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Numerical conformal mapping methods based on function conjugation

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Abstract: A unifying treatment of methods for computing conformal maps from the unit disk onto a Jordan region is presented. Integral and integro-differential equations (involving the conjugation operator) for the boundary correspondence function are first derived using an arbitrary auxiliary function having certain properties. Then various iteration methods for solving these equations are described in this generality, so that the basic ideas become manifest. Specific methods are then treated as examples of the general theory. Among them are, in particular, the successive conjugation methods of Theodorsen, Melentiev and Kulisch, Timman, and Friberg, the projection method of Bergström, and the Newton methods of Vertgeim, Wegmann, and Hübner (which make use of the easy construction of the solutions of Riemann–Hilbert problems). Many of these methods are treated in greater generality than in the literature. The connections with the methods of Fornberg, Menikoff–Zemach, Chakravarthy–Anderson, and Challis–Burley are also outlined.

Keywords: Numerical conformal mapping, conjugate function, conjugation operator, method of successive conjugation, Riemann–Hilbert problem.

Mathematics Subject Classifications: 30C30, 42A50, 65R20.

0. Introduction

Let g be a conformal map of the unit disk D in the w-plane onto a given Jordan domain Δ with boundary Γ in the z-plane. The function g can be extended to the closure $\overline{D}$ of D in such a way that g is a homeomorphism of $\overline{D}$ onto $\overline{\Delta}$. We assume that $0 \in \Delta$ and that g is normalized either by

$$g(0) = 0, \quad g'(0) > 0 \tag{0.1}$$

or by

$$g(0) = 0, \quad g(1) = z_0 \in \Gamma. \tag{0.2}$$

Most numerical methods for computing g are in one or another sense based on the *conjugation* of periodic functions, i.e. on the possibility of constructing for a (usually real-valued) function u defined on the unit circle S another function v such that the values of $u + \mathrm{i}v$ are the boundary values of a function analytic in D. This fact is often applied not to the mapping function g itself,

Table 1
Classification of mapping methods. (i/e: method for the interior or the exterior mapping problem, respectively.)

type of method	auxiliary function	i/e	method of	Section no.
Successive conjugation (Section 5)	$\log(g(w)/w)$	i, e	Theodorsen	6.1
	$g(w)/w$	i	Melentiev–Kulisch	6.2
	$\log g'(w)$	e	Timman	6.5
	$\log g'(w)-2\log(g(w)/w)$	i	–	6.5
	$\log g'(w)-\log(g(w)/w)$	i, e	Friberg	6.6
Projection (Bergström) (Section 5)	$g(w)/w$	i, e	Bergström	6.3
	$\log(g(w)/w)$	i, e	–	6.3
Newton (Vertgeim–Hübner) (Section 7)	$\log(g(w)/w)$	i, e	Vertgeim–Hübner	8.1
	$g(w)/w$	i, e	–	8.2
Newton (Wegmann) (Section 7)	$\log(g(w)/w)$	i, e	–	8.1
	$g(w)/w$	i, e	Wegmann	8.2
Various related methods	$g(w)/w$	i, e	Fornberg	7
	$\log(g(w)/w)$	i, e	Menikoff–Zemach	9.1
	$g(w)$	i	Chakravarthy–Anderson	9.2
	$\log(g(w)/w)$	i	Challis–Burley	9.3

but instead to an *auxiliary function* h that is related to g. It leads readily to various integral equations for the boundary correspondence function (defined in Section 4).

However, this construction is also fundamental for solving Riemann–Hilbert problems on the disk, which is the key step in the very efficient methods of Vertgeim [62], Wegmann [69,70] and Hübner [34].

The efficiency of these and many other methods based on conjugation is due in the first place to the fact that in practice the conjugate function (or rather its approximate values on a regularly spaced set of points) can be computed by just two *fast Fourier transforms* (FFTs).

The aim of this paper is to describe various basic principles that can be and have been used to find such numerical methods, and to classify classical, recent, and new methods according to these principles, cf. Table 1. The description of particular methods in the literature is often obscured by the use of a specific auxiliary function and a specific representation of the boundary Γ, not to speak of the wide variety of notation in use. Our treatment is based on a general definition of the auxiliary function h as the image of an operator having certain properties. The basic ideas of the various methods are then first described in terms of this general auxiliary function. Later, most of the methods that have appeared in the literature are presented in detail as examples of the general theory.

However, this paper is neither a complete survey of all the work that has been done, nor a serious judgement and evaluation of methods currently available. (We hope to present a comparison of numerical results for many of the methods discussed here in the near future.) A comprehensive survey of results known in 1964 is Gaier's well-known book [13]. An excellent introduction to the subject is given by Henrici [30]; it includes, for example, elegant presentations of the methods of Theodorsen, Timman, and Wegmann. A general definition of the auxiliary function was proposed by Jeltsch [37] in his diploma thesis; his definition is similar in spirit to ours, but the details are quite different. (In particular, he did not introduce an operator.)

We partially also discuss the corresponding *exterior mapping problem*, where g maps the exterior of the unit circle S (including the point at infinity) onto the exterior of the curve Γ. The normalizations (0.1) and (0.2) are then replaced by

$$g(\infty)=\infty, \quad g'(\infty)>0 \tag{0.3}$$

and

$$g(\infty)=\infty, \quad g(1)=z_0 \in \Gamma, \text{ respectively.} \tag{0.4}$$

However, the related methods for doubly connected or multiply connected regions are not treated. (Some references are given in Section 9.)

The paper is organized as follows: In Sections 1–3 we summarize some mathematical background material on function spaces, the conjugation of functions, and the Riemann–Hilbert problem on the disk. In Section 4 we then present our general definition of the auxiliary function and derive the corresponding integral or integro-differential equation for the boundary correspondence function. Associated direct iteration methods (namely, the method of successive conjugation and a projection method due to Bergström) are described in Sections 5 and 6, first in our general framework, then for specific auxiliary functions. In the next two sections Newton methods, where each step mainly consists of solving a Riemann–Hilbert problem, are discussed in a similar way. In particular, we show that the methods of Vertgeim–Hübner [62,34] and Wegmann [69,70] are equivalent in the sense that they produce the same iterates, if they are applied undiscretized to the same auxiliary function. Finally, in Section 9, we briefly mention some further related methods. In the whole paper we mainly consider known methods, but it should become clear that our general approach also suggests a number of new methods or at least variants of old ones. Only a few of these are mentioned explicitly.

Concerning applications of conformal mapping we refer, e.g., to [1,25,35,36,45] and the references in [43], and note that conformal mapping is a competitive tool for grid generation, which is surveyed in [47,58,59,60].

1. Function spaces

Let us fix some of our notation now and define a number of function spaces that are either relevant for the problem or helpful for understanding the background material.

2π-periodic functions are written either in terms of a variable $t \in T$ (where T is the quotient space $T := \mathbb{R}/2\pi\mathbb{Z}$) or in terms of $w = e^{it} \in S$. The complex function spaces $L^p(T)$ $(1 \leqslant p \leqslant \infty)$, $C(T)$, $C^m(T)$ $(m \in \mathbb{Z}^+)$ are defined as usual, see, e.g. [40,53]. For $1 \leqslant p < \infty$

$$\|f\|_p := \left[(2\pi)^{-1}\int_T |f(t)|^p \mathrm{d}t\right]^{1/p}.$$

If the variable is $w = e^{it}$ instead of t, we write $L^p(S)$, etc., and if we want to stress that a function $f \in L^p(T)$, say, is real-valued, we write $f \in L^p(T, \mathbb{R})$. Equalities between values of L^p-functions are in general assumed to hold a.e. only.

The *Fourier coefficients* of a function $f \in L^1(T)$ [or $f \in L^1(S)$] are denoted by $\hat{f}_k$:

$$\hat{f}_k := \frac{1}{2\pi} \int_T f(t)\, \mathrm{e}^{-ikt} \mathrm{d}t \quad \left[\text{or } \hat{f}_k := \frac{1}{2\pi} \int_T f(\mathrm{e}^{it})\, \mathrm{e}^{-ikt} \mathrm{d}t\right], \quad k \in \mathbb{Z} \tag{1.1}$$

By using the Fourier coefficients with nonnegative index as Taylor coefficients we can associate with $f \in L^1(T)$ [or $L^1(S)$] the function

$$f^+(z) := \sum_{k=0}^{\infty} \hat{f}_k z^k, \quad z \in D, \tag{1.2}$$

which is analytic in D. (According to the Riemann–Lebesgue lemma $\hat{f}_k \to 0$ as $|k| \to \infty$, hence the radius of convergence of (1.2) is at least 1.)

The *Hardy space* H^p $(1 \leqslant p \leqslant \infty)$ can be defined as a subspace of $L^p(S)$:

$$H^p := \{ f \in L^p(S);\ \hat{f}_k = 0\ (\forall k < 0) \}. \tag{1.3}$$

However, it is clear from the above that the domain of the functions in H^p can be extended to D by setting $f(z) := f^+(z)$ $(z \in D)$. One can then show that

$$f(\mathrm{e}^{it}) = \lim_{\rho \uparrow 1} f(\rho\, \mathrm{e}^{it}) \quad \text{a.e. on } S \tag{1.4}$$

[16, p. 57; 40, p. 86; 53, p. 368]. (More generally, this holds with any nontangential limit.) Therefore, a better definition of H^p is

$$H^p := \{ f\colon \overline{D} \to \mathbb{C};\ f|_D \in A(D),\ f|_S \in L^p(S),\ (1.4) \text{ holds} \}. \tag{1.5}$$

Here $f|_D$ and $f|_S$ are the restrictions of f to D and S, respectively, and $A(D)$ is the space of functions analytic in D. By $A(\overline{D})$ we denote the space of functions analytic in D and continuous in $\overline{D}$. Of course, $A(\overline{D})$ is a subspace of H^∞, which is often introduced as the space of bounded analytic functions in D. (In fact, a third, analogous definition of H^p is the most satisfactory [9,16,40,53]).

Since $L^q(S) \subset L^p(S)$ if $1 \leqslant p < q \leqslant \infty$, we have also $H^q \subset H^p$. Furthermore, it is worth noting that Cauchy's integral formula holds for $f \in H^p$ [9, p. 40; 53, p. 369].

For the discussion of the exterior mapping problem we denote by Z^c the complement with respect to the extended plane of any set $Z \subset \mathbb{C}$, and we consider spaces of functions analytic in $\overline{D}^c$ (including ∞), such as $A(D^c)$ and $H^p(D^c)$.

There is often the need to indicate that the values of some $f \in C(S)$ [or $f \in C(T)$] are boundary values of some function in $A(\overline{D})$, and we simply write $f \in A(\overline{D})|_S$ [or $f \in A(\overline{D})|_T$] then.

Further function spaces playing an important role here are the *Lipschitz classes* $\mathrm{Lip}^\alpha(T)$ $(0 < \alpha \leqslant 1)$,

$$\mathrm{Lip}^\alpha(T) := \{ f \in C(T);\ \nu_\alpha(f) < \infty \}, \tag{1.6a}$$

with the semi-norm

$$\nu_\alpha(f) := \sup_{\substack{\tau \in T \\ \delta \neq 0}} \frac{|f(\tau + \delta) - f(\tau)|}{|\delta|^\alpha}, \tag{1.6b}$$

the *Hölder spaces* $C^{m,\alpha}(T)$ $(m \in \mathbb{Z},\ m \geqslant 0,\ 0 < \alpha \leqslant 1)$,

$$C^{m,\alpha}(T) := \{ f \in C^m(T);\ f^{(m)} \in \mathrm{Lip}^\alpha(T) \}, \tag{1.7}$$

and the *Sobolev spaces* $W^{m,p}(T)$ ($m \in \mathbb{Z}$, $m \geqslant 0$, $1 \leqslant p \leqslant \infty$),

$$W^{m,p}(T) := \left\{ f \in C^{m-1}(T);\ f^{(m-1)}\text{abs.cont.},\ f^{(m)} \in L^p(T) \right\}. \tag{1.8}$$

In particular, $C^{0,\alpha}(T) = \mathrm{Lip}^\alpha(T)$ and $W^{0,p}(T) = L^p(T)$. Several norms are in use for (1.6), (1.7), and (1.8), e.g.,

$$\| f \| := \| f \|_\infty + \nu_\alpha\left(f^{(m)}\right) \tag{1.9}$$

for $C^{m,\alpha}(T)$ ($m \geqslant 0$, $0 < \alpha \leqslant 1$) and

$$\| f \| := \max\left\{ \| f \|_\infty, \| f' \|_\infty, \dots, \| f^{(m-1)} \|_\infty, \| f^{(m)} \|_p \right\} \tag{1.10}$$

or

$$\| f \| := \begin{cases} \left[| \hat{f}_0 |^p + \| f^{(m)} \|_p^p \right]^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \max\left\{ | \hat{f}_0 |, \| f^{(m)} \|_\infty \right\} & \text{if } p = \infty \end{cases} \tag{1.11}$$

for $W^{m,p}(T)$ ($m > 0$) [1]. With these norms the spaces are Banach spaces. However, we will often make use of the fact that apart from addition and scalar multiplication a number of further operations do not lead out of these spaces:

Lemma 1.1. *Let* $c \in \mathbb{C}$ *and* f, $g \in C^{m,\alpha}(T)$ ($m \geqslant 0$, $0 < \alpha \leqslant 1$) [*or* f, $g \in W^{m+1,p}(T)$ ($m \geqslant 0$, $1 \leqslant p \leqslant \infty$)]. *Then the following functions lie in* $C^{m,\alpha}(T)$ [*or* $W^{m+1,p}(T)$] *also*:

$$cf \pm g, \tag{1.12a}$$

$$f \cdot g, \tag{1.12b}$$

$$f/g \quad \text{if } 0 \notin g(T), \tag{1.12c}$$

$$\mathrm{Re}\, f,\ \mathrm{Im}\, f, \tag{1.12d}$$

$$\exp(f), \tag{1.12e}$$

$$| f | \quad \text{if } 0 \notin f(T) \quad \text{or} \quad m = 0, \tag{1.12f}$$

$$\log f \quad \text{if } 0 \notin f(T),\quad \#f(T) = 0, \tag{1.12g}$$

$$\begin{aligned} h \circ f \quad & \text{if either } m \geqslant 1 \text{ and } h \in C^{m,\alpha}(f(T)) \\ & \text{or } m = 0 \text{ and } h \in \mathrm{Lip}^1(f(T)) \\ & \left[\text{if } h \in W^{m+1,\infty}(f(T))\right], \end{aligned} \tag{1.12h}$$

$$\begin{aligned} f(g(\ ) + (\ )) \quad & \text{if } g \text{ is real-valued and either} \\ & m \geqslant 1 \text{ or } m = 0 \text{ and } f \in \mathrm{Lip}^1(T) \\ & \left[\text{if } g \text{ is real-valued and } f \in W^{m+1,\infty}(T)\right]. \end{aligned} \tag{1.12i}$$

In (1.12g) $\#f(T)$ denotes the winding number of $f(t)$, $t \in T$, with respect to 0. Since it is 0, there exists a continuous branch of the logarithm. In (1.12i) we used an empty bracket to denote the

[1] One could replace the sum in (1.9) by a maximum and the maximum in (1.10) by a sum. For Sobolev spaces of non-periodic functions the norm $(\Sigma_{j=0}^m \| f^{(j)} \|_p^p)^{1/p}$ is widely used.

identity function (following a proposal of C. de Boor):

$$(\ ):\ t \mapsto t. \tag{1.13}$$

In particular, $g(\ )+(\ )$ is the function $t \mapsto g(t)+t$. The composition $f(g(\ )+(\ ))$ involves an implicit equivalence modulo 2π.

The rules (1.12a)–(1.12h) have been compiled by Bernhardsgrütter [3], except that he restricted $W^{m+1,p}$ to $W^{1,2}$; our assumptions in (1.12h) are also less restrictive than his. The crucial properties (1.12b) and (1.12h) of $C^{m,\alpha}(T)$ are, e.g., also proven in Hörmander [31], where many results on multivariate Hölder spaces are summarized.

Outline of the proof of Lemma 1.1. All the functions involved are continuous on a compact set. Therefore $0 \notin f(T)$ implies that $f(T)$ is bounded away from zero, hence (1.12e)–(1.12g) are special cases of (1.12h). The rules (1.12a) and (1.12d) are trivial.

For the case $[0, \alpha]$ the somewhat surprising results (1.12b) and (1.12c) are proved in [48, p. 13]. (1.12h) and (1.12i) are readily verified. In particular,

$$\nu_\alpha(h \circ f) \leqslant \nu_1(h)\nu_\alpha(f). \tag{1.14}$$

In the cases $[m, \alpha]$ with $m > 0$ [and $(m+1, p)$ with $m \geqslant 0$] (1.12b) is verified by applying the Leibniz product rule for differentiation, and for (1.12c) one has just to insert the derivatives of $1/g$ instead of those of g. For (1.12h) one proves first by induction that

$$f \in C^{m,\alpha}(T), \quad h \in C^{m-1,\alpha}(f(T)) \ \Rightarrow\ h \circ f \in C^{m-1,\alpha}(T) \tag{1.15}$$

[and

$$f \in W^{m+1,p}(T), \quad h \in W^{m,\infty}(f(T)) \ \Rightarrow\ h \circ f \in W^{m,p}(T), \tag{1.16}$$

respectively]. For $m = 1$ one has in particular

$$\nu_\alpha(h \circ f) \leqslant \nu_\alpha(h)\,\|f'\|_\infty^\alpha \tag{1.17}$$

[and, since $\|(h \circ f)'\|_p \leqslant \|h'\|_\infty \|f'\|_p$,

$$\|h \circ f\| \leqslant \max\{\|h\|_\infty,\ \|h'\|_\infty \|f'\|_p\} \leqslant \|h\|(1 + \|f\|) \tag{1.18}$$

in the case $(1, p)$]. The induction step is then an immediate consequence of the chain rule $(h \circ f)'(t) = h'(f(t))f'(t)$, and (1.12h) follows from the chain rule and (1.15) [(1.16)].

Of course, (1.12i) is proved analogously. □

One may wonder how the spaces $C^{m,\alpha}(T)$ and $W^{m,p}(T)$ relate to each other. Clearly,

$$C^{m+1}(T) \subset C^{m,\alpha}(T) \subset C^{m,\tilde\alpha}(T) \subset C^m(T) \quad \text{if } 0 < \tilde\alpha < \alpha \leqslant 1 \tag{1.19}$$

and

$$W^{m+1,1}(T) \subset W^{m,p}(T) \subset W^{m,\tilde p}(T) \subseteq W^{m,1}(T) \quad \text{if } 1 \leqslant \tilde p < p \leqslant \infty. \tag{1.20}$$

Moreover, it is easy to see that

$$W^{m+1,\infty}(T) = C^{m,1}(T) \tag{1.21}$$

and from a result of W.H. Young [75, p. 105] one knows that

$$W^{m+1,p}(T) \subseteq C^{m,1-1/p}(T) \quad \text{if } p > 1. \tag{1.22}$$

As mentioned, functions defined on S can be considered as functions on T, and vice versa. Corresponding to this substitution we define operators I_{ST} and $I_{TS} = I_{ST}^{-1}$ by

$$h = I_{ST} f, \quad f = I_{TS} h, \quad \text{if } f(t) = h(\mathrm{e}^{it}). \tag{1.23}$$

2. The conjugate periodic function

The *conjugation operator* K: $L^2(T) \to L^2(T)$ is defined by associating to $f \in L^2(T)$ with the Fourier series

$$f \sim \sum_{k \in \mathbf{Z}} \hat{f}_k \, \mathrm{e}^{ikt} \tag{2.1}$$

the *conjugate periodic function* Kf with the Fourier series

$$Kf \sim -i \sum_{k \in \mathbf{Z}} \operatorname{sign}(k) \hat{f}_k \, \mathrm{e}^{ikt}. \tag{2.2}$$

Since $|\hat{f}_k| = |-\mathrm{i}\operatorname{sign}(k)\hat{f}_k|$ $(k \neq 0)$, since $f \in L^2(T)$ iff $\Sigma|\hat{f}_k|^2 < \infty$, and since $\|f\|_2^2 = \Sigma|\hat{f}_k|^2$, it is clear that $Kf \in L^2(T)$ and $\|Kf\|_2^2 = \|f\|_2^2 - |\hat{f}_0|^2$. Hence K is a bounded linear operator on $L^2(T)$, and its norm is

$$\|K\|_2 = 1. \tag{2.3}$$

The same remark applies to the Sobolev space $W^{m,2}(T)$ with norm (1.11) since its elements can also be characterized by a growth condition on the Fourier coefficients, $\Sigma k^{2m}|\hat{f}_k|^2 < \infty$, and since the norm depends only on the moduli of the Fourier coefficients. Thus

$$K\left(W^{m,2}(T)\right) \subset W^{m,2}(T), \quad \|K\| = 1 \tag{2.4}$$

if the norm (1.11) is used. Wegmann [70] has shown with an ingenious construction that in the case of $W^{1,2}$ (i.e. $m = 1$) with norm (1.10) one has $\|K\| \leqslant \pi/\sqrt{3}$. (Actually, the main estimate $\|Kf\|_\infty \leqslant \pi/\sqrt{3}\,\|f'\|_2$ of the proof was also given by Friberg [12].)

If we consider only functions with $\hat{f}_0 = 0$, which form subspaces of codimension 1 of the Hilbert spaces $L^2(T)$ and $W^{m,2}(T)$, the restrictions K_0 of K to these subspaces are unitary operators, and

$$K_0^{-1} = K_0^* = -K_0. \tag{2.5}$$

On $L^2(T)$ and $W^{m,2}(T)$, K is still skew-symmetric,

$$K^* = -K, \tag{2.6}$$

but no longer invertible, since its kernel is the set of constant functions.

