Finding Clusters of Primes, II

Progress Report 2006 - 2011

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1 Review of Earlier Results.

This project was initiated in the year 2000 with the development of the software system PMP, "Poor Man's Parallelizer" by Peter Leikauf. PMP is a simple but robust system for parallelizing large computational efforts for workstations connected to the internet and/or for cluster computers. It is particularly well suited for easily parallelizable tasks with low data traffic. For details see the link `Projects/pmp.pdf` on the website [10].

In the current application of our parallelization software we are using an algorithm involving sieving techniques for locating and counting clusters of prime numbers. Whereas the distribution of primes seems to be fairly regular, the distribution of twin primes and longer clusters is largely unknown and is characterized by large-scale anomalies. Collecting experimental data on these anomalies is one of the reasons for the interest in clusters of primes. Research in the theory of prime numbers has again become fashionable with the invention of the RSA coding scheme [9] in 1978, and a world-wide competition for prime number records is still going on [7].

Another challenge of finding clusters of prime numbers is the unproven prime k-tuple hypothesis, which is concerned with patterns of natural numbers that occur repeatedly with all elements being prime. The hypothesis states that any pattern that is not forbidden by simple divisibility considerations occurs infinitely often in the sequence of primes. Collecting data in support of this hypothesis is our current goal.

The discovery of a dense cluster of 18 primes among 71 consecutive integers in the range of $3 \cdot 10^{24}$ on November 13, 2000 was the first success of the project. This computation is about 50 times harder than finding the three 23-digit clusters of 17 primes among 67 consecutive integers, discovered by Tony Forbes [3], and independently by J. Waldvogel [10], both in 1998. Our discovery was immediately announced in the number theory press [6], and also got coverage in the web journal of ETH (ETH Life, December 6, 2000, [2]). For previous reports and more details see the links

```
Projects/clprimes01.pdf          Projects/cl18.pdf
Projects/clprimes03.pdf          Projects/paricode.gp
```

on the website [10]. Dense clusters of primes receive particular attention on the website [4], continuously actualized by Tony Forbes. The successes of our implementation bear the danger of monopolizing this site and taking away all the fun!

2 Some Particular Patterns

In this section we pick a few “exotic” admissible patterns with no obvious prime instances. If the Hardy-Littlewood (HL) estimates indicate any chance of reoccurrences in the range accessible with the current hardware and software, we tried to locate a few prime instances of the patterns.

We will consider four patterns of 16 or 18 elements, all of them being tight repetitions of well-known short patterns, as well as a remarkable “quadratic” pattern of 21 primes. In all cases the search for prime instances was successful, thus accumulating more experimental data on the prime k-tuple hypothesis. To begin, we briefly state this famous conjecture in its strong form put forth by Hardy and Littlewood [5].

Let $\mathbf{c} = [c_1, c_2, \ldots, c_k\,]$ be a monotonically increasing sequence of $k \in \mathbb{N}$ integers, exclusively odd or exclusively even, called a *pattern*. The counting function $\pi_{\mathbf{c}}(x)$ of the pattern $\mathbf{c}$ is defined as the number of shifts s, $0 \le s \le x$, such that every element of $\mathbf{c}_s := [s + c_1, s + c_2, \ldots, s + c_k\,]$ is a prime (called a *prime instance* of $\mathbf{c}$). The conjecture states the asymptotic relationship

$$\pi_{\mathbf{c}}(x) \sim h_{\mathbf{c}} \cdot R_k(x) \quad \text{as} \quad x \to \infty\,.$$

Instead of the conventional logarithmic integrals we prefer to use the generalized Riemann functions

$$R_k(x) := \sum_{j=0}^{\infty} \frac{(\log x)^j}{(k-1+j)!\, j\, \zeta(1+j)} \,.$$

The number $h_{\mathbf{c}}$ is the Hardy-Littlewood constant of the pattern $\mathbf{c}$,

$$h_{\mathbf{c}} := 2^{k-1} \cdot \prod_q \frac{1 - q^{-1}\, r_{\mathbf{c}}(q)}{(1-q^{-1})^k} \,,$$

where the product is taken over all odd primes q, and $r_{\mathbf{c}}(q)$ is the number of distinct residue classes mod q in the pattern $\mathbf{c}$. Note that for $k = 1$ we have $h_{[0]} = 1$, and non-amissible patterns $\mathbf{c}$ yield $h_{\mathbf{c}} = 0$. Despite the slow convergence of the above infinite product Hardy-Littlewood constants may be efficiently computed to arbitrary precision by means of the prime zeta function, see, e.g. [8].

