Structured eigenvalue condition numbers and linearizations for matrix polynomials*

Bibhas Adhikari[†] Rafikul Alam[‡] Daniel Kressner[§]

Abstract. This work is concerned with eigenvalue problems for structured matrix polynomials, including complex symmetric, Hermitian, even, odd, palindromic, and anti-palindromic matrix polynomials. Most numerical approaches to solving such eigenvalue problems proceed by linearizing the matrix polynomial into a matrix pencil of larger size. Recently, linearizations have been classified for which the pencil reflects the structure of the original polynomial. A question of practical importance is whether this process of linearization significantly increases the eigenvalue sensitivity with respect to structured perturbations. For all structures under consideration, we show that this cannot happen if the matrix polynomial is well scaled: There is always a structured linearization for which the structured eigenvalue condition number does not differ much. This implies, for example, that a structure-preserving algorithm applied to the linearization fully benefits from a potentially low structured eigenvalue condition number of the original matrix polynomial.

Keywords. Eigenvalue problem, matrix polynomial, linearization, structured condition number.

AMS subject classification(2000): 65F15, 15A57, 15A18, 65F35.

1 Introduction

Consider an $n \times n$ matrix polynomial

$$\mathrm{P}(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^m A_m, \tag{1}$$

with $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$. An eigenvalue $\lambda \in \mathbb{C}$ of P, defined by the relation $\det(\mathrm{P}(\lambda)) = 0$, is called simple if λ is a simple root of the polynomial $\det(\mathrm{P}(\lambda))$.

This paper is concerned with the sensitivity of a simple eigenvalue λ under perturbations of the coefficients A_i. The condition number of λ is a first-order measure for the worst-case effect of perturbations on λ. Tisseur [35] has provided an explicit expression for this condition number. Subsequently, this expression was extended to polynomials in homogeneous form by Dedieu and Tisseur [10], see also [1, 5, 9], and to semi-simple eigenvalues in [24]. In the more general context of nonlinear eigenvalue problems, the sensitivity of eigenvalues and eigenvectors has been investigated in, e.g., [3, 26, 27, 28].

Loosely speaking, an eigenvalue problem (1) is called *structured* if there is some distinctive structure among the coefficients $A_0, \ldots, A_m$. For example, much of the recent research on structured polynomial eigenvalue problems was motivated by the second-order T-palindromic eigenvalue problem [20, 29]

$$A_0 + \lambda A_1 + \lambda^2 A_0^T,$$

where A_1 is complex symmetric: $A_1^T = A_1$. In this paper, we consider the structures listed in Table 1. To illustrate the notation of this table, consider a T-palindromic polynomial

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[†]Department of Mathematics, Indian Institute of Technology Guwahati, India, E-mail: bibhas@iitg.ernet.in

[‡]Department of Mathematics, Indian Institute of Technology Guwahati, India, E-mail: rafik@iitg.ernet.in, rafikul@yahoo.com, Fax: +91-361-2690762/2582649.

[§]Seminar for Applied Mathematics, ETH Zurich, Switzerland. E-mail: kressner@sam.math.ethz.ch

	Structured Polynomial $\mathrm{P}(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$	
Structure	**Condition**	$\mathbf{m = 2}$
symmetric	$\mathrm{P}^T(\lambda) = \mathrm{P}(\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A_0 + \lambda A_1 + A_2$, $A_0^T = A_0,\ A_1^T = A_1,\ A_2^T = A_2$
Hermitian	$\mathrm{P}^H(\lambda) = \mathrm{P}(\overline{\lambda})$	$\mathrm{P}(\lambda) = \lambda^2 A_0 + \lambda A_1 + A_2$, $A_0^H = A_0,\ A_1^H = A_1,\ A_2^H = A_2$
$\star$-even	$\mathrm{P}^\star(\lambda) = \mathrm{P}(-\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + C$, $A^\star = A, B^\star = -B, C^\star = C$
$\star$-odd	$\mathrm{P}^\star(\lambda) = -\mathrm{P}(-\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + C$, $A^\star = -A, B^\star = B, C^\star = -C$
$\star$-palindromic	$\mathrm{P}^\star(\lambda) = \lambda^m \mathrm{P}(1/\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + A^\star, B^\star = B$
$\star$-anti-palindromic	$\mathrm{P}^\star(\lambda) = \lambda^m \mathrm{P}(-1/\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B - A^\star, B^\star = -B$

Table 1: Overview of structured matrix polynomials discussed in this paper. Note that $\star \in \{T, H\}$ may denote either the complex transpose ($\star = T$) or the Hermitian transpose ($\star = H$).

characterized by the condition $\mathrm{P}^T(\lambda) = \lambda^m \mathrm{P}(1/\lambda)$. For even m, P takes the form

$$\mathrm{P}(\lambda) = A_0 + \cdots + \lambda^{m/2-1} A_{m/2-1} + \lambda^{m/2} A_{m/2} + \lambda^{m/2+1} A_{m/2+1}^T + \cdots + \lambda^m A_0^T,$$

with complex symmetric $A_{m/2}$, and for odd m, P takes the form

$$\mathrm{P}(\lambda) = A_0 + \cdots + \lambda^{(m-1)/2} A_{(m-1)/2} + \lambda^{(m+1)/2} A_{(m+1)/2}^T + \cdots + \lambda^m A_0^T.$$

In certain situations, it is reasonable to expect that perturbations of the polynomial respect the underlying structure. For example, if a strongly backward stable eigenvalue solver was applied to a palindromic matrix polynomial then the computed eigenvalues would be the exact eigenvalues of a slightly perturbed *palindromic* eigenvalue problems. Also, structure-preserving perturbations are physically more meaningful in the sense that the spectral symmetries induced by the structure are not destroyed. Restricting the admissible perturbations might have a positive effect on the sensitivity of an eigenvalue. This question has been studied for linear eigenvalue problems in quite some detail recently [8, 14, 23, 21, 22, 24, 31, 32, 33]. It often turns out that the desirable positive effect is not very remarkable: in many cases the worst-case eigenvalue sensitivity changes little or not at all when imposing structure. Notable exceptions can be found among symplectic, skew-symmetric, and palindromic eigenvalue problems [23, 24]. Bora [7] has identified situations for which the structured and unstructured eigenvalue condition numbers for matrix polynomials are equal, see also Section 2.

Throughout this paper, we consider *complex* structured perturbations despite the fact that the coefficient matrices A_i are typically real. This is mainly for convenience; the expressions for *real* structured condition numbers can be expected to be quite technical, which would complicate the subsequent analysis. Moreover, existing results [8, 24, 33] indicate that there is often no or little difference between real and complex structured condition numbers.

Due to the lack of a robust genuine polynomial eigenvalue solver, the eigenvalues of P are usually computed by first reformulating (1) as an $mn \times mn$ linear generalized eigenvalue problem and then applying a standard method such as the QZ algorithm [13] to the linear problem. This process of linearization introduces unwanted effects. Besides the obvious increase of dimension, it may also happen that the eigenvalue sensitivities deteriorate. Fortunately, one can use the freedom in the choice of linearization to minimize this deterioration for the eigenvalue region of interest, as proposed for quadratic eigenvalue problems in [11, 19, 35]. For the general polynomial eigenvalue problem (1), Higham et al. [18, 16] have identified linearizations with minimal eigenvalue condition number/backward error among the set of linearizations described in [30]. For structured polynomial eigenvalue problems, rather than using *any* linearization it is of course advisable to use one which has a similar structure. For example, it was shown in [29] that a palindromic matrix polynomial can usually be linearized into a palindromic or anti-palindromic matrix pencil, offering the possibility to apply structure-preserving algorithms to the linearization. It is natural to ask whether there is also a structured linearization that has no adverse effect on the structured condition number. For a small subset of structures from Table 1, this question has already been discussed in [18]. In the second part of this paper, we extend the discussion to all structures from Table 1.

The rest of this paper is organized as follows. In Section 2, we first review the derivation of the unstructured eigenvalue condition number for a matrix polynomial and then provide explicit expressions for structured eigenvalue conditions numbers. In Section 4, we apply these results to find good choices from the set of structured linearizations described in [29].

2 Structured condition numbers for matrix polynomials

Before discussing the effect of structure on the sensitivity of an eigenvalue, we briefly review existing results on eigenvalue condition numbers for matrix polynomials. Assume that λ is a *simple finite* eigenvalue of the matrix polynomial P defined in (1) with normalized right and left eigenvectors x and y:

$$\mathrm{P}(\lambda)x = 0, \quad y^H\mathrm{P}(\lambda) = 0, \quad \|x\|_2 = \|y\|_2 = 1. \tag{2}$$

The perturbation

$$(\mathrm{P} + \triangle\mathrm{P})(\lambda) = (A_0 + E_0) + \lambda(A_1 + E_1) + \cdots + \lambda^m(A_m + E_m)$$

moves λ to an eigenvalue $\widehat{\lambda}$ of $\mathrm{P} + \triangle\mathrm{P}$. A useful tool to study the effect of $\triangle\mathrm{P}$ is the first order *perturbation expansion*

$$\widehat{\lambda} = \lambda - \frac{1}{y^H\mathrm{P}'(\lambda)x} y^H \triangle\mathrm{P}(\lambda)x + O(\|\triangle\mathrm{P}\|^2), \tag{3}$$

which can be derived, e.g., by applying the implicit function theorem to (2), see [10, 35]. Note that $y^H\mathrm{P}'(\lambda)x \neq 0$ because λ is simple [3].

To measure the sensitivity of λ we first need to specify a way to measure $\triangle\mathrm{P}$. Given a matrix norm $\|\cdot\|_{\mathsf{M}}$ on $\mathbb{C}^{n\times n}$, a monotone vector norm $\|\cdot\|_{\mathsf{V}}$ on $\mathbb{C}^{m+1}$ and non-negative weights $\omega_0, \ldots, \omega_m$, we define

$$\|\triangle\mathrm{P}\| := \left\| \left[\frac{1}{\omega_0}\|E_0\|_{\mathsf{M}}, \, \frac{1}{\omega_1}\|E_1\|_{\mathsf{M}}, \, \ldots, \, \frac{1}{\omega_m}\|E_m\|_{\mathsf{M}} \right] \right\|_{\mathsf{V}}. \tag{4}$$

A relatively small weight ω_i means that $\|E_i\|_{\mathsf{M}}$ will be small compared to $\|\triangle\mathrm{P}\|$. In the extreme case $\omega_i = 0$, we define $\|E_i\|_{\mathsf{M}}/\omega_i = 0$ for $\|E_i\|_{\mathsf{M}} = 0$ and $\|E_i\|_{\mathsf{M}}/\omega_i = \infty$ otherwise. If all ω_i are positive then (4) defines a norm on $\mathbb{C}^{n\times n} \times \cdots \times \mathbb{C}^{n\times n}$, see [2, 1] for more on norms of matrix polynomials.

We are now ready to introduce a condition number for the eigenvalue λ of P with respect to the choice of $\|\triangle\mathrm{P}\|$ in (4):

$$\kappa_{\mathrm{P}}(\lambda) := \lim_{\epsilon\to 0} \sup \left\{ \frac{|\hat{\lambda} - \lambda|}{\epsilon} : \ \|\triangle\mathrm{P}\| \leq \epsilon \right\}, \tag{5}$$

where $\hat{\lambda}$ is the eigenvalue of $\mathrm{P} + \triangle\mathrm{P}$ closest to λ. An explicit expression for $\kappa_{\mathrm{P}}(\lambda)$ can be found in [35, Thm. 5] for the case $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_\infty$ and $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$. In contrast, the approach used in [10] requires an accessible geometry on the perturbation space and thus facilitates the norms $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_2$ and $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$. Lemma 2.1 below is more general and includes both settings. For stating our result, we recall that the dual to the vector norm $\|\cdot\|_{\mathsf{V}}$ is defined as

$$\|w\|_{\mathsf{D}} := \sup_{\|z\|_{\mathsf{V}} \leq 1} |w^T z|,$$

see, e.g., [15].

Lemma 2.1 *Consider the condition number $\kappa_{\mathrm{P}}(\lambda)$ defined in (5) with respect to (4). For any unitarily invariant norm $\|\cdot\|_{\mathsf{M}}$ we have*

$$\kappa_{\mathrm{P}}(\lambda) = \frac{\|[\omega_0,\, \omega_1|\lambda|,\, \ldots, \omega_m|\lambda|^m]\|_{\mathsf{D}}}{|y^H \mathrm{P}'(\lambda) x|} \tag{6}$$

where $\|\cdot\|_{\mathsf{D}}$ denotes the vector norm dual to $\|\cdot\|_{\mathsf{V}}$.