The domain of K can be extended to $L^1(T)$: $f \in L^1(T)$ is replaced by a function f_r smoothed by the Poisson kernel; then, for almost every t the limit of

$$Kf_r(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sign}(k) r^{|k|} \hat{f}_k \, \mathrm{e}^{ikt} \tag{2.7}$$

as $r \uparrow 1$ can be seen to exist; however, the limit function Kf need not be in $L^1(T)$ [40, p. 64]. If it is, then formula (2.2) still holds, and only in this case is the series in (2.2) a Fourier series of some function in $L^1(T)$ [40, p. 64]. For $1 < p < \infty$ it was shown by M. Riesz that K is a bounded linear operator from $L^p(T)$ into itself:

$$K(L^p(T)) \subset L^p(T), \quad \|K\|_p < \infty, \qquad 1 < p < \infty \tag{2.8a}$$

(see, e.g., [16, p. 113], [40, p. 68], or [53, p. 380]). However, $\|K\|_p = O(1/(p-1))$ as $p \to 1$, and $\|K\|_p = O(p)$ as $p \to \infty$ [16, p. 113]. Earlier, Privalov had already shown that

$$K(\mathrm{Lip}^\alpha(T)) \subset \mathrm{Lip}^\alpha(T), \quad \|K\| < \infty, \qquad 0 < \alpha < 1 \tag{2.8b}$$

[16, p. 106]. From the fact that conjugation and differentiation commute as long as all functions involved remain in $L^1(T)$, it follows then that in any of the norms mentioned

$$K(W^{m,p}(T)) \subset W^{m,p}(T), \quad \|K\| < \infty, \qquad m \geqslant 0, \quad 1 < p < \infty, \tag{2.8c}$$

$$K(C^{m,\alpha}(T)) \subset C^{m,\alpha}(T), \quad \|K\| < \infty, \qquad m \geqslant 0, \quad 0 < \alpha < 1. \tag{2.8d}$$

Unfortunately, some other important spaces besides $L^1(T)$ are *not* mapped into themselves by K. There exist continuous functions whose conjugate functions are unbounded [16, p. 105]; hence

$$K(C(T)) \not\subseteq C(T), \tag{2.9a}$$

$$K(L^\infty(T)) \not\subseteq L^\infty(T). \tag{2.9b}$$

Other examples (e.g., [75, p. 157]) show that

$$K(\mathrm{Lip}^1(T)) \not\subseteq \mathrm{Lip}^1(T). \tag{2.9c}$$

A fortiori, (2.8c) does not hold for $p = 1$ and $p = \infty$, and (2.8d) does not hold for $\alpha = 1$. The so-called Dini-continuity of f suffices to guarantee that $Kf \in C(T)$ [16, p. 106]; this is a weaker condition than assuming that $f \in \mathrm{Lip}^\alpha(T)$ for some $\alpha > 0$, but it is rather impractical.

In the basis $\{e^{ikt}\}_{k \in \mathbf{Z}}$ of $L^2(T)$ the conjugation operator as defined by (2.1), (2.2) is represented by the biinfinite diagonal matrix

$$\hat{K} := \mathrm{diag}(-\mathrm{i}\ \mathrm{sign}(k))_{k \in \mathbf{Z}}. \tag{2.10}$$

Hence, once a function is represented in this basis, conjugation is nearly trivial. Concerning the numerical implementation of the conjugation process we are therefore motivated to approximate f (which in practice may be given on a discrete point set only) by a function whose Fourier coefficients can be computed rapidly; in addition, it should also be possible to evaluate the conjugate series fast. Trigonometric interpolation on a set of N equispaced points is a natural choice since both the (Fourier) coefficients of the interpolating trigonometric polynomial (Fourier analysis) and the values of the conjugate trigonometric polynomial at the same points (Fourier synthesis) can be computed by a fast Fourier transform (FFT), so that the costs are only $O(N \log N)$ operations, see [20,28] for details. Surprisingly, except for a larger overhead, conjugation of a periodic spline interpolant can be implemented equally efficiently [24]. Both the trigonometric and the spline interpolant are optimal approximations in certain appropriately chosen function spaces [6,24]. Unfortunately, trigonometric interpolation suffers often from Gibbs oscillations. In practice this occurs even when f is analytic but has singularities close to T. The effect of these oscillations in numerical conformal mapping is often devastating [22]. A simple but effective remedy is smoothing, which amounts to multiplying the Fourier coefficients by certain constants. Step type singularities of f or its derivatives can be taken into account analytically [24]. For some situations rational trigonometric approximation (or interpolation) of f seems promising, cf. [23].

The importance of the conjugation operator for our conformal mapping problem (or, generally, for functions analytic in the unit disk) is apparent from the following two theorems.

Theorem 2.1. *Let $h \in H^1$, and set*

$$\xi(t) := \operatorname{Re} h(e^{it}), \quad \eta(t) := \operatorname{Im} h(e^{it}), \qquad t \in T. \tag{2.11}$$

Then,

$$\eta(t) - \hat{\eta}_0 = K\xi(t) \quad (a.e.\ on\ T), \tag{2.12a}$$

$$\xi(t) - \hat{\xi}_0 = -K\eta(t) \quad (a.e.\ on\ T), \tag{2.12b}$$

and

$$\hat{\xi}_0 = \operatorname{Re} h(0), \quad \hat{\eta}_0 = \operatorname{Im} h(0). \tag{2.12c}$$

Proof. By our definition of H^1, $h \in L^1(S)$; hence, trivially, $\xi \in L^1(T)$ and $\eta \in L^1(T)$. For any $h \in L^1(S)$ the relations

$$\xi(t) = \tfrac{1}{2}\left[h(e^{it}) + \overline{h(e^{it})}\right], \quad \eta(t) = \frac{1}{2i}\left[h(e^{it}) - \overline{h(e^{it})}\right] \tag{2.13}$$

are transformed into

$$\hat{\xi}_k = \tfrac{1}{2}\left[\hat{h}_k + \overline{\hat{h}_{-k}}\right], \quad \hat{\eta}_k = \frac{1}{2i}\left[\hat{h}_k - \overline{\hat{h}_{-k}}\right]. \tag{2.14}$$

But here $h \in H^1$, hence

$$\hat{\xi}_k = \begin{cases} \tfrac{1}{2}\hat{h}_k, & k > 0, \\ \operatorname{Re} \hat{h}_0, & k = 0, \\ \tfrac{1}{2}\overline{\hat{h}_{-k}}, & k < 0, \end{cases} \qquad \hat{\eta}_k = \begin{cases} -\dfrac{i}{2}\hat{h}_k, & k > 0, \\ \operatorname{Im} \hat{h}_0, & k = 0, \\ \dfrac{i}{2}\overline{\hat{h}_{-k}}, & k < 0. \end{cases} \tag{2.15}$$

In view of (2.10) the relations (2.12) are now readily verified. □

Theorem 2.1 of course applies in particular to $h \in H^p$ $(1 \leqslant p \leqslant \infty)$ and to $h \in A(\overline{D})$. In the latter case $\xi, \eta \in C(T)$.

For our application to conformal mapping it is also important that the converse of Theorem 2.1 is true:

Theorem 2.2. *Assume $\xi, \eta \in L^p(T, \mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ satisfy* (2.12a) *or* (2.12b). *Then the function h defined by*

$$h(e^{it}) := \xi(t) + i\eta(t) \tag{2.16}$$

is in H^p, and all relations (2.12a)–(2.12c) *hold.*

If $\xi, \eta \in C(T, \mathbb{R})$, then $h \in A(\overline{D})$.

Proof. Clearly, $h \in L^p(S)$ and $\hat{h}_k = \hat{\xi}_k + i\hat{\eta}_k$. From (2.10) it follows that $\hat{h}_k = 0$, $\forall k < 0$; hence $h \in H^p$, and Theorem 2.1 holds.

If $\xi, \eta \in C(T, \mathbb{R})$, then $\operatorname{Re} h$ and $\operatorname{Im} h$ are both solutions of a Dirichlet problem (with continuous boundary values), hence $h \in A(\bar{D})$. □

Naturally there are also versions of Theorems 2.1 and 2.2 that apply to functions analytic *outside* the disk (including at ∞).

Theorem 2.1′. *Let $h \in H^1(D^c)$, and define ξ and η by* (2.11). *Then*

$$\eta(t) - \hat{\eta}_0 = -K\xi(t) \quad \text{(a.e. on } T\text{)}, \tag{2.12a′}$$

$$\xi(t) - \hat{\xi}_0 = K\eta(t) \quad \text{(a.e. on } T\text{)}, \tag{2.12b′}$$

and

$$\hat{\xi}_0 = \operatorname{Re} h(\infty), \quad \hat{\eta}_0 = \operatorname{Im} h(\infty). \tag{2.12c′}$$

Theorem 2.2′. *Assume $\xi, \eta \in L^p(T, \mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ satisfy* (2.12a′) *or* (2.12b′). *Then h defined by* (2.16) *is in $H^p(D^c)$, and all relations* (2.12a′)–(2.12c′) *hold. If $\xi, \eta \in C(T, \mathbb{R})$, then $h \in A(D^c)$.*

By simply computing a conjugate function, Theorems 2.2 and 2.2′ allow us to construct for a given real part ξ (or an imaginary part η) two functions $L_R^+\xi$ and $L_R^-\xi$ (or $L_I^+\eta$ and $L_I^-\eta$) that are analytic inside and outside the unit circle, respectively. If $1 < p < \infty$,

$$L_R^+: L^p(T, \mathbb{R}) \to H^p, \qquad L_R^+\xi(e^{it}) = ((I + iK)\xi)(t), \tag{2.17a}$$

$$L_R^-: L^p(T, \mathbb{R}) \to H^p(D^c), \quad L_R^-\xi(e^{it}) = ((I - iK)\xi)(t), \tag{2.17b}$$

$$L_I^+: L^p(T, \mathbb{R}) \to H^p, \qquad L_I^+\eta(e^{it}) = ((-K + iI)\eta)(t), \tag{2.17c}$$

$$L_I^-: L^p(T, \mathbb{R}) \to H^p(D^c), \quad L_I^-\eta(e^{it}) = ((K + iI)\eta)(t), \tag{2.17d}$$

and

$$[L_R^+\xi]_0^\wedge, \quad [L_R^-\xi]_0^\wedge, \quad i[L_I^+\eta]_0^\wedge, \quad i[L_I^-\eta]_0^\wedge \in \mathbb{R}. \tag{2.18}$$

Similarly, for every $\alpha \in (0, 1]$,

$$L_R^+, L_I^+: \operatorname{Lip}^\alpha(T) \to A(\bar{D}), \quad L_R^-, L_I^-: \operatorname{Lip}^\alpha(T) \to A(D^c). \tag{2.19}$$

Often, in particular when solving the Riemann–Hilbert problem of Section 3, we are interested in these operators rather than in the conjugation operator, which serves here as a tool. In practice, the Fourier analysis of ξ (or η) furnishes us with the Taylor coefficients of L_R^+ and L_R^- (or L_I^+ and L_I^-) directly.

Finally, we should mention that the conjugation operator is a singular integral operator: if $f \in L^1(T)$, then for almost every $t \in T$

$$\begin{aligned} Kf(t) &= \frac{1}{2\pi} \mathrm{PV} \int_T \cot\left(\frac{t-s}{2}\right) f(s) \mathrm{d}s \\ &:= \frac{1}{2\pi} \lim_{\delta \downarrow 0} \int_{\substack{T \\ |t-s| > \delta}} \cot\left(\frac{t-s}{2}\right) f(s) \mathrm{d}s \end{aligned} \tag{2.20}$$

[16, p. 103; 40, p. 79]. Often this principal value is used to define the conjugate function. But we believe that definition (2.2), which is linked to Fourier analysis, is preferable since some of the main applications (such as Theorems 2.1, 2.2) and one of the best numerical implementations (trigonometric interpolation using the FFT) become immediate. There are a few instances where (2.20) is actually used in applications. One is the method of Menikoff and Zemach [46], see Section 9; another is Berrut's integral equation [5].

Using (2.20) it is not difficult to prove

$$Kf(-(\ ))(t) = -Kf(-t) \quad \text{(a.e. on } T). \tag{2.21}$$

Similarly, the operators in (2.17) can be written in integral form. For example, $L_{\mathbb{R}}^{+}$ is given by Schwarz's integral [16, p. 102]:

$$L_{\mathbb{R}}^{+}\xi(w) = \frac{1}{2\pi}\int_T \frac{e^{is}+w}{e^{is}-w}\xi(s), \quad w \in D. \tag{2.22}$$

The formula remains correct a.e. on S if we replace the integral by a principal value integral.

3. The Riemann–Hilbert problem on the disk

The most promising methods for mapping the disk conformally onto a Jordan region with smooth boundary make use of the fact that Riemann–Hilbert problems on the disk can be solved very efficiently. In theory, the solution can be written explicitly in terms of two conjugate functions; in practice basically just four FFTs are needed to compute it. In our nonstandard setting the basic result is

Theorem 3.1. *Let $a \in C(T)$ be nonvanishing on T, so that it has the form*

$$a(t) = \alpha(t)\, e^{-i(mt+\phi(t))} \tag{3.1}$$

with $\alpha \in C(T, \mathbb{R}^{+})$, $\phi \in C(T, \mathbb{R})$, and $m \in \mathbb{Z}$ ($-m$ is the winding number of $a(T)$ with respect to 0). Assume that $K\phi \in C(T, \mathbb{R})$, and let $\beta \in L^{p}(T, \mathbb{R})$ ($1 < p < \infty$). The Riemann–Hilbert problem consists in finding a function $f \in H^{p}$ such that

$$\operatorname{Re}\{a(t)f(e^{it})\} = \beta(t) \quad (a.e.\ on\ T). \tag{3.2}$$

If $m \geqslant 0$, this problem has the general solution

$$f(w) = [iq(w) + w^{m}L_{\mathbb{R}}^{+}\sigma(w)]\, e^{L_{i}^{+}\phi(w)}, \tag{3.3}$$

where

$$\sigma(t) := \frac{\beta(t)}{\alpha(t)} e^{K\phi(t)} \tag{3.4}$$

and q is an arbitrary self-reciprocal polynomial of degree $2m$ (i.e. q satisfies $q(w) = w^{2m}\bar{q}(1/w)$). [2]

If $m < 0$, (3.2) has no solution $f \in H^{p}$ unless

$$\hat{\sigma}_k = 0, \quad k = 0, \ldots, -m-1, \tag{3.5}$$

in which case the function f of (3.3) with $q = 0$ is the unique solution.

[2] $\bar{q}$ denotes the polynomial with the conjugate complex coefficients.

Proof. Assume a solution $f \in H^p$ of (3.2) exists. Since ϕ, $K\phi \in C(T, \mathbb{R})$, we have $L_I^+\phi \in A(\overline{D})$, and hence s defined by

$$s(w) := w^{-m} f(w)\, e^{-L_I^+\phi(w)} \tag{3.6}$$

is in $w^{-m}H^p$ (i.e. $w \mapsto w^m s(w) \in H^p$). According to (3.2) and (2.17c)

$$\begin{aligned}\beta(t) &= \mathrm{Re}\{\alpha(t) s(e^{it}) \exp[L_I^+\phi(e^{it}) - i\phi(t)]\} \\ &= \alpha(t)\, e^{-K\phi(t)} \mathrm{Re}\, s(e^{it}),\end{aligned}$$

i.e.

$$\mathrm{Re}\, s(e^{it}) = \sigma(t). \tag{3.7}$$

On the other hand, it is easily verified that every solution $s \in w^{-m}H^p$ of (3.7) leads by inversion of (3.6), i.e. by setting

$$f(w) := w^m s(w)\, e^{L_I^+\phi(w)}, \tag{3.8}$$

to a solution $f \in H^p$ of (3.2). Hence we have to study the set of solutions of (3.7).

Clearly $\sigma \in L^p(T, \mathbb{R})$, and thus $L_R^+\sigma \in H^p$, cf. (2.17a). Therefore $s := L_R^+\sigma$ is an admissible solution of (3.7) if $m \geqslant 0$, and in the case $m = 0$ this solution is unique up to an additive imaginary constant $i\hat{q}_0$. If $m > 0$, the dimension of the solution space of the linear system (3.7) can increase by at most the real dimension $2m$ since H^p has complex codimension m as a subspace of $w^{-m}H^p$. If $\mathscr{P}_{2m}$ denotes the set of polynomials of degree at most $2m$, then the set

$$\{w \mapsto iw^{-m}q(w);\; q \in \mathscr{P}_{2m},\; w^{-m}q(w) = w^m\bar{q}(1/w)\} \subset w^{-m}H^p, \tag{3.9}$$

which forms a real linear space of dimension $2m+1$, consists of solutions of the homogeneous equation $\mathrm{Re}\, s(e^{it}) = 0$. By our dimension argument it must be the full solution space of the homogeneous equation. Hence (3.7) has the general solution $s(w) = L_R^+\sigma(w) + iw^{-m}q(w)$ with q as in (3.9). Inserting it into (3.8) finally yields (3.3).

If $m < 0$, (3.2) can have a solution f in H^p only if the solution $L_R^+\sigma \in H^p$ of (3.7) happens to be in $w^{-m}H^p$. In view of (2.2) and (2.17a), the condition (3.5) is clearly necessary and sufficient for this. (The additive imaginary constant is no longer allowed since it is not in $w^{-m}H^p$.) □

The integer $2m$ is called the *index* of the Riemann–Hilbert problem (3.2).

For the purpose of our applications to conformal mapping we are interested in certain particular solutions of problems with index 0 or 2:

Theorem 3.2. *Under the assumptions of Theorem* 3.1 *the following holds*:

(i) *If* $m = 0$ *and* $\hat{\phi}_0 - \frac{1}{2}\pi \notin \pi\mathbb{Z}$, (3.2) *has a unique solution* $f \in H^p$ *satisfying* $\mathrm{Im}\, \hat{f}_0 = 0$, *namely the one where in* (3.3)

$$q(w) \equiv \hat{q}_0 = -\hat{\sigma}_0 \tan \hat{\phi}_0. \tag{3.10}$$

(ii) *If* $m = 1$ *and* $\hat{\phi}_0 - \frac{1}{2}\pi \notin \pi\mathbb{Z}$, (3.2) *has a unique solution* $f \in H^p$ *satisfying* $\hat{f}_0 = 0$ *and* $\mathrm{Im}\, \hat{f}_1 = 0$, *namely the one where in* (3.3)

$$q(w) = \hat{q}_1 w = -w\hat{\sigma}_0 \tan \hat{\phi}_0. \tag{3.11}$$

Proof. (i) By the mean value principle for analytic functions and by (2.2) and (2.17a, c), we see that

$$\left[e^{L_I^+\phi}\right]_0^\wedge = e^{L_I^+\phi(0)} = e^{i\hat{\phi}_0},$$

$$[L_R^+\sigma]_0^\wedge = \hat{\sigma}_0,$$

$$\hat{f}_0 = [iq + L_R^+\sigma]_0^\wedge \left[e^{L_I^+\phi}\right]_0^\wedge = (i\hat{q}_0 + \hat{\sigma}_0)\, e^{i\hat{\phi}_0}. \tag{3.12}$$

The assumption on $\hat{\phi}_0$ implies that $e^{i\hat{\phi}_0} \in S\setminus\{i, -i\}$. Since q is a self-reciprocal polynomial of degree 0, $q(w) \equiv \hat{q}_0 \in \mathbb{R}$. Hence, if $\hat{\sigma}_0 = 0$, clearly $\operatorname{Im} \hat{f}_0 = 0$ iff $\hat{q}_0 = 0$. If $\hat{\sigma}_0 \neq 0$, we need to determine $\hat{q}_0$ such that

$$\arg\{i\hat{q}_0 + \hat{\sigma}_0\} \equiv -\hat{\phi}_0 \pmod{\pi},$$

i.e. $\arctan(\hat{q}_0/\hat{\sigma}_0) = -\hat{\phi}_0$.

(ii) If $m = 1$, the additional factor w in (3.3) has the effect that (3.12) is replaced by $\hat{f}_0 = i\hat{q}_0\, e^{i\hat{\phi}_0}$. Hence $\hat{f}_0 = 0$ iff $\hat{q}_0 = 0$. But then

$$\hat{f}_1 = \left(i\hat{q}_1 + [L_R^+\sigma]_0^\wedge\right)\left[e^{L_I^+\phi}\right]_0^\wedge = (i\hat{q}_1 + \hat{\sigma}_0)\, e^{i\hat{\phi}_0},$$

where $\hat{q}_1 \in \mathbb{R}$ since q is self-reciprocal of degree 2. Obviously the determination of $\hat{q}_1$ becomes identical to the determination of $\hat{q}_0$ in (i). □

The smoothness of a and β has of course an effect on the smoothness of f. Assuming

$$\begin{aligned} &t \mapsto e^{imt}a(t) \in C^{l,\tilde{\alpha}}(T) \quad \left[\text{or } W^{l+1,p}(T)\right], \\ &\beta \in C^{l,\tilde{\alpha}}(T, \mathbb{R}) \quad \left[\text{or } W^{l+1,p}(T, \mathbb{R}) \text{ respectively}\right] \end{aligned} \tag{3.13}$$

for some integer $l \geqslant 0$ and some $\tilde{\alpha} \in (0, 1)$ [or $p \in (1, \infty)$], we conclude from the rules (1.12) and the results (2.8) that the functions α, ϕ, $K\phi$, $e^{K\phi}$, σ are all in $C^{l,\tilde{\alpha}}(T, \mathbb{R})$ $[W^{l+1,p}(T, \mathbb{R})]$ and that $L_R^+\sigma$, $L_I^+\phi$, $e^{L_I^+\phi}$, and finally f lie in $C^{l,\tilde{\alpha}}(S)$ $[W^{l+1,p}(S)]$. Thus we get

Theorem 3.3. *In addition to the assumptions of Theorem* 3.1 *suppose that* (3.13) *holds for some integer* $l \geqslant 0$ *and some* $\tilde{\alpha} \in (0, 1)$ [*or some* $p \in (1, \infty)$, *respectively*]. *Then the solutions* f *of the Riemann–Hilbert problem* (3.2) *satisfy* $f|_S \in C^{l,\tilde{\alpha}}(S)$ [*or* $W^{l+1,p}(S)$, *respectively*].