2.1 Nine Close Prime Twins

Except for the singular situation of the overlapping twins (3,5) and (5,7), two twin primes may only occur at a distance of $6\,n-2,\ n = 1, 2, \ldots,$ as is seen by a simple consideration of residue classes mod 3. The closest admissible arrangement of 9 prime twins spans an interval of length 104 and comes in 4 types represented by the patterns

```
c1=[-31,-29, -19,-17, -13,-11, 11, 13, 17,19, 29,31, 41,43, 59,61, 71,73]
c2=[-43,-41, -31,-29, -19,-17,-13,-11, -1, 1, 11,13, 17,19, 41,43, 59,61]
```

and their mirror images $\mathbf{c_1}'$ and $\mathbf{c_2}'$. The Hardy-Littlewood constants are found to be

$$h_1 = 23041978.71272\,, \quad h_2 = 18433582.97017 = \frac{4}{5} \cdot h_1\,;$$

the corresponding estimated frequencies $\mathrm{HL}_1(x)$, $\mathrm{HL}_2(x)$ for the patterns $\mathbf{c_1}, \mathbf{c_2}$, respectively, are

x	10^{23}	$5 \cdot 10^{23}$	10^{24}	$2 \cdot 10^{24}$
$\mathrm{HL}_1(x)$	0.33	0.94	1.50	2.38
$\mathrm{HL}_2(x)$	0.26	0.75	1.20	1.90

We extended the search up to 10^{24}, except for the case $\mathbf{c_2}'$, where the range had to be doubled. The initial primes of the existing 9 instances are given in the table below; the total HL count is 6.1. We also indicate the differences in the patterns; the occurrence of an additional prime between two twins is indicated by a plus sign +.

```
        Initial Prime                Differences in the Pattern
c1                                                             (to 1e24)
1    3958297190183074938251 9  2   10 2 4   2 22 2 4 2 10 2 10 2 16 2 10 2
```

```
        Initial Prime                Differences in the Pattern
c1                                                             (to 1e24)
1    39582971901830749382519  2   10 2 4   2 22 2 4 2 10 2 10 2 16 2 10 2
2    85002283332977673203069  2  6+4 2 4   2 22 2 4 2 10 2 10 2 16 2 10 2
3   111871787983521806791079  2   10 2 4   2 22 2 4 2 10 2 10 2 16 2 10 2
4   237848512839161942948759  2   10 2 4   2 22 2 4 2 10 2 10 2 16 2 10 2
c1'                                                            (to 1e24)
1    35902987875008630158997  2   10 2 16 2 10 2 10 2 4 2 22 2  4 2 10 2
2   681954129502801901959427  2   10 2 16 2 10 2 10 2 4 2 22 2  4 2 10 2
c2                                                             (to 1e24)
1   670962238726376003928317  2   10 2 10 2  4 2 10 2 10 2 4 2 22 2 10+6 2
2   785878473089354651160797  2   10 2 10 2  4 2 10 2 10 2 4 2 22 2 16 2
c2'                                                            (to 2e24)
1 1063660630652819772482009  2 16 2 18+4 2 4 2 10 2 10 2 4 2 10 2 10 2
```

2.2 Four Close Quadruplets

A striking short pattern is the arrangement of two twins in the same decade, such as $[11, 13, 17, 19]$, referred to as a *prime quadruplet*. Its HL constant is $h_4 = 4.1511808632$; in a large interval Δ near a much larger x we expect to find about $h_4 \cdot \Delta/(\log x)^4$ quadruplets.

