Theory of Kepler Motion by Regularization

Jörg Waldvogel

Seminar for Applied Mathematics, ETH Zürich, Switzerland

CELMEC 5

San Martino al Cimino, Viterbo, Italy

September 6 - 12, 2009

Abstract

The talk is concerned with aspects of teaching the theory of Kepler motion. As opposed to the usual technique of directly solving the equations of motion,

$$\boxed{\frac{d^2x}{dt^2} + \mu \frac{x}{|x|^3} = 0},$$

we propose to use regularization and the harmonic oscillator as the basis for developing this theory.

Outline

1. Levi-Civita regularization
2. The harmonic oscillator
3. The eccentric anomaly
4. The Keplerian orbit
5. Energy
6. Time
7. Polar coordinates
8. Angular momentum

1. Levi-Civita Regularization

The two-body problem in relative coordinates x

$$\ddot{x} + \mu \frac{x}{|x|^3} = 0\,, \quad x = \binom{x_0}{x_1} \in \mathbb{R}^2 \quad \text{or} \quad \mathbf{x} = x_0 + i\,x_1 \in \mathbb{C}$$

$$\dot{(\;)} = \frac{d}{dt}(\;), \quad t = \text{time}$$

μ = gravitational parameter = gravit.const. times total mass

Regularization procedure

Step 1. Time transformation: $dt = r\,d\tau, \quad r = |\mathbf{x}|$

(Sundman transformation, τ = fictitious time)

Step 2. Conformal squaring: $\mathbf{x} \;=\; \mathbf{u}^2 \in \mathbb{C}$

Step 3. Use the energy integral for eliminating $\mathbf{u}'$

The formal regularization procedure

$$\ddot{\mathbf{x}} + \mu \frac{\mathbf{x}}{r^3} = 0, \quad r = |\mathbf{x}|, \qquad \frac{1}{2}|\dot{\mathbf{x}}|^2 - \frac{\mu}{r} = -h = \text{energy constant} \quad (1)$$

Step 1:

$$\frac{d}{dt} = r^{-1}\frac{d}{d\tau}, \qquad \frac{d^2}{dt^2} = r^{-2}\frac{d^2}{d\tau^2} - r'\,r^{-3}\frac{d}{d\tau}, \qquad (\ )' = \frac{d}{d\tau}(\ )$$

$$\Longrightarrow\ r\,\mathbf{x}'' - r'\,\mathbf{x}' + \mu\,\mathbf{x} = 0\,, \qquad \frac{1}{2}r^{-2}|\mathbf{x}'|^2 - \frac{\mu}{r} = -h$$

Step 2:

$$\mathbf{x} = \mathbf{u}^2,\ x' = 2\mathbf{u}\mathbf{u}',\ \mathbf{x}'' = 2(\mathbf{u}\mathbf{u}'' + \mathbf{u}'^2),\ r = \mathbf{u}\bar{\mathbf{u}},\ r' = \mathbf{u}'\bar{\mathbf{u}} + \mathbf{u}\bar{\mathbf{u}}'$$

$$\Longrightarrow\ r\cdot 2\mathbf{u}\mathbf{u}'' + \underbrace{r\cdot 2\mathbf{u}'^2 - \mathbf{u}'\bar{\mathbf{u}}\cdot 2\mathbf{u}\mathbf{u}'}_{0} - \mathbf{u}\bar{\mathbf{u}}'\cdot 2\mathbf{u}\mathbf{u}' + \mu\mathbf{u}^2 = 0$$

Energy equation $\Longrightarrow\ \frac{1}{2}r^{-2}\cdot 4\mathbf{u}\mathbf{u}'\bar{\mathbf{u}}\bar{\mathbf{u}}' - \frac{\mu}{r} = -h$ or $2\,\mathbf{u}'\,\bar{\mathbf{u}}' = \mu - r\,h$ (2)

The formal regularization procedure, continued

Step 3: Elimination of $\mathbf{u}'$ from the last two equations and division by $2\,r\mathbf{u}$ yields

$$\boxed{\mathbf{u}'' + \frac{h}{2}\,\mathbf{u} = 0\,, \quad \mathbf{u} \in \mathbb{C}}$$