The case $l = 0$, $\tilde{\alpha} \in (0, 1)$ (i.e. $e^{im(\ )}a \in \mathrm{Lip}^{\tilde{\alpha}}(T)$, $\beta \in \mathrm{Lip}^{\tilde{\alpha}}(T, \mathbb{R})$ corresponds to the standard treatment of the Riemann–Hilbert problem [30,48].

Naturally, there is also a version of the Riemann–Hilbert problem where the function sought is analytic in the exterior $\overline{D}^c$ of the unit circle:

Theorem 3.4. *Under the assumptions of Theorem* 3.1 *the functions* $f \in H^p(D^c)$ *satisfying* (3.2) *are in the case* $m \leqslant 0$ *given by*

$$f(w) = \left[iq(1/w) + w^m L_R^- \sigma(w)\right] e^{L_I^-\phi(w)}, \tag{3.14}$$

where

$$\sigma(t) := \frac{\beta(t)}{\alpha(t)} e^{-K\phi(t)} \tag{3.15}$$

and q is an arbitrary self-reciprocal polynomial of degree $-2m$. In the case $m > 0$, the problem has no solution except when $\hat{\sigma}_k = 0$, $k = -m+1, \ldots, 0$; then $q = 0$ yields the only solution.

The proof is nearly word-for-word identical to that of Theorem 3.1. The substitution of $q(w)$ in (3.3) by $q(1/w)$ in (3.14) is due to the fact that (3.9) is replaced by

$$\{ w \mapsto \mathrm{i} w^m q(w);\ q \in \mathscr{P}_{-2m},\ w^m q(w) = w^{-m} \bar{q}(1/w) \} \subset w^{-m} H^p(D^c).$$

The analogues of Theorems 3.2 and 3.3 for the exterior problem are also easily established.

4. A general approach to deriving integral equations for the boundary correspondence function

Now we are ready to attack the conformal mapping problem. The boundary Γ of the Jordan region Δ is assumed to be rectifiable and given in parametric form

$$\Gamma := \{ \gamma(t);\ t \in T \}. \tag{4.1}$$

γ is assumed to be a regular parametrization of Γ, so that $\gamma \in W^{1,\infty}(T)$, $1/\gamma' \in L^\infty$. Further smoothness assumptions will be made later.

We first concentrate on the interior mapping problem, but at the end of this section we indicate the few modifications required for the exterior problem, which will also be covered in some of the examples of Section 6.

One of the key facts in numerical conformal mapping is that the mapping function, which we now call g_i (i for interior), is uniquely determined by its boundary values. (Since $g_\mathrm{i} \in A(\overline{D})$, Cauchy's integral formula is valid.) Hence, it suffices to compute first the *reduced boundary correspondence function* $\tau_\mathrm{i} \in C(T, \mathbb{R})$ satisfying

$$g_\mathrm{i}(\mathrm{e}^{\mathrm{i}t}) := \gamma(\tau_\mathrm{i}(t) + t), \tag{4.2}$$

which is determined up to an irrelevant additive multiple of 2π, and thus may be normalized by

$$-\pi < [\tau_\mathrm{i}]_0^\wedge \leqslant \pi \tag{4.3a}$$

or

$$-\pi < \tau_\mathrm{i}(0) \leqslant \pi. \tag{4.3b}$$

Many formulas involving γ are simpler if one works instead with the *boundary correspondence function* θ_i related to τ_i by

$$\theta_\mathrm{i}(t) := \tau_\mathrm{i}(t) + t;$$

which however is not periodic and therefore is not directly accessible to Fourier analysis. (θ_i could be considered as an element of $C(T, T)$, but then θ_i and τ_i would be related by an equality modulo 2π, which we prefer to take care of implicitly when composing γ with θ_i, cf. (4.2).)

Our basic notation for the interior mapping problem is summarized in Fig. 1.

Of course, the smoothness of Γ and γ is related to the smoothness of g_i and τ_i. In fact, many results on this connection have been obtained in the past [18, pp. 417–428; 41,44,51,52,54,55, 63–65,67,68]. We cannot go into the details here, but we assume that one of the following typical

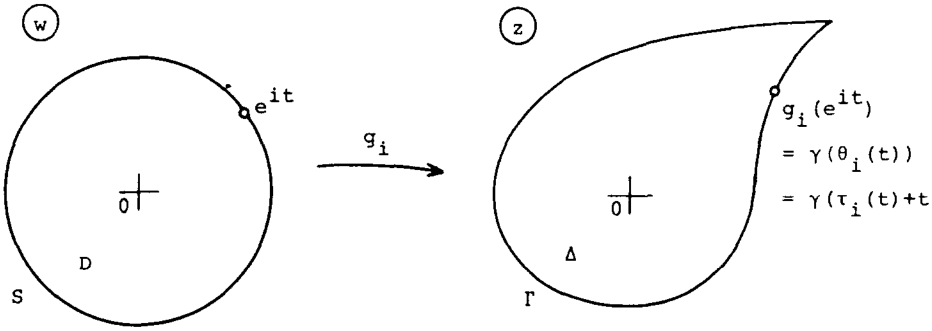

Fig. 1. Notation for the interior mapping problem.

situations applies to our problem:

Case $(m+1,\ p)$ $(m \geqslant 0,\ 1 < p < \infty)$:

$$\begin{aligned} &\gamma \in W^{m+1,\infty}(T, \mathbb{R}) = C^{m,1}(T, \mathbb{R}), \\ &\tau_i \in W^{m+1,p}(T, \mathbb{R}), \quad \gamma \circ \theta_i \in W^{m+1,p}(T), \\ &g_i|_S \in W^{m+1,p}(S), \quad g_i^{(m)} \in A(\bar{D}). \end{aligned} \tag{4.4a}$$

Case $[m,\ \alpha]$ $(m \geqslant 0,\ 0 < \alpha < 1)$:

$$\begin{aligned} &\gamma \in \begin{cases} C^{m,\alpha}(T, \mathbb{R}) & \text{if } m \geqslant 1, \\ \mathrm{Lip}^1(T, \mathbb{R}) & \text{if } m = 0, \end{cases} \\ &\tau_i \in C^{m,\alpha}(T, \mathbb{R}), \quad \gamma \circ \theta_i \in C^{m,\alpha}(T), \\ &g_i|_S \in C^{m,\alpha}(S), \quad g_i^{(m)} \in A(\bar{D}). \end{aligned} \tag{4.4b}$$

Note that by (1.12i) the assumptions on γ and τ_i imply the statement on $\gamma \circ \theta_i$, which in turn implies trivially the statements on g_i.

For example, if Γ is piecewise analytic and $|\gamma'(t)|$ is constant a.e., and if $\alpha\pi \in (0, \pi)$ is a lower bound for the smallest interior angle and $1 < p < (1-\alpha)^{-1}$, then the cases $(1, p)$ and $[0, \alpha]$ are known to obtain under mild additional assumptions, see [44,22] and [68], respectively. If $\gamma \in C^{m,\alpha}(T, \mathbb{R})$ with $m \geqslant 1$, case $[m, \alpha]$ is implied by a result of Kellog and Warschawski [18, p. 414; 63,64,67].

We now introduce an operator H such that the auxiliary function mentioned in the introduction is the image of g_i under H. $\mathscr{D}_H$ and $\mathscr{R}_H$ denote the domain and the range of H.

Definition of the operator H: Assuming case $(m+1, p)$ or case $[m, \alpha]$ and $0 \leqslant l \leqslant m$, let

$$H\colon\ \mathscr{D}_H \subseteq C^l(S) \to \mathscr{R}_H \subseteq C(S) \tag{4.5a}$$

be a (possibly nonlinear) operator of the form

$$Hg(w) := h\big(g(w),\ g'(w), \ldots, g^{(l)}(w);\ w\big), \quad w \in S, \tag{4.5b}$$

with the following properties:

(i) $\mathscr{D}_H^p := \mathscr{D}_H \cap W^{l+1,p}(S) \;\Rightarrow\; \mathscr{R}_H^p := H(\mathscr{D}_H^p) \subseteq W^{1,p}(S)$

or

$\mathscr{D}_H^\alpha := \mathscr{D}_H \cap C^{l,\alpha}(S) \;\Rightarrow\; \mathscr{R}_H^\alpha := H(\mathscr{D}_H^\alpha) \subseteq \mathrm{Lip}^\alpha(S)$,

respectively;

(ii) $g_i|_S \in \mathscr{D}_H$ and $Hg_i|_S \in A(\overline{D})|_S$;

(iii) H is continuous at $g_i|_S$ with respect to both the spaces in (4.5a) and those in (i);

(iv) H is invertible, i.e. given $r \in \mathscr{R}_H$, there is a unique $H^{-1}r = g \in \mathscr{D}_H$ such that $Hg = r$;

(v) $r \in \mathscr{R}_H \cap A(\overline{D})|_S \;\Rightarrow\; H^{-1}r \in A(\overline{D})|_S$;

(vi) H^{-1} is continuous at $Hg_i|_S$ with respect to both the spaces in (4.5a) and those in (i).

We are particularly interested in cases where both H and H^{-1} are given by simple formulas, e.g.

$$Hg(w) = \log\frac{g(w)}{w}, \qquad g(w) = w\,\mathrm{e}^{Hg(w)}. \tag{4.6}$$

Basically, iterative methods for the mapping problem iteratively modify some given function $g \in \mathscr{D}_H$ in such a way that its image under H approaches $Hg_i|_S$. (The reader may wonder why the trivial choice $Hg = g$ is not the best. One reason is that, for example, the function Hg in (4.6) makes it easier to take care of the normalization (0.1).)

A crucial point is that $Hg_i|_S$ can be written in terms of h (known), γ (known), and θ_i or τ_i (unknown, but real-valued). More generally, whenever

$$\theta(t) := \tau(t) + t \tag{4.7}$$

and $\tau \in C^l(T, \mathbb{R})$, the function g defined by

$$g(\mathrm{e}^{\mathrm{i}t}) := \gamma(\theta(t)) = \gamma(\tau(t) + t) \tag{4.8}$$

satisfies

$$g'(\mathrm{e}^{\mathrm{i}t}) = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\gamma'(\theta(t))\theta'(t), \ldots, g^{(l)}(\mathrm{e}^{\mathrm{i}t}) = \ldots\,. \tag{4.9}$$

If $g \in \mathscr{D}_H$, we get

$$\begin{aligned} Hg(\mathrm{e}^{\mathrm{i}t}) &= h\big(\gamma(\theta(t)),\ -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\gamma'(\theta(t))\theta'(t), \ldots;\ \mathrm{e}^{\mathrm{i}t}\big) \\ &= h\big(\gamma(\tau(t)+t),\ -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\gamma'(\tau(t)+t)(\tau'(t)+1), \ldots;\ \mathrm{e}^{\mathrm{i}t}\big) \\ &=: G\tau(t). \end{aligned} \tag{4.10}$$

Using our operator I_{TS}, which redefines a function of w in terms of t, we can write this simply as $G\tau := I_{TS}Hg$. Thus (4.10) defines an operator

$$G\colon\ \mathscr{D}_G \to \mathscr{R}_G \subseteq I_{TS}(\mathscr{R}_H), \qquad \tau \mapsto G\tau := I_{TS}Hg \tag{4.11a}$$

on some $\mathscr{D}_G \subseteq C^l(T, \mathbb{R})$ containing

$$\mathscr{D}_G^0 := \big\{\tau \in C^l(T, \mathbb{R});\ (4.8) \text{ holds for some } g \in \mathscr{D}_H\big\}. \tag{4.11b}$$

From Lemma 1.1 and assumption (4.4) we know that

$$\tau \in W^{l+1,p}(T, \mathbb{R}) \ \rightarrow \ \gamma \circ \theta \in W^{l+1,p}(T), \tag{4.12a}$$

$$\tau \in C^{l,\alpha}(T, \mathbb{R}) \ \Rightarrow \ \gamma \circ \theta \in C^{l,\alpha}(T), \tag{4.12b}$$

respectively. Hence, property (i) implies that for $\mathscr{D}_G = \mathscr{D}_G^0$

$$\mathscr{D}_G^p := \mathscr{D}_G \cap W^{l+1,p}(T, \mathbb{R}) \ \Rightarrow \ \mathscr{R}_G^p := G(\mathscr{D}_G^p) \subseteq W^{1,p}(T), \tag{4.13a}$$

$$\mathscr{D}_G^\alpha := \mathscr{D}_G \cap C^{l,\alpha}(T, \mathbb{R}) \ \Rightarrow \ \mathscr{R}_G^\alpha := G(\mathscr{D}_G^\alpha) \subseteq \mathrm{Lip}^\alpha(T), \tag{4.13b}$$

respectively, and we require that this hold still if $\mathscr{D}_G \supset \mathscr{D}_G^0$.

Now, $G\tau_i = I_{TS} Hg_i|_S \in A(\overline{D})|_T$ by property (ii), and hence Theorem 2.1 applies. More generally, whenever $G\tau \in A(\overline{D})|_T$, the relations (2.12a) and (2.12b) hold for

$$\xi := \mathrm{Re}\, G\tau, \qquad \eta := \mathrm{Im}\, G\tau, \tag{4.14}$$

and they yield two basically equivalent equations of the form

$$\Psi\tau(t) := \psi(\tau(t), \tau'(t), \ldots, \tau^{(l)}(t);\ t) \equiv 0, \tag{4.15}$$

where either

$$\Psi\tau(t) := \eta(t) - \hat{\eta}_0 - K\xi(t) \tag{4.16a}$$

or

$$\Psi\tau(t) := \xi(t) - \hat{\xi}_0 + K\eta(t). \tag{4.16b}$$

Note that in the case $l = 0$ we have simply

$$\xi(t) = \mathrm{Re}\, h(\gamma(\theta(t));\ \mathrm{e}^{it}), \quad \eta(t) = \mathrm{Im}\, h(\gamma(\theta(t));\ \mathrm{e}^{it}).$$

Since $\Psi\tau$ contains the conjugation operator K, (4.15) is in view of (2.20) a nonlinear singular integral equation for τ if $l = 0$ and an integro-differential equation for τ if $l > 0$.

The operator Ψ is naturally defined for any $\tau \in \mathscr{D}_G$, but in general $\Psi\tau$ need not be a continuous function (cf. (2.9a)):

$$\Psi\colon\ \mathscr{D}_\Psi := \mathscr{D}_G \to \mathscr{R}_\Psi \subseteq L^2(T, \mathbb{R}). \tag{4.16c}$$

However, in view of (2.8b), (2.8c), and (4.13) it is clear that

$$\mathscr{R}_\Psi^p := \Psi(\mathscr{D}_G^p) \subseteq W^{1,p}(T, \mathbb{R}), \tag{4.17a}$$

$$\mathscr{R}_\Psi^\alpha := \Psi(\mathscr{D}_G^\alpha) \subseteq \mathrm{Lip}^\alpha(T, \mathbb{R}). \tag{4.17b}$$

Theorem 4.1. *If* $\tau \in \mathscr{D}_\Psi^0 := \mathscr{D}_G^0$ *is a solution of* (4.15), *then*

$$(H^{-1} I_{ST} G\tau)(\mathrm{e}^{it}) = \gamma(\tau(t) + t), \quad t \in T, \tag{4.18}$$

and these are the boundary values of some $g \in A(D)$ *which maps* D *conformally onto* Δ. *There exists exactly one solution* τ *normalized by* (4.3a) *or* (4.3b) *such that* g *satisfies the normalization* (0.1) *or* (0.2), *respectively.*

Proof. (A generalization of Gaier's proof for Theodorsen's equation [15, p. 55].) The existence of a solution τ with the stated properties is an immediate consequence of the existence of the solution g_i of our mapping problem and our derivation of (4.15); in particular, $\tau_i \in \mathscr{D}_G^0$.

$I_{ST}G\tau_i = Hg_i$, hence by (4.2) τ_i satisfies (4.18).

Let us in turn assume that $\tau \in \mathscr{D}_G^0$ is any solution of (4.15). Then $G\tau \in A(\overline{D})|_T$ by Theorem 2.2, and according to properties (iv) and (v) of H,

$$g := H^{-1}I_{ST}G\tau \in \mathscr{D}_H \cap A(\overline{D})|_S,$$

and this function satisfies (4.8) and (4.10); hence (4.18) holds. From (4.8) it is clear that g maps S onto Γ and that $g(\mathrm{e}^{\mathrm{i}t})$ winds around Γ once while $\mathrm{e}^{\mathrm{i}t}$ winds around S. By the argument principle g is therefore a one-to-one map of D onto Δ, and hence conformal. Finally, it is well known that the mapping function is uniquely normalized by (0.1) or (0.2). □

Note that properties (i), (iii) and (vi) of H have not been used in the proof of Theorem 4.1.

In Section 6 we will apply the theorem to particular operators H, i.e. to particular integral and integro-differential equations. It will turn out that the assertion still holds under the weaker assumption $\tau \in \mathscr{D}_\Psi$ (instead of $\mathscr{D}_\Psi^0$).

When presenting examples in Section 6 it will be our policy to restrict the domain $\mathscr{D}_H$ so that (4.15) has only one solution, which takes account of the normalization (0.1) or (0.2). (In some cases $g'(0) > 0$ is actually replaced by the weaker conditon $g'(0) \in \mathbb{R}$, so that there are two solutions.)

Another simple but useful result is

Theorem 4.2. *If $l < m$, the operators G and Ψ, restricted to $\mathscr{D}_G^0 \cap W^{l+1,p}(T, \mathbb{R})$, are continuous at $\tau = \tau_i$ with respect to the norm of $W^{l+1,p}(T)$ in the domain and the norm of $W^{1,p}(T)$ in the range.*

For the proof we need

Lemma 4.3. *If $\gamma \in W^{m+1,\infty}(T)$ and $l < m$, $1 \leqslant p \leqslant \infty$, the nonlinear operator F: $W^{l+1,p}(T, \mathbb{R}) \to W^{l+1,p}(T)$ defined by F: $\tau \mapsto \gamma(\tau(\ ) + (\ ))$ is continuous.*

Proof. Let $\tilde{\tau} = \tau + \delta$, $\tilde{\theta} = \theta + \delta$, and $j \leqslant l+1$. Note that

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}t^j}(\gamma \circ \tilde{\theta} - \gamma \circ \theta)(t) &= \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}[(\gamma' \circ \tilde{\theta})\tilde{\theta}' - (\gamma' \circ \theta)\theta'](t) \\
&= \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}[(\gamma' \circ \tilde{\theta} - \gamma' \circ \theta)\tilde{\theta}' - (\gamma' \circ \theta)\delta'](t) \\
&= \sum_{s=0}^{j-1} \binom{j-1}{s} \left\{ \frac{\mathrm{d}^s}{\mathrm{d}t^2}(\gamma' \circ \tilde{\theta} - \gamma' \circ \theta)(t)\tilde{\theta}^{j-s}(t) + \frac{\mathrm{d}^s}{\mathrm{d}t^s}(\gamma' \circ \theta)(t)\delta^{(j-s)}(t) \right\}.
\end{aligned}$$

Here, $\gamma' \circ \theta \in W^{l+1,p}$ by (1.12i). If $\|\cdot\|$ denotes the norm (1.10) of $W^{l+1,p}(T)$, we conclude that for $j \leqslant l$ the L^∞-norm and for $j = l+1$ the L^p-norm of the above function is bounded by

$$\sum_{s=0}^{j-1} \binom{j-1}{s} \left\{ \left\| \frac{\mathrm{d}^s}{\mathrm{d}t^s}(\gamma' \circ \tilde{\theta} - \gamma' \circ \theta) \right\|_\infty \|\tilde{\theta}\| + \left\| \frac{\mathrm{d}^s}{\mathrm{d}t^s}(\gamma \circ \theta) \right\|_\infty \|\delta\| \right\}.$$

We claim that $\|\mathrm{d}^s(\gamma' \circ \tilde{\theta} - \gamma' \circ \theta)/\mathrm{d}t^s\|_\infty = \mathrm{O}(\sum_{k=0}^s \|\delta^{(k)}\|_\infty)$ whenever $\gamma' \in W^{s+1,\infty}$ and θ, $\tilde{\theta} \in W^{s,\infty}$. This can indeed be shown by induction, using the same arguments as above. □

Proof of Theorem 4.2. By Lemma 4.3, $\tau \mapsto \gamma \circ \theta$ is continuous; by (iii), $g := \gamma \circ \theta \mapsto Hg = I_{ST} G\tau$ is continuous; clearly, $G\tau \mapsto \xi := \operatorname{Re} G\tau$ and $G\tau \mapsto \eta := \operatorname{Im} G\tau$ are continuous; and by (2.8c) K is continuous on $W^{1,p}(T)$. $\square$

Our approach is readily modified for the *exterior mapping problem*. The properties required for H remain basically the same; of course, g_i is replaced by g_e, and in (ii) $A(\overline{D})|_S$ is replaced by $A(D^c)$. Moreover, (v) becomes

$$\text{(v}'\text{)} \qquad r \in \mathscr{R}_H \cap A(D^c)|_S \quad \Rightarrow \quad w \mapsto \frac{H^{-1} r(w)}{w} \in A(D^c)|_S.$$

No further modifications are necessary up to the definition (4.16) of $\Psi\tau$, where the sign changes between Theorems 2.1 and 2.1′ have their effect: now, either

$$\Psi\tau(t) := \eta(t) - \hat{\eta}_0 + K\xi(t) \tag{4.19a}$$

or

$$\Psi\tau(t) := \xi(t) - \hat{\xi}_0 - K\eta(t). \tag{4.19b}$$

There is always a trivial second approach to the exterior mapping problem: by inversions $z \mapsto 1/z$ and $w \mapsto 1/w$ in both planes it can be transformed into an interior problem for the boundary curve $\Gamma_i := \{z;\ 1/z \in \Gamma\}$. Let g_i, θ_i be the solution of the latter. Then we have

$$\gamma_i(\theta_i(t)) = g_i(e^{it}) = 1/g_e(e^{-it}) = 1/\gamma(\theta_e(-t)). \tag{4.20}$$

If $\Psi_i\tau = 0$ is an equation of type (4.15) with $l = 0$ for the interior problem, substituting (4.20) into it leads to an equation of the same type for the exterior problem. If $l > 0$, substitution formulas for the derivatives γ_i' and θ_i' have to be derived by differentiating (4.20). Examples for this second approach are also given in Section 6.

