

# Theory and Numerics of Model Reduction

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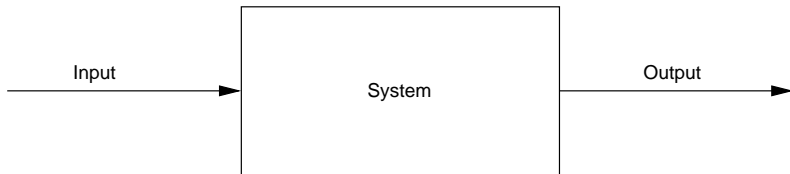
<http://www.math.ethz.ch/~kressner/modred.php>

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# Contents

- ▶ Continuous- and discrete-time control systems
- ▶ Laplace transform and transfer matrices
- ▶  $H^p$  norms
- ▶ Exponential matrix function and state space representation
- ▶ Stability, controllability, and observability

Systems are separated from their outer environment.



**Input variables:**

External quantities acting on the system.

**State variables:**

Describe condition or state of the system.

**Output variables:**

System quantities that can be observed.

# Continuous time control systems

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t)), & x(0) &= x_0, \\ y(t) &= h(t, x(t), u(t)),\end{aligned}$$

where

$$\begin{aligned}t \in [0, \infty) &= \text{time,} \\ x(t) \in \mathbb{R}^n &= \text{state vector at time } t, \\ u(t) \in \mathbb{R}^m &= \text{input vector at time } t, \\ y(t) \in \mathbb{R}^p &= \text{output vector at time } t,\end{aligned}$$

and  $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ .

System = state equation (parameterized first-order ODE)  
+ output equation (function in time/state/input)

# Extensions

This is *not* the most general form of a continuous time control systems.

- ▶ **higher order systems** (state equation involves higher order derivatives of  $x$ );
- ▶ **differential-algebraic systems** (implicit state equation  $F(t, x(t), \dot{x}(t), u(t)) = 0$ );
- ▶ **infinite-dimensional systems** (e.g., state equation is PDE).

# Linear time-invariant (LTI) continuous time control systems

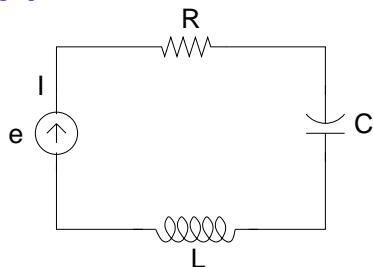
$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0, \\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where

- $A \in \mathbb{R}^{n \times n}$  = state matrix,
- $B \in \mathbb{R}^{n \times m}$  = input matrix,
- $C \in \mathbb{R}^{p \times n}$  = output matrix,
- $D \in \mathbb{R}^{p \times m}$  = feedthrough matrix (often zero).

$$\Sigma = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right]$$

## Linear RLC circuit



Kirchhoff's law states that the total voltage must be balanced:

$$e(t) - V_R(t) - V_C(t) - V_L(t) = 0.$$

Combined with

$$V_R(t) = I(t)R, \quad V_C(t) = Q(t)/C, \quad V_L(t) = L\dot{I}(t), \quad I(t) = \dot{Q}(t),$$

gives

$$e(t) = L\ddot{Q}(t) + R\dot{Q}(t) + Q(t)/C.$$

## Linear RLC circuit, ctd.

Second order  $\rightarrow$  first order by introducing  $x_1 = Q$ ,  $x_2 = \dot{Q}$ :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

Difficult to observe  $Q$  directly, instead current  $I = \dot{Q} = x_2$ :

$$y = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Typical question of **observability**:

Is it possible to determine the complete state from this observation?

# Discrete time control systems

If time evolves in discrete steps  $0, 1, 2, 3, \dots$ :

$$\begin{aligned}x(k+1) &= f(k, x(k), u(k)), & x(0) &= x_0, \\y(k) &= h(k, x(k), u(k)).\end{aligned}$$

LTI case:

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k), & x(0) &= x_0, \\y(k) &= Cx(k) + Du(k).\end{aligned}$$

