THEODORSEN'S INTEGRAL EQUATION

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Theodorsen's integral equation [7] is a well-known tool for computing numerically the conformal mapping g of the unit disk D onto a star-like region Δ given by the polar coordinates τ, $\rho(\tau)$ of its boundary Γ. The mapping g is assumed to be normalized by $g(0) = 0$, $g'(0) > 0$. It is uniquely determined by its boundary correspondence function θ, which is implicitly defined by

$$g(e^{it}) = \rho(\theta(t))\, e^{i\theta(t)} \quad (\forall t \in \mathbb{R}), \qquad \int_0^{2\pi} \theta(t) dt = 2\pi^2.$$

Theodorsen's equation follows from the fact that the function $h(w) = log(g(w)/w)$ is analytic in D and can be extended to a homeomorphism of the closure $\bar{D}$ onto the closure $\bar{\Delta}$. It simply states that the 2π-periodic function $y : t \mapsto \theta - t$ is the conjugate periodic function of $x : t \mapsto \log \rho(\theta(t)))$, that is, $y = Kx$, where K is the conjugation operator defined on $L[0, 2\pi]$ by the principal value integral

$$(Kx)(t) = \frac{1}{2\pi} \text{P.V.} \int_0^{2\pi} x(s) \cot \frac{t-s}{2} ds \qquad (a.e.)$$

When restricted to $L_2[0, 2\pi]$, K is a skew-symmetric endomorphism of norm 1 with a very simple diagonal representation in Fourier space: when x has the real Fourier coefficients a_0, a_1, a_2, ..., b_1, b_2, ..., then y has the coefficients 0, $-b_1$, $-b_2$, ..., a_1, a_2, ...

Hence, while Theodorsen's integral equation is normally written as

$$\theta(t) - t = \frac{1}{2\pi} \text{P.V.} \int_0^{2\pi} \log \rho(\theta(s))) \cot \frac{t-s}{2} ds,$$

for practical purposes the conjugation is executed by transformation to Fourier space: x is approximated by a trigonometric polynomial of degree N, whose Fourier coefficients are quickly found by the Fast Fourier Transform (FFT), which then can also be applied to determine values at $2N$ equispaced points of the trigonometric polynomial that approximates $y = Kx$. Before the FFT became the standard tool for this discrete conjugation process, the transition from the values of x to those of y was based on the multiplication with a matrix, called Wittich matrix in [1]. The FFT meant a cost reduction from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$ per iteration.

Until the end of the 1970ies the recommendation was to solve a so obtained discrete version of Theodorsen's equation by fixed point (Picard) iteration, an approach that is limited to Jordan regions with piecewise differentiable boundary satisfying $|\rho'/\rho| < 1$, and is vey slow when the bound 1 is nearly attained. Other regions, like those from airfoil design, which was the standard application targeted by Theodorsen, could be handled by applying first a suitable preliminary conformal mapping, which turned the exterior of the wing cross-section into the exterior of a Jordan curve that is close to a circle; see Chapter 10 of [6]. Moreover, for this application, the equation has to be modified slightly to map the exterior of the disk onto the exterior of a Jordan curve.

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Gutknecht [2, 3] extended the applicability of Theodorsen's equation by applying more refined iterative methods and discretizations, and Hübner [5] improved the convergence order from linear to quadratic by adapting Wegmann's treatment of a similar equation obtained by using $h(w) = g(w)/w$ instead. Wegmann's method [9, 10] applies Newton's method and solves the linear equation for the corrections by interpreting it as a Riemann-Hilbert problem that can be solved with four FFTs.

A common framework for conformal mapping methods based on function conjugation is given in [4]; Theodorsen's restriction to regions given in polar coordinates can be lifted. Both Theodorsen's [8] and Wegmann's [11] equations and methods can be extended to the doubly connected case.

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