Prime quadruplets may accumulate as closely as in the pattern $\mathbf{c_8} = [-19, -17, -13, -11, 11, 13, 17, 19]$, since $\mathbf{c_8}$ is admissible. With its HL constant $h_8 = 288.1001283176$, $\mathbf{c_8}$ is expected to reoccur in terms of positive primes in the range of 4 000 000. In fact, the first pair of quadruplets occurs somewhat prematurely at 1 006 301, and up to $x = 2 \cdot 10^8$ there are 10 instances, whereas the HL estimate is 6.

Two pairs $\mathbf{c_8}$ of quadruplets in their closest arrangement are separated by a gap of length 382, i.e. the pattern

$$\mathbf{c_{16}} = [-229, -227, -223, -221, -199, -197, -193, -191, 191, 193, 197, 199, 221, 223, 227, 229]$$

is admissible. In this case, however, it is not possible to represent the pattern by a sequence of small primes: $221 = 13 \cdot 17$! The HL constant is found to be

$h_{16} = 9350784.9154303126$. A search up to $x = \frac{4}{3} \cdot 10^{19}$ turned up two fairly close occurrences near $1.1 \cdot 10^{19}$ and $1.2 \cdot 10^{19}$, whereas the Hardy-Littlewood count yields $\mathrm{HL}(x) = h_{16} \cdot R_{16}(x) = 1.003$. Below we show the initial primes of these instances; we again indicate the sequences of differences in the pattern; Q stands for [2,4,2], and a plus sign + indicates additional primes in the large gaps of the pattern.

Initial Prime		Differences in the Pattern	
11281963036964038421	Q 22 Q	42+42+196+12+2+70+18	Q 22 Q
12114914563464663491	Q 22 Q	22+32+28+8+34+14+54+162+28	Q 22 Q

Four quadruplets in their *closest* arrangement span an interval of only 218 and occur in the pattern

$$\mathbf{C_{16}} = [-19, -17, -13, -11, 11, 13, 17, 19, 101, 103, 107, 109, 191, 193, 197, 199]$$

or in its mirror image $\mathbf{C_{16}}'$; the common HL constant is $H_{16} = 6042367.03907$. For $x = \frac{4}{3} \cdot 10^{19}$ the Hardy-Littlewood count yields 0.648 for each pattern. Both patterns have early first occurrences, namely $\mathbf{C_{16}}$ at the HL count of 0.262, $\mathbf{C_{16}}'$ even at 0.066. In the table below we use the same coding scheme as above.

Patt.	Initial Prime		Differences in the Pattern					
C16	3051450534439926131	Q	22	Q	82	Q	12+16+6+8+40	Q
C16'	300000224101777931	Q	10+8+4+26+12+22	Q	4+54+24	Q	12+10	Q

2.3 Three Close Sixtuplets and Two Close 9-Tuplets

Being able to detect occurrences of clusters of 18 primes, we have a chance of displaying instances of three sixtuplets [97, 101, 103, 107, 109, 113] as well as two 9-tuplets [13, 17, 19, 23, 29, 31, 37, 41, 43] in their closest arrangements. The patterns are $\mathbf{c_6}$ and $\mathbf{c_9}$ below, spanning intervals of length 436 and 90, having the HL constants $h_6 = 92680700.00031$ and $h_9 = 19311372.63543$, respectively.

```
c6=[-113,-109,-107,-103,-101,-97,97,101,103,107,109,113,307,311,313,317,319,323]
c9=[-47,-43,-41,-37,-31,-29,-23,-19,-17, 13,17,19,23,29,31,37,41,43].
```

As it happened in the pattern $\mathbf{c_{16}}$ of Section 2.2, $\mathbf{c_6}$ cannot be represented in terms of small primes: $319 = 11 \cdot 29$, $323 = 17 \cdot 19$! The two patterns were searched up to 10^{23}, corresponding to HL counts of 1.33 and 0.277; at the first occurrences given below the actual HL counts are 0.85 and 0.0026, respectively We were lucky to capture the extremely early occurrence of $\mathbf{c_9}$!

```
Patt.    Initial Prime            Differences in the Pattern
c6  50038627250687303646277  4 2 4 2 4 56+138 4 2 4 2 4 38+90+10+56 4 2 4 2 4
c9     54014646858393564377  4 2 4 6 2 6 4 2        30        4 2 4 6 2 6 4 2
```