Proof. Inserting the perturbation expansion (3) into (5) yields

$$\kappa_{\mathrm{P}}(\lambda) = \frac{1}{|y^H \mathrm{P}'(\lambda) x|} \sup\left\{ |y^H \triangle\mathrm{P}(\lambda) x| : \ \|\triangle\mathrm{P}\| \leq 1 \right\}. \tag{7}$$

Defining $b = \left[\|E_0\|_{\mathsf{M}}/\omega_0, \ldots, \|E_m\|_{\mathsf{M}}/\omega_m\right]^T$, we have $\|\triangle\mathrm{P}\| = \|b\|_{\mathsf{V}}$. By the triangular inequality,

$$|y^H \triangle\mathrm{P}(\lambda) x| \leq \sum_{i=0}^{m} |\lambda|^i |y^H E_i x|. \tag{8}$$

With a suitable scaling of E_i by a complex number of modulus 1, we can assume without loss of generality that equality holds in (8). Hence,

$$\sup_{\|\triangle\mathrm{P}\| \leq 1} |y^H \triangle\mathrm{P}(\lambda) x| = \sup_{\|b\|_{\mathsf{V}} \leq 1} \sum_{i=0}^{m} |\lambda|^i \sup_{\|E_i\|_{\mathsf{M}} = \omega_i b_i} |y^H E_i x|. \tag{9}$$

Using the particular perturbation $E_i = \omega_i b_i y x^H$, it can be easily seen that the inner supremum is $\omega_i b_i$ and hence

$$\sup_{\|\triangle\mathrm{P}\| \leq 1} |y^H \triangle\mathrm{P}(\lambda) x| = \sup_{\|b\|_{\mathsf{V}} \leq 1} \left| [\omega_0, \omega_1|\lambda|, \ldots, \omega_m|\lambda|^m] b \right| = \|[\omega_0,\, \omega_1|\lambda|,\, \ldots, \omega_m|\lambda|^m]\|_{\mathsf{D}},$$

which completes the proof. □

We refer to [1] for a general setting which allows some of the matrices A_i to remain unperturbed. From a practical point of view, measuring the perturbations of the individual coefficients of the polynomial separately makes a lot of sense and thus the choice $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_\infty$ seems to be most natural. However, it turns out – especially when considering structured condition numbers – that more elegant results are obtained with the choice $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_2$, which we will use throughout the rest of this paper. In this case, the expression (6) takes the form

$$\kappa_{\mathrm{P}}(\lambda) = \frac{\|[\omega_0,\, \omega_1\lambda,\, \ldots, \omega_m\lambda^m]\|_2}{|y^H \mathrm{P}'(\lambda) x|}, \tag{10}$$

see also [1, 4].

If $\lambda = \infty$ is a simple eigenvalue of P, a suitable condition number can be defined as

$$\kappa_{\mathrm{P}}(\infty) := \lim_{\epsilon \to 0} \sup\{1/|\hat{\lambda}\epsilon| : \ \|\triangle\mathrm{P}\| \leq \epsilon\},$$

and, following the arguments above,

$$\kappa_{\mathrm{P}}(\infty) = \omega_m / |y^H A_{m-1} x|$$

for any (Hölder) p-norm $\|\cdot\|_{\mathsf{V}}$. Note that this discrimination between finite and infinite eigenvalues disappears when homogenizing P as in [10] or measuring the distance between perturbed eigenvalues with the chordal metric as in [34]. In order to keep the presentation simple, we have decided not to use these concepts.

The rest of this section is concerned with quantifying the effect on the condition number when the perturbation $\triangle \mathrm{P}$ is restricted to a subset $\mathbb{S}$ of the space of all $n \times n$ matrix polynomials of degree at most m.

Definition 2.2 *Let λ be a simple finite eigenvalue of a matrix polynomial* P *with normalized right and left eigenvectors x and y. Then the* structured condition number *of λ with respect to $\mathbb{S}$ is defined as*

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) := \lim_{\epsilon \to 0} \sup \left\{ \frac{|\hat{\lambda} - \lambda|}{\epsilon} : \ \triangle \mathrm{P} \in \mathbb{S}, \ \|\triangle \mathrm{P}\| \leq \epsilon \right\} \tag{11}$$

For a simple infinite eigenvalue λ, $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) := \lim_{\epsilon \to 0} \sup \{1/|\hat{\lambda}\epsilon| : \ \triangle \mathrm{P} \in \mathbb{S}, \ \|\triangle \mathrm{P}\| \leq \epsilon\}$.

If $\mathbb{S}$ is a star-shaped set [12] with respect to 0, the expansion (3) can be used to show

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{|y^H \mathrm{P}'(\lambda) x|} \sup \left\{ |y^H \triangle \mathrm{P}(\lambda) x| : \ \triangle \mathrm{P} \in \mathbb{S}, \ \|\triangle \mathrm{P}\| \leq 1 \right\} \tag{12}$$

and

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \frac{1}{|y^H A_{m-1} x|} \sup \left\{ |y^H E_m x| : \ \triangle \mathrm{P} \in \mathbb{S}, \ \|E_m\|_{\mathsf{M}} \leq \omega_m \right\}. \tag{13}$$

The formulation (12) is the starting point to derive explicit expressions for $\kappa_{\mathrm{P}}^{\mathbb{S}}$ under particular choices of $\mathbb{S}$. To proceed, one can employ results by Karow [21] on the geometry of the set $\{y^H E x : E \in \mathbb{E}, \ \|E\|_{\mathsf{M}} \leq 1\}$ with respect to some matrix structures $\mathbb{E} \subseteq \mathbb{C}^{n \times n}$ induced by the polynomial structure $\mathbb{S}$. Such an approach was proposed by Bora [7], who also derived explicit expressions and bounds on $\kappa_{\mathrm{P}}^{\mathbb{S}}$ for the structures considered in this paper. Our expressions are of a similar nature and we therefore defer the derivations to Appendix A. The major difference is that we use $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_2$ while [7] uses $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_\infty$ for combining the norm of perturbations in the polynomial coefficients, see (4). We deliberately choose the 2-norm setting as this allows simpler explicit expressions for structured eigenvalue condition numbers. This in turn enables easy comparison of structured eigenvalue condition numbers of structured polynomials with those of the structured linearizations discussed in Section 4.

2.1 Complex symmetric matrix polynomials

No or only an insignificant decrease of the condition number can be expected when imposing complex symmetries on the perturbations of a matrix polynomial.

Lemma 2.3 *Let $\mathbb{S}$ denote the set of complex symmetric matrix polynomials. Then for a finite or infinite, simple eigenvalue λ of a matrix polynomial* P,

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$, *and*

2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{\sqrt{1+|y^T x|^2}}{\sqrt{2}} \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.

2.2 T-even and T-odd matrix polynomials

To describe the structured condition numbers for T-even and T-odd polynomials in a convenient manner, we introduce the vector

$$\Lambda_\omega = \left[\omega_m \lambda^m, \, \omega_{m-1} \lambda^{m-1}, \, \ldots, \, \omega_1 \lambda, \, \omega_0\right]^T \tag{14}$$

along with the even coefficient projector

$$\Pi_e : \Lambda_\omega \mapsto \Pi_e(\Lambda_\omega) := \begin{cases} \big[\omega_m \lambda^m,\, 0,\, \omega_{m-2}\lambda^{m-2},\, 0,\, \ldots,\, \omega_2\lambda^2,\, 0,\, \omega_0\big]^T, & \text{if } m \text{ is even,} \\ \big[0, \omega_{m-1}\lambda^{m-1},\, 0,\, \omega_{m-3}\lambda^{m-3},\, \ldots,\, 0,\, \omega_0\big]^T, & \text{if } m \text{ is odd.} \end{cases} \tag{15}$$

The odd coefficient projection is defined analogously and satisfies $\Pi_o(\Lambda_\omega) = \Lambda_\omega - \Pi_e(\Lambda_\omega)$.

Lemma 2.4 *Let $\mathbb{S}$ denote the set of all T-even matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial* P*,*

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sqrt{1 - |y^T x|^2 \frac{\|\Pi_o(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$*, and*
2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}}\sqrt{1 - |y^T x|^2 \frac{\|\Pi_o(\Lambda_\omega)\|_2^2 - \|\Pi_e(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.

For an infinite, simple eigenvalue,

3. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \kappa_{\mathrm{P}}(\infty), & \textit{if } m \textit{ is even,} \\ \sqrt{1 - |y^T x|^2}\, \kappa_P(\infty), & \textit{if } m \textit{ is odd,} \end{cases}$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$*, and*
4. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{1 + |y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \textit{if } m \textit{ is even,} \\ \frac{1}{\sqrt{2}}\sqrt{1 - |y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \textit{if } m \textit{ is odd,} \end{cases}$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.

Remark 2.5 *Note that the statement of Lemma 2.4 does not assume that* P *itself is T-even. If we impose this condition then, for odd m,* P *has a* simple *infinite eigenvalue only if also the size of* P *is odd, see, e.g., [24]. In this case, the skew-symmetry of A_m forces the infinite eigenvalue to be preserved under arbitrary structure-preserving perturbations. This is reflected by $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = 0$.*

Lemma 2.4 reveals that the structured condition number can only be significantly lower than the unstructured one if $|y^T x|$ *and* the ratio

$$\frac{\|\Pi_o(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2} = \frac{\sum\limits_{i \text{ odd}} \omega_i^2 |\lambda|^{2i}}{\sum\limits_{i=0,\ldots,m} \omega_i^2 |\lambda|^{2i}} = 1 - \frac{\sum\limits_{i \text{ even}} \omega_i^2 |\lambda|^{2i}}{\sum\limits_{i=0,\ldots,m} \omega_i^2 |\lambda|^{2i}}$$

are close to one. The most likely situation for the latter ratio to become close to one is when m is odd, ω_m does not vanish, and $|\lambda|$ is large.

Example 2.6 ([33]) *Let*

$$\mathrm{P}(\lambda) = I + \lambda 0 + \lambda^2 I + \lambda^3 \begin{bmatrix} 0 & 1-\phi & 0 \\ -1+\phi & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{bmatrix}$$

with $0 < \phi < 1$. This matrix polynomial has one eigenvalue $\lambda_\infty = \infty$ because of the highest coefficient, which is – as any odd-sized skew-symmetric matrix – singular. The following table additionally displays the eigenvalue $\lambda_{\max}$ of largest magnitude, the eigenvalue $\lambda_{\min}$ of smallest magnitude, as well as their unstructured and structured condition numbers for the set $\mathbb{S}$ of T-even matrix polynomials. We have chosen $\omega_i = \|A_i\|_2$ and $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$.

ϕ	10^0	10^{-3}	10^{-9}
$\kappa(\lambda_\infty)$	1	1.4×10^3	1.4×10^9
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_\infty)$	0	0	0
$\lvert\lambda_{\max}\rvert$	1.47	22.4	2.2×10^4
$\kappa_{\mathrm{P}}(\lambda_{\max})$	1.12	3.5×10^5	3.5×10^{17}
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\max})$	1.12	2.5×10^4	2.5×10^{13}
$\lvert\lambda_{\min}\rvert$	0.83	0.99	1.00
$\kappa_{\mathrm{P}}(\lambda_{\min})$	0.45	5.0×10^2	5.0×10^8
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\min})$	0.45	3.5×10^2	3.5×10^8

The entries $0 = \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_\infty) \ll \kappa_{\mathrm{P}}(\lambda_\infty)$ reflect the fact that the infinite eigenvalue stays intact under structure-preserving but not under general perturbations. For the largest eigenvalues, we observe a significant difference between the structured and unstructured condition numbers as $\phi \to 0$. In contrast, this difference becomes negligible for the smallest eigenvalues.

Remark 2.7 *For* even m, *the structured eigenvalue condition number of a T-even polynomial is usually close to the unstructured one. For example if all weights are equal, $\|\Pi_o(\Lambda)\|_2^2 \leq \|\Lambda\|_2^2/2$ implying $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \geq \kappa_{\mathrm{P}}(\lambda)/\sqrt{2}$ for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$.*

For T-odd polynomials, we obtain the following analogue of Lemma 2.4 by simply exchanging the roles of odd and even in the proof.

Lemma 2.8 *Let $\mathbb{S}$ denote the set of all T-odd matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial* P*,*

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$, *and*

2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}}\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_\omega)\|_2^2 - \|\Pi_o(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.

For an infinite, simple eigenvalue,

3. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even,} \end{cases}$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$, *and*

4. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{1 + |y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even,} \end{cases}$ *for* $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.

Similar to the discussion above, the only situation for which $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$ can be expected to become significantly smaller than $\kappa_{\mathrm{P}}(\lambda)$ when $|y^T x| \approx 1$ and $\lambda \approx 0$.

2.3 T-palindromic and T-anti-palindromic matrix polynomials

For a T-palindromic polynomial it is sensible to require that the weights in the choice of $\|\triangle \mathrm{P}\|$, see (4), satisfy $\omega_i = \omega_{m-i}$. This condition is tacitly assumed throughout the entire section. The Cayley transform for polynomials introduced in [29, Sec. 2.2] defines a mapping between palindromic/anti-palindromic and odd/even polynomials. As already demonstrated in [24] for the case $m = 1$, this idea can be used to transfer the results from the previous section to the (anti-)palindromic case. For the mapping to preserve the underlying norm we have to restrict ourselves to the case $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$. The coefficient projections appropriate for palindromic polynomials are given by $\Pi_\pm : \Lambda_\omega \mapsto \Pi_\pm(\Lambda_\omega)$ with

$$\Pi_\pm(\Lambda_\omega) := \begin{cases} \left[\omega_0 \frac{\lambda^m \pm 1}{\sqrt{2}}, \ldots, \omega_{m/2-1} \frac{\lambda^{m/2+1} \pm \lambda^{m/2-1}}{\sqrt{2}}, \omega_{m/2} \frac{\lambda^{m/2} \pm \lambda^{m/2}}{2}\right]^T & \text{if } m \text{ is even,} \\ \left[\omega_0 \frac{\lambda^m \pm 1}{\sqrt{2}}, \ldots, \omega_{(m-1)/2} \frac{\lambda^{(m+1)/2} \pm \lambda^{(m-1)/2}}{\sqrt{2}}\right]^T, & \text{if } m \text{ is odd.} \end{cases} \tag{16}$$

Note that $\|\Pi_+(\Lambda_\omega)\|_2^2 + \|\Pi_-(\Lambda_\omega)\|_2^2 = \|\Lambda_\omega\|_2^2$.

