Here the $M \times M$ system matrix $\mathbf{T}$ is symmetric tridiagonal and can be seen to be positive definite. (It is weakly diagonally dominant.) The system could be solved easily by Gaussian elimination, that is by LU factorization, by LDL^T factorization, or by Cholesky factorization. The cost of such a factorization is only $\mathcal{O}(M)$ since the factors are bidiagonal. There exist yet other efficient recursive algorithms for solving such a system, for example, cyclic reduction or fast Toeplitz solvers, since $\mathbf{T}$ has also Toeplitz structure (*i.e.*, t_{ij} only depends on $i - j$).

The corresponding two-dimensional example is the **Poisson problem**

$$\begin{aligned} \Delta u &= f \qquad \text{on} \quad S :\equiv (0,1) \times (0,1)\,, \\ u(x,y) &= 0 \qquad \text{on} \quad \partial S\,, \end{aligned} \tag{1.4}$$

where

$$\Delta u :\equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,.$$

Here ∂S denotes the boundary of the unit square S. In analogy to (1.2) we apply the approximation

$$\Delta u(x_j) = \frac{1}{h^2}\left[(u_{j+1} - 2u_j + u_{j-1}) + (u_{j+M} - 2u_j + u_{j-M})\right] + O(h^2)\,, \tag{1.5}$$

where we assume that the $N :\equiv M^2$ interior points are numbered row by row from the upper left corner to the lower right one. This transforms (1.4) into the linear system

$$\mathbf{Au} = -\,h^2\mathbf{f} \tag{1.6}$$

with the block matrix $\mathbf{A}$ of size $N \times N = M^2 \times M^2$ defined by

$$\mathbf{A} :\equiv \begin{pmatrix} \mathbf{T} & -\mathbf{I} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ -\mathbf{I} & \mathbf{T} & -\mathbf{I} & \ddots & & \vdots \\ \mathbf{O} & -\mathbf{I} & \mathbf{T} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\mathbf{I} & \mathbf{O} \\ \vdots & & \ddots & -\mathbf{I} & \mathbf{T} & -\mathbf{I} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} & -\mathbf{I} & \mathbf{T} \end{pmatrix}. \tag{1.7}$$

Here, $\mathbf{I}$ is the $M \times M$ unit matrix and

$$\mathbf{T} :\equiv \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix}$$

is also of size $M \times M$. The system matrix $\mathbf{A}$ is still symmetric positive definite and banded, but the total bandwidth is now $2M+1$. Clearly, in three dimensions we would have $N = M^3$ and obtain a matrix with total bandwidth $2M^2 + 1$ and with at most seven nonzeros per row, namely one number six and at most six minus ones.

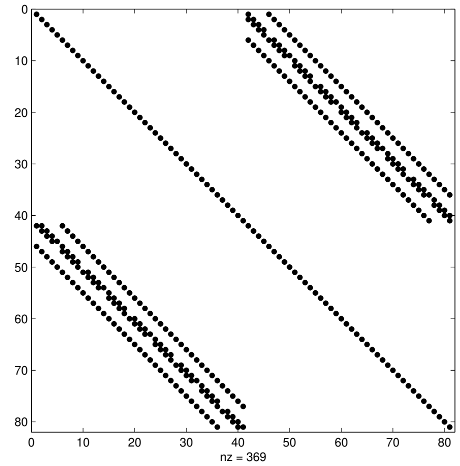
Note that the matrices depend on the numbering or order of the points. For example, if in the first problem (1.3) we take all the currently odd numbered points first, followed by all even numbered points — a one-dimensional version of what is called **red-black ordering** — the matrix $\mathbf{T}$ is transformed into

$$\widetilde{\mathbf{T}} :\equiv \begin{pmatrix} 2\,\mathbf{I} & -\mathbf{B}_1 \\ -\mathbf{B}_1^{\mathsf{T}} & 2\,\mathbf{I} \end{pmatrix}, \tag{1.8}$$

where $\mathbf{B}_1$ is a lower bidiagonal matrix of ones of size $\frac{1}{2}M$ if M is even. Likewise, by using red-black ordering in the two-dimensional problem (1.6) and assuming M odd we can replace $\mathbf{A}$ by

$$\widetilde{\mathbf{A}} :\equiv \begin{pmatrix} 4\mathbf{I}_1 & -\mathbf{B}_2 \\ -\mathbf{B}_2^{\mathsf{T}} & 4\mathbf{I}_2 \end{pmatrix}, \tag{1.9}$$

where $\mathbf{I}_1$ is the unit matrix of size[1] $\lceil \frac{1}{2}M^2 \rceil$ and $\mathbf{I}_2$ is the unit matrix of size $\lfloor \frac{1}{2}M^2 \rfloor$, while $\mathbf{B}_2$ is a (non-square) Toeplitz matrices with four co-diagonals of ones (but not all next to each other). The nonzeros of $\widetilde{\mathbf{A}}$ are shown in the following "spy plot" (with $M = 9$, $N = 81$):



It was generated with the following Matlab commands:

$M = 9;\ N = M^2;$
$A = gallery('poisson', M);$
$p = [1:2:N, 2:2:N];$
$Atilde = A(p,p);$
$spy(Atilde)$ ♦

In most applications the data are real, *i.e.,* $A \in \mathbb{R}^{N\times N}$ and $b \in \mathbb{R}^N$. But, *e.g.,* in electrical engineering there are also problems where $A \in \mathbb{C}^{N\times N}$ and $b \in \mathbb{C}^N$. Most of what we say is true for both cases. To handle them at once we will write $A \in \mathbb{E}^{N\times N}$, $b \in \mathbb{E}^N$ with $\mathbb{E} :\equiv \mathbb{R}$ or $\mathbb{C}$, and use $\mathbf{x}^\star :\equiv \mathbf{x}^{\mathsf{T}}$ if $\mathbb{E} = \mathbb{R}$ and $\mathbf{x}^\star :\equiv \mathbf{x}^{\mathsf{H}}$ if $\mathbb{E} = \mathbb{C}$.

Except in the sections on least squares we will assume that $\mathbf{A}$ is nonsingular. In many applications $\mathbf{A}$ is real symmetric or Hermitian and positive definite (or, briefer: **spd** or **Hpd**) — in which case all the eigenvalues are positive, and there is an orthonormal basis of eigenvectors. We will see that solving the linear system $\mathbf{Ax} = \mathbf{b}$ is then much easier than when $\mathbf{A}$ is indefinite or even nonsymmetric.

[1] $\lceil y \rceil$ denotes the smallest integer larger or equal to y, and $\lfloor y \rfloor$ denotes the largest integer smaller or equal to y.