- A harmonic oscillator in 2 dimensions, frequency $\omega = \sqrt{h/2}$
- All Kepler formulas may be conveniently derived from the above ODE and the transformation rules
- Initial conditions from a complex square root, e.g.

$$\mathbf{u} = \sqrt{\mathbf{x}} = \frac{\mathbf{x} + |\mathbf{x}|}{\sqrt{2\,(|\mathbf{x}| + \mathrm{Re}\,\mathbf{x})}}, \qquad \mathbf{u}' = \frac{1}{2}\;\bar{\mathbf{u}}\;\dot{\mathbf{x}}$$

2. The Harmonic Oscillator in $n = 2$ Dimensions

$$\ddot{\mathbf{u}} + \omega^2 \, \mathbf{u} = 0, \quad \mathbf{u} \in \mathbb{C}, \qquad \omega = \sqrt{\frac{h}{2}} \tag{3}$$

Initial-value problem, $\mathbf{u}(\tau_0) = \mathbf{u}_0 \in \mathbb{C}, \quad \dot{\mathbf{u}}(\tau_0) = \dot{\mathbf{u}}_0 \in \mathbb{C}$:

$$\mathbf{u}(\tau) = \mathbf{u}_0 \, \cos\omega(\tau - \tau_0) + \frac{\dot{\mathbf{u}}_0}{\omega} \, \sin\omega(\tau - \tau_0) \, .$$

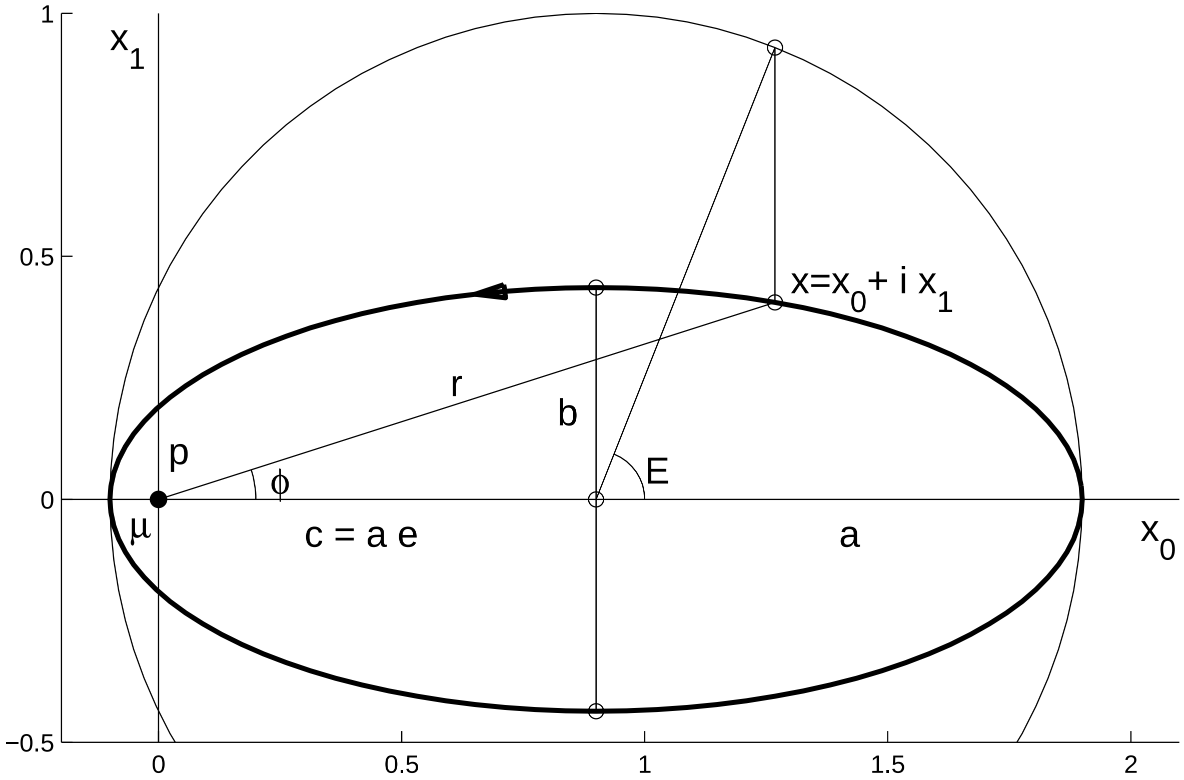
The orbit is an ellipse centered at the origin.

