\epsilon)$.

Proof: We choose ϵ so small, that there is in the trapezoid $K(0,x-\epsilon,\epsilon,x+\epsilon)$ a symmetrizer A having all properties in the Lemma 3.1. Then we use the Lemma 4.3 and obtain

$$J(\tau) \le J(0) - J(\Gamma_l,0,\tau) + J(\Gamma_r,0,\tau)$$

for any $\tau \in [0,\epsilon]$, where $\Gamma_l : x = Mt + x - \epsilon$ and $\Gamma_r : x = -Mt + x + \epsilon$ are the two side line of the trapezoid $K(0,x-\epsilon,\epsilon,x+\epsilon)$. Since $U = \bar{U}$ on the ground line $t = 0$, we have

$$J(\tau) \le -J(\Gamma_l,0,\tau) + J(\Gamma_r,0,\tau).$$

We investigate the sign of the integrand of $J(\Gamma_l,0,\tau)$ and find

$$\begin{aligned} &< W, A[\gamma_l'(t)\cdot E - (\bigtriangledown f)]W > \phi\Big|_{x=Mt+x-\epsilon} \\ &= W^\tau L^\tau \begin{pmatrix} M-\sigma_1 & 0 \\ 0 & M-\sigma_2 \end{pmatrix} LW\Big|_{x=Mt+x-\epsilon} \\ &\ge 0. \end{aligned}$$

It follows that

$$-J(\Gamma_l,0,\tau) \le 0.$$

Similarly

$$J(\Gamma_r, 0, \tau) \leq 0.$$

Thus $J(\tau) \leq 0$. This implies that $U = \bar{U}$ in the trapezoid $K(0, x - \epsilon, \epsilon, x + \epsilon)$. ∎

In order to show that K has the property T_0, we have to show furthermore that U and $\bar{U}$ are identical in a neighhood of $(0,0)$. Before we do this we consider two small trapezoides on the right and left side of $(0,0)$.

Let $\gamma_r(t) := \min\{\alpha(t), \bar{\alpha}(t)\}$, $\gamma_l(t) := \max\{\beta(t), \bar{\beta}(t)\}$,

$$G_\epsilon^1 := \left\{(x,t) \in K \,\middle|\, \begin{matrix} \gamma_r - \epsilon \leq x \leq \gamma_r \\ 0 \leq t \leq \epsilon \end{matrix} \right\}$$

and

$$G_\epsilon^2 := \left\{(x,t) \in K \,\middle|\, \begin{matrix} \gamma_l \leq x \leq \gamma_l + \epsilon \\ 0 \leq t \leq \epsilon \end{matrix} \right\}.$$

Remark: From the Lemma 2.5 it follows that $\gamma_r(t) \leq \gamma_l(t)$ for $t \in [0, c]$.

Lemma 5.2 : There exists $\epsilon > 0$, so that $U = \bar{U}$ holds in G_ϵ^1 and G_ϵ^2.

Proof: We show here only that $U = \bar{U}$ holds in the trapezoid G_ϵ^1 and choose ϵ so small, that there is in the trapezoid G_ϵ^1 a symmetrizer A having all properties in the Lemma 3.1. Then we use the Lemma 4.3 and obtain

$$J(\tau) \leq J(0) - J(\gamma_r - \epsilon, 0, \tau) + J(\gamma_r, 0, \tau)$$

for any $\tau \in [0, \epsilon]$. Similarly as in the proof of the last lemma we have $J(0) = 0$, $-J(\gamma_r - \epsilon, 0, \tau) \leq 0$. Then

$$J(\tau) \leq J(\gamma_r, 0, \tau).$$

It remains to show that

(5.2.1) $$J(\gamma_r, 0, \tau) \leq 0.$$

As in the last lemma we investigate the sign of the integrand of $J(\gamma_r, 0, \tau)$ and find

$$\begin{aligned} &< W, A[\gamma_r'(t) \cdot E - (\bigtriangledown f)]W > \phi \Big|_{x=\gamma_r(t)} \\ &= W^\tau L^\tau \begin{pmatrix} \gamma_r'(t) - \sigma_1 & 0 \\ 0 & \gamma_r'(t) - \sigma_2 \end{pmatrix} LW \Big|_{x=\gamma_r(t)}. \end{aligned}$$