5. Direct iteration: methods of successive conjugation and Bergström type methods

Theorem 4.1 motivates us to try to solve (4.15). A first approach to this is *direct iteration* or, as it may be called here, *successive conjugation*: Assuming that $\tau^{(l)}$ is actually present in (4.15), we rewrite (4.15) in the form

$$\tau^{(l)}(t) = \Phi\tau(t) =: \phi(\tau(t), \tau'(t), \ldots, \tau^{(l)}(t);\ t) \tag{5.1}$$

with an operator Φ defined in some $\mathscr{D}_\Phi \subseteq \mathscr{D}_\Psi = \mathscr{D}_G$ whose values on $\mathscr{D}_\Phi^p := \mathscr{D}_\Phi \cap W^{l+1,p}$ and $\mathscr{D}_\Phi^\alpha := \mathscr{D}_\Phi \cap C^{l,\alpha}$ are supposed to satisfy

$$\mathscr{R}_\Phi^p := \Phi(\mathscr{D}_\Phi^p) \subseteq \{\sigma \in W^{1,p}(T, \mathbb{R});\ \sigma = \tau^{(l)} \text{ for some } \tau \in \mathscr{D}_G^p\}, \tag{5.2a}$$

$$\mathscr{R}_\Phi^\alpha := \Phi(\mathscr{D}_\Phi^\alpha) \subseteq \{\sigma \in \operatorname{Lip}^\alpha(T, \mathbb{R});\ \sigma = \tau^{(l)} \text{ for some } \tau \in \mathscr{D}_G^\alpha\}. \tag{5.2b}$$

In particular, $\Phi\tau$ must be the lth derivative of a periodic function, i.e. we must have

$$[\Phi\tau]_0^\wedge = 0 \quad \text{if } l > 0. \tag{5.3}$$

We assume that (5.1) is equivalent to (4.15) in the sense that

$$\Phi\tau = \tau^{(l)} \quad \text{iff } \Psi\tau = 0, \tag{5.4}$$

and we also suppose that the continuity of Ψ as expressed in Theorem 4.2 is reflected in continuity properties of Φ. In specific cases, such as when $\Psi\tau = \Phi\tau - \tau$, this is trivial.

The following iteration suggests itself for solving (5.1):

$$\tau_{n+1} := \Phi\tau_n \quad \text{if } l = 0, \tag{5.5a}$$

$$\tau_{n+1}(t) := \int^t \int^{t_1} \cdots \int^{t_{l-1}} \Phi\tau_n(t_l)\mathrm{d}t_l \ldots \mathrm{d}t_1 \quad t \in T, \text{ if } l > 0, \tag{5.5b}$$

$n = 0, 1, \ldots$. Of course, an initial approximation $\tau_0 \in \mathscr{D}_G^p$ (or $\mathscr{D}_G^\alpha$) must be given. Note that there is only one free constant in the integration in (5.5b) (all the others are eliminated by the periodicity requirement), and this constant is easily determined in the case of normalization (0.2), which we assume to imply that $\tau_{n+1} \in \mathscr{D}_G^p$ (or $\mathscr{D}_G^\alpha$, respectively) in accordance with assumption (5.2). Then iteration (5.5) is well defined.

Since $\tau^{(l)}$ may appear in Φ also, there are infinitely many ways to transform (4.15) into (5.1), but often there is a 'natural' one. The hope is to find one for which (5.5) converges fast. If convergence takes place, it is typically linear.

In practice one of course has to discretize. If sufficiently many points are used, this usually has little influence on the local convergence behavior of (5.5), but discretization may create additional solutions of (4.15) for which θ may not even be monotone [22,33].

Direct iteration methods include, e.g., Theodorsen's method ($l = 0$), the Melentiev–Kulisch method ($l = 0$), Timman's method ($l = 1$), and Friberg's method ($l = 1$), see Section 6.

In Theodorsen's method convergence can be improved drastically by applying suitable convergence acceleration techniques [21,22,32], cf. Section 6; these, on the other hand, can be considered sometimes as direct iteration methods corresponding to another version of (5.1).

The standard way to establish global convergence of (5.5a) for arbitrary initial approximations τ_0 in $\mathscr{D}_\Phi$ consists in proving that Φ is a contraction on $\mathscr{D}_\Phi$, i.e. $\Phi(\mathscr{D}_\Phi) \subseteq \mathscr{D}_\Phi$ and in some norm

$$\| \Phi\tau - \Phi\tilde{\tau} \| \leqslant L \| \tau - \tilde{\tau} \|, \quad \tau, \tilde{\tau} \in \mathscr{D}_\Phi \tag{5.6}$$

with a fixed $L \in (0, 1)$. If this holds only for some (possibly small) neighborhood $\mathscr{D}'_\Phi$ of τ_{i} we get local convergence. In particular, the Lipschitz condition (5.6) follows if Φ is Fréchet-differentiable at τ_{i}, Φ'_τ is uniformly bounded on $\mathscr{D}_\Phi$, and any two points τ and $\tilde{\tau}$ of $\mathscr{D}_\Phi$ can be connected by a rectifiable arc whose length is $\mathrm{O}(\tau - \tilde{\tau})$. According to Ostrowski's theorem [50, p. 300], the local convergence of discretized versions of (5.5a) can be proved by showing that the moduli of the eigenvalues of the discretized F-derivative $\Phi'_{\tau_{\mathrm{i}}}$ are bounded by some $L < 1$.

The same remarks apply to (5.5b) except that Φ is replaced by the composition of an l-fold integration operator with Φ.

Unfortunately, it is often impossible to prove global convergence, and even local convergence can sometimes only be proved for curves Γ that are close to a circle with center at the origin. For specific relations between Ψ and Φ it is easy to show in view of Theorem 4.2 and property (vi) that convergence of τ_n implies that the limit τ solves (4.15) and that $H^{-1}I_{ST}G\tau_n$ approaches a mapping function if $\{\tau_n\} \subset \mathscr{D}_G$.

Another direct iteration method, completely different from (5.5), can be tried if in a suitable neighborhood of $I_{TS}Hg_{\mathrm{i}} = G\tau_{\mathrm{i}}$ a continuous inverse of the operator G exists and can be evaluated easily. Then we have for τ (close to τ_{i}) satisfying $G\tau \in A(\overline{D})\,|_T$

$$\tau = G^{-1}P^+G\tau, \tag{5.7}$$

where P^+ denotes a suitable projection of $\mathscr{R}_G$ onto $\mathscr{R}_G \cap A(\bar{D})$, which has no effect on $G\tau$ though. Conversely however, if $\tau \in C(T, \mathbb{R})$ is any solution of (5.7), then $G\tau = P^+ G\tau$, i.e. $G\tau \in A(\bar{D})|_T$, and we can complete the proof of Theorem 4.1 as before:

Theorem 5.1. *Assume $\tau \in \mathscr{D}_G$ is such that G^{-1} is (uniquely) defined at $P^+G\tau$ and* (5.7) *holds. Then* (4.18) *and the other assertions of Theorem* 4.1 *hold.*

Equation (5.7) is a nonlinear integral equation with Cauchy kernel (due to P^+). It clearly suggests the iteration

$$\tau_{n+1} := G^{-1}P^+ G\tau_n, \tag{5.8}$$

which we call a *Bergström type method* since Bergström's method is of this kind (see section 6.2). Unfortunately, we can in general not assert that τ_{n+1} is well defined by (5.8) since $P^+ G\tau_n$ need not lie in the domain of G^{-1}.

Evaluation of g_i at interior points. Once τ_i has been constructed as the limit of iteration (5.5) or (5.8), the values of the mapping function are known on S or, in practice, at the Nth roots of unity. For computing values of g_i at points in D one could of course make use of Cauchy's integral formula, but it is much more appropriate to remember that g_i has been obtained via an approximation of $Hg_i = I_{ST}G\tau_i$. In all our examples of Section 6 the inversion formula $g_i = H^{-1}(Hg_i)$ is still correct at points in D, and at least in the examples with $l = 0$ it is very easy to evaluate. (If $l > 0$, one could switch to another H with $l = 0$ at this point.) There remains the question how to evaluate Hg_i. Depending on the number and the location of the points where this has to be done there exist several good algorithms. Usually, the iteration process (5.5) automatically yields the Fourier oefficients, i.e. the Taylor coefficients, of Hg_i. Evaluation on a set of equispaced points on a circle with center 0 can then be done with an FFT. To compute the values at a single point, Horner's algorithm can be applied. To compute such a single value directly from the known values at the roots of unity, there exist special barycentric formulas for the interpolation polynomial.

Let $t_k := 2\pi k/N$, $w_k := \exp(\mathrm{i}t_k)$, $\eta_k := (Hg_i)(w_k)$ $(k = 1, \ldots, N)$, and let $q(w)$ be the interpolation polynomial of degree $N-1$ for the data $(w_k, \eta_k)_{k=1,\ldots,N}$. The Lagrange representation of q is readily found to be

$$q(w) = \frac{w^N - 1}{N} \sum_{k=1}^{N} \frac{w_k}{w - w_k}\eta_k, \tag{5.9}$$

which, as usual, is brought into barycentric form by noting that $q(w) \equiv 1$ if $\eta_k = 1$ $(\forall k)$:

$$q(w) = \sum_{k=1}^{N} \frac{w_k}{w - w_k}\eta_k \Big/ \sum_{k=1}^{N} \frac{w_k}{w - w_k}. \tag{5.10}$$

If $w = \mathrm{e}^{\mathrm{i}t} \in S$, this can be written as

$$q(\mathrm{e}^{\mathrm{i}t}) = \sum_{k=1}^{N} \eta_k\, \mathrm{e}^{\mathrm{i}t_k/2} \operatorname{cosec} \tfrac{1}{2}(t - t_k) \Big/ \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}t_k/2} \operatorname{cosec} \tfrac{1}{2}(t - t_k) \tag{5.11}$$

[$\operatorname{cosec} t = (\sin t)^{-1}$]. As in the trigonometric case [4,29], the formulas (5.10) and (5.11) are very stable even if w is close to a w_k. In contrast, Cauchy's formula evaluated with the trapezoidal rule yields

$$c(w) = \frac{1}{N} \sum_{k=1}^{N} \frac{w_k \eta_k}{w_k - w} = \frac{q(w)}{1 - w^N}, \tag{5.12}$$

which deviates considerably from $q(w)$ if w is very close to or on S.

6. Examples of auxiliary operators and associated direct iteration methods

6.1. Theodorsen's method

Theodorsen's method makes use of the operator

$$Hg(w) := \log(g(w)/w), \quad w \in S, \tag{6.1a}$$

$$\mathscr{D}_H := \left\{ g \in C(S);\ 0 \notin g(S),\ \#g(S) = 1,\ \hat{\eta}_0 := [\arg\{g(\ )/(\ )\}]_0^\wedge = 0 \right\}, \tag{6.1b}$$

where $\#g(S)$ again denotes the winding number of $g(S)$ with respect to 0. Since $\#g(S) = 1$, a continuous branch of $\arg(g(w)/w)$ and hence of $\log(g(w)/w)$ can be defined on S, uniquely up to a multiple of π and $\mathrm{i}\pi$, respectively. We choose it in such a way that $|\hat{\eta}_0| < \pi$ and restrict the domain of H so that $\hat{\eta}_0 = 0$ for $g \in \mathscr{D}_H$. If $g \in \mathscr{D}_H \cap A(\bar{D})$ and $Hg \in A(\bar{D})$ we have then by the mean value theorem

$$0 = \hat{\eta}_0 = \operatorname{Im}[\log\{g(\ )/(\ )\}]_0^\wedge = \operatorname{Im}[\log g'(0)] = \arg g'(0), \tag{6.2}$$

which is satisfied by $g = g_{\mathrm{i}}$ if normalization (0.1) is used; hence property (ii) holds. Property (i) follows from the rules (1.12). Furthermore, it is easy to see that

$$\mathscr{R}_H = \left\{ r \in C(S);\ [\operatorname{Im} r]_0^\wedge = 0 \right\}, \tag{6.3}$$

and that for any $r \in \mathscr{R}_H$

$$H^{-1}r(w) = w\,\mathrm{e}^{r(w)}, \quad w \in S \tag{6.4}$$

(property (iv)). Clearly, (v), (vi) and (iii) hold also.

Following the developments of Section 4 we further get

$$G\tau(t) = \log\frac{\gamma(\tau(t) + t)}{\mathrm{e}^{\mathrm{i}t}} = \log\frac{\gamma(\theta(t))}{\mathrm{e}^{\mathrm{i}t}}, \tag{6.5a}$$

$$\mathscr{D}_G := C(T, \mathbb{R}), \tag{6.5b}$$

$$\mathscr{D}_G^0 = \left\{ \tau \in C(T, \mathbb{R});\ \hat{\eta}_0 := \left[\arg\{\gamma(\theta(\ ))\,\mathrm{e}^{-\mathrm{i}(\ )}\}\right]_0^\wedge = 0 \right\}. \tag{6.5c}$$

(Note that $\#\gamma \circ \theta(T) = 1$ whenever $\tau \in C(T, \mathbb{R})$.) Using version (4.16a) of Ψ, we obtain

$$\Psi\tau(t) := \arg\{\gamma(\theta(t))\,\mathrm{e}^{-\mathrm{i}t}\} - K[\log|\gamma \circ \theta|](t) \tag{6.6}$$

(with $\mathscr{D}_\Psi = \mathscr{D}_G$), and (4.15) becomes the *generalized Theodorsen integral equation*

$$\arg\{\gamma(\theta(t))\,\mathrm{e}^{-\mathrm{i}t}\} = K[\log|\gamma \circ \theta|](t), \quad t \in T. \tag{6.7}$$

Our Theorem 4.1 readily leads to

Theorem 6.1. *The generalized Theodorsen integral equation* (6.7) *has (up to an irrelevant additive constant* $2k\pi$) *exactly one solution* θ *for which* $\tau \in C(T, \mathbb{R})$, *namely* $\theta = \theta_i$, *the boundary correspondence function of the conformal map* g_i: $D \to \Delta$ *normalized by* (0.1).

Proof. If θ is any solution, then $\hat{\eta}_0 = 0$ since by (6.7) $\arg\{\cdots\}$ is a conjugate function; hence $\tau \in \mathscr{D}_G^0$. In view of Theorem 4.1 it remains to prove uniqueness. We know already that $g(e^{it}) = \gamma(\theta(t))$ are the boundary values of a conformal map of D onto Δ and that $g = H^{-1} I_{ST} G\tau$. By (6.4) it is clear that $g(0) = 0$, and (6.2) yields $g'(0) > 0$. Hence, $g = g_i$. □

The classical assumption for Theodorsen's method is that the region Δ is starlike with respect to the origin and that the boundary Γ is given in polar coordinates, i.e.

$$\gamma(t) = \rho(t)\, e^{it}, \tag{6.8}$$

with $\rho \in W^{1,\infty}(T) = \mathrm{Lip}^1(T)$. Then (6.5) and (6.6) become

$$G\tau(t) = \log \gamma(\tau(t) + t) + i\tau(t), \tag{6.9a}$$

$$\mathscr{D}_G := C(T, \mathbb{R}), \tag{6.9b}$$

$$\mathscr{D}_G^0 = \{\tau \in C(T, \mathbb{R});\ \hat{\tau}_0 = 0\}, \tag{6.9c}$$

$$\Psi\tau(t) = \tau(t) - K[\log \rho(\tau(\ ) + (\ ))](t), \tag{6.10}$$

and (6.7) specializes to the classical *Theodorsen integral equation* [56;57;13, p. 65]

$$\tau(t) = \Phi\tau(t) := K[\log \rho(\tau(\ ) + (\ ))](t). \tag{6.11}$$

Theodorsen's equation has exactly one continuous solution, since it follows from (6.11) that any continuous solution τ is a conjugate function, hence periodic with $\hat{\tau}_0 = 0$.

Theodorsen's method is the direct iteration method (5.5a) associated with (6.11):

$$\tau_{n+1}(t) := \Phi\tau_n(t) = K[\log \rho(\tau_n(\ ) + (\ ))](t). \tag{6.12}$$

Among the numerical methods for mapping the disk onto a given Jordan region Theodorsen's method is the most thoroughly investigated and the best understood. Gaier's book [13, pp. 64–105] summarizes the work up to 1964; newer contributions include [19–22,32,33,49].

Under the assumption $\|\rho'/\rho\|_\infty < 1$ it is easy to prove global convergence of $\{\tau_n\}$ in the L_2-norm. In fact, from $\|K\|_2 = 1$ and

$$\begin{aligned} |\log \rho(\tilde{\theta}(t)) - \log \rho(\theta(t))| &= \left| \int_{\theta(t)}^{\tilde{\theta}(t)} \frac{\rho'(\phi)}{\rho(\phi)}\, d\phi \right| \\ &\leqslant \|\rho'/\rho\|_\infty\, |\tilde{\theta}(t) - \theta(t)| \end{aligned} \tag{6.13}$$

we see that $L := \|\rho'/\rho\|_\infty$ is a Lipschitz constant for Φ (as required in (5.6)).

Geometrically, L is equal to the tangent of the largest angle between the radius vector $\gamma(t) = \rho(t)\, e^{it}$ and the outward normal of Γ, whose direction is $-i\gamma'(t) = (\rho(t) - i\rho'(t))\, e^{it}$. Hence, e.g., for a square with center 0 we get $L = 1$, but $L > 1$ for any other rectangle or for a square with another center. In practice, divergence is likely even when L is only slightly larger than 1.

However, in 1965, Niethammer [49] made two new proposals for solving the discretized Theodorsen equation. One was to apply the nonlinear SOR iteration (after having permuted the equations in order to attain a consistently ordered system). He conjectured that local convergence occurs and that

$$\omega_S := 2/\left(1 + \sqrt{1 + L^2}\right) \quad (< 1) \tag{6.14}$$

is a nearly optimal underrelaxation factor. In [21] we were able to prove that this conjecture is indeed true for a class of symmetric curves Γ satisfying

Assumption (SD_c). Γ is symmetric about the real axis and, in addition, ν-fold [3] rotationally symmetric about 0, where $\nu \geqslant 1$. The function $\rho \in W^{1,\infty}(T, \mathbb{R})$ is continuously differentiable and weakly monotone in $(0, \pi/\nu)$.

Some further assumptions on the discretization and the (discrete) solution of the discretized Theodorsen equation have to be made, see [21, p. 411] and [32]. In particular, the nonlinear SOR method can only be applied together with the standard discretization of K based on trigonometric interpolation. In contrast, the nonlinear second order Euler method (or second-order Richardson iteration) proposed in [21],

$$\tau_{n+1}(t) := \omega_S \Phi \tau_n(t) + (1 - \omega_S)\tau_{n-1}(t) \tag{6.15}$$

(with the same relaxation factor ω_S) is conjectured to converge locally for a wide range of discretizations (but otherwise the same assumptions), the asymptotic convergence factor being

$$\sigma_E = \frac{L}{1 + \sqrt{1 + L^2}} = \frac{\sqrt{1 + L^2} - 1}{L} \quad (< 1). \tag{6.16}$$

The nonlinear SOR method is known to converge twice as fast: $\sigma_S = \sigma_E^2$. For example, if $L = \|\rho'/\rho\|_\infty = 1$, we obtain $\sigma_E \doteq 0.4142$, $\sigma_S \doteq 0.1716$. Thus convergence is quite fast. Experimental evidence presented in [22] suggests that in practice the convergence rate of these methods, when applied to nonsymmetric curves, is usually still close to σ_E or σ_S, respectively. However, it is known that this is not true in certain pathological situations [33].

In contrast to Theodorsen's equation (6.11), its generalized version (6.7) seems to have received no attention in the Western literature except for Gaier's references to Vertgeim [62] and to a paper by Batyrev in [13]. We will return to it in Section 8.