If $\tau = 0$ corresponds to an apex of the ellipse, we have

$$\boxed{\mathbf{u}(\tau) = A \, \cos(\omega\tau) + i \, B \, \sin(\omega\tau)} \quad \text{with} \quad A \in \mathbb{R}, \; B \in \mathbb{R}. \tag{4}$$

3. The Eccentric Anomaly

The planar elliptic Kepler motion, basic geometry and notation (e=0.9)



a, b semi-axes, c focal distance, p semi-latus rectum, E eccentric anomaly (w.r.t. apocenter), μ gravitational parameter, r radial distance, ϕ polar angle.

Eccentric anomaly, continued

The square of Equation (4), p. 7,

$$\mathbf{x} = \mathbf{u}^2 = \frac{A^2 - B^2}{2} + \frac{A^2 + B^2}{2}\cos(2\omega\tau) + i\,A\,B\,\sin(2\omega\tau)\,, \qquad (5)$$

parametrizes the Keplerian orbit, an ellipse. Comparison with the figure on p. 8 yields the geometric meaning of the angle E, referred to as the eccentric anomaly:

$$\boxed{E := 2\,\omega\,\tau = \sqrt{2\,h}\;\tau}\,, \qquad (6)$$

a natural way of introducing this fundamental parameter.

4. The Keplerian Orbit

Express the geometric parameters a, b, c of p. 8 in terms of A, B:

$$a = \frac{A^2 + B^2}{2}\,, \quad b = A\,B\,, \qquad c = a\,e = \frac{A^2 - B^2}{2}\,. \tag{7}$$

Since $a^2 = b^2 + c^2$, the origin is a focus of the ellipse. Eccentricity:

$$e := \frac{c}{a} = \frac{A^2 - B^2}{A^2 + B^2}\,, \quad |e| < 1.$$

In terms of a, e the parameters A, B may now be written as

$$A = \sqrt{a\,(1 + e)}\,, \quad B = \sqrt{a\,(1 - e)}\,. \tag{8}$$

Keplerian orbit, continued

Therefore, the parametrization (5) of the orbit in terms of E, in view of $\mathbf{x} = x_0 + i\,x_1$, becomes

$$\boxed{x_0 = a\Big(e + \cos(E)\Big)\,, \quad x_1 = a\sqrt{1-e^2}\,\sin(E)}\,. \tag{9}$$

Furthermore, this directly yields

$$\boxed{r = |\mathbf{x}| = \sqrt{x_0^2 + x_1^2} = a\Big(1 + e\,\cos(E)\Big)}\,. \tag{10}$$

5. Energy

Express the energy constant h (=negative energy) from Equ. (1_3) in terms of the major semi-axis a:

Energy equation in the form (2), together with (10) for r and (3_2) for ω :

$$2\,|\mathbf{u}'|^2 = \mu - a\,h\Big(1 + e\,\cos(E)\Big).$$

Derivative $\mathbf{u}'$ of (4), together with (7) for a, e and (6) for τ yields

$$2\,|\mathbf{u}'|^2 = a\,h\Big(1 - e\,\cos(E)\Big).$$

These equations are identical in μ, e, E if

$$a\,h = \frac{\mu}{2} \qquad \text{or} \qquad \boxed{h = \frac{\mu}{2\,a}}\,. \tag{11}$$

6. Time

Write Sundman's transformation (p.4) in terms of E by using (6,10,11):

$$dt = \frac{a}{\sqrt{2\,h}} \Big(1 + e\ \cos(E)\Big)\, dE = \frac{1}{n} \Big(1 + e\ \cos(E)\Big)\, dE\,. \qquad (12)$$