The curve $\gamma_r(\cdot)$ consists of two curves of discontinuities: $\alpha(\cdot)$ and $\bar{\alpha}(\cdot)$. In the case that $(\gamma_r(t), t) = (\alpha(t), t)$ it follows from the entropy condition (1.8) that

$$\begin{aligned} 0 &> [\gamma_r'(t) - \sigma_1(U)]\Big|_{x=\gamma_r(t)-0} \\ &> [\gamma_r'(t) - \sigma_2(U)]\Big|_{x=\gamma_r(t)-0}. \end{aligned}$$

and

$$J(\gamma_r, 0, \tau) \leq 0.$$

In the case that $(\gamma_r(t), t) = (\bar{\alpha}(t), t)$ we have also according to the entropy condition (1.8)

$$\begin{aligned} 0 &> [\gamma_r'(0) - \sigma_1(\bar{U})]\Big|_{x=\gamma_r(0)-0} \\ &> [\gamma_r'(0) - \sigma_2(\bar{U})]\Big|_{x=\gamma_r(0)-0}. \end{aligned}$$

Since $\bar{U}(\bar{\alpha}(0) - 0, 0) = U(\alpha(0) - 0, 0) = U(\bar{\alpha}(0) - 0, 0)$, we have

$$\begin{aligned} 0 &> [\gamma_r'(0) - \sigma_1(U)]\Big|_{x=\gamma_r(0)-0} \\ &> [\gamma_r'(0) - \sigma_2(U)]\Big|_{x=\gamma_r(0)-0}. \end{aligned}$$

Then we can find $\epsilon > 0$, so that for $t \in (0, \epsilon)$

$$\begin{aligned} 0 &> [\gamma_r'(t) - \sigma_1(U)]\Big|_{x=\gamma_r(t)-0} \\ &> [\gamma_r'(t) - \sigma_2(U)]\Big|_{x=\gamma_r(t)-0}. \end{aligned}$$

This implies also (5.2.1).∎

Let $\gamma_1(t) := \max\{\alpha(t), \bar{\alpha}(t)\}$ and $\gamma_2(t) := \min\{\beta(t), \bar{\beta}(t)\}$.

Lemma 5.3 : For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there exists $\epsilon > 0$, so that $\gamma_1(t) < \gamma_2(t)$ hold for $t \in (0, \epsilon)$.

Proof: Let $V_m := \lim_{t\to 0} U(\alpha(t)+0,t) = \lim_{t\to 0} U(\beta(t)-0,t)$ and $\bar{V}_m := \lim_{t\to 0} \bar{U}(\bar{\alpha}(t)+0,t) = \lim_{t\to 0} \bar{U}(\bar{\beta}(t)-0,t)$. We shall show here that

$$V_m = \bar{V}_m. \tag{5.3.1}$$

Then it follows from the Rankine-Hugoniot jump condition that $\alpha'(0) = \bar{\alpha}'(0)$ and $\beta'(0) = \bar{\beta}'(0)$. This implies the conclusion because of the Lemma 2.5.

We show now (5.3.1). Let

$$U_l := U(\alpha(0)-0,0) = \bar{U}(\bar{\alpha}(0)-0,0)$$

and

$$U_r := U(\beta(0)+0,0) = \bar{U}(\bar{\beta}(0)+0,0).$$

We obtain from the Rankine-Hugoniot jump condition that V_m and $\bar{V}_m$ satisfy the equation with the unknown V:

$$\begin{cases} < L_l(V), V - U_l >= 0 \\ < L_r(V), V - U_r >= 0, \end{cases}$$

where $L_l(V)$ is the left eigenvector corresponding to the second eigenvalue of the matrix