Lemma 2.9 *Let $\mathbb{S}$ denote the set of all T-palindromic matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial* P*, with $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$,*

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}}\sqrt{1 + |y^T x|^2 \frac{\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda).$$

For an infinite, simple eigenvalue, $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \kappa_{\mathrm{P}}(\infty)$.

From the result of Lemma 2.9 it follows that a large difference between the structured and unstructured condition numbers for T-palindromic matrix polynomials may occur when $|y^T x|$ is close to one, *and* $\|\Pi_+(\Lambda_\omega)\|_2$ is close to zero. Assuming that all weights are positive, the latter condition implies that m is odd and and $\lambda \approx -1$. An instance of such a case is given by a variation of Example 2.6.

Example 2.10 *Consider the T-palindromic matrix polynomial*

$$\mathrm{P}(\lambda) = \begin{bmatrix} 1 & 1-\phi & 0 \\ -1+\phi & 1 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{bmatrix} + \lambda I + \lambda^2 I - \lambda^3 \begin{bmatrix} 1 & 1-\phi & 0 \\ -1+\phi & 1 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{bmatrix}$$

with $0 < \phi < 1$. An odd-sized T-palindromic matrix polynomial, P *has the eigenvalue $\lambda_{-1} = -1$. The following table additionally displays one eigenvalue λ_{close} closest to -1, an eigenvalue $\lambda_{\min}$ of smallest magnitude, as well as their unstructured and structured condition numbers for the set $\mathbb{S}$ of T-palindromic matrix polynomials. We have chosen $\omega_i = \|A_i\|_F$ and $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.*

ϕ	10^{-1}	10^{-4}	10^{-8}
$\kappa(\lambda_{-1})$	20.9	2.2×10^4	2.2×10^8
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{-1})$	0	0	0
$\|1+\lambda_{\text{close}}\|$	0.39	1.4×10^{-2}	1.4×10^{-4}
$\kappa_{\mathrm{P}}(\lambda_{\text{close}})$	11.1	1.1×10^4	1.1×10^8
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\text{close}})$	6.38	2.5×10^2	2.6×10^4
$\|1+\lambda_{\min}\|$	1.25	1.41	1.41
$\kappa_{\mathrm{P}}(\lambda_{\min})$	7.92	7.9×10^3	7.9×10^7
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\min})$	5.75	5.6×10^3	5.6×10^7

The entries $0 = \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{-1}) \ll \kappa_{\mathrm{P}}(\lambda_{-1})$ reflect the fact that the eigenvalue -1 remains intact under structure-preserving but not under general perturbations. Also, eigenvalues close to -1 benefit from a significantly lower structured condition numbers as $\phi \to 0$. In contrast, only a practically irrelevant benefit is revealed for the eigenvalue $\lambda_{\min}$ not close to -1.

Results analogous to Lemma 2.9 hold for T-anti-palindromic matrix polynomials.

Lemma 2.11 *Let $\mathbb{S}$ denote the set of all T-anti-palindromic matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial* P*, with $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$,*

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}}\sqrt{1 - |y^T x|^2 \frac{\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda).$$

For an infinite, simple eigenvalue, $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \kappa_{\mathrm{P}}(\infty)$.

2.4 Hermitian matrix polynomials

For reasons explained in Section A.2, the structured eigenvalue condition numbers for Hermitian matrix polynomials do not admit a simple explicit expression. Therefore, the following lemma rather presents a bound implying that the unstructured and structured eigenvalue condition numbers are nearly the same.

Lemma 2.12 *Let $\mathbb{S}$ denote the set of all Hermitian matrix polynomials. Then for a finite or infinite, simple eigenvalue of a matrix polynomial* P,

1. $\sqrt{1 - \frac{1}{2}|y^H x|^2}\, \kappa_{\mathrm{P}}(\lambda) \le \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \le \kappa_{\mathrm{P}}(\lambda)$ *for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$, and*

2. $\kappa_{\mathrm{P}}(\lambda)/\sqrt{2} \le \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \le \kappa_{\mathrm{P}}(\lambda)$ *for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.*

Remark 2.13 *Since Hermitian and skew-Hermitian matrices are related by multiplication with* i*, which simply rotates the first-order perturbation set by* 90 *degrees, a slight modification of the proof shows that the statement of Lemma 2.12 remains true when* $\mathbb{S}$ *denotes the space of* H*-odd or* H*-even polynomials. This can in turn be used – as in the proof of Lemma 2.9 – to show that also for* H*-(anti-)palindromic polynomials there is at most an insignificant difference between the structured and unstructured eigenvalue condition numbers.*

3 Condition numbers for linearizations

As already mentioned in the introduction, polynomial eigenvalue problems are often solved by linearizing the matrix polynomial into a larger matrix pencil. Of the classes of linearizations proposed in the literature, the vector spaces introduced in [30] are particularly amenable to further analysis, while offering a degree of generality that is often sufficient in applications.

Definition 3.1 *Let* $\Lambda_{m-1} = [\lambda^{m-1}, \lambda^{m-2} \ldots \lambda, 1]^T$ *and let* P *be a matrix polynomial of degree* m*. Then a matrix pencil* $L(\lambda) = \lambda X + Y \in \mathbb{C}^{mn\times mn}$ *is in* $\mathbb{DL}(\mathrm{P})$ *if there is a so called* ansatz vector $v \in \mathbb{C}^m$ *satisfying*

$$L(\lambda) \cdot (\Lambda_{m-1} \otimes I) = v \otimes P(\lambda) \quad \textit{and} \quad (\Lambda_{m-1}^T \otimes I) \cdot L(\lambda) = v^T \otimes P(\lambda).$$

It is easy to see that the ansatz vector v is uniquely determined by $L \in \mathbb{DL}(\mathrm{P})$. In [30, Thm. 6.7] it has been shown that $L \in \mathbb{DL}(\mathrm{P})$ is a linearization of P if and only if none of the eigenvalues of P is a root of the polynomial

$$\mathsf{p}(\mu; v) = v_1\mu^{m-1} + v_2\mu^{m-2} + \cdots + v_{m-1}\mu + v_m \tag{17}$$

associated with the ansatz vector v. If P has eigenvalue ∞, this condition should be read as $v_1 \neq 0$. Apart from this elegant characterization, probably the most important property of $\mathbb{DL}(\mathrm{P})$ is that it leads to a simple one-to-one relation between the eigenvectors of P and $L \in \mathbb{DL}(\mathrm{P})$. To keep the notation compact, we define Λ_{m-1} as in Definition 3.1 for finite λ but let $\Lambda_{m-1} = [1, 0, \ldots, 0]^T$ for $\lambda = \infty$.

Theorem 3.2 ([30]) *Let* P *be a matrix polynomial and* $L \in \mathbb{DL}(\mathrm{P})$ *with ansatz vector* v*. Then* $x \neq 0$ *is a right eigenvector of* P *associated with an eigenvalue* λ *if and only if* $\Lambda_{m-1} \otimes x$ *is a right eigenvector of* L *associated with* λ*. Similarly,* $y \neq 0$ *is a left eigenvector of* P *associated with an eigenvalue* λ *if and only if* $\overline{\Lambda}_{m-1} \otimes y$ *is a left eigenvector of* L *associated with* λ*.*

As a matrix pencil $L(\lambda) = \lambda X + Y$ is a special case of a matrix polynomial, we can use the results of Section 2 to study the (structured) eigenvalue condition numbers of L. To simplify the analysis, we will assume that the weights $\omega_0, \ldots, \omega_m$ in the definition of $\|\triangle \mathrm{P}\|$ are all equal to 1 for the rest of this paper. This assumption is only justified if P is not badly scaled, i.e., the norms of the coefficients of P do not vary significantly. To a certain extent, bad scaling can be overcome by rescaling the matrix polynomial before linearization, see [6, 11, 16, 18, 19]. Moreover, we assume that $\|\cdot\|_{\mathsf{M}}$ is an arbitrary but fixed unitarily invariant matrix norm. The same norm is used for measuring perturbations $\triangle L(\lambda) = \triangle X + \lambda \triangle Y$ to the linearization L. To summarize

$$\|\triangle P\| = \sqrt{\|E_0\|_{\mathsf{M}}^2 + \|E_1\|_{\mathsf{M}} + \cdots + \|E_m\|_{\mathsf{M}}^2}, \tag{18}$$

$$\|\triangle L\| = \sqrt{\|\triangle X\|_{\mathsf{M}}^2 + \|\triangle Y\|_{\mathsf{M}}^2}, \tag{19}$$

for the rest of this paper. For unstructured eigenvalue condition numbers, Lemma 2.1 together with Theorem 3.2 imply the following formula.

Lemma 3.3 *Let λ be a finite, simple eigenvalue of a matrix polynomial P with normalized right and left eigenvectors x and y. Then the eigenvalue condition number $\kappa_L(\lambda)$ for a linearization $L \in \mathbb{DL}(\mathrm{P})$ with ansatz vector v satisfies*

$$\kappa_L(\lambda) = \frac{\sqrt{1+|\lambda|^2}}{|\mathsf{p}(\lambda;v)|} \cdot \frac{\|\Lambda_{m-1}\|_2^2}{|y^H P'(\lambda)x|} = \frac{\sqrt{1+|\lambda|^2}\,\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda;v)|\,\|\Lambda_m\|_2} \kappa_{\mathrm{P}}(\lambda).$$

Proof. A similar formula for the case $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_1$ can be found in [18, Section 3]. The proof for our case $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_2$ is almost identical and therefore omitted. □

To allow for a simple interpretation of the result of Lemma 3.3, we define the quantity

$$\delta(\lambda;v) := \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda;v)|} \tag{20}$$

for a given ansatz vector v and with $\mathsf{p}(\lambda;v)$ defined as in (17). Obviously $\delta(\lambda;v) \geq 1$. Since L is assumed to be a linearization, $\mathsf{p}(\lambda;v) \neq 0$ and hence $\delta(\lambda;v) < \infty$. Using the straightforward bound

$$1 \leq \frac{\sqrt{1+|\lambda|^2}\|\Lambda_{m-1}\|_2}{\|\Lambda_m\|_2} \leq \sqrt{2}, \tag{21}$$

the result of Lemma 3.3 yields

$$\delta(\lambda;v) \leq \frac{\kappa_L(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{2}\,\delta(\lambda;v). \tag{22}$$

This shows that the process of linearizing P invariably increases the condition number of a simple eigenvalue of P at least by a factor of $\delta(\lambda;v)$ and at most by a factor of $\sqrt{2}\delta(\lambda;v)$. In other words, $\delta(\lambda;v)$ serves as a growth factor for the eigenvalue condition number.

Since $\mathsf{p}(\lambda;v) = \Lambda_{m-1}^T v$, it follows from the Cauchy-Schwarz inequality that among all ansatz vectors with $\|v\|_2 = 1$ the vector $v = \overline{\Lambda}_{m-1}/\|\Lambda_{m-1}\|_2$ minimizes $\delta(\lambda;v)$ and, hence, for this particular choice of v we have $\delta(\lambda;v) = 1$ and

$$\kappa_{\mathrm{P}}(\lambda) \leq \kappa_L(\lambda) \leq \sqrt{2}\,\kappa_{\mathrm{P}}(\lambda).$$

Let us emphasize that this result is primarily of theoretical interest as the optimal choice of v depends on the (typically unknown) eigenvalue λ. A practically more useful recipe is to choose $v = [1,0,\ldots,0]^T$ if $|\lambda| \geq 1$ and $v = [0,\ldots,0,1]^T$ if $|\lambda| \leq 1$. In both cases, $\delta(\lambda;v) = \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda;v)|} \leq \sqrt{m}$ and therefore $\kappa_{\mathrm{P}}(\lambda) \leq \kappa_L(\lambda) \leq \sqrt{2m}\,\kappa_{\mathrm{P}}(\lambda)$.

In the following section, the discussion above shall be extended to structured linearizations and condition numbers.

4 Structured condition numbers for linearizations

If the polynomial P is structured then it is desirable that its linearization also reflects this structure. It is a fact that structured linearizations impose conditions on ansatz vectors. These conditions can be found in [17, Thm 3.4] for symmetric polynomials, in [17, Thm. 6.1] for Hermitian polynomials, and in [29, Tables 6.1, 6.2] for $\star$-even/odd, $\star$-palindromic/anti-palindromic polynomials with $\star \in \{T,H\}$.

If, for example, a structure-preserving method is used for computing the eigenvalues of a structured linearization L then the structured condition number of L is an appropriate measure for the influence of roundoff error on the accuracy of the computed eigenvalues. It is therefore of interest to choose L such that the structured condition number is minimized.