2.4 21 Primes in a Quadratic Pattern

Leonhard Euler (1707-1783), the mathematical giant of the eighteenth century, made the stunning discovery that the quadratic polynomial $f(n) := n^2 + n + c$ with $c = 41$ has only prime values for $n = 0, 1, 2, \ldots, 39$. Also for $n \geq 40$ prime values of $f(n)$ are extraordinarily abundant. One of the reasons for this is the obvious symmetry relation

$$f(n) \bmod p = f(p - n - 1) \bmod p \quad \text{for every prime } p\,.$$

This implies that a finite pattern $[f(n), n = 0, 1, \ldots, N-1]$ of arbitrary length N occupies at most ceil($p/2$) residue classes modulo any prime p. Therefore at least floor($p/2$) residue classes mod p are empty, hence the pattern is admissible.

As a consequence, *quadratic patterns*

$$[n^2 + n + c, \quad n = 0, 1, \ldots, N - 1]$$

of N elements are candidats to yield prime instances of rather long patterns of this preassigned structure. In fact, it was established that the first reoccurrence of the quadratic pattern

$$\mathbf{C} = [41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461]$$

of $N = 21$ (non-consecutive) primes begins at

$$x = 2\,34505\,01594\,32353\,29417\,.$$

The HL constant of $\mathbf{C}$ is found to be $H = 3916679971309.28168$, and for the HL count we obtain $H \cdot R_{21}(x) = 0.0139$, hence we again profited from a rather early occurrence.

It is unlikely that the linearly growing sequence [2,4,6,8,...] persists throughout the entire pattern as the sequence of differences between consecutive primes. Rather, it will occasionally be broken by additional primes in the ever longer gaps of the pattern. The above instance has the difference pattern

```
2 4 6 8 10 12 14 16 18 20 4+18 14+6+4 26 10+18 30 32 22+12 36 24+14 40
```

where plus signs + again denote additional primes interrupting the gaps of the preassigned pattern.

If the prime k-tuple hypothesis is true, instances of **C** in terms of *consecutive* primes must exist. To locate one of these, however, is beyond the possibilities of our current technique.

3 Migrating from Asgard to Hreidar

After almost five years of continuous operation on Asgard the more powerful hardware Hreidar became available in Spring 2005. Migration from the ailing Asgard to Hreidar, including tests of the adapted software, took place from May 11 to May 18, 2005. The authors thank the steering committee for again generously allotting idle computing power to this project. We also express our sincere thanks to George Sigut for his competent help during the migration process.

Our system consisting of the parallelizing software PMP and the searching algorithm (coded in PARI [1], the innermost loop in C++) proved to be very easy to migrate. E.g., the doubling of the word length from 32 to 64 bits could be handled by merely redfining a parameter called `bits`.

Migration took place while a long run was under way: the first block of length 10^{24} for the pattern C_2, see Section 4 below. No data were lost; after migration, and also after a subsequent change of configuration, the search resumed exactly where it was stopped. The only losses (besides the week of migration) were the 337 tasks (or 0.07 %) that were interrupted. More details of the long run are given in the table below.

Dates 2005	Days	%	Tasks	%	Tsk/Day	Machine	Nodes
01/06 - 05/11	124.9	62.0	250425	50.17	2005	Asgard	100
05/18 - 07/19	61.9	30.7	95686	19.17	1545	Hreidar	32
07/19 - 08/03	14.8	7.3	153426	30.73	10362	Hreidar	32+32
Total	201.6	100.0	499537	100.07	2476		