$$\int_0^1 \nabla f(\delta(V - U_l) + U_l)\, d\delta$$

and $L_r(V)$ the left eigenvector corresponding to the first eigenvalue of the matrix

$$\int_0^1 \nabla f(\delta(V - U_r) + U_r)\, d\delta.$$

Then we consider a mapping

$$\Phi(V) := -\begin{pmatrix} L_l^T(U_l) \\ L_r^T(U_l) \end{pmatrix}^{-1} \begin{pmatrix} < L_l(V), V - U_l > \\ < L_r(V), V - U_r > \end{pmatrix} + U$$

and estimate $|d\Phi|$ by use of the estimations in the Lemma 3.1. If $|V - U_l| < \theta$, $|V - U_r| < \theta$ and

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|V\right| < C$$

then $|d\Phi| \leq konst\ \theta$ holds, where the constant depends on f and C. We choose θ so small that $|d\Phi| \leq 1/2$ holds. It follows that

$$|\Phi(V_m) - \Phi(\bar{V}_m)| < \frac{1}{2}|V_m - \bar{V}_m|.$$

Then (5.3.1) follows from $\Phi(V_m) = V_m$ and $\Phi(\bar{V}_m) = \bar{V}_m$.∎

In order to continue the proof we require the following lemma.

Lemma 5.4 : Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there exists $\epsilon > 0$, so that $J(\gamma_1 + 0, 0, t) \geq 0$ and $J(\gamma_2 - 0, 0, t) \leq 0$ hold for $t \in (0, \epsilon)$.

We shall show this lemma in the next section.

Lemma 5.5 : Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there exists $\epsilon > 0$, so that $U = \bar{U}$ in

$$G_\epsilon^3 := \left\{ (x,t) \in K \,\middle|\, \begin{array}{c} \gamma_1 \leq x \leq \gamma_2 \\ 0 \leq t \leq \epsilon \end{array} \right\}$$

This conclusion is a direct corollary of the last lemma and the Lemma 4.3.

It remains to show the identity of U and $\bar{U}$ in the regions between α and $\bar{\alpha}$ and between β and $\bar{\beta}$. In the following lemma we prove that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$.

Lemma 5.6 : Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there exists $\epsilon > 0$, so that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ hold for $t \in [0, \epsilon]$.

Proof: If $\alpha \neq \bar{\alpha}$, there is a region

$$G := \left\{ (x,t) \in K \middle| \begin{array}{c} \gamma_l(t) \le x \le \gamma_r(t); \\ t_1 \le t \le t_2 \end{array} \right\}$$

with the following properties:

(5.6.1) The two side lines γ_l und γ_r are either α or $\bar{\alpha}$.

(5.6.2) The weak solutions U und $\bar{U}$ are identical on the left side of γ_l and on the right side of γ_r, i.e., for $t \in [t_1, t_2]$

$$U(\gamma_l(t) - 0, t) = \bar{U}(\gamma_l(t) - 0, t)$$

and

$$U(\gamma_r(t) + 0, t) = \bar{U}(\gamma_r(t) + 0, t)$$

hold.

(5.6.3) The left and right side lines don't intersect for $t \in (t_1, t_2)$, i.e., for $t \in (t_1, t_2)$

$$\gamma_l(t) < \gamma_r(t)$$

holds.

(5.6.4) U and $\bar{U}$ are Lipschitz continuous in G.

(5.6.5) $\gamma_l(t_1) = \gamma_r(t_1)$.