Let us recall our choice of norms (18)–(19) for measuring perturbations. A first general result can be obtained from combining the identity $\frac{\kappa_L^{\mathbb{S}_L}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_{\mathrm{P}}}(\lambda)} = \frac{\kappa_L^{\mathbb{S}_L}(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_{\mathrm{P}}}(\lambda)} \frac{\kappa_L(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}$ with (22):

$$\kappa_{\mathrm{P},L}^{\mathsf{ratio}}(\lambda) \cdot \delta(\lambda;v) \leq \frac{\kappa_L^{\mathbb{S}_L}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_{\mathrm{P}}}(\lambda)} \leq \sqrt{2}\,\kappa_{\mathrm{P},L}^{\mathsf{ratio}}(\lambda) \cdot \delta(\lambda;v), \qquad \kappa_{\mathrm{P},L}^{\mathsf{ratio}}(\lambda) := \frac{\kappa_L^{\mathbb{S}_L}(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_{\mathrm{P}}}(\lambda)}. \tag{23}$$

We will make frequent use of (23) to obtain tight bounds for specific structures. The general strategy will be to first show $\kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda) \approx 1$, if possible. Then the vector v determining the linearization is chosen to minimize $\delta(\lambda; v)$, provided that there is freedom in the choice of v. All bounds presented in the following are only shown for the case of a simple finite eigenvalue λ. However, since the bounds will not depend on λ, they carry over to a simple infinite eigenvalue by a continuity argument.

Finally, we let the symmetric matrices $\Sigma \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times m}$ be defined by

$$\Sigma = \operatorname{diag}\big\{(-1)^{m-1}, (-1)^{m-2}, \ldots, (-1)^0\big\}, \qquad R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}. \tag{24}$$

4.1 Complex symmetric matrix polynomials

For a complex symmetric matrix polynomial P, any ansatz vector v yields a complex symmetric linearization $L \in \mathbb{DL}(\mathrm{P})$, see [17, Thm 3.4]. Thus, we are free to use the optimal choice $v = \overline{\Lambda}_{m-1}/\|\Lambda_{m-1}\|_2$ from Section 3.

Theorem 4.1 *Let $\mathbb{S}$ denote the set of complex symmetric matrix polynomials. Let λ be a finite or infinite, simple eigenvalue of a matrix polynomial* P*. Then for the linearization $L \in \mathbb{DL}(\mathrm{P})$ corresponding to an ansatz vector v, we have*

$$\delta(\lambda; v) \le \frac{\kappa^{\mathbb{S}}_L(\lambda)}{\kappa^{\mathbb{S}}_{\mathrm{P}}(\lambda)} \le \sqrt{2}\,\delta(\lambda; v)$$

for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$ and $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$. In particular, for $v = \overline{\Lambda}_{m-1}/\|\Lambda_{m-1}\|_2$, we have

$$\kappa^{\mathbb{S}}_{\mathrm{P}}(\lambda) \le \kappa^{\mathbb{S}}_L(\lambda) \le \sqrt{2}\,\kappa^{\mathbb{S}}_{\mathrm{P}}(\lambda).$$

Proof. Lemma 2.3 shows we have $\kappa^{\mathbb{S}}_{\mathrm{P}}(\lambda) = \kappa_{\mathrm{P}}(\lambda)$ and $\kappa^{\mathbb{S}}_L(\lambda) = \kappa_L(\lambda)$ for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$. Hence the result follows directly from (23). For $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$, the additional factors appearing in Lemma 2.3 are the same for $\kappa^{\mathbb{S}}_{\mathrm{P}}(\lambda)$ and $\kappa^{\mathbb{S}}_L(\lambda)$. This can be seen as follows. According to Theorem 3.2, the normalized right and left eigenvectors of the linearization take the form $\tilde{x} = \Lambda_{m-1} \otimes x/\|\Lambda_{m-1}\|_2$, $\tilde{y} = \overline{\Lambda}_{m-1} \otimes y/\|\Lambda_{m-1}\|_2$. Thus,

$$\tilde{y}^T\tilde{x} = \frac{\overline{\Lambda}_{m-1}^T \Lambda_{m-1}}{\|\Lambda_{m-1}\|_2^2} y^T x = y^T x,$$

concluding the proof. □

4.2 T-even and T-odd matrix polynomials

For T-even and T-odd polynomials it is in general not possible to find a structure-preserving linearization within the class $\mathbb{DL}(\mathrm{P})$. Following [29], we instead require for a structured linearization L that $(\Sigma \otimes I)L \in \mathbb{DL}(\mathrm{P})$. Further conditions need to be imposed on the ansatz vector Σv for $(\Sigma \otimes I)L$, see Table 2.

As a consequence of Theorem 3.2, we have that $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ are right and left eigenvectors of P belonging to an eigenvalue λ if and only if $\widetilde{x} = \Lambda_{m-1} \otimes x$ and $\widetilde{y} = \Sigma\overline{\Lambda}_{m-1} \otimes y$ are right and left eigenvectors of L belonging to the same eigenvalue. In particular,

$$|\widetilde{y}^T\widetilde{x}| = \frac{|\Lambda^H_{m-1}\Sigma\Lambda_{m-1}|}{\|\Lambda_{m-1}\|_2^2} |y^T x|. \tag{25}$$

Note that $\kappa_L(\lambda) = \kappa_{(\Sigma\otimes I)L}(\lambda)$ because unstructured eigenvalue condition numbers of matrix pencils do not change under orthogonal transformations. Hence, we obtain from (23),

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda)\cdot\delta(\lambda; \Sigma v) \le \frac{\kappa^{\mathbb{S}_L}_L(\lambda)}{\kappa^{\mathbb{S}_{\mathrm{P}}}_{\mathrm{P}}(\lambda)} \le \sqrt{2}\,\kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda)\cdot\delta(\lambda; \Sigma v), \qquad \kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda) := \frac{\kappa^{\mathbb{S}_L}_L(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa^{\mathbb{S}_{\mathrm{P}}}_{\mathrm{P}}(\lambda)}. \tag{26}$$

Structure of P	Structure of L	Condition on Σv
$\star$-even	$\star$-even	$\Sigma v = (v^\star)^T$
	$\star$-odd	$\Sigma v = -(v^\star)^T$
$\star$-odd	$\star$-even	$\Sigma v = -(v^\star)^T$
	$\star$-odd	$\Sigma v = (v^\star)^T$

Table 2: Conditions on ansatz vector Σv for $(\Sigma \otimes I)L \in \mathbb{DL}(\mathrm{P})$ such that L is $\star$-even / $\star$-odd for a $\star$-even / $\star$-odd polynomial P. Taken from [29, Table 6.2].

These results will be instrumental in proving the following bounds on the ratio between the structured eigenvalue condition numbers for P and L.

Theorem 4.2 *Let $\mathbb{S}_e$ and $\mathbb{S}_o$ denote the sets of T-even and T-odd polynomials, respectively. Let λ be a finite or infinite, simple eigenvalue of a T-even matrix polynomial P of degree m. Then the following statements hold for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$.*

1. *If L_e is a T-even linearization corresponding to the ansatz vector $\Sigma v = v$ then*

$$\begin{aligned} &\textit{for odd } m \textit{ and } |\lambda| \le 1: && \delta(\lambda; v) \le \frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \le 2\,\delta(\lambda; v) \\ &\textit{for odd } m \textit{ and } |\lambda| \ge 1: && \delta(\lambda; v) \le \frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \le \sqrt{10}\,\delta(\lambda; v) \\ &\textit{for even } m \textit{ and } |\lambda| \le 1: && \delta(\lambda; v) \le \frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \le 2\,\delta(\lambda; v). \end{aligned}$$

2. *If L_o is a T-odd linearization corresponding to the ansatz vector $\Sigma v = -v$ then*

$$\textit{for even } m \textit{ and } |\lambda| \ge 1: \qquad \delta(\lambda; v) \le \frac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \le 2\,\delta(\lambda; v).$$

Proof. The proof makes use of $\delta(\lambda; \Sigma v) = \delta(\lambda; v)$ when $\Sigma v = \pm v$ and the basic relation

$$\frac{|\lambda|^2}{1+|\lambda|^2} \ge \frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}, \quad \text{with equality for odd } m. \tag{27}$$

1 (a). Let m be odd. Then (27) implies – together with Lemma 2.4 and (25) – the equality

$$\begin{aligned} \kappa_{\mathrm{P},L_e}^{\mathsf{ratio}}(\lambda) &= \frac{\sqrt{1 - |y^T x|^2 \frac{|\lambda|^2}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \\ &= \frac{\sqrt{1 - |y^T x|^2 \frac{|\lambda|^2}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 - |y^T x|^2 \frac{|\lambda|^2}{1+|\lambda|^2}}}. \end{aligned} \tag{28}$$

The inequality $|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}| \le \|\Lambda_{m-1}\|_2^2$ implies, on the one hand, $\kappa_{\mathrm{P},L_e}^{\mathsf{ratio}}(\lambda) \ge 1$ and, on the other hand,

$$\kappa_{\mathrm{P},L_e}^{\mathsf{ratio}}(\lambda) \le \frac{\sqrt{1 - \frac{|\lambda|^2}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 - \frac{|\lambda|^2}{1+|\lambda|^2}}} = \sqrt{1 + |\lambda|^2 - |\lambda|^2 \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}.$$

For $|\lambda| \le 1$, we clearly obtain $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\lambda) \le \sqrt{2}$. For $|\lambda| \ge 1$, a tedious algebraic manipulation is necessary to show

$$1 + |\lambda|^2 - |\lambda|^2 \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4} = 5 - \frac{9\sum_{i=1}^{m-1} |\lambda|^{4i-2} + 10 \sum_{i=1}^{m-2} |\lambda|^{4i} + 4}{\|\Lambda_{m-1}\|_2^4},$$

which implies $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\lambda) \le \sqrt{5}$.

1 (b). Let m be even and $|\lambda| \le 1$. Inserting $\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \le \frac{|\lambda|^2}{1+|\lambda|^2} \le \frac{1}{2}$ from (27) into (28) yields $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\lambda) \le \sqrt{2}$. For the other direction, we note that once again (27) implies

$$\frac{\Lambda_{m-1}^H \Sigma \Lambda_{m-1}}{\|\Lambda_{m-1}\|_2^2} = \frac{\|\Lambda_{m-1}\|_2^2 - 2\|\Pi_o(\Lambda_{m-1})\|_2^2}{\|\Lambda_{m-1}\|_2^2} = \frac{1-|\lambda|^2}{1+|\lambda|^2} \qquad \text{for even } m. \tag{29}$$

Combined with

$$\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} = \frac{\|\Pi_o(\Lambda_{m-1})\|_2^2}{\|\Lambda_m\|_2^2} = \frac{|\lambda|^2}{1+|\lambda|^2}\frac{\|\Lambda_{m-1}\|_2^2}{\|\Lambda_m\|_2^2} \ge \frac{|\lambda|^2}{(1+|\lambda|^2)^2} \ge \frac{|\lambda|^2(1-|\lambda|^2)^2}{(1+|\lambda|^2)^3}, \tag{30}$$

this shows

$$\frac{|\lambda|^2}{1+|\lambda|^2}\frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4} \le \frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2},$$

which implies $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\lambda) \ge 1$ by (28).

2. Now, let m be even, $|\lambda| \ge 1$, and suppose that a T-odd linearization is used. Then Lemma 2.8 combined with (29) yields

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) = \frac{\sqrt{1 - |y^T x|^2 \frac{1}{1+|\lambda|^2}\frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1-|y^Tx|^2 \frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} = \frac{\sqrt{1-|y^Tx|^2\frac{(1-|\lambda|^2)^2}{(1+|\lambda|^2)^3}}}{\sqrt{1-|y^Tx|^2\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}.$$

Using $\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \le \frac{1}{1+|\lambda|^2} \le \frac{1}{2}$, we immediately obtain $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \le \sqrt{2}$. The other direction, $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \ge 1$, is shown similarly as in 1 (b) from

$$\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} = \frac{|\lambda|^2}{1+|\lambda|^2}\frac{\|\Lambda_{m-1}\|_2^2}{\|\Lambda_m\|_2^2} \ge \frac{|\lambda|^2}{(1+|\lambda|^2)^2} \ge \frac{(1-|\lambda|^2)^2}{(1+|\lambda|^2)^3}.$$

□

By Theorem 4.2, obtaining a nearly optimally conditioned linearization requires finding the maximum of $|\mathsf{p}(\lambda; v)| = |\Lambda_{m-1}^T v|$ among all v with $\Sigma v = \pm v$ and $\|v\|_2 \le 1$. This maximization problem can be addressed by the following basic linear algebra result.

Proposition 4.3 *Let $\Pi_{\mathcal{V}}$ be an orthogonal projector onto a linear subspace $\mathcal{V}$ of $\mathbb{F}^m$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then for $A \in \mathbb{F}^{l\times m}$,*

$$\max_{\substack{v\in\mathcal{V}\\ \|v\|_2\le 1}} \|Av\|_2 = \|A\Pi_{\mathcal{V}}\|_2.$$

For a T-even linearization we have $\mathcal{V} = \{v \in \mathbb{C}^m : \Sigma v = v\}$ and the orthogonal projector onto $\mathcal{V}$ is given by the even coefficient projector Π_e defined in (15). Hence, by Proposition 4.3,

$$\max_{\substack{v=\Sigma v\\ \|v\|_2\le 1}} |\mathsf{p}(\lambda; v)| = \max_{\substack{v=\Sigma v\\ \|v\|_2\le 1}} |\Lambda_{m-1}^T v| = \|\Pi_e(\Lambda_{m-1})\|_2$$

where the maximum is attained by $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$. Similarly, for a T-odd linearization,

$$\max_{\substack{v=-\Sigma v\\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; v)| = \|\Pi_o(\Lambda_{m-1})\|_2$$

with the maximum attained by $v = \Pi_o(\overline{\Lambda}_{m-1})/\|\Pi_o(\Lambda_{m-1})\|_2$.

