The Feet of the Altitudes
of a Simplex

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Abstract

We present an elegant algorithm for simultaneously computing all feet $F_0, F_1, \ldots, F_n \in \mathbb{R}^n$ of the altitudes of a simplex from the coordinates of its vertices $A_0, A_1, \ldots, A_n \in \mathbb{R}^n$. The algorithm which is based on projective geometry and matrix algebra, is valid for every $n \geq 1$. Coded in MATLAB, it may be expressed by means of three simple statements using no more than 50 characters.

Key words: Feet of altitudes, orthic triangle, projective spaces, homogeneous coordinates, simplex, inverse matrix.

1 Introduction

The construction of the feet of the altitudes of a triangle or a tetrahedron is a standard task in elementary geometry or solid geometry. The planar case is of particular interest: The feet $F_0, F_1, F_2 \in \mathbb{R}^2$ of the altitudes of a triangle with vertices $A_0, A_1, A_2 \in \mathbb{R}^2$ form the so-called orthic triangle. This triangle is the solution of Fagnano's problem (1775), see, e.g., [1]: Among all triangles P_0, P_1, P_2 with P_k

on the side opposite to $A_k \, (k = 0, 1, 2)$ or on its extension, find the triangle with minimal perimeter.

In $n{=}2$ dimensions the computation of all feet $F_0, F_1, \ldots, F_n \in \mathbb{R}^n$ of a simplex in $\mathbb{R}^n$ from the coordinates of its vertices $A_0, A_1, \ldots, A_n \in \mathbb{R}^n$ is therefore of interest. In this note we describe a surprisingly compact algorithm for this task that is valid in every dimension $n \geq 1$. The basis of our approach is the use of homogeneous coordinates that will be reviewed in the next section. Then, the faces of the simplex will be represented by means of an inverse matrix. Finally, the feet are obtained by intersecting the altitudes with the faces. An elegant code in MATLAB [4] will be given at the end.

2 Homogeneous Coordinates

In homogeneous coordinates the point $X = (X_1, X_2, \ldots, X_n)^T \in \mathbb{R}^n$ is represented by the $(n + 1)$-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_n)^T \in \mathbb{R}^{n+1} \setminus \{(0, 0, \ldots, 0)^T\}$, where the superscript T denotes transposition and

$$X_k = \frac{x_k}{x_0}, \quad k = 1, \ldots, n \tag{1}$$

for any $x_0 \neq 0$. Any multiple $c\,\mathbf{x}$ of $\mathbf{x}$ with $c \neq 0$ represents the same point. A projective space is obtained by allowing $x_0 = 0$: $(0, x_1, \ldots, x_n)^T$ represents the infinitely remote point, or *absolute point*, in the direction of $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$.

All points $X = (X_1, X_2, \ldots, X_n)^T \in \mathbb{R}^n$ satisfying the linear relation

$$e_0 + e_1 X_1 + \cdots + e_n X_n = 0 \tag{2}$$

form an $(n - 1)$-dimensional hyperplane $E \subset \mathbb{R}^n$. The elements e_k of the $(n + 1)$-tuple $\mathbf{e} = (e_0, e_1, \ldots, e_n)^T \in \mathbb{R}^{n+1} \setminus \{(0, 0, \ldots, 0)^T\}$ are defined as the hyperplane coordinates, or *Plücker coordinates*, of E; again, any multiple $c\,\mathbf{e}$ of $\mathbf{e}$ with $c \neq 0$ represents the same hyperplane. Equ. (2) with (1) yields the condition of incidence,

$$\sum_{k=0}^{n} e_k \, x_k = 0 \quad \text{or} \quad \mathbf{e}^T \mathbf{x} = 0 \,; \tag{3}$$

i.e. the hyperplane E contains the point X if and only if the dot product of the homogeneous vectors $\mathbf{e}$ and $\mathbf{x}$ vanishes.

From Equ. (2) it is seen that the vector $(e_1, e_2, \ldots, e_n)^T \in \mathbb{R}^n$ is orthogonal to every vector in the hyperplane E. Therefore, the direction of $(e_1, e_2, \ldots, e_n)^T$, i.e. its absolute point

$$\mathbf{e}^0 := (0, \, e_1, e_2, \ldots, e_n)^T \in \mathbb{R}^{n+1} \,, \tag{4}$$

corresponds to the direction orthogonal to E. For a more detailed introduction to projective spaces and homogeneous coordinates see, e.g., [1] or [3].