According to the Lemma 4.3 we have in G

$$J(\tau) \leq J(t_1) - J_\triangle(\Gamma_l, t_1, \tau) + J_\triangle(\Gamma_r, t_1, \tau).$$

It follows from (5.6.5) that $J(t_1) = 0$. By using (5.6.1), (5.6.2) and the Rankine-Hugoniot jump condition we have

$$J_\triangle(\Gamma_l, t_1, \tau) = J_\triangle(\Gamma_r, t_1, \tau) = 0.$$

This implies that $U = \bar{U}$ in G. Then it follows from (5.6.2) that γ_l and γ_r are not the discontinuities of U or $\bar{U}$. This is a contradiction to (5.6.1).∎

Proof of Theorem 1.3 : By putting the Lemma 5.2, the Lemma 5.5 and the Lemma 5.6 together we show that there is a neighhood of (0,0), in which $U = \bar{U}$. This means that the trapezoid K has the property T_0. Then the theorem follows from Lemma 2.3.∎

Proof of Theorem 1.4 : We show here the identity of the discontinuities, i.e., $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. The rest is similar with the proof of the last theorem.

Proposition. *For every weak solution U in the class 2 there exists a number $\theta > 0$ depending on f und U with the following property.*

If $\bar{U}$ is a solution in the class 1 with

$$\max_{(x,t)\in R\times R^+} \left|U(x,t) - \bar{U}(x,t)\right| < \theta,$$

then the discontinuity of U is also the discontinuity of $\bar{U}$.

Let $U^+ := U(\alpha(t)+0, t)$ and $U^- := U(\alpha(t)-0, t)$. The strong entropy condition (1.9) implies that

$$\sigma_1(U^-) - \sigma_1(U^+) > \sigma > 0.$$

We assume that $\bar{U}$ is continuous across the curve $\alpha(\cdot)$.

Then $\bar{U}(\alpha(t), t) = \bar{U}(\alpha(t)+0, t) = \bar{U}(\alpha(t)-0) = U(\alpha(t)-0)$ and

$$\left|\sigma_1(U^-) - \sigma_1(\bar{U})\right|\Big|_{(x,t)=(\alpha(0),0)} = 0.$$

Since $\sigma_1(\cdot)$ is Lipschitz continuous, there is a constant depending on f und U, such that

$$\begin{aligned}\left|\sigma_1(U^+) - \sigma_1(\bar{U})\right|\Big|_{(x,t)=(\alpha(0),0)} &\leq const\left|U^+ - \bar{U}\right|\Big|_{(x,t)=(\alpha(0),0)} \\ &= \left|U^+ - U^-\right|\Big|_{(x,t)=(\alpha(0),0)} \\ &\leq const\ \theta\end{aligned}$$

holds. Then we have

$$
\begin{aligned}
0 < \sigma < [\sigma_1(U^-) - \sigma_1(U^+)]\Big|_{(x,t)=(\alpha(0)-0,0)} \\
= \big|\sigma_1(U^+) - \sigma_1(\bar{U})\big|\Big|_{(x,t)=(\alpha(0)-0,0)} \\
\leq const\ \theta.
\end{aligned}
$$

Then there exists $\epsilon > 0$, so that for $t \in [0, \epsilon]$

$$\sigma < const\ \theta$$

holds. We choose $\theta < \frac{\sigma}{const}$ and obtain a contradiction. Thus the proposition is shown.

We continue showing that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. According to this proposition $\bar{\alpha}$ is identical either with α or with β.

If $\bar{\alpha}$ and β are identical, the curve $\bar{\beta}$ is on the right side of β because of the Lemma 2.5. Applying the Lemma 2.5 again we obtain that $\bar{\beta}$ is on the right side of α. This is a contradiction to the proposition, that $\bar{\beta}$ is identical either with α or with β. This implies that α and $\bar{\alpha}$ must be identical.