Here,

$$\boxed{n := \sqrt{\frac{\mu}{a^3}} = \frac{2\,\pi}{T}}\,, \quad T \text{ the period of revolution,}$$

is the mean angular velocity of the Kepler motion (called mean motion in astronomy), satifying Kepler's third law, $n^2\, a^3 = \mu\,.$

Integration of Equ. (12) (normalized for $t = 0$ at the apocenter) yields Kepler's equation,

$$\boxed{t = \frac{1}{n} \Big(E + e\ \sin(E)\Big)}\,. \qquad (13)$$

7. Polar coordinates, $\mathbf{x} = r\,e^{i\,\phi}$

Rewrite Equ. (4) in terms of E, a, e by means of (6), (8):

$$\mathbf{u} = \sqrt{\mathbf{x}} = \sqrt{a\,(1+e)}\,\cos\left(\frac{E}{2}\right) + i\,\sqrt{a\,(1-e)}\,\sin\left(\frac{E}{2}\right) = \sqrt{r}\,e^{i\,\phi/2}\,.$$

This immediately implies the famous relation

$$\boxed{\tan\left(\frac{\phi}{2}\right) = \sqrt{\frac{1-e}{1+e}}\,\tan\left(\frac{E}{2}\right)}\,. \tag{14}$$

Solving (14) for $\tan(E/2)$ and passing over to $\cos(E)$ yields

$$\cos(E) = \frac{\cos(\phi) - e}{1 - e\,\cos(\phi)}\,.$$

Polar coordinates, continued

Substituting this into (10) yields

$$\boxed{r = \frac{p}{1 - e\,\cos(\phi)} \quad \text{with} \quad p = a\,(1 - e^2)}\,. \tag{15}$$

p is called the semi-latus rectum; it is the value of r at $\phi = \pi/2$.

8. Angular momentum

Invariance of angular momentum vector $C \in \mathbb{R}^3$ may be derived directly from the equations of motion by means of the vector product $C = x \times \dot{x}$.

Alternate approach: Explicit computations in $\mathbb{R}^2$, $\mathbf{x} = x_0 + i\, x_1$:

$$C = \mathrm{Im}(\bar{\mathbf{x}}\ \dot{\mathbf{x}}) \quad \text{(scalar)}$$

Use the orbit (9) as well as r from (10) and $a\,(1 - e^2) = p$:

$$\mathrm{Im}\left(\bar{\mathbf{x}}\ \frac{d\mathbf{x}}{dE}\right) = \sqrt{a\,p} \cdot r\,.$$

Transforming this to time derivatives by means of $dt = r\sqrt{\frac{a}{\mu}}\,dE$ yields

$$\boxed{C = \mathrm{Im}\,(\bar{\mathbf{x}}\ \frac{d\mathbf{x}}{dt}) = \sqrt{\mu\,p} = \text{const}}\,.$$

A few references

1. T. Levi-Civita, 1920, *Sur la régularisation du problème des trois corps*. Acta Math. **42**, 99-144.
2. E. L. Stiefel, G. Scheifele, 1971, *Linear and Regular Celestial Mechanics*. Springer, 301 pp., in particular 36-42.
3. K. F. Sundman, 1907, *Recherches sur le problème des trois corps*. Acta Societatis Scientificae Fennicae **34**, 6.
4. J. Waldvogel, 2008, *Quaternions for regularizing Celestial Mechanics: the right way*. Celest Mech Dyn Astr. **102**, 149-162.

Conclusions

- An unconventional approach to deriving and teaching the classical theory of Kepler motion has been suggested. It is based on the harmonic oscillator in two dimensions as the solution of the regularized equations of motion.
- Besides leading to an elegant and concise theory this approach may also simplify further applications such as perturbation theories, orbit determination, etc.
- The generalization to three dimensions via Kustaanheimo-Stiefel regularization and quaternions as well as including the Laplace-Runge-Lenz vector may lead to an extension of the present sketch.