Corollary 4.4 *Under the assumptions of Theorem 4.2, consider the specific T-even and T-odd linearizations corresponding to the ansatz vectors $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$ and $v = \Pi_o(\overline{\Lambda}_{m-1})/\|\Pi_o(\Lambda_{m-1})\|_2$, respectively. Then the following statements hold.*

1. *If m is odd and $|\lambda| \leq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda) \leq \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq 2\sqrt{2}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.
2. *If m is odd and $|\lambda| \geq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda) \leq \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq \sqrt{20}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.
3. *If m is even and $|\lambda| \leq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda) \leq \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq 2\sqrt{2}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.
4. *If m is even and $|\lambda| \geq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda) \leq \kappa_{L_o}^{\mathbb{S}_o}(\lambda) \leq 2\sqrt{2}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.

Proof. Note that, by definition, $\delta(\lambda; v) \geq 1$ and hence all lower bounds are direct consequences of Theorem 4.2. To show $\delta(\lambda; v) \leq \sqrt{2}$ for the upper bounds of statements 1 and 2, we make use of the inequalities

$$\|\Pi_e(\Lambda_{m-1})\| \leq \|\Lambda_{m-1}\|_2 \leq \sqrt{2}\|\Pi_e(\Lambda_{m-1})\|, \tag{31}$$

which hold if either m is odd or m is even and $|\lambda| \leq 1$. For statement 4, the bound $\delta(\lambda; v) \leq \sqrt{2}$ is a consequence of (27). □

The morale of Theorem 4.2 and Corollary 4.4 is quickly told: There is always a "good" T-even linearization (in the sense that the linearization increases the structured condition number at most by a modest factor) if either m is odd, or m is even and $|\lambda| \leq 1$. In the exceptional case, when m is even and $|\lambda| \geq 1$, there is always a "good" T-odd linearization. Intuitively, the necessity of such an exceptional case becomes clear from the fact that there exists no T-even linearization for a T-even polynomial with even m and infinite eigenvalue. Even though there are T-even linearization for even m and large but finite λ, it is not advisable to use them for numerical computations.

In practice, one does not know λ in advance and hence the linearizations used in Corollary 4.4 for which $\delta(\lambda; v) \leq \sqrt{2}$ are mainly of theoretical interest. Table 3 provides practically more feasible recommendations on the choice of v, such that there is still at worst a slight increase of the structured condition number. The bounds in this table follow from Theorem 4.2 combined with $\delta(\lambda; v) \leq \sqrt{m}$ for all displayed choices of v. The example linearizations are taken from [29, Tables 3.4–3.6].

m	λ of interest	v	Bound on struct. cond. of linearization	Example
odd or even	$\lvert\lambda\rvert \leq 1$	e_m	$\kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq 2\sqrt{m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} 0 & -A_3 & 0 \\ A_3 & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & A_3 \\ 0 & -A_3 & -A_2 \\ A_3 & A_2 & A_1 \end{bmatrix}$
odd	$\lvert\lambda\rvert \geq 1$	e_1	$\kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq \sqrt{10\,m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} A_2 & A_1 & A_0 \\ -A_1 & -A_0 & 0 \\ A_0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_1 & A_0 \\ 0 & -A_0 & 0 \end{bmatrix}$
even	$\lvert\lambda\rvert \geq 1$	e_1	$\kappa_{L_o}^{\mathbb{S}_o}(\lambda) \leq 2\sqrt{m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} A_2 & 0 \\ 0 & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_1 & A_0 \\ -A_0 & 0 \end{bmatrix}$

Table 3: Recipes for choosing the ansatz vector v for a T-even or T-odd linearization L_e or L_o of a T-even matrix polynomial of degree m. Note that e_1 and e_m denote the 1st and mth unit vector of length m, respectively.

We extend Theorem 4.2 and Corollary 4.4 to T-odd polynomials.

Theorem 4.5 *Let λ be a finite or infinite, simple eigenvalue of a T-*odd *matrix polynomial* P *of degree m. Then the following statements hold for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$.*

1. *If L_o is a T-odd linearization corresponding to the ansatz vector $\Sigma v = v$ then*

$$\begin{array}{lcccc} \text{for odd } m \text{ and } |\lambda| \le 1: & \delta(\lambda; v) & \le & \dfrac{\kappa^{\mathbb{S}_o}_{L_o}(\lambda)}{\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)} & \le & \sqrt{10}\,\delta(\lambda; v) \\ \text{for odd } m \text{ and } |\lambda| \ge 1: & \delta(\lambda; v) & \le & \dfrac{\kappa^{\mathbb{S}_o}_{L_o}(\lambda)}{\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)} & \le & 2\,\delta(\lambda; v) \\ \text{for even } m \text{ and } |\lambda| \le 1: & \delta(\lambda; v) & \le & \dfrac{\kappa^{\mathbb{S}_o}_{L_o}(\lambda)}{\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)} & \le & \sqrt{10}\,\delta(\lambda; v). \end{array}$$

2. *If L_e is a T-even linearization corresponding to the ansatz vector $v = -\Sigma v$ then*

$$\text{for even } m \text{ and } |\lambda| \ge 1: \qquad \delta(\lambda; v) \le \frac{\kappa^{\mathbb{S}_e}_{L_e}(\lambda)}{\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)} \le \sqrt{10}\,\delta(\lambda; v).$$

Proof. Recall that $\delta(\lambda; v) = \delta(\lambda; \Sigma v)$ holds for $\Sigma v = \pm v$.

1 (a). Let m be odd. Then Lemma 2.8 combined with (25) yields

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) = \frac{\sqrt{1 - |y^T x|^2 \frac{1}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}. \tag{32}$$

Let $\theta = \lambda^{-1}$ and define $\Theta_{m-1} = [\theta^{m-1}, \theta^{m-2} \dots \theta, 1]^T$. Then it is not hard to see that

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) = \kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\theta^{-1}) = \frac{\sqrt{1 - |y^T x|^2 \frac{|\theta|^2}{1+|\theta|^2} \frac{|\Theta_{m-1}^H \Sigma \Theta_{m-1}|^2}{\|\Theta_{m-1}\|_2^4}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_o(\Theta_m)\|_2^2}{\|\Theta_m\|_2^2}}}.$$

The right-hand side coincides with the starting expression (28) in the proof of Theorem 4.2.1 (a), only with λ replaced by θ. In particular, this implies $1 \le \kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \le \sqrt{5}$ for $|\theta| \ge 1$ and $1 \le \kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \le \sqrt{2}$ for $|\theta| \le 1$.

1 (b). Let m be even and $|\lambda| \le 1$. Using (29), we obtain

$$\begin{aligned} 1 - \frac{1}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4} &= 1 - \frac{(1-|\lambda|^2)^2}{(1+|\lambda|^2)^3} = \frac{5|\lambda|^2 + 2|\lambda|^4 + |\lambda|^6}{(1+|\lambda|^2)^3} \\ &\le 5\frac{|\lambda|^2(1+|\lambda|^2)}{(1+|\lambda|^2)^3} = 5\frac{|\lambda|^2}{(1+|\lambda|^2)^2} \\ &\overset{(30)}{\le} 5\frac{\|\Pi_o(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} = 5\Big(1 - \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}\Big). \end{aligned}$$

From (32), we therefore obtain

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \le \frac{\sqrt{1 - \frac{1}{1+|\lambda|^2} \frac{|\Lambda_{m-1}^H \Sigma \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 - \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \le \sqrt{5}.$$

The other direction, $\kappa^{\mathsf{ratio}}_{\mathrm{P},L_o}(\lambda) \ge 1$, follows from combining (32) with $\frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \ge \frac{1}{1+|\lambda|^2}$, see (27).

2. Now, let m be even, $|\lambda| \geq 1$, and suppose that a T-even linearization L_e is used. Then Lemmas 2.4 and 2.8 imply

$$\begin{aligned}\kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\lambda) &= \frac{\sqrt{1-|y^Tx|^2\frac{|\lambda|^2}{1+|\lambda|^2}\frac{|\Lambda^H_{m-1}\Sigma\Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1-|y^Tx|^2\frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}\\ = \kappa^{\mathsf{ratio}}_{\mathrm{P},L_e}(\theta^{-1}) &= \frac{\sqrt{1-|y^Tx|^2\frac{1}{1+|\theta|^2}\frac{|\Theta^H_{m-1}\Sigma\Theta_{m-1}|^2}{\|\Theta_{m-1}\|_2^4}}}{\sqrt{1-|y^Tx|^2\frac{\|\Pi_e(\Theta_m)\|_2^2}{\|\Theta_m\|_2^2}}}\end{aligned}$$

and hence the result follows from 1 (b).

□

Corollary 4.6 *Under the assumptions of Theorem 4.5, consider the specific T-odd and T-even linearizations L_o, L_e corresponding to the ansatz vectors $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$ and $v = \Pi_o(\overline{\Lambda}_{m-1})/\|\Pi_o(\Lambda_{m-1})\|_2$, respectively. Then the following statements hold.*

1. *If m is odd and $|\lambda| \leq 1$:* $\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda) \leq \kappa^{\mathbb{S}_o}_{L_o}(\lambda) \leq \sqrt{20}\,\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)$.
2. *If m is odd and $|\lambda| \geq 1$:* $\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda) \leq \kappa^{\mathbb{S}_o}_{L_o}(\lambda) \leq 2\sqrt{2}\,\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)$.
3. *If m is even and $|\lambda| \leq 1$:* $\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda) \leq \kappa^{\mathbb{S}_o}_{L_o}(\lambda) \leq \sqrt{20}\,\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)$.
4. *If m is even and $|\lambda| \geq 1$:* $\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda) \leq \kappa^{\mathbb{S}_e}_{L_e}(\lambda) \leq \sqrt{20}\,\kappa^{\mathbb{S}_o}_{\mathrm{P}}(\lambda)$.

Proof. The proof is similar to that of Corollary 4.4 and follows from Theorem 4.5 and (31).
□

Finally, we mention that Table 3 has a corresponding analogue for a T-odd matrix polynomial.

4.3 T-palindromic matrix polynomials

Turning to (anti-)palindromic matrix polynomials, we require that a structured linearization L satisfies $(R \otimes I)L \in \mathbb{DL}(\mathrm{P})$ with the flip permutation matrix R, see (24). Again, further conditions need to be imposed on the ansatz vector Rv for $(R \otimes I)L$, see Table 4.

Structure of P	Structure of L	Condition on Rv
$\star$-palindromic	$\star$-palindromic	$Rv = (v^\star)^T$
	$\star$-anti-palindromic	$Rv = -(v^\star)^T$
$\star$-anti-palindromic	$\star$-palindromic	$Rv = -(v^\star)^T$
	$\star$-anti-palindromic	$Rv = (v^\star)^T$

Table 4: Conditions on ansatz vector Rv for $(R \otimes I)L \in \mathbb{DL}(\mathrm{P})$ such that L is $\star$-(anti)-palindromic for a $\star$-(anti)-palindromic polynomial P. Taken from [29, Table 6.1].

As above, we obtain from Theorem 3.2

$$|\widetilde{y}^T\widetilde{x}| = \frac{|\Lambda^H_{m-1}R\Lambda_{m-1}|}{\|\Lambda_{m-1}\|_2^2}|y^Tx| = \frac{(m+1)|\lambda|^m}{\|\Lambda_{m-1}\|_2^2}|y^Tx|, \tag{33}$$

where $\widetilde{x} = \Lambda_{m-1} \otimes x$ and $\widetilde{y} = R\overline{\Lambda}_{m-1} \otimes y$ are the right/left eigenvectors of L corresponding to right/left eigenvectors x, y of P. Also, (26) has an (anti-)palindromic analogue:

$$\kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda)\cdot\delta(\lambda;Rv) \leq \frac{\kappa^{\mathbb{S}_L}_L(\lambda)}{\kappa^{\mathbb{S}_\mathrm{P}}_\mathrm{P}(\lambda)} \leq \sqrt{2}\,\kappa^{\mathsf{ratio}}_{\mathrm{P},L}(\lambda)\cdot\delta(\lambda;Rv). \tag{34}$$

This result is again instrumental for bounding the ratio between the structured eigenvalue condition numbers for P and L.