Similarly one show that β and $\bar{\beta}$ are identical.∎

6. Proof of the Lemma 5.4.

We show here that $J(\gamma_1 + 0, 0, t) \geq 0$ holds for $t \in (0, \epsilon)$ and one can obtain the other estimate in the Lemma 5.4: $J(\gamma_2 - 0, 0, t) \leq 0$, in the same way. We investigate the sign of the integrand of $J(\gamma_1 + 0, 0, t)$:

$$< W, A(U)[\gamma_1' E - \nabla f(U)]W > \Big|_{x=\gamma_1+0}.$$

The curve $\gamma_1(\cdot)$ consists of two discontinuities: $\alpha(\cdot)$ and $\bar{\alpha}(\cdot)$. From now on we assume that $\gamma_1(t) = \bar{\alpha}(t)$, $\alpha(t) < \bar{\alpha}(t)$ hold for $t \in [t_1, t_2]$ and $\alpha(t_1) = \bar{\alpha}(t_1)$. Then we shall show that for $t \in [t_1, t_2]$

$$\int_{t_1}^{t} < W, A(U)[\gamma_1' E - \nabla f(U)]W > \Big|_{x=\bar{\alpha}+0} \geq 0 \tag{6.1}$$

holds, if t_1 and t_2 are sufficiently small. In the other case where $\gamma_1(t) = \alpha(t)$ we can obtain

$$\int_{t_1}^{t} < W, A(U)[\gamma_1' E - \nabla f(U)]W > \Big|_{x=\alpha+0} \geq 0$$

in the same way.

We introduce the following notations:

$$U_{\bar\alpha\pm} := U(\bar\alpha(t) \pm 0, t)$$

$$U_{\alpha\pm} := U(\alpha(t) \pm 0, t)$$

$$\bar U_{\bar\alpha\pm} := \bar U(\bar\alpha(t) \pm 0, t)$$

$$\bar U_{\alpha\pm} := \bar U(\alpha(t) \pm 0, t)$$

$$W_{\bar\alpha\pm} := \bar U_{\bar\alpha_\pm} - U_{\bar\alpha_\pm}$$

and

$$W_{\alpha\pm} := \bar U_{\alpha_\pm} - U_{\alpha_\pm},$$

Obviously $U_{\bar\alpha} := U_{\bar\alpha+} = U_{\bar\alpha-}$ and $\bar U_\alpha := \bar U_{\alpha+} = \bar U_{\alpha-}$ hold. According to the Lemma 5.2 there exists $\epsilon > 0$, so that $U_{\alpha-} = \bar U_{\alpha-} = \bar U_{\alpha+}$ and $W_{\alpha\_} = 0$ hold, if $0 \le t_1 \le t \le t_2 \le \epsilon$.

We introduce a new function $\psi : R^2 \times R^2 \to R$

$$\psi(V_1, V_2) :=< V_1 - V_2, A(U_{\bar\alpha+})[\bar\alpha' \cdot E - \bigtriangledown f(U_{\bar\alpha+})](V_1 - V_2) > .$$

Instead of the proving (6.1) we prove that for $t \in [t_1, t_2]$

$$\int_{t_1}^{t} \Psi(U_{\bar\alpha+}, \bar U_{\bar\alpha+})\, d\tau \ge 0 \tag{6.2}$$

holds, if t_1 and t_2 are sufficiently small.

Let $V_0 \in R^2$ and $S(V_0)$ be the *shock set* [Liu]

$$S(V_0) := \left\{ V \in R^2 \Big| \frac{f_1(V) - f_1(V_0)}{v_1 - v_{01}} = \frac{f_2(V) - f_2(V_0)}{v_2 - v_{02}} \right\},$$

where $V = (v_1, v_2)^\tau$ and $V_0 = (v_{01}, v_{02})^\tau$. Let

$$S(V, V_0) := \frac{f_1(V) - f_1(V_0)}{v_1 - v_{01}} = \frac{f_2(V) - f_2(V_0)}{v_2 - v_{02}}.$$