Theodorsen's auxiliary operator (6.1) can also be applied to the exterior mapping problem normalized by (0.3). In particular, $g_e|_S \in \mathscr{D}_H$. The definition (4.19a) of Ψ then leads to the exterior versions of (6.7) and (6.11), which differ from the interior ones only by a minus sign in front of K. (Henrici [30] presents a detailed derivation for the case where Γ is given in polar coordinates.)

When we approach the exterior problem in the second way mentioned at the end of Section 4 and insert (4.20) into (6.7), we end up with the same integral equation after having used relation (2.21).

6.2. *The Melentiev–Kulisch method*

The method of Melentiev and Kulisch [13, pp. 107–109; 42] is based on the operator

$$Hg(w) := g(w)/w, \quad w \in S, \tag{6.17a}$$

$$\mathscr{D}_H := \{g \in C(S);\ \hat{g}_1 \geqslant 0\}. \tag{6.17b}$$

[3] The case $\nu = 1$ is included. The > sign on p. 411, line 15, of [21] is a misprint.

Clearly,

$$\mathscr{R}_H := \{r \in C(s);\ \hat{r}_0 \geqslant 0\}, \tag{6.18}$$

$$H^{-1}r(w) = wr(w), \qquad w \in S, \quad r \in \mathscr{R}_H. \tag{6.19}$$

Let $\mathscr{D}_G := C(T, \mathbb{R})$ again; then in view of $G\tau(t) = \gamma(\theta(t))\ \mathrm{e}^{-\mathrm{i}t}$

$$\mathscr{D}_G^0 = \{\tau \in C(T, \mathbb{R});\ [\gamma \circ \theta]_1^\wedge \geqslant 0\}. \tag{6.20}$$

If $r \in \mathscr{R}_H \cap A(\overline{D})$, $g := H^{-1}r \in \mathscr{D}_H \cap A(\overline{D})$ satisfies the normalization (0.1). When choosing $\Psi\tau$ as in (4.16a) we obtain in view of $\hat{\eta}_0 = [\operatorname{Im} G\tau]_0^\wedge = [\operatorname{Im} Hg]_0^\wedge = \operatorname{Im} \hat{g}_1 = 0$ the integral equation

$$\operatorname{Im}\{\gamma(\theta(t))\ \mathrm{e}^{-\mathrm{i}t}\} = K[\operatorname{Re}\{\gamma(\theta(\ ))\ \mathrm{e}^{-\mathrm{i}(\ )}\}](t), \tag{6.21}$$

which in the case of polar coordinates simplifies to

$$\rho(\tau(t) + t) \sin \tau(t) = K[\rho(\tau(\ ) + (\ )) \cos \tau(\ )](t). \tag{6.22}$$

On the other hand, from Im$\{\cdots\}$ and Re$\{\cdots\}$ we can of course retrieve arg$\{\cdots\}$ modulo 2π. (E.g., with the FORTRAN function ATAN2 and a continuation procedure if values beyond $(-\pi, \pi)$ occur.) Now, in polar coordinates, $\arg\{\gamma(\theta(t))\ \mathrm{e}^{-\mathrm{i}t}\} = \tau(t)$, hence we obtain the equation

$$\tau(t) \equiv \Phi\tau(t) := \arg(\xi(t) + \mathrm{i}K\xi(t)) \pmod{2\pi} \tag{6.23a}$$

with

$$\xi(t) := \rho(\tau(t) + t) \cos \tau(t). \tag{6.23b}$$

Conversely, assume $\tau \in C(T, \mathbb{R})$ is a solution of (6.23). First, (6.23a) yields

$$\sin \tau = K\xi / |\xi + \mathrm{i}K\xi|, \qquad \cos \tau = \xi / |\xi + \mathrm{i}K\xi|,$$

hence, by (6.23b), $|\xi(t) + \mathrm{i}K\xi(\mathrm{t})| = \rho(\tau(t) + t)$, and it follows that (6.22) holds. Defining $r(\mathrm{e}^{\mathrm{i}t}) := \xi(t) + \mathrm{i}K\xi(t) = \rho(\tau(t) + t)\ \mathrm{e}^{\mathrm{i}\tau(t)}$ we conclude from (6.22) and Theorem 2.2 that $r \in A(\overline{D})|_S$ and $\hat{r}_0 \in \mathbb{R}$. If $\hat{r}_0 < 0$, we replace $\tau(t)$ by $\tilde{\tau}(t) := \tau(t - \pi) + \pi$ and obtain another solution of (6.23), for which $\tilde{\xi}(t) := \xi(t - \pi)$, so that $\hat{r}_0 > 0$. Hence, we may assume that $\tau \in \mathscr{D}_G^0$, i.e. $I_{ST}G\tau \in \mathscr{R}_H$, and apply our Theorem 4.1. It follows that $g := H^{-1}I_{ST}G\tau$ is a conformal map of D onto Δ, and from (6.17)–(6.20) it is clear that the normalization (0.1) holds for g. Clearly, τ is only determined up to an additive multiple of 2π. If we choose this multiple appropriately, we must obtain the same solution as for Theodorsen's equation, namely one for which $\hat{\tau}_0 = 0$, cf. (6.9c). Summarizing, we get

Theorem 6.2. *Equations* (6.23) *have exactly one solution* $\tau \in C(T, \mathbb{R})$ *for which* $\hat{\tau}_0 = 0$, *namely* $\tau = \tau_{\mathrm{i}}$. *Any other solution is obtained by adding a constant* πk $(k \in \mathbb{Z})$ *and yields either the same mapping function* g_{i} (*if k is even*) *or the mapping function normalized by* $g(0) = 0$, $g'(0) < 0$.

Kulisch [42] (and similarly Melentiev earlier, cf. [39, pp. 451–478]) proposed the iteration of (6.23). However, $\Phi\tau$ is not defined for every $\tau \in \mathscr{D}_G = C(T, \mathbb{R})$; one has to make sure that a continuous argument exists:

$$\mathscr{D}_\Phi := \{\tau \in C(T, \mathbb{R});\ K\xi \in C(T, \mathbb{R}),\ \#(\xi + \mathrm{i}K\xi)(S) = 0\}, \tag{6.24}$$

where ξ is defined by (6.23b). Given $\tau_n \in \mathscr{D}_\Phi^\beta$ (or $\mathscr{D}_\Phi^\alpha$), we can then define τ_{n+1} by

$$\tau_{n+1} \equiv \Phi\tau_n \pmod{2\pi} \tag{6.25}$$

and the condition $[\tau_{n+1}]_0^\wedge \in (-\pi, \pi]$. It follows that $\tau_{n+1} \in W^{1,p}(T, \mathbb{R})$ (or $\mathrm{Lip}^\alpha(T, \mathbb{R})$), cf. (1.12), (2.8b), (2.8c), but unfortunately we cannot exclude the possibility that $\tau_{n+1} \notin \mathscr{D}_\Phi$, since the winding number condition in (6.24) may fail to hold. It clearly holds if τ_{n+1} is close enough to the solution τ_i, however. But little seems to be known about the convergence of (6.25).

In practice, $|\tau_\mathrm{i}(t)|$ is rarely greater than π, so the principal branch of arg will usually be the correct one in (6.23) and (6.25).

The two versions for the exterior problem are left to the reader.

6.3. Bergström's method

Bergström's projection method [2;13, pp. 109–110] looks similar to Kulisch's method. It also makes use of the auxiliary operator (6.17). Assuming polar coordinates we have

$$G^{-1}(G\tau)(t) = \tau(t) = \arg\{\rho(\tau(t)+t)\,\mathrm{e}^{\mathrm{i}\tau(t)}\} = \arg\{G\tau(t)\}. \tag{6.26}$$

If P^+ denotes now the orthogonal projection of L^2 onto H^2, which is easily implemented with FFTs, (5.8) becomes

$$\tilde{\tau}_{n+1}(t) := \arg\left\{\sum_{k=0}^{\infty}\left[\rho(\tau_n(\ )+(\ ))\,\mathrm{e}^{\mathrm{i}\tau_n(\ )}\right]_k^\wedge \mathrm{e}^{\mathrm{i}kt}\right\}. \tag{6.27a}$$

The second condition $g'(0) > 0$ (or $\hat{g}_1 > 0$) has not yet been taken care of. From Theodorsen's method we know that in polar coordinates it translates into $\hat{\tau}_0 = 0$. Hence, we suggest adjusting τ_{n+1} by subtracting $[\tilde{\tau}_{n+1}]_0^\wedge$:

$$\tau_{n+1}(t) := \tilde{\tau}_{n+1}(t) - [\tilde{\tau}_{n+1}]_0^\wedge. \tag{6.27b}$$

Bergström's idea can also be applied to Theodorsen's auxiliary operator (6.1). Then (6.27a) is to be replaced by

$$\tilde{\tau}_{n+1}(t) := \arg\left\{\exp\sum_{k=0}^{\infty}\left[\log\rho(\tau_n(\ )+(\ )) + \mathrm{i}\tau_n(\ )\right]_k^\wedge \mathrm{e}^{\mathrm{i}kt}\right\}. \tag{6.27c}$$

Note that both iterations (6.27a, b) and (6.27c, b) may be undefined for functions τ_n not close to τ_i since no continuous branch of arg needs to exist. (In contrast, in (6.26), before the projection, there is no problem with arg.)

Not much seems to be known about the convergence of Bergström's method either. Wöhner [73] gave a formula for a Lipschitz constant L for the iteration operator, but no condition implying $L < 1$.

It is not difficult to switch to the exterior problem, to which Bergström actually applied this method originally.

6.4. The identity operator

The identity operator

$$Hg(w) = g(w) \tag{6.28a}$$

also satisfies the assumptions of Section 4. To take the normalization (0.1) into account, it should be restricted to

$$\mathscr{D}_H := \{g \in C(S);\ \hat{g}_0 = 0,\ \hat{g}_1 > 0\}. \tag{6.28b}$$

Clearly, $\mathscr{R}_H = \mathscr{D}_H$, so that $[Hg]_0^\wedge = 0$. However, whenever $\mathscr{R}_H$ is restricted by the condition $[Hg]_0^\wedge = 0$, we urge the replacement of H by

$$\tilde{H}g(w) := Hg(w)/w, \tag{6.29}$$

since $\tilde{H}g \in A(\tilde{D})|_S$ implies then that this condition is satisfied. In particular, this means that (6.17) is preferable to the identity operator (6.28).

Many methods that have been described by their authors directly in terms of g make actually use of the operator (6.17) if interpreted in the framework of our theory, e.g., those in [2,10,69,70].

6.5. Timman's method

Timman's method [61] is widely used for airfoil analysis and design, usually after a preliminary Kármán–Trefftz map has been applied to the airfoil profile [1,13,25,36,74]. Its popularity for this problem seems due partly to historical reasons and partly to the fact that it allows one to handle airfoils with an open trailing edge.

The airfoil analysis problem is an *exterior* problem normalized by (0.3), but it is easier to aim at the solution g_e satisfying (0.4) first, and to rotate it afterwards. The second condition in (0.4) is now written as

$$g_e(1) = \gamma(0), \quad \text{i.e. } \tau_e(0) = 0. \tag{6.30}$$

We assume $m \geqslant 1$ in the exterior version of (4.4), so that $g_e' \in A(D^c)$.

The auxiliary operator used here is

$$Hg(w) := \log g'(w), \quad w \in S, \tag{6.31a}$$

$$\mathscr{D}_H := \{ g \in C^1(S);\ 0 \notin g'(S),\ \#g'(S) = 0,\ g(1) = \gamma(0) \}. \tag{6.31b}$$

In (6.31a) any continuous branch of the logarithm can be chosen. In order to make the operator H continuous we may consider Hg modulo the constant function $2\pi i$, but we do not write this explicitly. If $r := Hg$, then $e^r = g'$ is a derivative of $g \in C^1(S)$, hence $[e^r]_{-1}^\wedge = 0$:

$$\mathscr{R}_H = \{ r \in C(S);\ [e^r]_{-1}^\wedge = 0 \}, \tag{6.32}$$

$$H^{-1}r(w) = \int_1^w e^{r(s)} ds + \gamma(0), \qquad w \in S, \quad r \in \mathscr{R}_H. \tag{6.33}$$

The mapping function g_e has a Laurent series of the form

$$g_e(w) = cw + a_0 + a_1 w^{-1} + a_2 w^{-2} + \cdots, \quad 1 < |w| \leqslant \infty. \tag{6.34}$$

The series for g_e' and $\log g_e'$ are therefore Taylor series in $1/w$ with no linear terms:

$$g_e'(w) = c - a_1 w^{-2} - 2a_2 w^{-3} - \cdots, \quad 1 < |w| \leqslant \infty, \tag{6.35}$$

$$\log g_e'(w) = \log c - (a_1/c) w^{-2} - \cdots, \quad 1 < |w| \leqslant \infty. \tag{6.36}$$

The nonexistence of these linear terms turns out to be crucial for convergence [38]. It can be expressed as

$$[g_e']_{-1}^\wedge = 0, \quad [\log g_e']_{-1}^\wedge = 0. \tag{6.37}$$

Note that the first condition is satisfied for every $g \in \mathscr{D}_H$. The second one has not been taken into account, however.

Following our construction in Section 4 we get

$$G\tau(t) = \log\{-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\gamma'(\theta(t))\theta'(t)\}, \tag{6.38a}$$

$$\mathscr{D}_G = \mathscr{D}_G^0 = \{\tau \in C^1(T, \mathbb{R});\ \theta' > 0\}. \tag{6.38b}$$

The condition $\theta' > 0$ asserts the continuity of the logarithm in (6.38a). Thus $\mathrm{e}^{G\tau} \in C(S)$, but

$$\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{e}^{G\tau(t)} = \gamma'(\theta(t))\theta'(t) = \mathrm{d}\gamma(\theta(t))/\mathrm{d}t \tag{6.39}$$

is the derivative of a periodic function, i.e. $[\mathrm{e}^{G\tau}]^{\wedge}_{-1} = 0$, and hence $\mathrm{e}^{G\tau} \in \mathscr{R}_H$. In view of

$$\xi(t) = \operatorname{Re} G\tau(t) = \log \theta'(t) + \log|\gamma'(\theta(t))|, \tag{6.40a}$$

$$\eta(t) = \operatorname{Im} G\tau(t) = \arg\{\gamma'(\theta(t))\,\mathrm{e}^{-\mathrm{i}t}\} - \tfrac{1}{2}\pi. \tag{6.40b}$$

Equation (4.15) with the choice (4.19b) finally yields the *generalized Timman integro-differential equation*

$$\log \theta'(t) = K\left[\arg\{\gamma'(\theta(\ ))\,\mathrm{e}^{-\mathrm{i}(\ )}\}\right](t) - \log|\gamma'(\theta(t))| + \hat{\xi}_0 \tag{6.41a}$$

with

$$\hat{\xi}_0 = [\log \theta' + \log|\gamma' \circ \theta|]^{\wedge}_0. \tag{6.41b}$$

Note that the constant $-\frac{1}{2}\pi$ in (6.40b) as well as the ambiguity in the definition of arg disappear when K is applied.

Timman and the other authors mentioned above assumed that Γ is parametrized by the arclength; in our notation this means that $|\gamma'(t)|$ is a known constant, namely 2π divided by the length of Γ, and this obviously simplifies (6.41).

As a corollary to the analog of Theorem 4.1 for the exterior mapping problem we again obtain a uniqueness result:

Theorem 6.3. *The generalized Timman equation* (6.41) *has exactly one solution* θ, *namely* $\theta = \theta_e$, *such that* $\tau \in \mathscr{D}_G^0$ (*i.e.* $\tau \in C(T, \mathbb{R})$, $\theta' > 0$) *and* $\tau(0) = 0$.

Proof. Let τ be such a solution. We want to show that the conformal map g given by (4.18) is unique. By construction, $G\tau \in A(D^c)|_T$, hence $\mathrm{e}^{G\tau} \in A(D^c)|_T$. But $g'|_S = I_{ST}\,\mathrm{e}^{G\tau}$, so g' cannot have a finite pole; hence the pole of g is at ∞. The condition $\tau(0) = 0$ asserts that $g(1) = \gamma(0)$. Therefore $\tau = \tau_e$. $\square$

Timman's equation can also be written as

$$\theta'(t) = \frac{\exp\{K\left[\arg\{\gamma'(\theta(\ ))\,\mathrm{e}^{-\mathrm{i}(\ )}\}\right](t) + \hat{\xi}_0\}}{|\gamma'(\theta(t))|}, \tag{6.42}$$

but before we start to iterate here, we must make sure to define Φ in such a way that $[\Phi\tau]^{\wedge}_0 = 0$, cf. (5.3). Since $[\tau']^{\wedge}_0 = 0$, $[\theta']^{\wedge}_0 = 1$, and hence the zeroth Fourier coefficient of the right-hand side of (6.42) is also 1 if θ is a solution. In general, in view of $\tau'(t) = \theta'(t) - 1$, let $\Phi\tau + 1$ be defined by this right-hand side devided by its zeroth Fourier coefficient, or, equivalently, let

$$\chi(t) := \arg\{\gamma'(\theta(t))\,\mathrm{e}^{-\mathrm{i}t}\}, \tag{6.43a}$$

$$\phi(t) := \mathrm{e}^{K\chi(t)}/|\gamma'(\theta(t))|, \tag{6.43b}$$

$$\Phi\tau(t) := \phi(t)/\hat{\phi}_0 - 1. \tag{6.43c}$$

Then $[\Phi\tau]_0^\wedge = 0$, so that iteration (5.5b) can be executed:

$$\tau_n(t) := \int_0^t \Phi\tau_{n-1}(s)\mathrm{d}s. \tag{6.44}$$

As we have seen in (6.39), the fact that θ_n' is the derivative of a periodic function θ_n implies that $[\mathrm{e}^{G\tau_n}]_{-1}^\wedge = 0$, and thus $g_n := H^{-1}I_{ST}G\tau_n$ satisfies

$$[g_n']_{-1}^\wedge = [\mathrm{e}^{Hg_n}]_{-1}^\wedge = [\mathrm{e}^{G\tau_n}]_{-1}^\wedge = 0 \tag{6.45a}$$

in accordance with the first condition in (6.37). If g_n were in $A(D^c)|_S$, it would follow as in (6.34)–(6.37) that also

$$[\log g_n']_{-1}^\wedge = 0, \tag{6.45b}$$

but this assumption is only true if g_n equals the solution g_e of our mapping problem. However, it seems that often the convergence can be improved if Φ is further modified so that condition (6.45b) is imposed approximately ([25] and private communications by N.D. Halsey and A. Kaiser). By definition

$$\begin{aligned}[\log g_n']_{-1}^\wedge &= [Hg_n]_{-1}^\wedge = [G\tau_n]_{-1}^\wedge = [\xi_n + \mathrm{i}\eta_n]_{-1}^\wedge \\ &= [\log \theta_n'(\ ) + \log |\gamma'(\theta_n(\ ))| + \mathrm{i}\arg\{\gamma'(\theta_n(\ ))\,\mathrm{e}^{-\mathrm{i}(\ )}\}]_{-1}^\wedge,\end{aligned} \tag{6.46}$$

and by construction

$$\begin{aligned}[\log \theta_n']_{-1}^\wedge &= [\log(\Phi\tau_{n-1} + 1)]_{-1}^\wedge = [\log \phi_{n-1}]_{-1}^\wedge \\ &= [K\chi_{n-1}]_{-1}^\wedge - [\log |\gamma' \circ \theta_{n-1}|]_{-1}^\wedge.\end{aligned}$$

Hence, if we modify the iteration by deleting $[\chi_{n-1}]_{-1}^\wedge$ and $[\chi_{n-1}]_1^\wedge$, the new function θ_n satisfies

$$[\log \theta_n' + \log |\gamma' \circ \theta_{n-1}|]_{\pm 1}^\wedge = 0,$$

and in a rough sense we have taken care of the requirement that the quantity in (6.46) should vanish. This modification means that (6.43b) is replaced by

$$\phi(t) := \frac{\exp\{K[\chi(\ ) - \hat{\chi}_1\,\mathrm{e}^{\mathrm{i}(\ )} - \hat{\chi}_{-1}\,\mathrm{e}^{-\mathrm{i}(\ )}](t)\}}{|\gamma'(\theta(t))|}. \tag{6.47}$$

The local convergence of iteration (6.44) with Φ defined by (6.43a), (6.47), and (6.43c) can be proved for certain nicely behaving curves Γ [38]. Halsey [25,26] and others [1,36] report favorably about the performance of this method in practice; their implementations require γ as a function of arclength, however.

At first sight, one is tempted to think that the operator (6.31) can be applied to the *interior* mapping problem with equal success. However, in repeating our derivation of Timman's equation (6.41) it becomes clear that every boundary correspondence function θ for a conformal map g of D onto Δ is a solution of the interior version of Timman's equation, differing from (6.41) only in a minus sign in front of K. While the normalization $g(1) = \gamma(0)$, i.e. $\tau(0) = 0$, is taken into account in (6.33) and (6.44), there seems to be in general no simple way to impose the condition $g(0) = 0$ in the iteration. But unless this condition is imposed the iteration cannot converge in general since it does not know to which solution it is supposed to converge. (The exterior map is different since $g' \in A(D^c)$ iff $g(\infty) = \infty$.)

However, if Γ is rotationally symmetric about 0, it follows again that the interior Timman equation has a unique symmetric solution and that the iteration converges locally under suitable assumptions [38]. (In fact, the higher the order of symmetry, the faster the convergence.)