# Predator-prey model

$x_1(k)$  = number of rabbits

$x_2(k)$  = number of foxes

$a$  = growth rate of rabbits in absence of predation

$b$  = death rate per encounter of rabbits due to predation

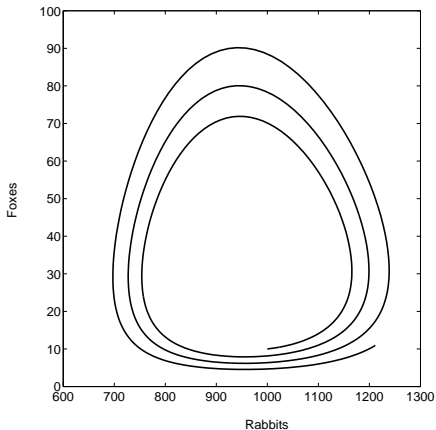
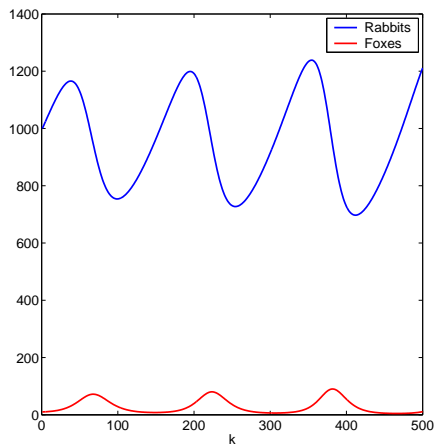
$c$  = death rate of foxes in absence of food

$e$  = efficiency of turning predated rabbits into foxes.

$$x_1(k+1) = x_1(k) + a x_1(k) - b x_1(k)x_2(k)$$

$$x_2(k+1) = x_2(k) + e b x_1(k)x_2(k) - c x_2(k)$$

## Predator-prey model, ctd.



Unstable situation! Typical question of **stabilizability**:

Is it possible to stabilize the system by applying control?

# System analysis

Introduce concepts of

- ▶ frequency domain approach
- ▶ time domain/state space approach
- ▶ stability
- ▶ controllability
- ▶ observability

In the following, we focus on:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0, \\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

# Laplace transform

For real-valued function  $f(t)$ , Laplace transform is defined as

$$\hat{f}(s) = (\mathcal{L}f)(s) = \int_0^{\infty} f(t) e^{-st} dt.$$

Some properties:

- ▶  $\mathcal{L}(\alpha f + \beta g) = \alpha(\mathcal{L}f) + \beta(\mathcal{L}g)$ ;
- ▶  $(\mathcal{L}\dot{f})(s) = s(\mathcal{L}f)(s) - f(0)$ ;
- ▶  $(\mathcal{L} \int f(\tau) d\tau)(s) = \frac{1}{s}(\mathcal{L}f)(s)$ .

Some special functions:

- ▶  $\hat{\delta}(s) \equiv 1$ , where  $\delta$  is the Dirac impulse at 0;
- ▶  $f(t) = e^{-\lambda t} \rightsquigarrow \hat{f}(s) = 1/(s + \lambda)$ ;
- ▶  $f(t) = \sin(bt) \rightsquigarrow \hat{f}(s) = b/(s^2 + b^2)$ ;
- ▶  $f(t) = \cos(bt) \rightsquigarrow \hat{f}(s) = s/(s^2 + b^2)$ .

# Transfer matrices

Applying Laplace transform to  $\dot{x}(t) = Ax(t) + Bu(t)$  yields

$$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s) \quad \Rightarrow \quad \hat{x}(s) = (sI - A)^{-1}B\hat{u}(s).$$

Inserting into  $\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$  gives

$$\hat{y}(s) = (C(sI - A)^{-1}B + D)\hat{u}(s).$$

The  $p \times m$  matrix  $G(s) = C(sI - A)^{-1}B + D$  is rational and called the **transfer function/matrix**.

**Poles** of  $G(\cdot)$  are eigenvalues of  $A$ . Zeros of  $G(\cdot)$  are **system zeros** (SISO).

For the linear RCL circuit:

$$G(s) = \frac{s/L}{s^2 + R/L \cdot s + 1/LC}$$

# Hardy spaces and $H^p$ norms

Appropriate spaces for “Laplace transformed” functions.