It is not difficult to show that $S(V, V_0)$ is either the first or the second eigenvalue of the matrix

$$\int_0^1 \bigtriangledown f(\delta(V - V_0) + V_0)\, d\delta,$$

if $V \ne V_0$. Let

$$S_i(V_0) := \left\{ V \in S(V_0) \Big| S(V, V_0) \text{ is the i} - \text{th eigenvalue.} \right\}$$

with $i = 1,2$. Then $S(V_0) = S_1(V_0) \cup S_2(V_0)$. By using of the standard method [Smo] we can show that $S_1(V_0)$ and $S_2(V_0)$ are two oneparameter families and for any $t \in [t_1, t_2]$ there is $\tilde{U} \in S_1(\bar{U}_{\bar{\alpha}-})$, so that

$$\alpha' - \sigma_1(U_{\alpha-}) = S(\tilde{U}, \bar{U}_{\bar{\alpha}-}) - \sigma_1(\bar{U}_{\bar{\alpha}-})$$

holds, if $|U^+ - U^-|$ and $|\bar{U}^+ - \bar{U}^-|$ are sufficiently small.

In the following lemmas we show the estimate (6.2) with the help of the intermediate state $\tilde{U}$. Without special explaination the positive constants in this section depend only on C, θ and f.

__Lemma 6.1.__ Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C;$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta;$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there is a constant, so that

$$\psi(\tilde{U}, \bar{U}_{\bar{\alpha}+}) \geq const|\tilde{U} - \bar{U}_{\bar{\alpha}+}|^2.$$

Proof: Let $R := \tilde{U} - U_{\bar{\alpha}+}$ and $L = (L_1, L_2)^\tau$ be the left eigenvectors of $\bigtriangledown f(U_{\bar{\alpha}+})$ with $|L_i| = 1, i = 1,2$.

From the Lemma 3.1 follows that

$$\begin{aligned}
&\psi(\tilde{U}, U_{\bar{\alpha}+}) \\
&=< R, [L(U_{\bar{\alpha}+})]^\tau \begin{pmatrix} \bar{\alpha}' - \sigma_1(U_{\bar{\alpha}+}) & 0 \\ 0 & \bar{\alpha}' - \sigma_2(U_{\bar{\alpha}+}) \end{pmatrix} L(U_{\bar{\alpha}+})R >, \\
&= |R|^2 \sum_{i=1}^{2} [\bar{\alpha}' - \sigma_i(U_{\bar{\alpha}+})](\cos\rho_i)^2,
\end{aligned}$$

where ρ_i with $i = 1,2$ is the angle between L_i and R.

In order to show the conclusion we show that

$$\sum_{i=1}^{2} [\bar{\alpha}' - \sigma_i(U_{\bar{\alpha}+})](\cos\rho_i)^2 > const > 0 \tag{6.1.1}$$

holds.

Since the system is genuinely nonlinear,

$$\bar{\alpha}' - \sigma_1(U_{\bar{\alpha}+})\Big|_{t=0} > konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0}.$$

Then it follows from the entropy condition that

$$\bar{\alpha}' - \sigma_1(U_{\bar{\alpha}+})\Big|_{t=0} > konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0} > 0$$

holds. The equality (5.3.1) means that $U_{\bar{\alpha}+}\Big|_{t=0} = \bar{U}_{\bar{\alpha}+}\Big|_{t=0}$ holds. It follows that

$$\bar{\alpha}' - \sigma_1(\bar{U}_{\bar{\alpha}+})\Big|_{t=0} > konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0} > 0.$$

A short estimate shows that there is a constant, so that

$$|\cos\rho_1|\Big|_{t=0} > const > 0.$$

Then we have

$$[\bar{\alpha}' - \sigma_1(U_{\bar{\alpha}+})](\cos\rho_1)^2\Big|_{t=0} \geq konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0} > 0.$$

By using the Rankine-Hugoniot jump condition we obtain

$$|\cos\rho_2|\Big|_{t=0} > konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0}.$$

Then we have

$$\sum_{i=1}^{2}[\bar{\alpha}' - \sigma_i(U_{\bar{\alpha}+})](\cos\rho_i)^2\Big|_{t=0}$$
$$\geq konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0} - konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|^2\Big|_{t=0}.$$

We choose θ so small, that

$$\sum_{i=1}^{2}[\bar{\alpha}' - \sigma_i(U_{\bar{\alpha}+})](\cos\rho_i)^2\Big|_{t=0} > konst|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}|\Big|_{t=0}$$

holds, if $|\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}| \leq \theta$.