4 The Quest for a Maximally Dense 19-Tuple

Both of the patterns

$$\mathbf{c}_1 = [13, 17, 19, \ldots, 79, 83, 89]\,, \quad \mathbf{c}_2 = [37, 41, 43, \ldots, 107, 109, 113]\,,$$

consisting of 19 consecutive primes and encompassing 77 consecutive natural numbers each, are admissible, i.e. their elements leave at least one residue class unoccupied modulo every prime. Together with their mirror images $\mathbf{c}_1^{'} = [-89.. - 13]$ and $\mathbf{c}_2^{'} = [-113.. - 37]$ they constitute the four different patterns in which 19 primes can be arranged as densly as possible. The Hardy-Littlewood constants associated with $\mathbf{c}_1$ (or $\mathbf{c}_1^{'}$) and $\mathbf{c}_2$ (or $\mathbf{c}_1^{'}$) are found to be

$$h_1 = 172\,98850.522\,, \quad h_2 = 585\,49955.614 = \frac{44}{13} \cdot h_1\,,$$

respectively; therefore the pattern $\mathbf{c}_2$ is expected to occur 3.4 times more often than $\mathbf{c}_1$. The average number of occurrences of $\mathbf{c}_2$ in a large interval Δ near a much larger x is about $h_2\,\Delta\,(\log\,x)^{-19}$. From this we infer that the chances of the pattern $\mathbf{c}_2$ reoccurring at an $x < 5 \cdot 10^{25}$ are about 93 %. This is also the chance of its mirror image ocurring in the same range.

In the Initial Report 2001, `Projects/clprimes01.pdf`, the idea of a search for a 19-tuple of maximum density was rejected because at that time the available hardware (the idle time of 128 nodes of Asgard) would have required an excessively long search time (up to five years).

With the installation of Hreidar the situation has changed. With the idle time of 64 nodes of Hreidar available, the speed increased about 5-fold. The quest for a 19-tuple of maximum density is on again!

We are again working in blocks of length $\Delta = 10^{24}$. In an initial attempt we were directly heading for one of the more abundant patterns, $\mathbf{c}_2$. This resulted in a computation time of about 28 days per block. Unfortunately, 7 blocks done in this way produced no output. Of course, these 7 blocks have been recorded as non-carriers of Pattern $\mathbf{c}_2$, but obviously a change of strategy was needed.

We decided to look for the sub-patterns

$$\mathbf{C}_2 = [41, 43, \ldots, 107, 109]\,, \quad \mathbf{C}_2^{'} = [-109, -107, \ldots, -43, -41]$$

of 17 elements, obtained by deleting the two boundary elements from the original patterns $\mathbf{c}_2$, $\mathbf{c}_2^{'}$. Deleting any other two elements from the original patterns would have resulted in a larger computational effort. From a list of subpatterns the original patterns may easily be sorted out by checking the deleted elements for primality.

One advantage of this strategy is that output is generated at a reasonable rate. With the Hardy-Littlewood constant $h = 31\,75403.027$ the estimated number $N(x) \approx h\,\Delta\,(\log x)^{-17}$ of occurrences in a block of length $\Delta = 10^{24}$ centered at $x = X \cdot 10^{24}$ is given in the following table.

X	+0.5	+1.5	+2.5	+3.5	+4.5	+5.5	+6.5	+7.5	+8.5	+9.5	sum
0	11.110	6.7601	5.7584	5.1955	4.8154	4.5338	4.3130	4.1330	3.9822	3.8531	54.45
10	3.7408	3.6417	3.5533	3.4738	3.4016	3.3357	3.2751	3.2192	3.1673	3.1190	88.38
20	3.0738	3.0314	2.9915	2.9538	2.9182	2.8845	2.8524	2.8219	2.7928	2.7650	117.47
30	2.7384	2.7130	2.6885	2.6651	2.6425	2.6208	2.5998	2.5796	2.5601	2.5412	143.82
40	2.5229	2.5053	2.4881	2.4715	2.4554	2.4397	2.4245	2.4097	2.3954	2.3813	168.31
50	2.3677	2.3544	2.3414	2.3288	2.3164	2.3044	2.2926	2.2811	2.2699	2.2589	191.43
60	2.2481	2.2376	2.2273	2.2172	2.2073	2.1976	2.1881	2.1788	2.1697	2.1607	213.46
70	2.1519	2.1433	2.1348	2.1264	2.1183	2.1102	2.1023	2.0945	2.0869	2.0794	234.61
80	2.0720	2.0647	2.0575	2.0505	2.0436	2.0367	2.0300	2.0234	2.0168	2.0104	255.01

The second advantage is the possibility of comparing actual counts with the HL estimates; in this way one might get an idea which pattern will occur earlier. Since the distribution of long patterns tends to have large-scale anomalies, this argument – although not well founded – may sometimes work. The price for these benefits is a somewhat larger computational effort: one block of length $\Delta = 10^{24}$ roughly takes 38 days (instead of 28 days for $\mathbf{c}_2$).