## Definition

$H^p(\mathbb{C}^m)$  is the space of all analytic  $u : \mathbb{C}^+ \rightarrow \mathbb{C}^m$  with  $\|u(\cdot)\|_{H^p} < \infty$  for

$$\|u(\cdot)\|_{H^p} = \begin{cases} \sup_{\beta > 0} \left( \int_{-\infty}^{\infty} \|u(\beta + i\omega)\|^p d\omega \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{s \in \mathbb{C}^+} \|u(s)\| & \text{if } p = \infty. \end{cases},$$

with some vector norm  $\|\cdot\|$ .

## Important property:

Define boundary function  $u^0(i\omega) = \lim_{\beta \downarrow 0} u(\beta + i\omega)$ . Then  $u^0 \in L^p(i\mathbb{R}, \mathbb{C}^m)$  and  $\|u\|_{H^p} = \|u^0\|_{L^p}$ . In particular,

$$\|u\|_{H^\infty} = \sup_{\omega \in \mathbb{R}} \|u(i\omega)\|.$$

## $H^p$ norms, ctd.

Plancherel-like equality (important for transferring results between time and frequency domain):

The normalized Laplace transform  $\sqrt{2\pi}\mathcal{L}$  is a linear *isometry* from  $L^2((0, \infty); \mathbb{C}^m)$  onto  $H^2(\mathbb{C}^m)$ .

$H^\infty$  is often the space easier to work with. Transfer function  $G(s) \in \mathbb{C}^{p \times m}$ :

$$\|G\|_{H^\infty} = \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|_2.$$

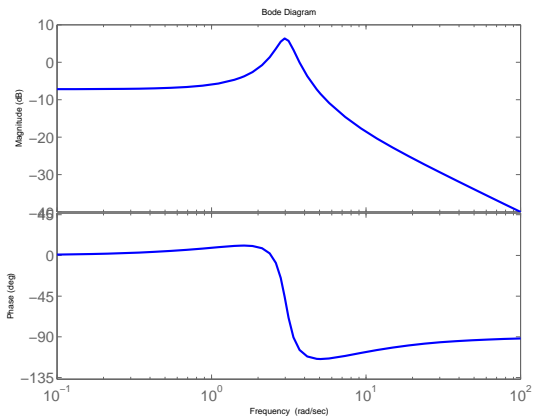
( $\exists$  reliable computational methods for finding global minimum)

# Bode plot

SISO case: For input  $u(t) = u_0 \sin(\omega t)$ ,  $\approx$  output for large  $t$

$$y^{SS}(t) = u_0 |G(j\omega)| \sin(\omega t + \arg(G(j\omega))).$$

$$\text{Example: } g(s) = \frac{(s-1)(s+4)}{(s-1)(s+0.4-3i)(s+0.4+3i)}.$$





# The exponential matrix function

Important tool for developing state space representations:

$$e^A := I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Some useful properties:

- ▶  $e^{A(t+s)} = e^{At} e^{As}$ ,  $e^{-At} = [e^{At}]^{-1}$ ,  $\frac{\partial}{\partial t} e^{At} = A e^{At}$ ;
- ▶  $e^{P^{-1}AP} = P^{-1} e^A P$ ;
- ▶ if  $\lambda$  is an eigenvalue of  $A$  then  $e^\lambda$  is an eigenvalue of  $e^A$ ;
- ▶ **But:**  $e^{A+B} \neq e^A e^B$  (unless  $A$  and  $B$  commute).

# Solution of state equation

The state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

is a linear ODE and thus *always* uniquely solvable. Solution:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau.$$

Inserting into output equation  $y(t) = Cx(t) + Du(t)$  yields

$$y(t) = C e^{At} x_0 + \int_0^t C e^{A(t-\tau)} Bu(\tau) d\tau + Du(t).$$

$\rightsquigarrow$  *input-output map*  $\mathcal{G} : u(\cdot) \rightarrow y(\cdot)$ .

# Stability

1. The system  $\dot{x}(t) = Ax(t)$  is called **asymptotically stable** if for each  $x_0$  the solution  $x(t)$  satisfies  $\lim_{t \rightarrow \infty} x(t) = 0$ .
2. If for each  $x_0$  there exists a constant  $C > 0$  so that  $\|x(t)\| \leq C$  for all  $t > 0$  then the system is called **stable**.
3. In any other case, the system is called **unstable**.