Thus there is $\epsilon > 0$, so that for $0 \leq t_1 \leq t_2 \leq \epsilon$ and $t \in [t_1, t_2]$ the estimate (6.1.1) holds, since the right is a continuous function of t.∎

<u>Lemma 6.2.</u> Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \big|U(x,t)\big| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \big|\bar{U}(x,t)\big| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there is a constant, so that

$$\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) \geq const|\tilde{U} - \bar{U}_{\bar{\alpha}+}|^2 - const\triangle^2,$$

where $\triangle(\cdot) := \bar{\alpha}(\cdot) - \alpha(\cdot)$.

Proof: We consider $\psi(\tilde{U}, \bar{U}_{\bar{\alpha}+})$ as a perturbation of $\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+})$ and have the estimate

$$\begin{aligned}&|\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) - \psi(\tilde{U}, \bar{U}_{\bar{\alpha}+})|\\ &\leq const|\tilde{U} - U_{\bar{\alpha}+}||\tilde{U} - \bar{U}_{\bar{\alpha}+}| + const|\tilde{U} - U_{\bar{\alpha}+}|^2\end{aligned}$$

It holds that

$$\begin{aligned}|\tilde{U} - U_{\alpha+}| \leq& |\bar{U}_{\bar{\alpha}-} - U_{\alpha-}|\\ =&|\bar{U}_{\bar{\alpha}-} - \bar{U}_{\alpha-}|\\ \leq& konst\triangle\end{aligned}$$

and

$$\begin{aligned}&|\tilde{U} - U_{\bar{\alpha}+}|^2\\ &\leq 2(|\tilde{U} - U_{\alpha+}|^2 + |U_{\alpha+} - U_{\bar{\alpha}+}|^2)\\ &\leq konst\triangle^2.\end{aligned}$$

Then we have by use of the last lemma that

$$\begin{aligned}&\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+})\\ &= \psi(\tilde{U}, \bar{U}_{\bar{\alpha}+}) + [\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) - \psi(\tilde{U}, \bar{U}_{\bar{\alpha}+})]\\ &\geq const|\tilde{U} - \bar{U}_{\bar{\alpha}+}|^2 - const\triangle|\tilde{U} - \bar{U}_{\bar{\alpha}+}| - const\triangle^2.\end{aligned} \tag{6.2.1}$$

For any positive number μ it holds that

$$\begin{aligned}&\triangle|\tilde{U} - \bar{U}_{\bar{\alpha}+}|\\ &\leq \mu^{-1}\triangle^2 + \mu|\tilde{U} - \bar{U}_{\bar{\alpha}+}|^2.\end{aligned}$$

We insert this estimation in (6.2.1). Obviously we can find a number μ, so that the conclusion holds.∎

As a corollary of this lemma we have

Lemma 6.3. Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there is a constant, so that

$$\psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) \geq const|U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2 - const\triangle^2.$$

Lemma 6.4. Suppose that the system is genuinely nonlinear. For every positive number C there exists a number $\theta > 0$ depending only on f and C with the following property.

If U und $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|U(x,t)\right| < C,$$

$$\operatorname*{esssup}_{(x,t)\in R\times R^+} \left|\bar{U}(x,t)\right| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there is a constant, so that for any $t \in [t_1, t_2]$

$$\int_{t_1}^{t} \triangle^2(\tau)\, d\tau \leq const \cdot (t - t_1)^2 \int_{t_1}^{t} |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2\, d\tau.$$

Proof: Taylor's expansion leads to

$$\triangle(t) \leq \int_{t_1}^{t} |\alpha'(\tau) - \bar{\alpha}'(\tau)|\, d\tau.$$