3 The Faces of a Simplex

Let the $n + 1$ vertices $A_k \, (k = 0, \ldots, n)$ of a nondegenerate simplex $\subset \mathbb{R}^n$ be given by their homogeneous coordinates, e.g.

$$\mathbf{a}_k = (1, A_{k1}, A_{k2}, \ldots, A_{kn})^T \,, \quad k = 0, \ldots, n \,, \tag{5}$$

where A_{kj} is the jth Cartesian component of the position vector of the kth vertex. Consider now the matrix

$$A = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{R}^{(n+1) \times (n+1)} \tag{6}$$

whose columns are the homogeneous coordinates of the vertices of the simplex. Assuming that A is invertible, we denote its *inverse* by S^T; hence

$$S^T \cdot A = I \quad \text{or} \quad S := (A^{-1})^T \,, \tag{7}$$

where I is the unit matrix of order $n + 1$. By partitioning S^T into its rows $\mathbf{s}_k^T = (s_{k0}, s_{k1}, \ldots, s_{kn})$ we have

$$S^T = \begin{pmatrix} \mathbf{s}_0^T \\ \vdots \\ \mathbf{s}_n^T \end{pmatrix} \quad \text{or} \quad S = (\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_n) \,, \tag{8}$$

and Equ. (7) may be written as

$$\mathbf{s}_k^T \mathbf{a}_l = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases} \,. \tag{9}$$

Therefore, as a consequence of (3), the hyperplane defined by the homogeneous vector $\mathbf{s}_k$ contains every vertex of the simplex except A_k; it is therefore the face opposite to A_k. Hence we have the following

Theorem 1. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be the matrix containing in its kth column the homogeneous coordinates of the vertex A_k ($k = 0, \ldots, n$) of a nondegenerate simplex $\subset \mathbb{R}^n$, $n \geq 1$. Then the transpose of the inverse of A, $S := (A^{-1})^T$, contains the homogeneous coordinates of the face opposite to A_k in its kth column. □

The matrix A in the above theorem is guaranteed to be invertible: Assume that A is singular; then we can express one of the columns of A, say the kth column, as a linear combination of the remaining columns. Hence the vertex A_k lies in the subspace spanned by the rest of the vertices, and the simplex is degenerate, in contradiction with the hypotheses of the theorem.

4 The Feet of the Altitudes

In the following, we restrict ourselves to proper simplices, i.e. no vertices are at infinity. Therefore the matrix A defined in (6) contains no zeros in its first row. The kth altitude of the simplex may be defined as the line connecting the vertex A_k with the absolute point $\mathbf{s}_k^0 := (0,\ s_{k1}, s_{k2}, \ldots, s_{kn})^T$ of the normal of the face opposite to A_k. In general, the line connecting two points A, B (given by their homogeneous coordinates $\mathbf{a}$, $\mathbf{b}$, respectively) is the set of points $\mathbf{c} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$ with $\lambda, \mu \in \mathbb{R}$, $(\lambda, \mu) \neq (0, 0)$. For the kth altitude we may choose $\mathbf{c} = \mathbf{f}_k := \mathbf{a}_k - \mu_k\,\mathbf{s}_k^0$, $\mu_k \in \mathbb{R}$, for simplicity. Its foot satisfies $\mathbf{s}_k^T\,\mathbf{f}_k = 0$ which implies

$$\mu_k = \frac{\mathbf{s}_k^T\,\mathbf{a}_k}{\mathbf{s}_k^T\,\mathbf{s}_k^0}\,. \tag{10}$$

Evaluating the numerator by means of Equ. (9) and using the explicit expression for the denominator yields

$$\mathbf{f}_k = \mathbf{a}_k - \mu_k\,\mathbf{s}_k^0 \quad \text{with} \quad \mu_k = \frac{1}{\sum_{j=1}^n s_{kj}^2}\,, \quad k = 0, 1, \ldots, n \tag{11}$$

for the foot of the kth altitude. In order to obtain all feet at once we consider the $n + 1$ first equations of (11) as the columns of a single matrix equation,

$$F = A - S^0 \cdot \mathrm{diag}(\mu_0, \mu_1, \ldots, \mu_n), \qquad (12)$$

where F is the matrix with the homogeneous coordinates of the feet as its columns. Furthermore, $\mathrm{diag}(\mu_0, \ldots, \mu_n)$ is the diagonal matrix with $\mu_0, \ldots, \mu_n$ as its diagonal elements. Finally, $S^0 = (\mathbf{s}_0^0, \mathbf{s}_1^0, \ldots, \mathbf{s}_n^0)$, the matrix of the absolute points of the vectors orthogonal to the faces, is obtained from S by zeroing out the first row and may be written as

$$S^0 = \mathrm{diag}(0, 1, \ldots, 1) \cdot (A^{-1})^T \qquad (13)$$

as a consequence of Equ. (7). We therefore have established

Theorem 2. Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be the matrix containing in its kth column the homogeneous coordinates of the vertex A_k ($k = 0, \ldots, n$) of a nondegenerate proper simplex $\subset \mathbb{R}^n$, $n \geq 1$. Then S^0 defined in (13) is the matrix of the absolute points of the vectors orthogonal to the faces, and F defined in (12), with μ_k from (11), has the homogeneous coordinates of the feet of the altitudes of the simplex as its columns. $\qquad \square$

In the case of dimension $n = 1$, when the simplex is a line segment, the algorithm simply exchanges the boundary points.

