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Concentration-cancellation and Hardy spaces

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Abstract

Let $v^\in$ a sequence of DiPerna-Majda approximate solutions to the 2-D incompressible Euler equations. We prove that if the vorticity sequence is weakly compact in the Hardy space $H^1(R^2)$ then a subsequence of $v^\in$ converges strongly in $L^2(R^2)$ to a solution of the Euler equations. This phenomenon is directly related to the cancellation effects exhibited by "phantom vortices".

Keywords: Riesz transform, equibounded, Dunford-Pettis theorem

Subject Classification: 35Q10 (76D05)

In their fundamental paper [4] DiPerna and Majda study the convergence of approximate solutions v^ϵ of the 2-D inviscid Euler equations as the regularization parameter ϵ goes to zero. They give several examples of sequences of compactly supported approximate solutions v^ϵ (as defined in Definition 1.1, [4]) whose vorticity ω^ϵ is bounded in L^1 which fail to be compact in L^2 so that in the limit concentration phenomena occur. Moreover in Th. 1.3 of [4] a criterion which rules concentrations out is proposed: it is shown that a uniform bound on a logarithmic Morrey norm of ω^ϵ yields strong L^2-convergence of the velocity field.In this note another criterion for compactness is introduced: we show that strong L^2-compactness of v^ϵ follows from weak compactness of ω^ϵ in the Hardy space $H^1(R^2)$. Since $H^1(R^2)$ is not rearrangement invariant the fine structure of the vorticity plays a crucial role in getting strong
L^2-convergence. We recall that by Dunford-Pettis theorem (see [5], VIII, Th.1.3) a necessary and sufficient condition for a subset Λ of $L^1(R^2)$ to be weakly pre-compact in $L^1(R^2)$ is that there exist a positive function $G(s) : R^+ \to R^+$ such that

$$\lim_{s\to+\infty} \frac{G(s)}{s} = +\infty \tag{1}$$

and

$$\sup_{f\in\Lambda} \int_{R^2} G(|f|)\, dx < +\infty \tag{2}$$

Let R_i $i = 1, 2$, denote the Riesz transforms:

$$R_j f(x) = \int_{R^2} \frac{x_j - y_j}{|x-y|^3}\, f(y)\, dx$$

We formulate our result as follows.**Theorem 1.** *Let v^ϵ be a sequence of approximate solutions such that for every $t \geq 0$*

$$\|\omega^\epsilon(.,t)\|_{H^1} < C \qquad 0 < \epsilon \leq \epsilon_0 \tag{3}$$

and ω^ϵ satisfies weak uniform control at infinity (cfr.[4],(3.5)). Moreover let there be a function $G(s) : R^+ \to R^+$ such that (1) and (2) hold for $\Lambda = \{\omega^\epsilon\}, \{R_i\omega^\epsilon\},\ i = 1, 2$.Then there is a subsequence of v^ϵ which converges strongly in L^2 to a weak classical solution v of the Euler equations. Moreover $v \in W^{1,1}(R^2)$. We recall (see [6]) that a function f belongs to the Hardy space $H^1(R^2)$ iff there is a sequence of numbers λ_j satisfying $\sum_1^\infty |\lambda_j| < \infty$ and a series of functions (atoms) a_j such that

$$f = \sum_1^\infty \lambda_j a_j \tag{4}$$

where the $a_j's$ have the following properties a) a_j is supported on a ball B_j and $\|a_j\|_\infty < \frac{1}{|B_j|}$b) $\int_{R^2} a_j(x)\, dx = 0$The H^1-norm of f can be defined as the infimum of the expressions $\sum_1^\infty |\lambda_j|$ on all possible representations of f as in (4).If condition b) were dropped the resulting space would be $L^1(R^2)$. It is the subtle cancellation

effect due to b) (cfr."phantom vortices"in [4], 1.A) together with (2) which yields strong L^2-compactness. *Proof of Theorem 1.* To prove the theorem we introduce the stream
function ψ^ϵ such that

$$\triangle\psi^\epsilon = \omega^\epsilon \tag{5}$$

and we proceed as in the proof of Th. 3.1 in [4]. It is known that for every f in $BMO(R^2)$ there are g_i in $L^\infty(R^2)$, $i = 0, 1, 2$, such that

$$f = g_0 + \sum_{i=1,2} R_i g_i$$

Hence

$$\int_{R^2} f\omega^\epsilon \, dx = \int_{R^2} \omega^\epsilon (g_0 + \sum_{i=1,2} R_i g_i) dx = \int_{R^2} \omega^\epsilon g_0 - \sum_{i=1,2} g_i R_i \omega^\epsilon \, dx$$

By our assumption (2) the sequence $\{\omega^\epsilon\}$ and its Riesz transforms admit a weakly convergent subsequence in $L^1(R^2)$. Therefore there is a subsequence such that

$$\omega^\epsilon \rightharpoonup \omega \quad \textit{weakly in } H^1(R^2) \tag{6}$$

The statement of Th.1 is guaranteed by showing that for all $\rho \in C_o^\infty(R^2)$

$$\lim_{\epsilon\to 0} \int_{R^2} \rho |v^\epsilon|^2 \, dx = \int_{R^2} \rho |v|^2 dx \tag{7}$$

Indeed after integrating by parts (7) is seen to hold iff (see [4], (3.7)-(3.10))

$$\lim_{\epsilon\to 0} \int_{R^2} \rho \psi^\epsilon \omega^\epsilon \, dx = \int_{R^2} \rho \psi d\omega \tag{8}$$

where ψ is the stream function corresponding to ω in (5). We recall that

$$\frac{\partial^2}{\partial x_j \partial x_k} f = -R_j R_k \triangle f$$

Hence

$$\frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon = -R_j R_k \omega^\epsilon$$