For  $\dot{x}(t) = Ax(t) + Bu(t)$ , let  $x(t)$  be solution corresponding to initial state  $x_0$  and let  $\hat{x}(t)$  be solution corresponding to perturbed  $\hat{x}_0$ . Then the error between both trajectories  $e(t) = \hat{x}(t) - x(t)$  satisfies  $\dot{e}(t) = Ae(t)$ .

# Eigenvalue criterion for stability

The system  $\dot{x}(t) = Ax(t)$  is

1. asymptotically stable  $\Leftrightarrow \lambda(A) \subset \mathbb{C}^-$ ;
2. stable  $\Leftrightarrow \lambda(A) \subset \mathbb{C}^- \cup i\mathbb{R}$  and each purely imaginary eigenvalue is semi-simple.

Examples:

$$A = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

## Definition of controllability

The system  $\dot{x}(t) = Ax(t) + Bu(t)$  is called **controllable** if there is a (continuous) control law  $u(t)$  that transfers any initial state  $x_0$  to any  $x_f$  at some time  $t_f > 0$ .

Remarks:

- ▶ State depends linearly on  $x_0 \rightsquigarrow$  w.l.o.g.  $x_f = 0$ .
- ▶ Time-invariance of system  $\rightsquigarrow x_0$  can be transferred in any time  $t_f > 0$ .

# Kalman test

$(A, B)$  is controllable if and only if

$$\text{rank}[B, AB, A^2B, \dots, A^{n-1}B] = n.$$

Note: Proof facilitates equivalent condition

$$P(t_f) = \int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau} d\tau$$

is p.d. for all/some  $t_f$ .  $t_f \rightarrow \infty$ : **Controllability Gramian**

$$P = \int_0^{\infty} e^{-A\tau} BB^T e^{-A^T\tau} d\tau$$

is unique solution of

$$AP + PA^T = -BB^T$$

(provided  $A$  is stable).

Direct application of Kalman test is numerically infeasible!

$$A = \begin{bmatrix} 1 & & & & & \\ 20 & 2 & & & & \\ & \ddots & \ddots & & & \\ & & 20 & 19 & & \\ & & & 20 & 20 & \end{bmatrix}, \quad B = \begin{bmatrix} 20 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$\mathcal{C} = [B, AB, \dots, A^{19}B]$  (MATLAB command `ctrb`) has singular values

$$2.6 \times 10^{28}, 2.2 \times 10^{28}, \dots, 78.2, 20.$$

$\mathcal{C}$  is numerically singular!

# Hautus test

Is  $p$  an eigenvector of  $A^T$ , then  $p^T B \neq 0$ .

Variations:

1.  $\text{rank}([A - \lambda I, B]) = n$  for all  $\lambda \in \Lambda(A)$ ;
2.  $\text{rank}([A - \lambda I, B]) = n$  for all  $\lambda \in \mathbb{C}$ .

For example from above,

$$\min_{\lambda \in \Lambda(A)} \sigma_{\min}([A - \lambda I, B]) \approx 5.9.$$

Far from being singular!

# System transformations

**State space transformation**  $x \rightarrow T^{-1}x$  for nonsingular  $T$  corresponds to system transformation

$$\frac{A \mid B}{C \mid D} \Rightarrow \frac{T^{-1}AT \mid T^{-1}B}{CT \mid D}.$$

Does not change the input-output map. To attain numerical stability, restrict to orthogonal coordinate transformations.

## Controllability form for SI systems

Consider single input case ( $m = 1$ ): Let  $m = 1$  and  $Q$  be an orthogonal matrix which maps  $B$  to multiple of  $e_1$ :

$$[A|B] \rightsquigarrow [Q^T A Q | Q^T B] = \left[ \begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \end{array} \right].$$

Reduce  $A$  further to Hessenberg form.

$$\rightsquigarrow \left[ \begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & 0 \\ \mathbf{0} & \times & \times & \times & 0 \\ \mathbf{0} & \times & \times & \times & 0 \end{array} \right].$$

## Controllability form for SI systems

$$\rightsquigarrow \left[ \begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & 0 \end{array} \right].$$

Controllability matrix  $\mathcal{C}$  of reduced system is **upper triangular** with diagonal elements

$$b_1, a_{21}b_1, a_{32}a_{21}b_1, a_{43}a_{32}a_{21}b_1, \dots$$

System is controllable  $\Leftrightarrow b_1 \neq 0, a_{21} \neq 0, a_{32} \neq 0, \dots$