Since α' is the eigenvalue of the matrix

$$\int_0^1 \nabla f(\delta(U_{\bar{\alpha}+} - U_{\bar{\alpha}-}) + U_{\bar{\alpha}-})\, d\delta$$

and $\bar{\alpha}'$ the eigenvalue of the matrix

$$\int_0^1 \nabla f(\delta(\bar{U}_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}-}) + \bar{U}_{\bar{\alpha}-})\, d\delta,$$

it isn't difficult to show that there is a constant, so that

$$\begin{aligned} |\alpha'(\tau) - \bar{\alpha}'(\tau)| &\leq const(|U_{\alpha+} - \bar{U}_{\bar{\alpha}+}| + |U_{\alpha-} - \bar{U}_{\bar{\alpha}-}|) \\ &\leq const(|U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}| + \triangle(t)). \end{aligned}$$

Then we have

$$\triangle(t) \leq const(\int_{t_1}^{t} \triangle(\tau)\, d\tau + \int_{t_1}^{t} |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|\, d\tau).$$

By use of Schwartz's inequality we obtain that

$$\triangle^2(t) \leq const \cdot (t - t_1) \int_{t_1}^{t} [\triangle^2(\tau) + |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2]\, d\tau.$$

Then the conclusion follows, if t_1 and t_2 are sufficiently small and $t \in [t_1, t_2]$. ∎

Proof of the estimation (6.2) :

Putting the last two lemmas together we obtain that

$$\begin{aligned} &\int_{t_1}^{t} \psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) \\ &\geq \left[const - const \cdot (t - t_1)^2\right] \int_{t_1}^{t} |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2]\, d\tau. \end{aligned}$$

If t_1 and t_2 are sufficiently small and $t \in [t_1, t_2]$, we have the estimation (6.2).∎

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Research Reports

No.	Authors	Title
93-03	R. Jeltsch, X. Wang	Uniqueness of Piecewise Lipschitz Continuous Solutions of the Cauchy-Problem for 2×2 Conservation Laws
93-02	W.-A. Yong	Difference approximations to the global $W^{1,\infty}$ solutions of the isentropic gas equations
93-01	Ch. Lubich, K. Nipp, D. Stoffer	Runge-Kutta solutions of stiff differential equations near stationary points
92-15	N. Botta	Is the transonic flow around a cylinder always periodic?
92-14	K. Nipp, D. Stoffer	Invariant manifolds of numerical integration schemes applied to stiff systems of singular perturbation type - Part I: RK-methods
92-13	K. Nipp	Smooth attractive invariant manifolds of singularly perturbed ODE's
92-12	D. Mao	A Shock Tracking Technique Based on Conservation in One Space Dimension
92-11	K. Nipp, D. Stoffer	Attractive invariant manifolds for maps: Existence, smoothness and continuous dependence on the map
92-10	M. Fey, R. Jeltsch	A Simple Multidimensional Euler Scheme
92-09	M. Fey, R. Jeltsch	A New Multidimensional Euler Scheme
92-08	M. Fey, R. Jeltsch, P. Karmann	Numerical solution of a nozzle flow
92-07	M. Fey, R. Jeltsch, P. Karmann	Special aspects of reacting inviscid blunt body flow
92-06	M. Fey, R. Jeltsch, S. Müller	The influence of a source term, an example: chemically reacting hypersonic flow
92-05	N. Botta, J. Sesterhenn	Deficiencies in the numerical computation of nozzle flow
92-04	Ch. Lubich	Integration of stiff mechanical systems by Runge-Kutta methods
92-03	M. Fey, R. Jeltsch, S. Müller	Stagnation point analysis
92-02	C. W. Schulz-Rinne, J. P. Collins, H. M. Glaz	Numerical Solution of the Riemann Problem for Two-Dimensional Gas Dynamics
92-01	R. J. LeVeque, K. M. Shyue	Shock Tracking Based on High Resolution Wave Propagation Methods
91-10	M. Fey, R. Jeltsch	Influence of numerical diffusion in high temperature flow
91-09	R. J. LeVeque, R. Walder	Grid Alignment Effects and Rotated Methods for Computing Complex Flows in Astrophysics