Since the Riesz transform maps $H^1(R^2)$ continuously into itself we get that

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon \right\|_{L^1} \le \left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon \right\|_{H^1} \le C \|\omega^\epsilon\|_{H^1} \tag{9}$$

and ψ^ϵ stays bounded in $W^{2,1}(R^2)$. We recall that for any bounded domain Ω in R^2 by the Gagliardo-Sobolev imbedding theorem $W^{2,1}(R^2)$ is continuously imbedded in $C(\overline{\Omega})$ .Therefore

$$\|\psi^\epsilon\|_{C(\overline{\Omega})} \le C \|\omega^\epsilon\|_{H^1} \tag{10}$$

Moreover (see [1], Lemma 5.8) if $u \in W^{2,1}(R^2)$ for any $P_o \in R^2$ we have that for $\delta > 0$ if $|\triangle P| < \frac{\delta}{2}$

$$|u(P_o + \triangle P) - u(P_o)| \leq C\Big(\frac{1}{\delta^2}\|u(P + \triangle P) - u(P)\|_{L^1(B_\delta(P_o))} \tag{11}$$

$$+\frac{1}{\delta}\sum_i\Big\|\frac{\partial}{\partial x_i}u(P + \triangle P) - \frac{\partial}{\partial x_i}u(P)\Big\|_{L^1(B_\delta(P_o))}$$

$$+\sum_{i,j}\Big\|\frac{\partial^2}{\partial x_j \partial x_i}u(P + \triangle P) - \frac{\partial^2}{\partial x_j \partial x_i}u(P)\Big\|_{L^1(B_\delta(P_o))}\Big)$$

By (weak) continuity of the Riesz transforms from $H^1(R^2)$ into itself there is a subsequence of $\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon$, that converges weakly in $H^1(R^2)$ to a $\phi_{i,j} \in H^1(R^2)$. On the other hand weak convergence in $H^1(R^2)$ implies weak convergence in $L^1(R^2)$ (indeed $L^\infty \subset BMO$) so that we have

$$\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon \rightharpoonup \phi_{i,j} \quad \textit{weakly in } L^1(R^2)$$

By the full version of Dunford-Pettis theorem for every $\kappa > 0$ there is a $\delta > 0$ such that for any $P \in \Omega$

$$\Big\|\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon\Big\|_{L^1(B_\delta(P))} < \kappa$$

uniformly in ϵ.We observe that if $|\triangle P| < \delta$

$$\Big\|\frac{\partial^2}{\partial x_j \partial x_k}[\psi^\epsilon(P + \triangle P) - \psi^\epsilon(P)]\Big\|_{L^1(B_\delta(P_o))} < C\Big\|\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon\Big\|_{L^1(B_{2\delta}(P_o))}$$

Therefore for every P_o given $\kappa_o > 0$ we can find a $\delta_o > 0$ such that if $|\triangle P| < \frac{\delta_o}{2}$

$$\sum_{i,j}\Big\|\frac{\partial^2}{\partial x_j \partial x_k}[\psi^\epsilon(P + \triangle P) - \psi^\epsilon(P)]\Big\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}$$

uniformly in ϵ. Moreover since $W^{1,1}(\Omega)$ is compactly imbedded in L^p for any $p < 2$ both $\{\psi^\epsilon\}$ and $\{\frac{\partial}{\partial x_i}\psi^\epsilon\}$ are compact in L^1. Hence by Kondratchev compactness criterion (see [1]) there is a $\delta_1 > 0$ such that if $|\triangle P| < \delta_1$

$$\frac{1}{{\delta_o}^2}\|u(P + \triangle P) - u(P)\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}$$

$$\frac{1}{\delta_o}\sum_i\Big\|\frac{\partial}{\partial x_i}u(P + \triangle P) - \frac{\partial}{\partial x_i}u(P)\Big\|_{L^1(B_\delta(P_o))} < \frac{\kappa_o}{3}$$

and by (11)

$$|\psi^\epsilon(P_o + \triangle P) - u(P_o)| < \kappa_o \tag{12}$$

uniformly in ϵ. The sequence ψ^ϵ is equibounded by (10) and equicontinuous by (12) and by Ascoli theorem we can extract a subsequence such that

$$\psi^\epsilon \to \psi \quad \textit{strongly in } C(\Omega) \tag{13}$$

By (6) we have that $\omega^\epsilon \rightharpoonup \omega$ weakly in $M(\Omega)$ so that (8) holds and the same argument as in Th. 1.3 of [4] yields the statement of the theorem.*Remark.* The first example in $(1, \S A)$ in [4] (phantom vortices) shows a sequence of vorticities which stays bounded in $H^1(R^2)$ whose velocity field fails to converge strongly in L^2; in the second example one has strong $L^1(R^2)$ convergence of the vorticity but the sequence does not lie in $H^1(R^2)$ and again concentrations occur. By looking at the proof of Delort's recent deep result ([3]), weak convergence of ω^ϵ in $L^1(R^2)$ is sufficient to pass to the limit in the quadratic terms of the Euler equations, due to their special structure. It is interesting that every bounded sequence in $H^1(R^2)$ admits a weakly(*) convergent subsequence whose limit stays in $H^1(R^2)$ (see [2], Lemma (4.2)). However, since $(VMO)^*$=$H^1(R^2)$ and $L^\infty \not\subset VMO$, this does not yield weak L^1-convergence.It is worth observing that condition (2) for ω^ϵ is rearrangement invariant and so in the time dependent case it is conserved by the particle trajectory map. On the other hand, as for the bounds (3.4) of Th. 3.1 in [4], it is not clear what happens to the H^1-norm as time goes by, since $H^1(R^2)$ is not rearrangement invariant.

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