Theorem 4.7 *Let $\mathbb{S}_p$ and $\mathbb{S}_a$ denote the sets of T-palindromic and T-anti-palindromic polynomials, respectively. Let λ be a finite or infinite, simple eigenvalue of a T-*palindromic *matrix polynomial* P *of degree m. Then the following statements hold for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.*

1. *If L_p is a T-palindromic linearization corresponding to the ansatz vector $Rv = v$ then*

$$\text{for odd } m \text{ and } \mathsf{Re}(\lambda) \geq 0: \qquad \frac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq 4\,\delta(\lambda; v)$$

$$\text{for odd } m \text{ and } \mathsf{Re}(\lambda) \leq 0: \qquad \frac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq 2\,\delta(\lambda; v)$$

$$\text{for even } m \text{ and } \mathsf{Re}(\lambda) \geq 0: \qquad \frac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq 4\,\delta(\lambda; v).$$

2. *If L_a is a T-anti-palindromic linearization corresponding to the ansatz vector $Rv = -v$ then*

$$\text{for even } m \text{ and } \mathsf{Re}(\lambda) \leq 0: \qquad \frac{\kappa_{L_a}^{\mathbb{S}_a}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq 4\,\delta(\lambda; v).$$

Proof. Clearly, $\delta(\lambda; v) = \delta(\lambda; Rv)$ for $Rv = \pm v$.

1 (a). Let m be odd and $\mathsf{Re}(\lambda) \geq 0$. Lemma 2.9 together with (33) imply

$$\kappa_{\mathrm{P},L_p}^{\mathsf{ratio}}(\lambda) = \frac{\sqrt{1 + |y^T x|^2 \frac{|1+\lambda|^2 - |1-\lambda|^2}{2(1+|\lambda|^2)} \frac{|\Lambda_{m-1}^H R \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 + |y^T x|^2 \frac{\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}. \tag{35}$$

For $\|\Pi_+(\Lambda_m)\|_2^2 \leq \|\Pi_-(\Lambda_m)\|_2^2$, we obtain from Lemma B.1.1 that

$$\kappa_{\mathrm{P},L_p}^{\mathsf{ratio}}(\lambda) \leq \frac{\sqrt{2}}{\sqrt{1 + \frac{\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} = \frac{\|\Lambda_m\|_2}{\|\Pi_+(\Lambda_m)\|_2} \leq 2\sqrt{2},$$

where we also used $\|\Pi_+(\Lambda_m)\|_2^2 + \|\Pi_-(\Lambda_m)\|_2^2 = \|\Lambda_m\|_2^2$. For $\|\Pi_+(\Lambda_m)\|_2^2 \geq \|\Pi_-(\Lambda_m)\|_2^2$, we obtain $\kappa_{\mathrm{P},L_p}^{\mathsf{ratio}}(\lambda) \leq \sqrt{2}$ trivially from (35). Hence, the desired bound follows from (34).

1 (b). Let m be odd and $\mathsf{Re}(\lambda) \leq 0$. With the notation of Lemma B.1.2, the relation (35) reads

$$\kappa_{\mathrm{P},L_p}^{\mathsf{ratio}}(\lambda) = \frac{\sqrt{1 + |y^T x|^2 \alpha}}{\sqrt{1 + |y^T x|^2 \beta}}.$$

Since $\beta > -1$ and $|y^T x| \leq 1$, the inequality $\alpha - 2\beta \leq 1$ shown in Lemma B.1.2 gives $|y^T x|^2 \alpha \leq 1 + 2|y^T x|^2 \beta$ which is equivalent to $\frac{1+|y^T x|^2 \alpha}{1+|y^T x|^2 \beta} \leq 2$. Hence the desired bound follows from (34).

1 (c). The case of even m and $\mathsf{Re}(\lambda) \geq 0$ follows – analogously as in 1 (a) – from Lemma B.1.3.

2. Let m be even, $\mathsf{Re}(\lambda) \leq 0$, and suppose that a T-anti-palindromic linearization L_a is used. Then Lemma 2.9, Lemma 2.11 and (33) imply

$$\kappa_{\mathrm{P},L_a}^{\mathsf{ratio}}(\lambda) = \frac{\sqrt{1 - |y^T x|^2 \frac{|1+\lambda|^2 - |1-\lambda|^2}{2(1+|\lambda|^2)} \frac{|\Lambda_{m-1}^H R \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}}}{\sqrt{1 + |y^T x|^2 \frac{\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}.$$

Since the sign in the numerator is not relevant for the arguments in 1 (a) and 1 (c), the result follows in the same way from Lemma B.1.3.

□

Remark 4.8 *Similar to the proof of Theorem 4.7, it can be shown that the bound for anti-palindromic linearization holds for* $\mathsf{Re}(\lambda) \geq 0$ *as well, provided that* m *is even. Further,* $\kappa^{\mathit{ratio}}_{\mathrm{P},L_p}(\lambda) \geq \frac{1}{\sqrt{2}}$ *when* $\mathsf{Re}(\lambda) \geq 0$ *and* $\kappa^{\mathit{ratio}}_{\mathrm{P},L_a}(\lambda) \geq \frac{1}{\sqrt{2}}$ *when* $\mathsf{Re}(\lambda) \leq 0$.

Motivated by the result of Theorem 4.7, a good linearization should belong to an ansatz vector that attains a small $\delta(\lambda; v)$ or, equivalently, a large $|\mathsf{p}(\lambda; v)|$. By Proposition 4.3,

$$\max_{\substack{v=Rv\\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; Rv)| = \|\Pi_+(\Lambda_{m-1})\|_2,$$

where the maximum is attained by v_+ defined as

$$v_\pm = \frac{\left[\frac{\lambda^{m-1}\pm 1}{2}, \ldots, \frac{\lambda^{m/2+1}\pm\lambda^{m/2}}{2}, \frac{\lambda^{m/2+1}\pm\lambda^{m/2}}{2}, \ldots, \frac{\lambda^{m-1}\pm 1}{2}\right]^T}{\|\Pi_\pm(\Lambda_{m-1})\|_2} \tag{36}$$

if m is even and as

$$v_\pm = \frac{\left[\frac{\lambda^{m-1}\pm 1}{2}, \ldots, \frac{\lambda^{(m-1)/2}\pm\lambda^{(m-1)/2}}{2}, \ldots, \frac{\lambda^{m-1}\pm 1}{2}\right]^T}{\|\Pi_\pm(\Lambda_{m-1})\|_2} \tag{37}$$

if m is odd. Similarly,

$$\max_{\substack{v=-Rv\\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; Rv)| = \|\Pi_-(\Lambda_{m-1})\|_2,$$

with the maximum attained by v_-.

Corollary 4.9 *Under the assumptions of Theorem 4.7, consider the specific* T*-palindromic and* T*-anti-palindromic linearizations* L_p, L_a *belonging to the ansatz vectors* v_+ *and* v_-*, respectively, defined in (36)–(37). Then the following statements hold.*

1. *If* m *is odd:* $\kappa^{\mathbb{S}_p}_{L_p}(\lambda) \leq 8\sqrt{2}\,\kappa^{\mathbb{S}_p}_{\mathrm{P}}(\lambda)$.

2. *If* m *is even and* $\mathsf{Re}(\lambda) \geq 0$*:* $\kappa^{\mathbb{S}_p}_{L_p}(\lambda) \leq 8\sqrt{2}\,\kappa^{\mathbb{S}_p}_{\mathrm{P}}(\lambda)$.

3. *If* m *is even and* $\mathsf{Re}(\lambda) \leq 0$*:* $\kappa^{\mathbb{S}_a}_{L_a}(\lambda) \leq 8\sqrt{2}\,\kappa^{\mathbb{S}_p}_{\mathrm{P}}(\lambda)$.

Proof. Recall that $\delta(\lambda; v) = \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda;v)|}$. All results then follow in a straightforward fashion from Theorem 4.7 and Lemma B.1. □

Again, Theorem 4.7 and Corollary 4.9 admit a simple interpretation. If either m is odd or m is even and λ has nonnegative real part, it is OK to use a T-palindromic linearization; there will be no significant increase of the structured condition number. In the exceptional case, when m is even and λ has negative real part, a T-anti-palindromic linearization should be preferred. This is especially true for $\lambda = -1$, in which case there is no T-palindromic linearization.

In practice, when λ is unknown, it is preferable to work with the heuristic choices listed in Table 5. The bounds listed in the table are proved in the following lemma. To provide recipes for even m larger than 2, one would need to discriminate further between $|\lambda|$ close to 1 and $|\lambda|$ far away from 1, similarly as for odd m.

Lemma 4.10 *The upper bounds on* $\kappa^{\mathbb{S}_p}_{L_p}(\lambda)$ *and* $\kappa^{\mathbb{S}_a}_{L_a}(\lambda)$ *listed in Table 5 are valid.*

m	λ of interest	v	Bound on struct. cond. of linearization	Example
odd	$\lvert\lambda\rvert \geq \alpha_m$ $\lvert\lambda\rvert \leq \alpha_m^{-1}$	$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 8\sqrt{m}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} A_0 & 0 & A_0 \\ A_1 - A_0^T & A_0 - A_1^T & 0 \\ A_1^T & A_1 - A_0^T & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T - A_0 & A_1 \\ 0 & A_0^T - A_1 & A_1^T - A_0 \\ A_0^T & 0 & A_0^T \end{bmatrix}$
odd	$\lvert\lambda\rvert \leq \alpha_m$ $\lvert\lambda\rvert \geq \alpha_m^{-1}$	$e_{\frac{m-1}{2}}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 4\sqrt{2m}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} 0 & A_0 & 0 \\ 0 & A_1 & A_0 \\ -A_0^T & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & -A_0 \\ A_0^T & A_1^T & 0 \\ 0 & A_0^T & 0 \end{bmatrix}$
$m = 2$	$\mathsf{Re}(\lambda) \geq 0$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 4\sqrt{2}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T - A_0 \\ A_0^T & A_0^T \end{bmatrix}$
$m = 2$	$\mathsf{Re}(\lambda) \leq 0$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\kappa_{L_a}^{\mathbb{S}_a}(\lambda) \leq 4\sqrt{2}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} -A_0 & A_0 \\ -A_1 - A_0^T & -A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T + A_0 \\ -A_0^T & A_0^T \end{bmatrix}$

Table 5: Recipe for choosing the ansatz vector v for a T-palindromic or T-anti-palindromic linearization L_e or L_o of a T-palindromic matrix polynomial of degree m. Note that $\alpha_m = 2^{1/(m-1)}$.

Proof. It suffices to derive an upper bound on $\delta(\lambda; v) = \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda;v)|}$. Multiplying such a bound by 4 then gives the coefficient in the upper bound on the structured condition number of the linearization, see Theorem 4.7.

1. For odd m and $|\lambda| \geq \alpha_m$ or $|\lambda| \leq 1/\alpha_m$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 8\sqrt{m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} \leq \frac{1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}}{|1 - \alpha_m^{m-1}|^2} = 1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2} \leq 4m.$$

2. For odd m and $1/\alpha_m \leq |\lambda| \leq \alpha_m$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} \leq \frac{1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}}{\alpha_m^{m-1}} = \frac{1}{2}(1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}) \leq 2m.$$

3. For $m = 2$ and $\mathsf{Re}(\lambda) \geq 0$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows for $|\lambda| \leq 1$ from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} = \frac{1 + |\lambda|^2}{|1 + \lambda|^2} \leq 2$$
and for $|\lambda| \geq 1$ from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} = \frac{|\lambda|^2}{|\lambda|^2} \frac{\frac{1}{|\lambda|^2} + 1}{|\frac{1}{\lambda} + 1|^2} \leq 2.$$

4. The proof for $m = 2$ and $\mathsf{Re}(\lambda) \leq 0$ is analogous to Part 3. □

For T-anti-palindromic polynomials, the implications of Theorems 4.7, Corollary 4.9, and Table 5 hold, but with the roles of T-palindromic and T-anti-palindromic exchanged. For example, if either m is odd or m is even and $\mathsf{Re}(\lambda) \geq 0$, there is always a good T-anti-palindromic linearization. Otherwise, if m is even and $\mathsf{Re}(\lambda) \leq 0$, there is a good T-palindromic linearization.

4.4 Hermitian matrix polynomials and related structures

The linearization of a Hermitian polynomial is also Hermitian if the corresponding ansatz vector v is real, see [17, Thm. 6.1]. The optimal v, which maximizes $|\mathsf{p}(\lambda; v)|$, could be found by finding the maximal singular value and the corresponding left singular vector of the real $m \times 2$ matrix $[\mathsf{Re}(\Lambda_{m-1}), \mathsf{Im}(\Lambda_{m-1})]$. Instead of invoking the rather complicated expression for this optimal choice, the following lemma uses a heuristic choice of v.

Lemma 4.11 *Let $\mathbb{S}_h$ denote the set of Hermitian polynomials. Let λ be a finite or infinite, simple eigenvalue of a Hermitian matrix polynomial* P. *Then the following statements hold for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$.*

1. *If $|\lambda| \geq 1$ then the linearization L corresponding to the ansatz vector $v = [1, 0, \ldots, 0]$ is Hermitian and satisfies $\kappa_L^{\mathbb{S}_h}(\lambda) \leq 2\sqrt{m}\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)$.*

2. *If $|\lambda| \leq 1$ then the linearization L corresponding to the ansatz vector $v = [0, \ldots, 0, 1]$ is Hermitian and satisfies $\kappa_L^{\mathbb{S}_h}(\lambda) \leq 2\sqrt{m}\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)$.*

Proof. Assume $|\lambda| \geq 1$. Lemma 2.12 together with Lemma 3.3 and (21) imply

$$\frac{\kappa_L^{\mathbb{S}_h}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)} \leq \sqrt{2}\,\frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} = \sqrt{2}\,\frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda; v)|} \leq 2\,\frac{\sqrt{m}|\lambda|^m}{|\lambda|^m} = 2\sqrt{m}.$$

The proof for $|\lambda| \leq 1$ proceeds analogously. □

H-even and H-odd matrix polynomials are closely related to Hermitian matrix polynomials, see Remark 2.13. In particular, Lemma 4.11 applies verbatim to H-even and H-odd polynomials. Note, however, that in the case of even m the ansatz vector $v = [1, 0, \ldots, 0]$ yields an H-odd linearization for an H-even polynomial, and vice versa. Similarly, the recipes of Table 5 can be extended to H-palindromic polynomials.