At the end of Section 4 we mentioned that there are two ways to transform a method for the interior problem into one for the exterior problem. Likewise, there is also a second way to transform Timman's method into one for the interior problem, namely by composing g_i with two inversions:

$$Hg_i(1/w) = \log \frac{d}{dw} \frac{1}{g_i(1/w)} = \log \frac{g_i'(1/w)}{w^2 g_i^2(1/w)}, \quad w \in \overline{D}. \tag{6.48}$$

This suggests the general definition

$$Hg(w) := \log g'(w) - 2 \log \frac{g(w)}{w}, \quad w \in S. \tag{6.49}$$

We do not want to go through all the details here, but it is worth mentioning that

$$H^{-1}r(w) = \left[\frac{1}{\gamma(0)} - \int_1^w \frac{e^{r(s)}}{s^2} ds \right]^{-1}. \tag{6.50}$$

The local convergence of the associated successive conjugation method to τ_i can be proved again under certain fairly restrictive assumptions on Γ [38].

6.6. Friberg's method

Friberg's method [12;13, pp. 113–114; 66, pp. 228–233] for the interior or exterior problem normalized by (0.2) or (0.4), respectively, is based on the operator

$$Hg(w) := \log g'(w) - \log \frac{g(w)}{w} = \log \frac{wg'(w)}{g(w)}, \quad w \in S, \tag{6.51a}$$

$$\mathscr{D}_H := \{ g \in C^1(S);\ 0 \notin g(S),\ \#g(S) = 1,\ 0 \notin g'(S),\ \#g'(S) = 0,\ g(1) = \gamma(0) \}. \tag{6.51b}$$

Its range and the inverse operator are given by

$$\mathscr{R}_H = \{ r \in C(S);\ [e^r]_0^{\wedge} = 1 \}, \tag{6.52}$$

$$H^{-1}r(w) = \gamma(0) w \exp\left(\int_1^w \frac{e^{r(s)} - 1}{s} ds \right), \quad w \in S, \quad r \in \mathscr{R}_H. \tag{6.53}$$

For the latter formula we have made use of

$$\left(\log \frac{g(w)}{w} \right)' = \frac{1}{w} \left(\frac{wg'(w)}{g(w)} - 1 \right) = \frac{1}{w} (e^{r(w)} - 1), \tag{6.54}$$

from which it also follows that the condition $[e^r]_0^{\wedge} = 1$ must occur in (6.52). To prove (6.52) we note that whenever $r \in C(S)$ satisfies this condition, the integral in (6.53) is a function $v \in C(S)$; then $\#\exp(v)(S) = 0$, hence $\#H^{-1}r(S) = 1$. Therefore, $\log(g(w)/w)$ is well-defined for $g := H^{-1}r$. Moreover, since $wg'(w)/g(w) = e^{r(w)}$, this function has winding number 0, and its logarithm is also well-defined up to irrelevant multiples of $2\pi i$. Consequently, $\log g'(w)$, being the sum of the two logarithms, is also well defined.

The operator (6.51) (as well as the operator (6.49)) is a linear combination of the operators in Theodorsen's and Timman's method. By combining (6.5) and (6.38) or by direct derivation we easily get

$$G\tau(t) = \log\left\{-i\frac{\gamma'(\theta(t))}{\gamma(\theta(t))}\theta'(t)\right\}, \tag{6.55a}$$

$$\mathscr{D}_G^0 = \{\tau \in C^1(T, \mathbb{R});\ \theta' > 0\}. \tag{6.55b}$$

Defining Ψ by (4.16b) or (4.19b) we obtain now the *generalized Friberg integro-differential equation*

$$\log \theta'(t) = \mp K\left[\arg\frac{\gamma' \circ \theta}{\gamma \circ \theta}\right](t) - \log\left|\frac{\gamma'(\theta(t))}{\gamma(\theta(t))}\right| + \hat{\xi}_0, \tag{6.56a}$$

with

$$\hat{\xi}_0 = \left[\log \theta' + \log\left|\frac{\gamma' \circ \theta}{\gamma \circ \theta}\right|\right]_0^\wedge; \tag{6.56b}$$

the minus sign belongs to the interior problem, the plus sign to the exterior one.

Note that (6.56) is just the sum of the generalized Timman equation (6.41) and an equation derived from the function $G\tau$ of Theodorsen's method by applying the second version (4.16b) of Ψ (instead of the first, as for Theodorsen's equation).

Friberg [12,13,67] derived (6.56) for the interior problem and polar coordinates.

If $r = I_{ST}G\tau \in A(\overline{D})$, it is clear from (6.53) that the normalization (0.2) holds with $z_0 = \gamma(0)$ for $g := H^{-1}r$. If $r = I_{ST}G\tau \in A(D^c)$, then the integrand $(e^{r(s)} - 1)/s$ in (6.53) lies in $A(D^c)$ also and vanishes at ∞; hence, the integral is in $A(D^c)$ too, and $g := H^{-1}r$ has a simple pole at ∞, as required in (0.4). Application of our Theorem 4.1 and its exterior version therefore yield

Theorem 6.4. *Friberg's equation* (6.56) *has exactly one solution* θ *such that* $\tau \in \mathscr{D}_G^0$ (*i.e.* $\tau \in C^1(T, \mathbb{R})$ *and* $\theta' > 0$) *and* $\tau(0) = 0$. *This solution is equal to* θ_i *in the case of a minus sign in* (6.56a) *and equal to* θ_e *in the case of a plus sign.*

The iterative solution of Friberg's method by successive conjugation parallels that of Timman's equation: (6.44) is now applied with

$$\chi(t) := \arg\frac{\gamma'(\theta(t))}{\gamma(\theta(t))}, \tag{6.57a}$$

$$\phi(t) := \left|\frac{\gamma(\theta(t))}{\gamma'(\theta(t))}\right| e^{\mp K\chi(t)}, \tag{6.57b}$$

$$\Phi\tau(t) := \phi(t)/\hat{\phi}_0 - 1. \tag{6.57c}$$

Friberg missed the last step, which makes his convergence result somewhat suspect [13, p. 113]. For the exterior problem a modification similar to (6.47), proposed by Kaiser [38], leads to an iteration whose convergence can be proved for well behaved curves Γ; it consists in replacing

(6.57b) by

$$\phi(t) := \left|\frac{\gamma(\theta(t))}{\gamma'(\theta(t))}\right| \exp\left\{K\left[\chi(\ ) - \hat{\chi}_1 e^{i(\ )} - \hat{\chi}_{-1} e^{-i(\ )}\right](t) - \hat{\omega}_1 e^{it} - \hat{\omega}_{-1} e^{-it}\right\}, \quad (6.58)$$

with $\omega(t) := \log |\gamma(\theta(t))|$.

While for Timman's method and for its equivalent interior method based on (6.49) the linear term in the power series expansion of Hg vanishes if $g \in A(D^c)|_S$ or $g \in A(\overline{D})|_S$, respectively, Friberg's choice (6.51) of the auxiliary operator yields a vanishing constant coefficient in this power series. Kaiser's detailed analysis [38] shows that if Γ is in a certain sense close to a circle, then the Lipschitz constant in (5.6) is close to 0 for Friberg's method, while it is close to 0.5 for Timman's method. This result suggests that at least for such curves Friberg's method is definitely superior.

7. Newton methods: Vertgeim–Hübner and Wegmann type methods

Instead of trying to solve our basic equation (4.15), $\Psi\tau = 0$, by direct iteration, we may attack it by Newton's method. We discuss the case $l = 0$ only. (Otherwise one has to solve a differential equation in each step.) As usual for Newton's method one has to assume that the function is differentiable, but, on the other hand, one can expect quadratic convergence for sufficiently smooth boundaries Γ.

To fix our minds we assume $\gamma \in W^{3,\infty}(T) = C^{2,1}(T)$ in this and the next section. In contrast to statement (4.4a) we will allow iterates $\tau_n \in W^{1,p}(T, \mathbb{R})$ and associated functions $g_n \in W^{1,p}(S)$, however, where $p \in (1, \infty)$ is arbitrary. As mentioned in Section 4, by a result of Kellog and Warschawski [18,63,64,67], $\gamma \in C^{2,1}(T) \subset C^{2,\alpha}(T)$ $(\forall \alpha \in (0, 1))$ can be seen to imply $\tau \in C^{2,\alpha}(T, \mathbb{R}) \subset W^{2,p}(T, \mathbb{R})$ $(\forall p \geqslant 1)$. (The last inclusion is of course very crude.) Hence, if the iterates will converge, the limit function will be smoother than the iterates. To simplify notation we set

$$W_S := W^{1,p}(S), \qquad W_T := W^{1,p}(T, \mathbb{R}). \quad (7.1a)$$

In accordance with (1.10) we choose in W_S and W_T the norm

$$\| f \| := \max\{ \| f \|_\infty, \| f' \|_p \}, \quad (7.1b)$$

which is easily seen to satisfy

$$\| fg \| \leqslant 2 \| f \| \, \| g \|. \quad (7.2)$$

We further assume that the operator H, which now has the simple form

$$Hg(w) = h(g(w); w), \quad w \in S, \quad (7.3)$$

has in addition to the properties (i)–(vi) postulated in Section 4 the following two:

(vii) The domain of H can be extended from $\mathscr{D}_H^p$ to an open set $\mathscr{D}_H^+ \supseteq \mathscr{D}_H^p$ of W_S in such a way that $\mathscr{R}_H^+ := H(\mathscr{D}_H^+) \subseteq W_S$ still holds and the function $h(u; v)$ has continuous partial derivatives up to $\partial^4 h/(\partial u^3 \partial v)$ on

$$\bigcup_{g \in \mathscr{D}_H^+} \{(u; v) \in \mathbb{C}^2;\ |u - g(v)| \leqslant \epsilon(g),\ v \in S\},$$

where ϵ: $\mathscr{D}_H^+ \to \mathbb{R}^+$ is a certain continuous function. (In particular, the function h_g defined by

$$h_g(w) := \frac{\partial h}{\partial u}(g(w);\ w)$$

belongs to $C^1(S) \subset W_S$.)

(viii) $h_g(S)$ has winding number $\# h_g(S) = -1$, and Hg satisfies

$$\operatorname{Im}[Hg]_0^\wedge = 0 \quad \text{if } g \in \mathscr{D}_H^p. \tag{7.4}$$

The case described in (viii) is not the only one that can be handled, but we want to avoid treating various cases in parallel in this section. For example, we could equally well require $\operatorname{Re}[Hg]_0^\wedge = 0$ in (7.4) or assume $\# h_g(S) = 0$, $\operatorname{Im}[Hg]_1^\wedge = 0$ if $g \in \mathscr{D}_H^p$. The latter case can be reduced to the one in (viii) by redefining H according to (6.29). Therefore, our examples with $l = 0$ in Section 6 are all covered.

The operators G and Ψ can now be defined on

$$\mathscr{D}_G^+ := \{\tau \in W_T;\ (4.8) \text{ holds for some } g \in \mathscr{D}_H^+\}, \tag{7.5a}$$

and it is advantageous to redefine $\mathscr{D}_G^0$ here by

$$\mathscr{D}_G^0 := \{\tau \in W_T;\ (4.8) \text{ holds for some } g \in \mathscr{D}_H^p\}, \tag{7.5b}$$

so that $\mathscr{D}_G^0 \subseteq \mathscr{D}_G^+$. In view of (7.4) it is natural to use definition (4.16a) of Ψ, since for $\tau \in \mathscr{D}_G^0$ we have

$$\hat{\eta}_0 = \operatorname{Im}[G\tau]_0^\wedge = \operatorname{Im}[Hg]_0^\wedge = 0; \quad \text{thus,}$$
$$\Psi\tau := \eta - K\xi, \quad \text{where } \xi + \mathrm{i}\eta = G\tau. \tag{7.6}$$

From now on we use this definition also on $\mathscr{D}_G^+$, though the term $\hat{\eta}_0$ does not in general vanish if $\tau \in \mathscr{D}_G^+ \setminus \mathscr{D}_G^0$. This has the effect that although we are solving $\Psi\tau = 0$ for $\tau \in \mathscr{D}_G^+$, every solution will satisfy (7.4).

Lemma 7.1. *The operators defined by* (7.3), (4.11), *and* (7.6), *restricted in domain according to*

$$H:\ \mathscr{D}_H^+ \subseteq W_S \to \mathscr{R}_H^+ \subseteq W_S, \tag{7.7a}$$
$$G:\ \mathscr{D}_G^+ \subseteq W_T \to \mathscr{R}_G^+ \subseteq W^{1,p}(T), \tag{7.7b}$$
$$\Psi:\ \mathscr{D}_G^+ \subseteq W_T \to \mathscr{R}_\Psi^+ \subseteq W_T, \tag{7.7c}$$

are Fréchet differentiable (with respect to the norm (7.1b) *in domain and range), and their F-derivatives are locally uniformly bounded and satisfy locally a Lipschitz condition. If we simply write $h_{\gamma\circ\theta}$ instead of $h_{I_{ST}(\gamma\circ\theta)}$, and if we define*

$$\zeta(t) := h_{\gamma\circ\theta}(\mathrm{e}^{\mathrm{i}t})\gamma'(\theta(t)), \tag{7.8}$$

the following formulas hold for the F-derivatives:

$$H_g' d(w) = h_g(w) d(w), \qquad d \in W_S, \quad w \in S, \tag{7.9}$$
$$G_\tau' \delta(t) = \zeta(t)\delta(t), \qquad \delta \in W_T, \qquad t \in T, \tag{7.10}$$
$$\Psi_\tau' \delta(t) = \delta(t) \operatorname{Im} \zeta(t) - K[\delta \operatorname{Re} \zeta](t), \qquad \delta \in W_T, \quad t \in T. \tag{7.11}$$

Proof. By straightforward but lengthy calculations involving the Taylor formula with remainder one can verify that indeed for H'_g defined in (7.9)

$$\| H(g+d) - Hg - H'_g d \| = \mathrm{O}(\| d \|^2) \quad \text{as } \| d \| \to 0 \tag{7.12}$$

uniformly on any sufficiently small neighborhood U_g of g and that there is a Liptschitz constant L such that for g, $\tilde{g} \in U_g$

$$\| H'_{\tilde{g}} d - H'_g d \| \leqslant L \| \tilde{g} - g \| \, \| d \|. \tag{7.13}$$

We omit the details. By (7.2), $\| H'_g \| \leqslant 2 \| h_g \|$.

Next we observe that according to the rules (1.12), δ, $\tau \in W_T$ implies $\gamma \circ (\theta + \delta)$, $\gamma \circ \theta$, $(\gamma' \circ \theta)\delta \in W^{1,p}(T)$. Moreover, one can verify that as a consequence of $\gamma \in W^{3,\infty}(T)$,

$$\| \gamma \circ (\theta + \delta) - \gamma \circ \theta - (\gamma' \circ \theta)\delta \| = \mathrm{O}(\| \delta \|^2), \tag{7.14a}$$

$$\| \gamma' \circ (\theta + \delta) - \gamma' \circ \theta \| = \mathrm{O}(\| \delta \|) \quad \text{as } \| \delta \| \to 0. \tag{7.14b}$$

Applying (7.12) to $g(\mathrm{e}^{\mathrm{i}t}) := \gamma(\theta(t))$, $d(\mathrm{e}^{\mathrm{i}t}) := \gamma(\theta(t) + \delta(t)) - \gamma(\theta(t))$ then yields in view of (7.2), (7.9), (7.10) and (7.14a)

$$\begin{aligned} \| G(\tau + \delta) - G\tau - G'_\tau \delta \| &\leqslant \| H(g+d) - Hg - H'_g d \| + \| G'_\tau \delta - I_{TS} H'_g d \| \\ &\leqslant \mathrm{O}(\| d \|^2) + \mathrm{O}(\| h_{\gamma \circ \theta} \| \, \| (\gamma' \circ \theta)\delta - I_{TS} d \|) \\ &\leqslant \mathrm{O}(\| \delta \|^2). \end{aligned} \tag{7.15}$$

Similarly, using (7.2), (7.9), (7.10), (7.13), and (7.14b) we get

$$\begin{aligned} \| G'_{\tilde{\tau}} \delta - G'_\tau \delta \| &\leqslant 2 \| H'_{\tilde{g}}(\gamma' \circ \tilde{\theta}) - H'_g(\gamma' \circ \theta) \| \, \| \delta \| \\ &\leqslant \left[2L \| \tilde{g} - g \| \, \| \gamma' \circ \tilde{\theta} \| + 4 \| h_g \| \, \| \gamma' \circ \tilde{\theta} - \gamma' \circ \theta \| \right] \| \delta \| \\ &\leqslant L' \| \tilde{\tau} - \tau \| \, \| \delta \| \end{aligned} \tag{7.16}$$

for some $L' > 0$ depending on the neighborhood U_g of g.

Finally, since $\| K \| < \infty$ (see 2.8c)), and since the operators associating to a function in $W^{1,p}(T)$ its real part, its imaginary part, and its zeroth Fourier coefficient (i.e. its mean value), respectively, all clearly have norm 1, (7.15) and (7.16) are easily seen to imply the corresponding inequalities for the operator Ψ:

$$\| \Psi(\tau + \delta) - \Psi\tau - \Psi'_\tau \delta \| = \mathrm{O}(\| \delta \|^2), \tag{7.17}$$

$$\| \Psi'_{\tilde{\tau}} \delta - \Psi'_\tau \delta \| \leqslant L'' \| \tilde{\tau} - \tau \| \, \| \delta \|. \quad \square \tag{7.18}$$

Assuming that the F-derivative Ψ'_τ is invertible we can now define the Newton method for (4.15), $\Psi\tau = 0$, as usual: In the nth step Ψ is linearized at the nth iterate $\tau_n \in \mathscr{D}_G^+$:

$$\Psi(\tau_n + \delta_n) \approx \Psi\tau_n + \Psi'_{\tau_n} \delta_n; \tag{7.19}$$

then the linearized function is equated to zero, and the resulting equation is solved for the correction δ_n:

$$\Psi'_{\tau_n} \delta_n + \Psi\tau_n = 0; \tag{7.20}$$

finally, one lets

$$\tau_{n+1} := \tau_n + \delta_n. \tag{7.21}$$

The important point, which in special situations was observed by Vertgeim [62] and Hübner [34], is that (7.20) can be solved explicitly by reducing it to a Riemann–Hilbert problem and the solution is unique (even when $\Psi\tau_n$ is replaced by any $\psi \in W_T$), so that Ψ_τ' is indeed invertible. We basically follow Hübner's treatment [34].

Inserting (7.11) into (7.21) (with the index n deleted for simplicity), we get

$$0 = \Psi_\tau'\delta + \Psi\tau = \delta \operatorname{Im} \zeta - K(\delta \operatorname{Re} \zeta) + \Psi\tau.$$

By Theorem 2.2 we conclude that

$$\begin{aligned} &\zeta\delta + \mathrm{i}\Psi\tau = \delta \operatorname{Re} \zeta + i(\delta \operatorname{Im} \zeta + \Psi\tau) \in A(\overline{D})\,|_T, \\ &\operatorname{Im}[\zeta\delta + \mathrm{i}\Psi\tau]_0^\wedge = [\delta \operatorname{Im} \zeta + \Psi\tau]_0^\wedge = 0 \end{aligned} \tag{7.22}$$

and that the function f defined on S by

$$f(\mathrm{e}^{\mathrm{i}t}) := \zeta(t)\delta(t) + \mathrm{i}\Psi\tau(t) \tag{7.23}$$

satisfies

$$f \in A(\overline{D}), \qquad \operatorname{Im} f(0) = 0, \tag{7.24}$$

when extended analytically to D. Consequently f is a solution of the Riemann–Hilbert problem

$$0 = \operatorname{Re}\{\mathrm{i}\,|\zeta(t)|^2\delta(t)\} = \operatorname{Re}\{\overline{\zeta(t)}[\mathrm{i}f(\mathrm{e}^{\mathrm{i}t}) + \Psi\tau(t)]\}, \tag{7.25}$$

whose index is 0 since $\#\zeta(T) = \#h_{\gamma\circ\theta}(S) + 1 = 0$, cf. (7.8) and property (viii). The function ϕ of (3.1) becomes $\phi(t) := \arg \zeta(t) - \frac{1}{2}\pi$. To formulate our result for (7.20) we of course let $\theta_n(t) = \tau_n(t) + t$ and

$$\zeta_n(t) := h_{\gamma\circ\theta_n}(\mathrm{e}^{\mathrm{i}t})\gamma'(\theta_n(t)). \tag{7.26}$$

Theorem 7.2. *If* $[\arg \zeta_n]_0^\wedge \notin \pi\mathbb{Z}$, *the Newton correction* δ_n *satisfying* (7.20) *can be computed by solving the Riemann–Hilbert problem*

$$\operatorname{Re}\{\mathrm{i}\overline{\zeta_n(t)}f_n(\mathrm{e}^{\mathrm{i}t})\} = -\Psi\tau_n \operatorname{Re}\{\zeta_n(t)\} \tag{7.27}$$

(*with index* 0) *for* $f_n \in A(\overline{D})$ *satisfying* $\operatorname{Im} f_n(0) = 0$. *Its unique solution is given by the Theorems* 3.1 *and* 3.2 *and satisfies* $f_n\,|_S \in W_S$. *In terms of* f_n,

$$\delta_n(t) = \frac{f_n(\mathrm{e}^{\mathrm{i}t}) - \mathrm{i}\Psi\tau_n(t)}{\zeta_n(t)}. \tag{7.28}$$

Proof. The result is an immediate consequence of the above and the Theorems 3.1–3.3, except that the assumptions of Theorem 3.3 have to be verified by tedious applications of the rules (1.12). □

By replacing $\Psi\tau_n$ in the above formulas by an arbitrary function $\psi \in W_T$ it can be seen that the equation

$$\Psi_\tau'\delta + \psi = 0$$

(with $\tau \in \mathscr{D}_G^+$) has a unique solution $\delta \in W_T$ whenever $\psi \in W_T$. Hence, the operator Ψ_τ': $W_T \to W_T$ is bijective. The closed graph theorem then implies that Ψ_τ' has a bounded inverse [17,

p. 221]. One can show further that the norm of Ψ_τ^{-1} is uniformly bounded for τ in a neighborhood of τ_i. This allows us to prove in a standard way

Theorem 7.3. *Assume that* $\gamma \in C^{2,1}(T)$ *and*

$$[\arg \zeta(t)]_0^\wedge \notin \pi\mathbb{Z} \quad \text{if } \tau = \tau_i. \tag{7.29}$$

Then the Newton iterates (defined by (7.20) *and* (7.21)) *converge in the norm* (7.1b) *locally and quadratically to* τ_i, *the reduced boundary correspondence function of the interior map* $g_i \in \mathcal{D}_H^p$ *normalized by* (7.4).