# Controllability form for SI systems

If subdiagonal element is zero, transformed system can be decomposed into

$$\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}$$

Two subsystems:

$$\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array} \quad \text{is controllable } (\lambda(A_{11}) \text{ are controllable modes});$$
$$\begin{array}{c|c} A_{22} & 0 \\ \hline C_2 & D \end{array} \quad \text{is uncontrollable } (\lambda(A_{22}) \text{ are uncontrollable modes}).$$

# Staircase form for MI systems

There exists an orthogonal matrix  $Q$  s.t.

$$[Q^T A Q | Q^T B] = \left[ \begin{array}{cccc|c|c} A_{1,1} & A_{1,2} & \cdots & A_{1,k} & A_{1,k+1} & X_1 \\ X_2 & A_{2,2} & & \vdots & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ & \ddots & X_k & A_{k,k} & A_{k,k+1} & 0 \\ \hline & & 0 & 0 & A_{k+1,k+1} & 0 \end{array} \right],$$

where

$A_{ij}$  are  $\rho_i \times \rho_i$  matrices;

$X_i$  are  $\rho_i \times \rho_{i-1}$  matrices of full row rank.

Eigenvalues of  $A_{k+1,k+1}$  are uncontrollable modes.

## Definition of observability

System  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t) + Du(t)$  is called **observable** if there is a  $t_f$  such that for arbitrary  $u(t)$  we can determine  $x(0)$  from knowledge of  $u(t)$  and  $y(t)$  in  $[0, t_f]$ .

Remark:

- ▶ For LTI systems, it is sufficient to show statement for one  $u$ .

Let  $g(t) = y(t) - \int_0^{t_f} Ce^{A(t-\tau)}Bu(\tau) d\tau - Du(t)$ , then

$$\int_0^{t_f} e^{A^T\tau} C^T Ce^{A\tau} d\tau x_0 = \int_0^{t_f} e^{A^T\tau} C^T g(\tau) d\tau$$

$\rightsquigarrow$  Observability equivalent to invertibility of **observability Gramian**

$$Q(t_f) = \int_0^{t_f} e^{A^T\tau} C^T Ce^{A\tau} d\tau.$$

# Duality to controllability

Compare both Gramians:

$$Q(t_f) = \int_0^{t_f} e^{A^T \tau} C^T C e^{A \tau} d\tau, \quad P(t_f) = \int_0^{t_f} e^{-A \tau} B B^T e^{-A^T \tau} d\tau$$

Replacing  $A$  by  $-A^T$  and  $B$  by  $C^T$  in  $P$  yields  $Q$ .

Gives equivalent conditions for observability:

1. Observability matrix  $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$  has full rank.
2.  $\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$  for all  $\lambda \in \lambda(C)$ .
3. Solution of  $A^T Q + QA = -C^T C$  is s.p.d. (provided  $A$  is stable).

## Observability form for SO systems

By a variation of algorithm for controllability form:

$$\left[ \begin{array}{c} Q^T A Q \\ C Q \end{array} \right] = \left[ \begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ \hline 0 & 0 & 0 & \times \end{array} \right]$$

If subdiagonal element is zero, decomposition

$$\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline 0 & C_2 & D \end{array}$$

Subsystem  $\begin{array}{c|c} A_{11} & B_1 \\ \hline 0 & D \end{array}$  is not observable  
( $\lambda(A_{11})$  **unobservable modes**).

# Combining controllability and observability forms

There is an orthogonal matrix  $Q$  s.t.

$$\left[ \begin{array}{c|c} Q^T A Q & Q^T B \\ \hline C Q & D \end{array} \right] = \left[ \begin{array}{ccc|c} A_{c\bar{o}} & \times & \times & B_{c\bar{o}} \\ 0 & A_{co} & \times & B_{co} \\ 0 & 0 & A_{\bar{c}} & 0 \\ \hline 0 & C_{co} & C_{\bar{o}} & D \end{array} \right]$$

where

$A_{c\bar{o}}$  contains unobservable modes;

$A_{\bar{c}}$  contains uncontrollable modes.

Subsystem  $\frac{A_{co} \mid B_{co}}{C_{co} \mid D}$  is **controllable and observable**.