5 Summary and conclusions

We have derived relatively simple expressions for the structured eigenvalue condition numbers of certain structured matrix polynomials. These expressions have been used to analyze the possible increase of the condition numbers when the polynomial is replaced by a structured linearization. At least in the case when all coefficients of the polynomial are perturbed to the same extent, the result is very positive: There is always a structured linearization such that the condition numbers increase at most by a factor linearly depending on m. We have also provided recipes for structured linearizations, which do not depend on the exact value of the eigenvalue, and for which the increase of the condition number is still negligible. Hence, the accuracy of a strongly backward stable eigensolver applied to the structured linearization will fully enjoy the benefits of structure on the sensitivity of an eigenvalue for the original matrix polynomial.

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A Derivation of results given in Section 2

This section contains the proofs of the results on structured eigenvalue condition numbers given in Section 2. As mentioned before, the starting point to derive explicit expressions for structured eigenvalue condition number is the formulation (12).

A.1 Structured first-order perturbation sets

To proceed from the characterization (12) of the structured eigenvalue condition number, we need to find the maximal absolute magnitude of elements from the set

$$\left\{y^H \triangle \mathrm{P}(\lambda) x = y^H E_0 x + \lambda y^H E_1 x + \cdots \lambda^m y^H E_m x : \ \triangle \mathrm{P} \in \mathbb{S}, \ \|\triangle \mathrm{P}\| \leq 1\right\} \tag{38}$$

It is therefore of interest to study the nature of the set $\{y^H Ex : E \in \mathbb{E}, \ \|E\|_{\mathsf{M}} \leq 1\}$ with respect to some $\mathbb{E} \subseteq \mathbb{C}^{n\times n}$. The following theorem by Karow [21] provides explicit descriptions of this set for certain $\mathbb{E}$. Note that the symbol $\cong$ is used to denote the natural isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$.

Theorem A.1 *Let $\mathbb{K}(\mathbb{E},x,y) := \{y^H E x : E \in \mathbb{E},\ \|E\|_{\mathsf{M}} \le 1\}$ for $x,y \in \mathbb{C}^n$ with $\|x\|_2 = \|y\|_2 = 1$ and some $\mathbb{E} \subseteq \mathbb{C}^{n\times n}$. Provided that $\|\cdot\|_{\mathsf{M}} \in \{\|\cdot\|_2, \|\cdot\|_F\}$, the set $\mathbb{K}(\mathbb{E},x,y)$ is an ellipse taking the form*

$$\mathbb{K}(\mathbb{E},x,y) \cong \mathbb{K}(\alpha,\beta) := \left\{K(\alpha,\beta)\xi :\ \xi \in \mathbb{R}^2, \|\xi\|_2 \le 1\right\}, \quad K(\alpha,\beta) \in \mathbb{R}^{2\times 2}, \tag{39}$$

for the cases that $\mathbb{E}$ consists of all complex ($\mathbb{E} = \mathbb{C}^{n\times n}$), real ($\mathbb{E} = \mathbb{R}^{n\times n}$), Hermitian ($\mathbb{E} = \mathsf{Herm}$), complex symmetric ($\mathbb{E} = \mathsf{symm}$), and complex skew-symmetric ($\mathbb{E} = \mathsf{skew}$), real symmetric (only for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$), and real skew-symmetric matrices. The matrix $K(\alpha,\beta)$ defining the ellipse in (39) can be written as

$$K(\alpha,\beta) = \begin{bmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha+|\beta|} & 0 \\ 0 & \sqrt{\alpha-|\beta|} \end{bmatrix} \tag{40}$$

with some of the parameter configurations α,β listed in Table 6, and $\phi = \arg(\beta)$.

$\mathbb{E}$	$\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$ α	β	$\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$ α	β
$\mathbb{C}^{n\times n}$	1	0	1	0
Herm	$1-\frac{1}{2}\|y^H x\|^2$	$\frac{1}{2}(y^H x)^2$	$\frac{1}{2}$	$\frac{1}{2}(y^H x)^2$
symm	1	0	$\frac{1}{2}(1+\|y^T x\|^2)$	0
skew	$1-\|y^T x\|^2$	0	$\frac{1}{2}(1-\|y^T x\|^2)$	0

Table 6: Parameters defining the ellipse (39).

Note that (39)–(40) describes an ellipse with semiaxes $\sqrt{\alpha+|\beta|}$, $\sqrt{\alpha-|\beta|}$, rotated by the angle $\phi/2$. The Minkowski sum of ellipses is still convex but in general not an ellipse [25]. Finding the maximal element in (38) is equivalent to finding the maximal element in the Minkowski sum.

Lemma A.2 *Let $\mathbb{K}(\alpha_0,\beta_0),\ldots,\mathbb{K}(\alpha_m,\beta_m)$ be ellipses of the form (39)–(40). Define*

$$\sigma := \sup_{\substack{b_0,\ldots,b_m\in\mathbb{R}\\ b_0^2+\cdots+b_m^2\le 1}} \sup\left\{\|s\|_2 :\ s \in b_0\mathbb{K}(\alpha_0,\beta_0)+\cdots+b_m\mathbb{K}(\alpha_m,\beta_m)\right\} \tag{41}$$

using the Minkowski sum of sets. Then

$$\sigma = \|[K(\alpha_0,\beta_0),\ldots,K(\alpha_m,\beta_m)]\|_2, \tag{42}$$

and

$$\sqrt{\alpha_0+\cdots+\alpha_m} \le \sigma \le \sqrt{2}\sqrt{\alpha_0+\cdots+\alpha_m}. \tag{43}$$

Proof. By the definition of $\mathbb{K}(\alpha_j,\beta_j)$, it holds that

$$\begin{aligned}
\sigma &= \sup_{\substack{b_i\in\mathbb{R}\\ b_0^2+\cdots+b_m^2\le 1}} \sup_{\substack{\xi_i\in\mathbb{R}^2\\ \|\xi_i\|_2\le 1}} \left\|b_0K(\alpha_0,\beta_0)\xi_0+\cdots+b_mK(\alpha_m,\beta_m)\xi_m\right\|_2 \\
&= \sup_{\substack{b_i\in\mathbb{R}\\ b_0^2+\cdots+b_m^2\le 1}} \sup_{\substack{\tilde\xi_i\in\mathbb{R}^2\\ \|\tilde\xi_i\|_2\le b_i}} \left\|K(\alpha_0,\beta_0)\tilde\xi_0+\cdots+K(\alpha_m,\beta_m)\tilde\xi_m\right\|_2 \\
&= \sup_{\substack{\tilde\xi_i\in\mathbb{R}^2\\ \|\tilde\xi_0\|_2^2+\cdots+\|\tilde\xi_m\|_2^2\le 1}} \left\|K(\alpha_0,\beta_0)\tilde\xi_0+\cdots+K(\alpha_m,\beta_m)\tilde\xi_m\right\|_2 \\
&= \left\|\left[K(\alpha_0,\beta_0),\ \ldots,\ K(\alpha_m,\beta_m)\right]\right\|_2,
\end{aligned}$$

applying the definition of the matrix 2-norm. The inequality (43) then follows from the well-known bound

$$\frac{1}{\sqrt{2}}\|[K(\alpha_0,\beta_0),\ldots,K(\alpha_m,\beta_m)]\|_F \leq \sigma \leq \|[K(\alpha_0,\beta_0),\ldots,K(\alpha_m,\beta_m)]\|_F$$

and using the fact that:

$$\|[K(\alpha_0,\beta_0),\ldots,K(\alpha_m,\beta_m)]\|_F^2 = \sum_{i=0}^{m} \|K(\alpha_i,\beta_i)\|_F^2 = \sum_{i=0}^{m} 2\alpha_i.$$

□

It is instructive to rederive the expression (10) for the unstructured condition number from Lemma A.2. Starting from Equation (7), we insert the definition (4) of $\|\triangle \mathrm{P}\|$ for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$, $\|\cdot\|_{\mathsf{V}} \equiv \|\cdot\|_2$, and obtain

$$\begin{aligned} \sigma_{\mathrm{P}}(\lambda) &:= \sup\left\{|y^H \triangle \mathrm{P}(\lambda)x| : \|\triangle \mathrm{P}\| \leq 1\right\} \\ &= \sup_{\substack{b_0^2+\cdots+b_m^2\leq 1 \\ \|E_0\|^2\leq b_0,\ldots,\|E_m\|^2\leq b_m}} \Big|\sum_{i=0}^{m}\omega_i\lambda^i y^H E_i x\Big| \\ &= \sup_{b_0^2+\cdots+b_m^2\leq 1} \sup\Big\{|s| :\ s\in\sum_{i=0}^{m} b_i\omega_i\lambda_i\mathbb{K}(\mathbb{C}^{n\times n},x,y)\Big\}. \end{aligned} \tag{44}$$

By Theorem A.1, $\mathbb{K}(\mathbb{C}^{n\times n},x,y) \cong \mathbb{K}(1,0)$ and, since a disk is invariant under rotation, $\omega_i\lambda^i\mathbb{K}(\mathbb{C}^{n\times n},x,y) \cong \mathbb{K}(\omega_i^2|\lambda|^{2i},0)$. Applying Lemma A.2 yields

$$\sigma_{\mathrm{P}}(\lambda) = \left\|\left[K(\omega_0^2,0),K(\omega_1^2|\lambda|^2,0),\ldots,K(\omega_m^2|\lambda|^{2m},0)\right]\right\|_2 = \left\|\left[\omega_0,\omega_1\lambda,\ldots,\omega_m\lambda^m\right]\right\|_2,$$

which together with (7) results in the known expression (10) for $\kappa_{\mathrm{P}}(\lambda)$.

In the following, it will be shown that expressions for structured condition numbers follow in a similar way from Lemma A.2. To keep the notation compact, we define

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup\left\{|y^H\triangle \mathrm{P}(\lambda)x| :\ \triangle \mathrm{P}\in\mathbb{S},\ \|\triangle \mathrm{P}\|\leq 1\right\}.$$

for a star-shaped set $\mathbb{S}$. By (12), $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda)/|y^H\mathrm{P}'(\lambda)x|$.

A.2 Proofs

Proof of Lemma 2.3 ($\mathbb{S}$ = complex symmetric matrix polynomials) Along the line of arguments leading to (44),

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2+\cdots+b_m^2\leq 1}\Big\{|s| :\ s\in\sum_{i=0}^{m} b_i\omega_i\lambda^i\mathbb{K}(\mathsf{symm},x,y)\Big\}$$

for finite λ. As in the unstructured case, $\mathbb{K}(\mathsf{symm},x,y) \cong \mathbb{K}(1,0)$ for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$ by Theorem A.1, and thus $\kappa_{\mathrm{P}}(\lambda) = \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$. For $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$ we have

$$\mathbb{K}(\mathsf{symm},x,y) \cong \mathbb{K}((1+|y^Tx|^2)/2,0) = \frac{\sqrt{1+|y^Tx|^2}}{\sqrt{2}}\mathbb{K}(1,0),$$

showing the second part of the statement. The proof for infinite λ is entirely analogous. □

Proof of Lemma 2.4 ($\mathbb{S} = T$-even matrix polynomials) By definition, the even coefficients of a T-even polynomial are symmetric while the odd coefficients are skew-symmetric. Thus, for finite λ,

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2+\cdots+b_m^2 \leq 1} \sup \Big\{ |s| : \ s \in \sum_{i \text{ even}} b_i \omega_i \lambda^i \mathbb{K}(\mathsf{symm}, x, y) + \sum_{i \text{ odd}} b_i \omega_i \lambda^i \mathbb{K}(\mathsf{skew}, x, y) \Big\}.$$

Applying Theorem A.1 and Lemma A.2 yields for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$,

$$\begin{aligned} \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \Big\| \Big[\Pi_e(\Lambda_\omega)^T \otimes K(1,0),\, \Pi_o(\Lambda_\omega)^T \otimes K(1-|y^Tx|^2, 0) \Big] \Big\|_2 \\ &= \Big\| \Big[\Pi_e(\Lambda_\omega)^T,\, \sqrt{1-|y^Tx|^2}\,\Pi_o(\Lambda_\omega)^T \Big] \Big\|_2 \\ &= \sqrt{\|\Lambda_\omega\|_2^2 - |y^Tx|^2 \|\Pi_o(\Lambda_\omega)\|_2^2}, \end{aligned}$$

once again using the fact that a disk is invariant under rotation. Similarly, it follows for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$ that

$$\begin{aligned} \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \frac{1}{\sqrt{2}} \Big\| \Big[\sqrt{1+|y^Tx|^2}\,\Pi_e(\Lambda_\omega)^T,\, \sqrt{1-|y^Tx|^2}\,\Pi_o(\Lambda_\omega)^T \Big] \Big\|_2 \\ &= \frac{1}{\sqrt{2}} \sqrt{\|\Lambda_\omega\|_2^2 + |y^Tx|^2 \big(\|\Pi_e(\Lambda_\omega)\|_2^2 - \|\Pi_o(\Lambda_\omega)\|_2^2 \big)}. \end{aligned}$$