Proof. In view of (7.17), (2.20) and $\Psi\tau_i = 0$,

$$\begin{aligned}\| \tau_{n+1} - \tau_i \| &= \| \tau_n + \delta_n - \tau_i \| = \| (\Psi'_{\tau_n})^{-1} (\Psi'_{\tau_n}(\tau_n - \tau_i) - \Psi\tau_n) \| \\ &\leqslant \| (\Psi'_{\tau_n})^{-1} \| \mathrm{O}(\| \tau_n - \tau_i \|^2),\end{aligned}$$

with a locally uniform O-term. Hence the assertion follows from the above remarks. □

Computing the Newton corrections according to formula (7.28) was proposed for the first time by Vertgeim in connection with the generalized Theodorsen equation (6.7) [62]. However, in his very brief article Vertgeim actually proposed application of the modified Newton iteration, $\tau_{n+1} := \tau_n + (\Psi'_{\tau_0})^{-1}\Psi\tau_n$, with a fixed inverse operator, leading to only linear local convergence, which he indeed proved. Moreover, it is likely that this method has never been seriously tested and used. Recently, when investigating Newton's method for the classical Theodorsen equation, Hübner [34] has reinvented this elegant and efficient approach and proved the quadratic local convergence of the Newton iteration. Note that in each step only three conjugations are needed, i.e., in practice, six real FFTs, plus calculations of the lower complexity $\mathrm{O}(N)$. The modified Newton iteration requires four real FFTs per step, and it is unlikely that this reduction can make up the loss in convergence speed.

We call a conformal mapping method based on (7.20), (7.21), and Theorem 7.2 a *Vertgeim-Hübner type method.*

Instead of applying Newton's method to $\Psi\tau = 0$ one can apply the idea of linearization to the conditions $G\tau \in A(\overline{D})\,|_T$, $\mathrm{Im}[G\tau]_0^\wedge = 0$ directly: In analogy to (7.19) we write

$$G(\tau_n + \delta_n) \approx G\tau_n + G'_{\tau_n}\delta_n, \tag{7.30}$$

but now we want to determine δ_n such that

$$G\tau_n + G'_{\tau_n}\delta_n = G\tau_n + \zeta_n\delta_n \in A(\overline{D})\,|_T, \tag{7.31a}$$

$$\mathrm{Im}\big[G\tau_n + G'_{\tau_n}\delta_n\big]_0^\wedge = \mathrm{Im}\big[G\tau_n + \zeta_n\delta_n\big]_0^\wedge = 0, \tag{7.31b}$$

i.e. such that h_n defined on S by

$$h_n(\mathrm{e}^{\mathrm{i}t}) := \zeta_n(t)\delta_n(t) + G\tau_n(t) \tag{7.32}$$

satisfies

$$h_n \in A(\overline{D}), \qquad \mathrm{Im}\ h_n(0) = 0. \tag{7.33}$$

This is again a Riemann–Hilbert problem:

$$0 = \mathrm{Re}\{\mathrm{i}\,|\zeta_n(t)|^2\delta_n(t)\} = \mathrm{Re}\{\mathrm{i}\overline{\zeta_n(t)}[h_n(\mathrm{e}^{\mathrm{i}t}) - G\tau_n(t)]\}. \tag{7.34}$$

Its function $a(t) = \mathrm{i}\bar{\zeta}_n(t)$ is the same as in (7.25), and in particular the index is 0 again.

The fact that (7.31) is equivalent to a Riemann–Hilbert problem we first made use of for numerical conformal mapping by Wegmann [69,70], who chose $Hg := g$, which leads to a Riemann–Hilbert problem with index 2 normalized as in part (ii) of Theorem 3.2. This problem can be reduced to one with index 0 normalized as in part (i) of Theorem 3.2 (cf. the proof of that theorem). In fact, one just has to make the substitution (6.29), cf. the remark following (7.4). Therefore, we call an iterative method based on solving a sequence of problems (7.31) as Riemann–Hilbert problems a *Wegmann type method.*

In analogy to Theorem 7.2 we have

Theorem 7.4. *If* $[\arg \zeta_n]_0^\wedge \notin \pi\mathbb{Z}$, *the corrections* δ_n *satisfying* (7.31) *can be computed by solving the Riemann–Hilbert problem*

$$\operatorname{Re}\{\mathrm{i}\overline{\zeta_n(t)}h_n(\mathrm{e}^{\mathrm{i}t})\} = -\operatorname{Im}\{\overline{\zeta_n(t)}G\tau_n(t)\} \tag{7.35}$$

(*with index* 0) *for* $h_n \in A(\bar{D})$ *satisfying* $\operatorname{Im} h_n(0) = 0$. *Its unique solution is given by the Theorems* 3.1 *and* 3.2 *and satisfies* $h_n|_S \in W_S$. *In terms of* h_n,

$$\delta_n(t) = \frac{h_n(\mathrm{e}^{\mathrm{i}t}) - G\tau_n(t)}{\zeta_n(t)}. \tag{7.36}$$

One may wonder whether the Vertgeim–Hübner type methods and the Wegmann type methods are more closely related than by the obvious similarity of the formulas. The answer is affirmative:

Theorem 7.5. *The Newton corrections* δ_n (*defined by* (7.20) *are identical with the corrections* δ_n *defined by* (7.31) *and constructed in the Wegmann type method which corresponds to the same auxiliary operator H and is started with the same initial approximation* τ_0. *The functions* f_n *of Theorem* 7.2 *and the functions* h_n *of Theorem* 7.4 *are related by*

$$h_n = f_n + L_{\mathrm{R}}^{+}\xi_n, \tag{7.37}$$

where $\xi_n := \operatorname{Re} G\tau_n$, *while* L_{R}^{+} *is defined by* (2.17a).

Proof. [4] In view of (7.6), adding

$$\xi_n + \mathrm{i}K\xi_n \in A(\bar{D}), \qquad \operatorname{Im}[\xi_n + \mathrm{i}K\xi_n]_0^\wedge = [K\xi_n]_0^\wedge = 0 \tag{7.38}$$

[4] When reading this manuscript Wegmann suggested a different exposition of this proof: First, the operator ψ of (7.6) is just the composition $\Psi = M \circ G$, where M is the operator

$M\colon\ W^{1,p}(T) \to W_T, \quad \phi \mapsto M\phi := \operatorname{Im}\phi - K(\operatorname{Re}\phi),$

which, in view of (2.8c), is Lipschitz continuous. According to Theorem 2.1 the kernel of M consists of the functions $\phi \in A(\bar{D})|_T$ satisfying $\operatorname{Im}\hat{\phi}_0 = 0$ (i.e., $\phi(t) = f(\mathrm{e}^{\mathrm{i}t})$, where $f \in A(\bar{D})$, $\operatorname{Im} f(0) = 0$). Hence, with the index n deleted, (7.31) can be written

$$M(G\tau + G_\tau'\delta) = 0, \tag{$*$}$$

Now, since $M\phi$ is real-valued, $M(\mathrm{i}M\phi) = M\phi$ (i.e. $\mathrm{i}M$ is a projector). Therefore, ($*$) is equivalent to

$M(\mathrm{i}MG\tau + G_\tau'\delta) = 0,$

which is equivalent to (7.23)–(7.24).

to (7.23) yields

$$\zeta_n\delta_n + \xi_n + \mathrm{i}\eta_n \in A(\bar{D})_T, \qquad \mathrm{Im}[\zeta_n\delta_n + \xi_n + \mathrm{i}\eta_n]_0^\wedge = 0,$$

which is identical with (7.31) since $G\tau_n = \xi_n + \mathrm{i}\eta_n$. Therefore, if δ_n is the unique solution of (7.23) and (7.24), it is also the unique solution of (7.32) and (7.33), and the functions f_n and h_n satisfy (7.37). □

Corollary 7.6. *Under the assumptions of Theorem* 7.3, *if* τ_0 *is sufficiently close to the solution* τ_i *of* $\Psi\tau = 0$, *the functions* f_n *of Theorem* 7.2 *converge quadratically to the zero function, and the functions* h_n *of Theorem* 7.4 *converge quadratically to* $Hg_\mathrm{i} = I_{ST}G\tau_\mathrm{i}$, *where* $g_i \in \mathscr{D}_H^p$ *is the interior map normalized by* (7.4).

Proof. From (7.17) (with $\tau = \tau_\mathrm{i}$, $\delta = \tau_n - \tau_\mathrm{i}$) we see that the quadratic convergence of δ to 0 (established in Theorem 7.3) implies the quadratic convergence of $\Psi\tau_n$ to 0. Solving (7.28) for f_n then leads to the claim concerning $\{f_n\}$, since the norm of the functions ζ_n is bounded in a neighborhood of the solution. Similarly, by (7.15), $G\tau_n \to G\tau_\mathrm{i}$ quadratically, and thus the claim concerning $\{h_n\}$ follows from $Hg_\mathrm{i} = I_{ST}G\tau_\mathrm{i}$ and (7.36) or (7.37). □

Remarks. The equivalence stated in Theorem 7.5 is no longer true for discretized methods. It is likely that the effect of discretization is not very different for the two methods, but that Vertgeim's formula (7.28) is superior with respect to roundoff, since $\Psi\tau_n$ and f_n are small if τ_n is close to the solution τ_i of $\Psi\tau = 0$, while this is not true for $G\tau_n$ and h_n, so that cancellation is inherent in (7.36). On the other hand, the evaluation of Wegmann's formula (7.36) requires only two applications of the conjugation operator ($G\tau_n$ does not involve K, in contrast to $\Psi\tau_n$), so that the costs are reduced by about one third.

In view of Theorem 3.4 both methods are easily adapted to the exterior mapping problem.

For the Riemann–Hilbert problems (7.27) and (7.35), the function β/α needed in (3.4) can be expressed in an elegant way in terms of ϕ defined by (3.1) or in terms of $\tilde{\phi}(t) = \phi(t) + \frac{1}{2}\pi$: We have

$$\alpha(t) = |\zeta_n(t)|, \tag{7.39}$$

$$\phi(t) = \arg\zeta_n(t) - \tfrac{1}{2}\pi, \quad \tilde{\phi}(t) := \arg\zeta_n(t), \tag{7.40}$$

$$\frac{\beta(t)}{\alpha(t)} = \begin{cases} \Psi\tau_n \sin\phi(t) = -\Psi\tau_n \cos\tilde{\phi}(t) & \text{for (7.27)}, \\ \mathrm{Re}\{G\tau_n\, \mathrm{e}^{-\mathrm{i}\phi(t)}\} = -\mathrm{Im}\{G\tau_n\, \mathrm{e}^{-\mathrm{i}\tilde{\phi}(t)}\} & \text{for (7.35)}. \end{cases} \tag{7.41}$$

Concerning the evaluation of g_i at interior points, most of what we said at the end of Section 5 still remains valid. In Vertgeim–Hübner type methods, where according to (7.37) and Corollary 7.6

$$Hg_i = \lim h_n = \lim L_\mathrm{R}^+\xi_n = L_\mathrm{R}^+ \lim \xi_n, \tag{7.42}$$

the Taylor coefficients of Hg_i are still a by-product of the iteration, since the Fourier coefficients of ξ_n are needed to compute $\Psi\tau_n$, cf. (7.6). In each step of a Wegmann type method solving the Riemann–Hilbert problem (7.35) yields an approximation h_n of Hg_i, and this h_n is repesented by a power series times the exponential of another power series, cf. (3.3). (In practice, if conjugation is performed via trigonometric interpolation, these power series are just polynomials.)

Independently of the work of Vertgeim and Wegmann, Fornberg [10] also proposed a method based on computing corrections δ_n such that (7.31a) holds. However, he made use of this condition by transforming it into the space of Fourier coefficients:

$$[G\tau_n + \zeta_n\delta_n]_k^\wedge = 0, \quad k < 0. \tag{7.43}$$

Actually, Fornberg was working with $Hg = g$, allowing $k \leqslant 0$ in (7.43), which is equivalent to working with $Hg(w) = g(w)/w$ and (7.43). He derived an ingenious and efficient method for solving finite systems (with $-N \leqslant k < 0$) of type (7.43). (From this system a positive definite system is derived, to which the conjugate gradient method is applied.) According to [10] about 30 complex FFTs with $\frac{1}{2}N$ points are typically needed to solve the system, i.e. to determine a discrete version of δ_n. This makes the method asymptotically about 7.5 times slower than Wegmann's, but, on the other hand, Wegmann [70] reports favorably on the numerical stability of Fornberg's method.

Fornberg's description of his method deviates from ours in that he does not update τ according to $\tau_{n+1} = \tau_n + \delta_n$. In fact he does not work with τ or θ at all. The boundary may be given either by γ or in the implicit form $F(z) = 0$. Fornberg then considers N points on Γ, and in each step these points are first moved along the tangent (as suggested by (7.30) with $G\tau_n = I_{TS}g_n$) and then projected back on the curve Γ (e.g., by solving $F(z) = 0$ approximately with Newton's method). This procedure is appropriate if Γ is given implicitly, but not if Γ is parameterized, as it is in most applications.

An explanation of the efficiency of Fornberg's method was first given by O. Widlund at the Workshop on Computational Problems in Complex Analysis at Stanford University, September 1981 (unpublished). After the completion of our present paper, Wegmann [72] has presented a more detailed analysis [5]. The non-discretized version of Fornberg's method, which is instrumental for this analysis, is derived along the following lines: As for Wegmann type methods we start from (7.31), but drop the index n for simplicity and replace ζ and δ by $\tilde{\zeta} := \zeta/|\zeta|$ and $\tilde{\delta} := \delta|\zeta|$. If P_+ and P_- denote the orthogonal projections of $L^2(T)$ onto $I_{TS}(H^2)$ and $I_{TS}((H^2)^\perp)$, respectively, then (7.31a) is clearly equivalent to $P_-(G\tau + \tilde{\zeta}\tilde{\delta}) = 0$, which implies

$$\operatorname{Re}\{\overline{\tilde{\zeta}(t)}P_-(G\tau + \tilde{\delta}\tilde{\zeta})(t)\} = 0. \tag{7.44}$$

Here, $P_-(G\tau + \tilde{\zeta}\tilde{\delta})$ can be thought of as a $(H^2)^\perp$ solution of a homogeneous exterior Riemann–Hilbert problem. By a variation of Theorem 3.4 the solution in $(H^2)^\perp$ is unique ($m = 0$, but $\hat{f}_0 = 0$ is required). Hence, (7.44) is equivalent to $P_-(G\tau + \tilde{\zeta}\tilde{\delta}) = 0$, and thus also to (7.31a).

Now, let $\phi := \arg\tilde{\zeta}$, and define the linear operator R: $W_T \to W_T$ by

$$Rf(t) := \operatorname{Re}\{e^{-\mathrm{i}t}\overline{\tilde{\zeta}(t)}([e^{\mathrm{i}(\ )}\tilde{\zeta}f]_0^\wedge - \mathrm{i}K[e^{\mathrm{i}(\ )}\tilde{\zeta}f](t))\}. \tag{7.45}$$

R has norm 1, and it is compact if $\phi \in \mathrm{Lip}^\alpha$ for some $\alpha > \frac{1}{2}$. In addition, it can be seen that R has a simple eigenvalue at -1, while the other eigenvalues have smaller modulus and lie symmetrically about 0. On the other hand,

$$(I \pm R)f = 2\operatorname{Re}\{\bar{\tilde{\zeta}}P_\mp(\tilde{\zeta}f)\}, \tag{7.46}$$

[5] This paragraph was added in December 1984.

so that (7.44) is equivalent to

$$(I+R)\tilde{\delta}=\kappa:=-(I+R)(\bar{\tilde{\zeta}}G\tau). \tag{7.47}$$

By various applications of Theorems 3.1 and 3.4 Wegmann shows further that $(I-R)$ is an automorphism of $L^2(T)$ and that under the assumption $[\mathrm{e}^{-K\phi}\kappa]_0^{\wedge}=0$, which is satisfied for the above κ, the equation $(I+R)\tilde{\delta}=\kappa$ and hence also the equation

$$(I-R^2)\tilde{\delta}=\tilde{\kappa}:=(I-R)\kappa=-(I-R^2)(\bar{\tilde{\zeta}}G\tau) \tag{7.48}$$

has a one-dimensional manifold of solutions, from which a unique solution can be chosen by requiring $\tilde{\delta}(0)=0$. Equation (7.48) defines a non-discretized version of Fornberg's method, which up to the difference in the normalization of the corrections ($\delta(0)=0$ in contrast to (7.31b)) is again mathematically equivalent to the Wegmann type method using the same operator H. In [72] Wegmann also discusses in detail the effects of discretization and suggests new variants of the method.

8. Examples of Vertgeim–Hübner and Wegmann type methods

In this section we briefly discuss the methods resulting from applying the ideas of the previous section to the auxiliary operators (6.1) and (6.17).

8.1. Application to the auxiliary function $Hg(w)=\log(g(w)/w)$

For Theodorsen's auxiliary function (6.1), $Hg(w)=h(g(w);\,w)=\log(g(w)/w)$, we let

$$\mathscr{D}_H^+:=\{g\in C(S);\,0\notin g(s),\ \#g(S)=1,\ |\hat{\eta}_0|<\pi\}, \tag{8.1}$$

so that $\mathscr{D}_H=\{g\in\mathscr{D}_H^+;\ \hat{\eta}_0=0\}$. If $g(w)$ is substituted for v, the derivatives required in property (vii), Section 7, are

$$\begin{gathered}h_g(w)=\frac{1}{g(w)},\qquad h_{gg}(w)=\frac{-1}{g^2(w)},\qquad h_{ggg}(w)=\frac{2}{g^3(w)},\\ \frac{\partial h}{\partial v}(g(w);\,w)=\frac{-1}{w},\qquad \frac{\partial^2 h}{\partial u\partial v}=\frac{\partial^3 h}{\partial u^2\partial v}=\frac{\partial^4 h}{\partial u^3\partial v}=0.\end{gathered} \tag{8.2}$$

Property (viii) is clearly satisfied. The function ζ defined in (7.8), which plays a prominent role in the formulas of the Theorems 7.2 and 7.4, is

$$\zeta(t)=\gamma'(\theta(t))/\gamma(\theta(t)). \tag{8.3}$$

In the case of polar coordinates (6.8) it becomes

$$\zeta(t)=\rho'(\theta(t))/\rho(\theta(t))+\mathrm{i}. \tag{8.4}$$

Inserting (8.3) or (8.4) into the Riemann–Hilbert problem (7.27), solving the latter according to Theorems 3.1 and 3.2, and inserting the solution f_n into (7.28) yields the *methods of Vertgeim* [62] *and Hübner* [34, Theorem 2] except that (as mentioned in Section 7) Vertgeim actually proposed a modified Newton iteration. The idea of applying instead the construction of Theorem 7.4 to this auxiliary function may be new in the case of an arbitrary parametrization, but the formulas resulting in the case of polar coordinates appear in Hübner [34, Theorem 4] as a second version of his method.