Remark. The use of projective spaces and homogeneous coordinates is widespread when it comes to deal with simplices. E.g. in [2] it can be seen how proofs of theorems get shortened and how the duality principle of projective geometry can enter the calculations.

5   An Elegant Implementation in Matlab

The concise algorithm (13), (12) may be coded in the language of Matlab [4] as follows below. Matlab is particularly well suited owing to its powerful instructions for the operations of matrix algebra.

```
% Given data: columns of A are the homogeneous coordinates
% of the vertices of the simplex. Example in 2 dimensions:
```

```
        A = [ 1   1   1
             -3   3   1
              0   1   4];
% Algorithm

        S = inv(A)'; S(1,:) = 0*S(1,:);
        F = A - S * diag(1./sum(S.^2)),

% Columns of F are the homogeneous coordinates of the feet.
% The top row of A (preferably ones) is inherited by F.

% Result for the above example of A:

% F = [1.000000000000   1.000000000000   1.000000000000
%       1.615384615385   0.500000000000   1.540540540541
%       3.076923076923   3.500000000000   0.756756756757];
```

Comments on the code of the algorithm. We use the symbol `S` for the matrix S^0. The prime causes transposition of a real matrix. The second statement replaces the top row of the current `S` by a row of zeros. The operator `diag` fills its argument vector into a diagonal matrix. The sequence `1./sum(S.^2)` computes all values μ_k for $k = 0, 1, \ldots, n$ of Equ. (11) by summation over the columns of the matrix of the squared elements of S^0.

References

[1] H. S. M. Coxeter, S. L. Greitzer: *Zeitlose Geometrie*. Klett, Stuttgart 1990, 187 pp. Original edition: *Geometry Revisited*. Random House, New York 1967, 193 pp.

[2] L. Gerber: *Associated and perspective simplices*. Trans. Amer. Math. Soc. **201** (1975), 43-55.

[3] H.-J. Kowalsky, G. O. Michler: *Lineare Algebra*. Walter de Gruyter, Berlin New York 2003, 416 pp.

[4] The MathWorks. URL `http://www.mathworks.com`

Kastentext.

Im Unterricht in Elementargeometrie gehören die Höhen im Dreieck zum Standardstoff. Die Verallgemeinerung auf das Tetraeder im Raum liefert gelegentlich Material für Maturitätsaufgaben, etwa *"Berechne den der Ecke A gegenüberliegenden Höhenfußpunkt F_A im Tetraeder ABCD aus den Koordinaten der Eckpunkte"*. In einer solchen Situation ist auf Lehrersseite diese kleine Arbeit entstanden. Es zeigt sich, dass es ein erstaunlich einfaches und kompaktes Verfahren für die simultane Berechnung *aller* Höhenfußpunkte eines n-dimensionalen Simplex gibt. Dieses funktioniert für alle Dimensionen $n \geq 1$ und beruht auf der Verwendung homogener Koordinaten und projektiver Räume. Nach einer kurzen Einführung dieser Begriffe werden zunächst alle Seitenflächen eines Simplex mit einer einzigen Matrixinversion berechnet. Der Text könnte auch Anregungen für den Unterricht geben.

Biographische Notizen.

Arthur Gut received his Ph.D. from ETH while working in the field of homological algebra as a student of Prof. Beno Eckmann. From 1974 onward he was a Visiting Assistant Professor at Ohio State University in Columbus, then at Universidad de las Américas, Cholula, Mexico, and he spent the academic year 1976 at Universidade Federal do Ceará, Fortaleza, Brazil. Later he became a Gymnasium teacher at the Kantonsschule Zürich-Oerlikon. For years and also after his retirement in 2005 his special interests have been geometry and computer algebra systems in education and in applications.

Jörg Waldvogel did his undergraduate studies (mathematics) and Ph.D. work (celestial mechanics) at ETH. Between 1967 and 1972 he worked in the USA, first as a Research Scientist, later as an Assistant Professor at the University of Alabama at Huntsville and at the University of Texas at Austin. Back at ETH, he became the head of the Numerical Algorithms Group of the Seminar for Applied Mathematics; in 1985 he was awarded the title of a professor. He has been a guest professor or academic guest at the Paris VI, Tampa, and Kent State Universities. Until his retirement in 2003 he taught engineering mathematics in all engineering departments of ETH. His continuing interests are numerical mathematics, celestial mechanics, and number theory.