The result for infinite λ follows in an analogous manner. □

Proof of Lemma 2.9 ($\mathbb{S} = T$-palindromic matrix polynomials) Assume m is odd. For $\triangle \mathrm{P} \in \mathbb{S}$,

$$\begin{aligned} \triangle \mathrm{P}(\lambda) &= \sum_{i=0}^{(m-1)/2} \lambda^i E_i + \sum_{i=0}^{(m-1)/2} \lambda^{m-i} E_i^T \\ &= \sum_{i=0}^{(m-1)/2} \frac{\lambda^i + \lambda^{m-i}}{\sqrt{2}} \frac{E_i + E_i^T}{\sqrt{2}} + \sum_{i=0}^{(m-1)/2} \frac{\lambda^i - \lambda^{m-i}}{\sqrt{2}} \frac{E_i - E_i^T}{\sqrt{2}}. \end{aligned}$$

Let us introduce the auxiliary polynomial

$$\triangle \widetilde{\mathrm{P}}(\mu) = \sum_{i=0}^{(m-1)/2} \mu^{2i} S_i + \sum_{i=0}^{(m-1)/2} \mu^{2i+1} W_i, \quad S_i = \frac{E_i + E_i^T}{\sqrt{2}},\ W_i = \frac{E_i - E_i^T}{\sqrt{2}}.$$

Then $\widetilde{\mathrm{P}} \in \widetilde{\mathbb{S}}$, where $\widetilde{\mathbb{S}}$ denotes the set of T-even polynomials. Since symmetric and skew-symmetric matrices are orthogonal to each other with respect to the matrix inner product $\langle A, B \rangle = \mathrm{trace}(B^H A)$, we have $\|A\|_F^2 + \|A^T\|_F^2 = \|(A+A^T)/\sqrt{2}\|_F^2 + \|(A-A^T)/\sqrt{2}\|_F^2$ for any $A \in \mathbb{C}^{n\times n}$ and hence $\|\triangle \mathrm{P}\| = \|\triangle \widetilde{\mathrm{P}}\|$ for $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$. This allows us to write

$$\begin{aligned}
\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \sup\left\{|y^H \triangle\mathrm{P}(\lambda)x| :\ \triangle\mathrm{P} \in \mathbb{S},\ \|\triangle\mathrm{P}\| \le 1\right\} \\
&= \sup\left\{\left|\sum \frac{\lambda^i + \lambda^{m-i}}{\sqrt{2}} y^H S_i x + \sum \frac{\lambda^i - \lambda^{m-i}}{\sqrt{2}} y^H W_i x\right| :\ \triangle\widetilde{\mathrm{P}} \in \widetilde{\mathbb{S}},\ \|\triangle\widetilde{\mathrm{P}}\| \le 1\right\} \\
&= \frac{1}{\sqrt{2}} \sup_{b_0^2+\cdots+b_m^2 \le 1} \Big\{ |s| :\ s \in \sum b_i \omega_i (\lambda^i + \lambda^{m-i}) \mathbb{K}(\mathsf{symm}, x, y) \\
&\qquad\qquad + \sum b_{(m-1)/2+i} \omega_i (\lambda^i - \lambda^{m-i}) \mathbb{K}(\mathsf{skew}, x, y) \Big\} \\
&= \frac{1}{2}\sqrt{(1+|y^T x|^2)\sum \omega_i^2 |\lambda^i + \lambda^{m-i}|^2 + (1-|y^T x|^2)\sum \omega_i^2 |\lambda^i - \lambda^{m-i}|^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(1+|y^T x|^2)\|\Pi_+(\Lambda_\omega)\|_2^2 + (1-|y^T x|^2)\|\Pi_-(\Lambda_\omega)\|_2^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\|\Lambda_\omega\|_2^2 + |y^T x|^2 (\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2)},
\end{aligned}$$

where we used Theorem A.1 and Lemma A.2.

For even m the proof is almost identical; with the only difference that the transformation leaves the complex symmetric middle coefficient $A_{m/2}$ unaltered.

For $\lambda = \infty$, observe that the corresponding optimization problem (13) involves only a single, unstructured coefficient of the polynomial and hence palindromic structure has no effect on the condition number. □

The derivations above were greatly simplified by the fact that the first-order perturbation sets under consideration were disks. For the set of Hermitian perturbations, however, $y^H E_i x$ forms truly an ellipse. Still, a computable expression is provided by (42) from Lemma A.2. However, the explicit formulas derived from this expression take a very technical form and provide little immediate intuition on the difference between the structured and unstructured condition number. Therefore, the result of Lemma 2.12 is based on the bound (43) instead.

Proof of Lemma 2.12 ($\mathbb{S}$ = Hermitian matrix polynomials) Let $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_F$. Then Theorem A.1 states

$$\mathbb{K}(\mathsf{Herm}, x, y) \cong \mathbb{K}(1/2, (y^H x)^2/2).$$

Consequently,

$$\omega_i \lambda^i \mathbb{K}(\mathsf{Herm}, x, y) \cong \mathbb{K}(\omega_i^2 |\lambda|^{2i}/2, \omega_i^2 \lambda^{2i} (y^H x)^2/2),$$

which implies

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2+\cdots+b_m^2 \le 1} \left\{ \|s\|_2 :\ s \in \sum_{i=0}^{m} b_i \mathbb{K}(\omega_i^2 \lambda^{2i}/2, \omega_i^2 \lambda^{2i}(y^H x)^2/2) \right\}.$$

By Lemma A.2,

$$\frac{1}{\sqrt{2}}\|\Lambda_\omega\|_2 \le \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) \le \|\Lambda_\omega\|_2.$$

The proof for the case $\|\cdot\|_{\mathsf{M}} \equiv \|\cdot\|_2$ is analogous. □

B Auxiliary results for T-palindromic matrix polynomials

The following lemma summarizes some auxiliary results needed in the proofs given in Section 4.3 concerning the condition number growth for linearizations of T-palindromic matrix polynomials.

Lemma B.1 *Let $\lambda \in \mathbb{C}$ and let $\Pi_\pm$ be defined as in (16). Then the following statements hold.*

1. *Assume m is odd. If $\mathsf{Re}(\lambda) \geq 0$ then $\frac{\|\Pi_+(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{8}$. If $\mathsf{Re}(\lambda) \leq 0$ then $\frac{\|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{8}$.*

2. *Assume m is odd and $\mathsf{Re}(\lambda) \leq 0$. Set*

$$\alpha = \frac{|1+\lambda|^2 - |1-\lambda|^2}{2(1+|\lambda|^2)} \frac{|\Lambda_{m-1}^H R \Lambda_{m-1}|^2}{\|\Lambda_{m-1}\|_2^4}, \quad \beta = \frac{\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}.$$

   Then $\alpha - 2\beta \leq 1$.

3. *Assume m is even. Then $\frac{\|\Pi_+(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{8}$.*

Proof. We will make use of the polar form $\lambda = |\lambda|(\cos\phi + \mathrm{i}\sin\phi)$, for which $\lambda^k = |\lambda|^k(\cos(k\phi) + \mathrm{i}\sin(k\phi))$.

1. From $|\lambda^{m-k} + \lambda^k|^2 = |\lambda|^{2(m-k)} + 2|\lambda|^m \cos\big((m-2k)\phi\big) + |\lambda|^{2k}$, it follows for odd m that

$$\begin{aligned} 2\|\Pi_+(\Lambda_m)\|_2^2 &= \sum_{k=0}^{(m-1)/2} |\lambda^{m-k} + \lambda^k|^2 = \|\Lambda_m\|_2^2 + 2|\lambda|^m \sum_{k=0}^{(m-1)/2} \cos\big((2k+1)\phi\big) \\ &= \|\Lambda_m\|_2^2 + |\lambda|^m \frac{\sin\big((m+1)\phi\big)}{\sin\phi}. \end{aligned} \tag{45}$$

   Now, assume $\mathsf{Re}(\lambda) \geq 0$, i.e., $\phi \in [-\pi/2, \pi/2]$. Without loss of generality, we may assume $\sin\phi \geq 0$, that is $\phi \in [0, \pi/2]$, and $\sin\big((m+1)\phi\big) \leq 0$. Then the inequality $\sin\phi \geq \frac{2}{\pi}\phi$ holds and implies

$$\frac{\sin\big((m+1)\phi\big)}{\sin\phi} \geq \frac{\pi}{2} \frac{\sin\big((m+1)\phi\big)}{\phi} = \frac{(m+1)\pi}{2} \mathrm{sinc}\big((m+1)\phi\big) \geq -\frac{3}{8}(m+1),$$

   where we used the rough lower bound $-3/(4\pi)$ for $\mathrm{sinc}(x) = (\sin x)/x$. Combined with (45) and the straightforward bound $|\lambda|^m \leq \frac{2}{m+1}\|\Lambda_m\|_2^2$, this inequality yields

$$2\|\Pi_+(\Lambda_m)\|_2^2 \geq \|\Lambda_m\|_2^2 - \frac{3}{8}(m+1)|\lambda|^m \geq \frac{1}{4}\|\Lambda_m\|_2^2$$

   which proves the first part of statement 1. The second part, when $\mathsf{Re}(\lambda) \leq 0$, follows similarly. In particular, instead of (45) we use

$$2\|\Pi_-(\Lambda_m)\|_2^2 = \|\Lambda_m\|_2^2 - |\lambda|^m \frac{\sin\big((m+1)\phi\big)}{\sin\phi}. \tag{46}$$

2. Assume $\mathsf{Re}(\lambda) \leq 0$, i.e., $\phi \in [\pi/2, 3\pi/2]$. Using $|\Lambda_{m-1}^H R \Lambda_{m-1}| = |\lambda|^{m-1}\left|\frac{\sin(m\phi)}{\sin\phi}\right|$ and the relations (45)–(46), we obtain

$$\alpha = \underbrace{\frac{|\lambda|}{(1+|\lambda|^2)} \frac{|\lambda|^{2(m-1)}}{\|\Lambda_{m-1}\|_2^4}}_{=:\alpha_{|\lambda|}} \underbrace{\frac{\sin(2\phi)}{\sin\phi}\left|\frac{\sin(m\phi)}{\sin\phi}\right|^2}_{=:\alpha_\phi}, \quad \beta = \underbrace{\frac{|\lambda|^m}{\|\Lambda_m\|_2^2}}_{=:\beta_{|\lambda|}} \underbrace{\frac{\sin\big((m+1)\phi\big)}{\sin\phi}}_{=:\beta_\phi}.$$

   Consider ϕ fixed . Then, by a simple calculation, it can be seen that the function

$$g_\phi(|\lambda|) = \alpha_{|\lambda|}\alpha_\phi - 2\beta_{|\lambda|}\beta_\phi$$

tends to zero for $|\lambda| \to 0$ and $|\lambda| \to \infty$. Moreover, $g_\phi(|\lambda|)$ is either completely zero or has precisely one extremum at $|\lambda| = 1$. Hence, $g_\phi(|\lambda|)$ is bounded from above by the maximum between zero and its value at 1:

$$g_\phi(1) = \frac{\sin(2\phi)}{2\sin\phi}\left|\frac{\sin(m\phi)}{m\sin\phi}\right|^2 - 2\frac{\sin\big((m+1)\phi\big)}{(m+1)\sin\phi}.$$

The maximal value of $g_\phi(1)$ is attained at $\phi = \pi$, for which $g_\pi(1) = 1$. This proves the desired result.

3. As in part 1, the expression

$$2\|\Pi_+(\Lambda_m)\|_2^2 = \|\Lambda_m\|_2^2 + |\lambda|^m \frac{\sin\big((m+1)\phi\big)}{\sin\phi}$$

can be shown to also hold for even m. As above, we have

$$\frac{\sin\big((m+1)\phi\big)}{\sin\phi} \geq -\frac{3}{8}(m+1), \quad \text{for } \phi \in [-\pi/2, \pi/2]. \tag{47}$$

Combined with the straightforward bound $|\lambda|^m \leq \frac{2}{m+2}\|\Lambda_m\|_2^2 \leq \frac{2}{m+1}\|\Lambda_m\|_2^2$, this shows the lower bound when $\mathsf{Re}(\lambda) \geq 0$, as in part 1. It remains to discuss the case $\mathsf{Re}(\lambda) \leq 0$, that is $\phi \in [\pi/2, 3\pi/2]$. Since m is even, it follows for $\tilde{\phi} := \phi - \pi \in [-\pi/2, \pi/2]$ from (47) that

$$\frac{\sin\big((m+1)\phi\big)}{\sin\phi} = \frac{-\sin\big((m+1)\tilde{\phi}\big)}{-\sin\tilde{\phi}} = \frac{\sin\big((m+1)\tilde{\phi}\big)}{\sin\tilde{\phi}} \geq -\frac{3}{8}(m+1).$$

The desired lower bound follows from this inequality as above.

□