Relation (8.4) clearly implies that $\arg \zeta(t) \in (0, \pi) \pmod{2\pi}$; thus $\phi(t) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi) \pmod{2\pi}$ and $[\arg \zeta]_0^\wedge = \hat{\phi}_0 - \frac{1}{2}\pi \notin \pi\mathbb{Z}$, as noticed by Hübner [34]. However, using polar coordinates is not essential for this property. Whenever Γ is starlike (and positively oriented, as we always assume),

$$\arg \zeta(t) = \arg \gamma'(\theta(t)) - \arg \gamma(\theta(t)) \in (0, \pi) \pmod{2\pi}.$$

Hence, we get

Theorem 8.1. *If $Hg(w) = \log(g(w)/w)$ and Γ is starlike with respect to the origin, then the Riemann–Hilbert problems* (7.27) *and* (7.35) *always have unique solutions f_n and h_n normalized by* $\operatorname{Im} f_n = 0$ *and* $\operatorname{Im} h_n = 0$, *respectively.*

If Γ is not necessarily starlike we can still show that

$$[\arg \zeta]_0^\wedge = \tfrac{1}{2}\pi \pmod{2\pi} \quad \text{if } \tau = \tau_i, \tag{8.5}$$

so that the assumption (7.29) of Theorem 7.3 holds. Therefore, in practice it is unlikely that the iteration will break down except if we are far off the true solution. To prove (8.5) we use (8.3) and (4.9) to get

$$\begin{aligned}\arg \zeta(t) &= \arg \frac{\gamma'(\theta(t))}{\gamma(\theta(t))} = \arg \frac{\mathrm{i}\, \mathrm{e}^{\mathrm{i}t} g'(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\\ &\equiv \operatorname{Im}\{\log g'(\mathrm{e}^{\mathrm{i}t}) - \log(\mathrm{e}^{-\mathrm{i}t} g(\mathrm{e}^{\mathrm{i}t}))\} + \tfrac{1}{2}\pi \pmod{2\pi}.\end{aligned}$$

Now, if $g = g_i$, then $\log g' \in A(\bar{D})$ and $\log[g(\ )/(\ )] \in A(\bar{D})$, so that by the mean-value theorem

$$[\arg \zeta]_0^\wedge \equiv \operatorname{Im}\{\log g'(0) - \log g'(0)\} + \tfrac{1}{2}\pi \equiv \tfrac{1}{2}\pi \pmod{2\pi}.$$

8.2. Application to the auxiliary function $Hg(w) = g(w)/w$

For the auxiliary operator (6.17) of the Melentiev–Kulisch method, $Hg(w) = g(w)/w$, we let $\mathscr{D}_H^+ := C(S)$, so that

$$\mathscr{D}_H = \{g \in \mathscr{D}_H^+;\ \hat{g}_1 \geqslant 0\} = \{g \in \mathscr{D}_H^+;\ \hat{\xi}_0 \geqslant 0,\ \eta_0 = 0\}.$$

We get

$$\begin{aligned}&h_g(w) = \frac{1}{w}, \qquad h_{gg}(w) = h_{ggg}(w) \equiv 0,\\ &\frac{\partial h}{\partial v}(g(w); w) = \frac{-g(w)}{w^2}, \quad \frac{\partial^2 h}{\partial u \partial v}(g(w); w) = \frac{-1}{w^2}, \quad \frac{\partial^3 h}{\partial u^2 \partial v} = \frac{\partial^4 h}{\partial u^3 \partial v} = 0.\end{aligned} \tag{8.6}$$

Again, the properties (vii) and (viii) are obviously satisfied. For ζ we obtain

$$\zeta(t) = \frac{\gamma'(\theta(t))}{\mathrm{e}^{\mathrm{i}t}}, \tag{8.7}$$

and in the case of polar coordinates,

$$\zeta(t) = [\rho'(\theta(t)) + i\rho(\theta(t))]\, e^{\tau(t)}. \tag{8.8}$$

Theorem 7.4 yields now *Wegmann's method* [69,70], while the Vertgeim–Hübner type method of Theorem 7.2 corresponding to this auxiliary function has not yet been proposed, at least to our knowledge.

There is again the question of whether the condition $[\arg \zeta]_0^{\wedge} \in \pi\mathbb{Z}$ is always satisfied. Theoretically, this is not true, but as shown by Wegmann [70], (8.5) holds again, so that the same remark as above can be made. To prove (8.5) now, we again make use of (4.9) and get

$$\begin{aligned}\arg \zeta(t) &= \arg\{e^{-it}\gamma'(\theta(t))\} = \arg\{ig'(e^{it})/\theta'(t)\}\\ &\equiv \arg g'(e^{it}) + \tfrac{1}{2}\pi = \operatorname{Im} \log g'(e^{it}) + \tfrac{1}{2}\pi \pmod{2\pi}.\end{aligned}$$

The result then follows as above.

Summarizing, we get as a corollary of Theorem 7.3:

Theorem 8.2. *Assume $\gamma \in C^{2,1}(T)$. Then for both the auxiliary operators $Hg(w) = \log(g(w)/w)$ and $Hg(w) = g(w)/w$, both the Vertgeim–Hübner type method of Theorem 7.2 and the Wegmann type method of Theorem 7.4 converge locally and quadratically in $W_T = W^{1,p}(T, \mathbb{R})$.*

The respective result for their particular methods was established by Hübner [34, Theorem 5] and Wegmann [70]. Wegmann gave further convergence results under weaker assumptions, and Hübner proved also global convergence in the case $\rho \in W^{2,\infty}$, $\|\rho'/\rho\|_\infty < \frac{1}{3}$.

9. Further related methods

9.1. The Menikoff–Zemach method

The method of Menikoff and Zemach [46] applies to the same situation as the classical Theodorsen method: $Hg(w) = \log(g(w)/w)$, $\gamma(t) = \rho(t)\, e^{it}$. However, version (4.16b) of Ψ is used, so that the integral equation (4.15) becomes

$$\xi(t) = \hat{\xi}_0 - K\eta(t) \tag{9.1a}$$

with

$$\xi(t) = \log \rho(\theta(t)), \qquad \eta(t) = \theta(t) - t = \tau(t). \tag{9.1b}$$

The basic idea is now to transform first the principal value integral (2.20) for $K\eta$ by subtraction of η and integration by parts into a nonsingular integral involving η' and then to get rid of this factor η' by the (unknown) variable substitution $t = \sigma(\theta)$, which has the effect that the given integral equation for θ becomes an integral equation for the *inverse boundary correspondence function* $\theta \mapsto \sigma(\theta)$ (which is the inverse function of θ, but also the boundary correspondence function of an inverse map from Δ to D). In the first step, using (2.20) and the fact that $Kc = 0$ for the constant $c = \eta(t)$ (t fixed), we have

$$K\eta(t) = \frac{1}{2\pi}\int_T \cot\left(\frac{t-\sigma}{2}\right)[\eta(\sigma) - \eta(t)]\,d\sigma,$$

which in view of

$$\cot \frac{t-\sigma}{2} = 2 \frac{\mathrm{d}}{\mathrm{d}\sigma} \log \left| \sin \frac{t-\sigma}{2} \right|$$

and $\eta'(\sigma)\mathrm{d}\sigma = (\theta'(\sigma) - 1)\mathrm{d}\sigma = \mathrm{d}\theta - \mathrm{d}\sigma$ transforms by integration by parts into

$$K\eta(t) = \frac{1}{\pi} \int_T \log \left| \sin \frac{t - \sigma(\theta)}{2} \right| \mathrm{d}\theta - \frac{1}{\pi} \int_T \log \left| \sin \frac{t-\sigma}{2} \right| \mathrm{d}\sigma. \tag{9.2}$$

From the periodicity of the sine function it is clear that the second integral must be a constant independent of t (its actual value is $-2 \log 2$); therefore we may replace t by θ there. Then, we call the integration variable in both integrals $\tilde{\theta}$, replace $K\eta(t)$ according to (9.1) and substitute t by $\sigma(\theta)$ (so that $\theta(t) = \theta(\sigma(\theta)) = \theta$) to obtain

$$\log \rho(\theta) = \hat{\xi}_0 - \frac{1}{\pi} \int_T \log \left| \frac{\sin \frac{1}{2}(\sigma(\theta) - \sigma(\tilde{\theta}))}{\sin \frac{1}{2}(\theta - \tilde{\theta})} \right| \mathrm{d}\tilde{\theta}, \tag{9.3}$$

which is an integral equation for σ and the unknown constant $\hat{\xi}_0$. Basically, Menikoff and Zemach [46] discretize this integral equation by applying Gauss quadrature to the integral, and then they solve the resulting nonlinear system of equations by Newton's method. In each step $O(N^3)$ operations seem to be necessary to solve the linear system for the corrections. Hence, the method is much slower for fixed N than the methods (using the FFT) considered so far, where each step requires only $O(N \log N)$ operations. On the other hand, as is known from other methods for computing σ, such as methods for solving the Symm–Gaier integral equation, σ often behaves much better than θ. In particular, this is true for flat ellipses and smooth curves of similar shape, where the Fourier series for σ converges very fast, while the series for θ converges slowly. In such cases a very coarse discretization (i.e. small N) may still yield a relative accurate solution σ of (9.3), while a method based on θ would require a very large N, at least if conjugation is based on trigonometric interpolation.

9.2. The Chakravarthy–Anderson method

The method of Chakravarthy and Anderson [7] can be understood as a method for solving a discretized version of $|\Psi_T|^2 = 0$ (with $Hg = g$) by a minimization method, such as the conjugate gradient or the damped Newton method [50]. However, the computation of the conjugate function needed for the evaluation of Ψ_T is done by multiplication with a certain 'influence' matrix, constructed in advance by solving N systems of discretized Cauchy–Riemann equations on a $O(N^2)$-point grid on D by cyclic reduction. This preliminary work alone requires $O(N^3 \log N)$ operations. The authors were obviously unaware of the fact that the explicitly known Wittich matrix [13, pp. 76–80] or, better, the now-standard conjugation process using two real FFTs [27,28], would have served the same purpose. (Actually, their conjugation procedure computes the odd-indexed components of $K\xi$ from the even-indexed components of ξ only. A fast algorithm for this task was presented in [20].)

9.3. The Challis–Burley method

The method of Challis and Burley [8] is closely related to Theodorsen's method (though the latter is not mentioned in [8]), but the setting is different: A rectangle of unknown side length ratio is to be mapped conformally onto a region of the form

$$\{z = x + \mathrm{i}y;\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \phi(x)\}, \tag{9.4}$$

so that the corners of the rectangle are mapped onto those of the region (ϕ is a given positive function). By adding to both regions their mirror images at the imaginary axis and by applying in both planes an exponential map, the problem can be seen to be equivalent to a special doubly connected mapping problem, where the outer boundary curve is exactly the unit circle. It turns out that for such a region Garrick's method [13, pp. 194–207; 30, 35] can be simplified in such a way that the iteration becomes formally similar to Theodorsen's iteration for simply connected regions. This method is fast in the sense that each iteration requires only 2 real FFTs plus $O(N)$ operations. If the region (9.4) is replaced by $\{z = x + \mathrm{i}y;\ 0 \leqslant x \leqslant 1,\ \phi_1(x) \leqslant y \leqslant \phi_2(x)\}$ (as in the problem of Wanstrath et al. referenced in [60, p. 12]), the equivalence with the standard doubly-connected problem persists and, e.g., Garrick's method can be applied.

9.4. Adaptations to the conformal mapping problem for doubly connected regions

Many of the methods we have discussed in previous sections have been or can be adapted to the conformal mapping problem for doubly connected regions. Best known and widely used is Garrick's extension of Theodorson's method [13, pp. 194–207; 30, 35]. Wegmann's [71] and Fornberg's [11] methods have also been adapted by their authors. Moreover, there is always the possibility (proposed and justified first by Komatu [13,14]) of solving this mapping problem via the construction of a sequence of maps for simply connected regions. For example, Halsey [25] has applied Timman's method in a similar way very successfully to the flow analysis of multielement airfoils.

Constructive aspects of the conformal mapping problem for doubly and multiply connected regions are discussed in detail in [13,14,30].

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References

[1] F. Bauer, P. Garabedian, D. Korn, and A. Jameson, *Supercritical Wing Sections II* (Springer, Berlin, Heidelberg, New York, 1975).

[2] H. Bergström, An approximation of the analytic function mapping a given domain inside or outside the unit circle, *Mém. Publ. Soc. Sci. Arts Lettr. Hainaut, Volume hors Série* (1958) 193–198.
[3] R. Bernhardsgrütter, Über ein iteratives Verfahren von Wegmann zur numerischen konformen Abbildung, Diplomarbeit, Seminar f. Angew. Math., ETH Zürich, 1983.
[4] J.-P. Berrut, Baryzentrische Formeln zur trigonometrischen Interpolation, *Z. Angew. Math. Phys.* **35** (1984) 91–105, and 193–205.
[5] J.-P. Berrut, A Fredholm integral equation of the second kind for conformal mapping, *J. Comput. Appl. Math.* **14** (this volume) 99–110.
[6] H. Brass, Zur numerischen Berechnung der konjugierten Funktion, in: L. Collatz, et al., Eds., Numerical Methods of Approximation Theory **6** (Birkhäuser, Basel, 1982) 43–62.
[7] S. Chakravarthy and D. Anderson, Numerical conformal mapping, *Math. Comp.* **33** (1979) 953–969.
[8] N.V. Challis and D.M. Burley, A numerical method for conformal mapping, *IMA J. Numer. Anal.* **2** (1982) 169–181.
[9] P.L. Duren, *Theory of H^p Spaces* (Academic Press, New York, 1970).
[10] B. Fornberg, A numerical method for conformal mapping, *SIAM J. Sci. Stat. Comput.* **1** (1980) 386–400.
[11] B. Fornberg, A numerical method for conformal mapping of doubly connected regions, *SIAM J. Sci. Stat. Comp.* **5** (1984) 771–783.
[12] M.S. Friberg, A new method for the effective determination of conformal maps, Thesis, Univ. Minnesota, 1951.
[13] D. Gaier, *Konstruktive Methoden der konformen Abbildung* (Springer, Berlin, Göttingen, Heidelberg, 1964).
[14] D. Gaier, Konforme Abbildung mehrfach zusammenhängender Gebiete, *Jahresber. Deutsch. Math.-Verein* **81** (1978) 25–44.
[15] D. Gaier, Numerical methods in conformal mapping, in: H. Werner, et al., Eds., *Computational Aspects of Complex Analysis* (Reidel, Dordrecht, 1983) 51–78.
[16] J.B. Garnett, *Bounded Analytic Functions* (Academic Press, New York, 1981).
[17] I. Gohberg and S. Goldberg, *Basic Operator Theory* (Birkhäuser, Boston, Basel, Stuttgart, 1981).
[18] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable* (American Mathematical Society, Providence, RI, 1969).
[19] M.H. Gutknecht, Existence of a solution to the discrete Theodorsen equation for conformal mappings, *Math. Comp.* **31** (1977) 478–480.
[20] M.H. Gutknecht, Fast algorithms for the conjugate periodic function, *Computing* **22** (1979) 79–91.
[21] M.H. Gutknecht, Solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods, *Numer. Math.* **36** (1981) 405–429.
[22] M.H. Gutknecht, Numerical experiments on solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods, *SIAM J. Sci. Stat. Comput.* **4** (1983) 1–30.
[23] M.H. Gutknecht, On the computation of the conjugate trigonometric rational function and on a related splitting problem, *SIAM J. Numer. Anal.* **20** (1983) 1198–1205.
[24] M.H. Gutknecht, Solving Theodorsen's integral equation for conformal maps with the fast Fourier transform, Part III: The evaluation of the conjugate function of a periodic spline on a uniform mesh, Report 79-05, Seminar für Angewandte Mathematik ETH Zürich, 1979.
[25] N.D. Halsey, Potential flow analysis of multielement airfoils using conformal mapping, *AIAA J.* **17** (1979) 1281–1288.
[26] N.D. Halsey, Comparison of the convergence characteristics of two conformal mapping methods, *AIAA J.* **20** (1982) 724–726.
[27] P. Henrici, Einige Anwendungen der schnellen Fourier transformation, in: J. Albrecht and L. Collatz, Eds., *Moderne Methoden der Numerischen Mathematik* (Birkhäuser, Basel/Stuttgart, 1976) 111–124.
[28] P. Henrici, Fast Fourier methods in computational complex analysis, *SIAM Rev.* **21** (1979) 481–527.
[29] P. Henrici, Barycentric formulas for trigonometric interpolation, *Numer. Math.* **33** (1979) 225–234.
[30] P. Henrici, *Applied and Computational Complex Analysis, Vol. III* (Wiley, New York, to appear).
[31] L. Hörmander, The boundary problems of physical geodesy, *Arch. Rational Mech. Anal.* **62** (1976) 1–52.
[32] O. Hübner, Zur Numerik der Theodorsenschen Integralgleichung in der konformen Abbildung, *Mitt. Math. Seminar Giessen* **140** (1979) 1–32.
[33] O. Hübner, Über die Anzahl der Lösungen der diskreten Theodorsen-Gleichung, *Numer. Math.* **39** (1982) 195–204.

[34] O. Hübner, The Newton method for solving the Theodorsen equation, *J. Comput. Appl. Math.* **14** (this volume) 19–30.
[35] D.C. Ives, A modern look at conformal mapping including multiply connected regions, *AIAA J.* **14** (1976) 1006–1011.
[36] R.M. James, A new look at two-dimensional incompressible airfoil theory, Report MDC-J0918/01, Douglas Aircraft Company, 1971.
[37] R. Jeltsch, Numerische konforme Abbildung mit Hilfe der Formel von Cisotti, Diplomarbeit, ETH Zürich, 1969.
[38] A. Kaiser, Die Konvergenz verschiedener Verfahren der sukzessiven Konjugation zur Berechnung konformer Abbildungen, in preparation.
[39] L.V. Kantorovich and V.I. Krylov, *Approximate Methods of Higher Analysis* (Interscience, New York, 1958; Noordhoff, Groningen, 1958).
[40] Y. Katznelson, *An Introduction to Harmonic Analysis* (Wiley, New York, 1968; Dover, New York, 1976).
[41] N. Kerzman, A Monge–Ampère equation in complex analysis, in: Proc. Symp. Pure Math. **30** (part 1) (American Mathematical Society, Providence, RI, 1977) 161–167.
[42] U. Kulisch, Ein Iterationsverfahren zur konformen Abbildung des Einheitskreises auf einen Stern, *Z. Angew. Math. Mech.* **43** (1963) 403–410.
[43] P.A.A. Laura, A survey of modern applications of the method of conformal mapping, *Rev. Un. Mat. Argentina* **27** (1975) 167–179.
[44] R.S. Lehman, Development of the mapping function at an analytic corner, *Pacific J. Math.* **7** (1957) 1437–1449.
[45] D.I. Meiron and S.A. Orszag, Applications of numerical conformal mapping, *J. Comput. Phys.* **40** (1981) 345–360.
[46] R. Menikoff and C. Zemach, Methods for numerical conformal mapping, *J. Comput. Phys.* **36** (1980) 366–410.
[47] G. Moretti, Grid generation using classical techniques, in: *Numerical Grid Generation*, NASA Conf. Publ. 2166, 1980, pp. 1–35.
[48] N.I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953).
[49] W. Niethammer, Iterationsverfahren bei der konformen Abbildung, *Computing* **1** (1966) 146–153.
[50] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, New York, 1970).
[51] Ch. Pommerenke, *Univalent Functions* (Vandenhoeck and Ruprecht, Göttingen, 1975).
[52] Ch. Pommerenke, Boundary behaviour of conformal mappings, in: D.A. Brannan and J.G. Clunie, Eds., *Contemporary Complex Analysis* (Academic Press, New York, 1980) 313–331.
[53] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1966).
[54] W. Seidel, Über die Ränderzuordnung bei konformen Abbildungen, *Math. Ann.* **104** (1931) 182–243.
[55] V. Smirnoff, Über die Ränderzuordnung bei konformer Abbildung, *Math. Ann.* **107** (1932) 313–323.
[56] T. Theodorsen, Theory of wing sections of arbitrary shape, NACA Report 411, 1931.
[57] T. Theodorsen and I.E. Garrick, General potential theory of arbitrary wing sections, NACA Report 452, 1933.
[58] J.F. Thompson, Ed., *Numerical Grid Generation* (North-Holland, Amsterdam, 1982).
[59] J.F. Thompson, A survey of grid generation techniques in computational fluid dynamics, AIAA-83-0447, AIAA 21st Aerospace Sciences Meeting, Reno, Nevada, 1983.
[60] J.F. Thompson, Z.U.A. Warsi, and C.W. Mastin, Boundary-fitted coordinate systems for numerical solution of partial differential equations—a review, *J. Comput. Phys.* **47** (1982) 1–108.
[61] R. Timman, The direct and the inverse problem of airofoil theory. A method to obtain numerical solutions, Nat. Luchtv. Labor. Amsterdam, Report F. 16, 1951.
[62] B.A. Vertgeim, Approximate construction of some conformal mappings (Russian), *Doklady Akad. Nauk SSSR* **119** (1958) 12–14.
[63] S.E. Warschawski, Über einen Satz von O.D. Kellog, *Göttinger Nachr., Math.-Phys. Klasse* (1932) 73–86.
[64] S.E. Warschawski, Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung, *Math. Z.* **35** (1932) 321–456.
[65] S.E. Warschawski, On a theorem of L. Lichtenstein, *Pacific J. Math.* **5** (1955) 835–840.
[66] S.E. Warschawski, Recent results in numerical methods of conformal mapping, in: Proc. Symp. Appl. Math. **6** (McGraw-Hill, New York, 1956) 219–250.
[67] S.E. Warschawski, On differentiability at the boundary in conformal mapping, *Proc. Amer. Math. Soc.* **12** (1961) 614–620.
[68] S.E. Warschawski and G.E. Schober, On conformal mapping of certain classes of Jordan domains, *Arch. Rat. Mech. Anal.* **22** (1966) 201–209.

[69] R. Wegmann, Ein Iterationsverfahren zur konformen Abbildung, *Numer. Math.* **30** (1978) 453–466, translated as: An iterative method for conformal mapping, *J. Comput. Appl. Math.* **14** (this volume) 7–18.
[70] R. Wegmann, Convergence proofs and error estimates for an iterative method for conformal mapping, *Numer. Math.* **44** (1984), 435–461.
[71] R. Wegmann, An iterative method for the conformal mapping of doubly connected regions, *J. Comput. Appl. Math.* **14** (this volume) 79–98.
[72] R. Wegmann, On Fornberg's numerical method for conformal mappings, Report MPA 154, Max-Planck-Institut für Physik und Astrophysik, Sep. 1984.
[73] W. Wöhner, Konvergenz- und Fehleruntersuchungen zu einem Näherungsverfahren der konformen Abbildung einfach zusammenhängender Gebiete, *Wiss. Z. Hochschule f. Verkehrswesen Dresden* **12** (1965) 17–20.
[74] L.C. Woods, *The Theory of Subsonic Plane Flow* (Cambridge Univ. Press, Cambridge, 1961).
[75] A. Zygmund, *Trigonometrical Series* (Monografje Matematyczne, Warsaw, 1935).