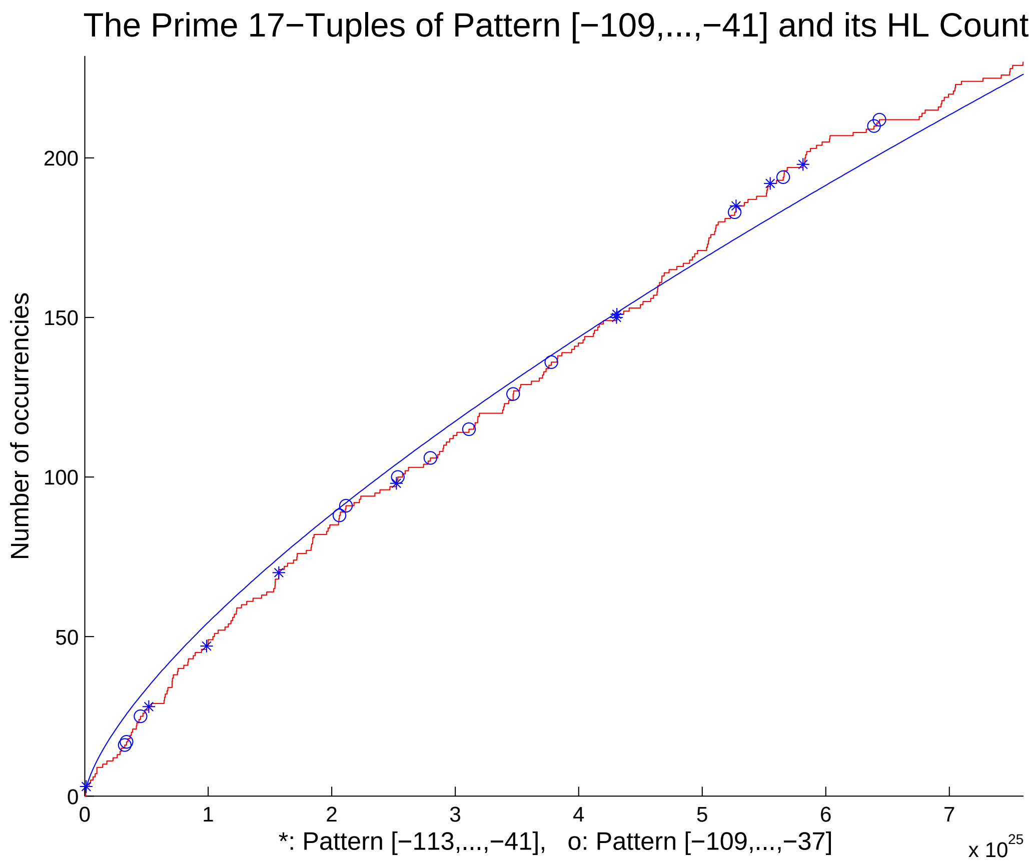
In the table below we summarize the current status (February 13, 2012) of the search for $\mathbf{C}_2$ and $\mathbf{C}_2'$ in the first seventy blocks. A digit in the table indicates the number of patterns $\mathbf{C}_2$ or $\mathbf{C}_2'$ located in the corresponding block. A question mark (?) means: the block has been screened for the 19-tuple $\mathbf{c}_2$ with a *negative* result; the number of 17-tuples $\mathbf{C}_2$ in the corresponding block is not known. Assuming an average turnout of 44 % (see below) we estimate the total number in the first 3 and the remaining 4 blocks of this type to be about 8 and 6, respectively. In all other cases (+) nothing has been done so far.

```
Blocks  1-5   6-10 11-15 16-20 21-25 26-30 31-35 36-40 41-45 46-50
  C2   6???3 1++++ ??+++ +++++ +++++ +++?? +++++ +++++ +++++ +++++
  C2'  92466 25654 34422 73353 43221 51325 16034 22533 25032 28223

Blocks 51-55 56-60 61-65 66-70 71-75 76-80 81-85 86-90
  C2   +++++ +++++ +++++ +++++ +++++ +++++ +++++ +++++
  C2'  64421 45062 20122 00215 40103 2++++ +++++ +++++
```

We infer that the first pattern is deficient in the regions considered; $\mathbf{C}_2$, $\mathbf{C}_2'$ produce about 44 % or 101.7 %, respectively, of the expected number of patterns. Not too bad a sign for continuing the search for a 19-tuple with the pattern $\mathbf{C}_2'$!

Being unable to report a 19-tuple (among 77 integers) at this time, we will at least report the initial primes of all 240 subpatterns (17 primes among 69 integers) found so far. In the figure below the number of patterns $\mathbf{C}_2'$ is plotted versus the initial prime x of the pattern, together with the Hardy-Littlewood estimate HL(x). The locations of Patterns [-113..-41] with an additional prime at a distance of 4 on the lower side are marked with an asterisk; a circle marks Patterns [-109..-37].



In the Appendix all initial primes of Patterns $\mathbf{C}_2$ and $\mathbf{C}_2'$ found so far are listed. The "near misses" with 18 primes among 73 integers are marked by asterisks. The position of the asterisk (left or right) indicates the side of the pattern (lower or upper) carrying the additional prime at a distance of 4. Hence the instance near $4.52 \cdot 10^{23}$ has the pattern of [41..113]; the one near $1.27 \cdot 10^{23}$ begins 4 units earlier and has the pattern of [-113..-41]. The instances near $3.24 \cdot 10^{24}$, $3.38 \cdot 10^{24}$ and $4.52 \cdot 10^{24}$ have the pattern of [-109..-37]. We are waiting for an entry framed by two asterisks.

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Appendix: Subpatterns found so far

	C2=[41..109]		C2'=[-109..-41]
1	41	1	71542018620258822095351
2	112650791633055206521031	2	110251111115724441042071
3	272064438851059718733911	3	*127372818047327460718001
4	452098732111641436261481*	4	359103301162581793637711
5	636277672291047674637791	5	472773823866578374260011
6	649410249467730311490971	6	668134176357714775428611
		7	842902600507150788234521
		8	979634834795806543814351
		9	980351128284287125225751
		10	1448301799940226082771811
		11	1791530392243173635753591
		12	2287676808438268517289911
		13	2638263731291524361295161
		14	2861160868941024590320271
		15	2965402830695807653891631
		16	3239304711211315285839191*
		17	3381058510131934598816741*
		18	3491478433645050966237011
		19	3663823986030922159656101
		20	3774602330694614789940461
		21	3878542221422020936683791
1	4169958545122711795353941	22	4184454523805320162938431
2	4262842381283978056404881	23	4225209451302652505772971
3	4355781833464360551222011	24	4386179912950387994201651
		25	4522878605388339260634461*
		26	4716202264464890619359831
		27	4869723749184137214270791
4	5332857791735927206954241	28	*5188248562214464094307851
		29	5413131850555132198264661
		30	6423359383060950425435231
		31	6462943397426544261913571
		32	6515331587478752597979311
		33	6657401626856646647681591
		34	6732639026163593595722561

C2'=[-109..-41]

35 7068848143689559074537581
36 7069902747669768688311521
37 7106614823259270051322481
38 7167564348232179592490681
39 7518303244972092457804811
40 7558395658997171017779641

41 8019983382743326416141611
42 8351014354384188300523601
43 8394208478031243473535821
44 8796828522725933171551961
45 8941097330542117721700611

46 9451163321791831501251281
47 *9867358577329746770637251
48 9980642183707047492649901
49 9982676186512737822143921

50 10384473055093816251062381
51 10508599120346583663997481
52 10809519785591093308761911

53 11360432123283992291743091
54 11633244241482020341414121
55 11836184125186489787548721
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