Asymptotic expansion of highly conductive thin sheets

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Sensitive measurement and control equipment are protected from disturbing electromagnetic fields by thin shielding sheets. Alternatively to discretisation of the sheets, the electromagnetic fields are modeled only in the surrounding of the layer taking them into account with the so called Generalised Impedance Boundary Conditions.

We study the shielding effect by means of the model problem of a diffusion equation with additional dissipation in the curved thin sheet. We use the asymptotic expansion techniques to derive a limit problem, when the thickness of the sheet ε tends to zero, as well as the models for contribution to the solution of higher order in ε. These problems are posed in limit area of vanishing ε with condition for the jump of the solution and it's normal derivative, which avoid to mesh the computational domain, even locally, at the scale of ε.

We derive the problems for arbitrary order and show their existence and uniqueness. Numerical experiments for the problems up to second order show the asymptotic convergence of the solution of right order in mean of the thickness parameter ε.

1 Outline of the problem

The diffusion-reaction equation in 2D

$$-\Delta u + cu = f \tag{1}$$

is a model problem for the magnetoquasistatic Eddy-current model with a thin conducting sheet [1], whereas $c(x)$ is vanishing besides the thin sheet Ω_C of constant thickness d. Domains with very thin sheets are hardly to mesh by todays grid generators. Thus, one would like to replace the sheet by an internal interface, where Generalised Impedance Boundary conditions (GIBCs) [2], [3] model the effect of the sheet. We will derive higher order GIBCs by asymptotic expansions [4] for smooth sheets and constant conductivity $c(x)$.

Even so we are looking for the solution for a sheet of a particular thickness d, we replace (1) by a family of problems (see Fig. 1), ordered by the thickness parameter ε

$$\begin{aligned} -\Delta u^\varepsilon_{\text{ext}} &= f \quad \text{in } \Omega^\varepsilon_{\text{ext}}, & u^\varepsilon_{\text{ext}} &= u^\varepsilon_{\text{int}} \quad \text{on } \partial\Omega^\varepsilon_{\text{int}},\\ -\Delta u^\varepsilon_{\text{int}} + \tfrac{c_0}{\varepsilon} u^\varepsilon_{\text{int}} &= 0 \quad \text{in } \Omega^\varepsilon_{\text{int}}, & \partial_n u^\varepsilon_{\text{ext}} &= \partial_n u^\varepsilon_{\text{int}} \quad \text{on } \partial\Omega^\varepsilon_{\text{int}}. \end{aligned} \tag{2}$$

We consider these equations for all $\varepsilon < \varepsilon_0$, such that the support of f is outside the sheet $\Omega^\varepsilon_{\text{int}}$. We choose in this model a conductivity scaled like $1/\varepsilon$. To solve for a particular thickness d, one select $c_0 = c\,d$, by what the problems (2) and (1) coincide for $\varepsilon = d$. Due to this scaling the limit solution for $\varepsilon \to 0$ is non-trivial, meaning that neither the effect of the sheet disappear nor the sheet gets a perfect conductor. The external solution $u^\varepsilon_{\text{ext}}(x)$ in the non-conducting subdomain $\Omega^\varepsilon_{\text{ext}}$ and the internal solution $u^\varepsilon_{\text{int}}(x)$ in $\Omega^\varepsilon_{\text{int}}$ are related by Dirichlet and Neumann transmission conditions.

2 Asymptotic expansions

External functions In the external domain $\Omega^\varepsilon_{\text{ext}}$ we are looking for an asymptotic of the solution with the form

$$u^\varepsilon_{\text{ext}}(x) = \sum_{i=0}^{\infty} \varepsilon^i u^i_{\text{ext}}(x) + \underset{\varepsilon\to\infty}{o}(\varepsilon^\infty) \quad . \tag{3}$$

Internal functions We introduce local coordinates (t, s) in the sheet, where t is the tangential coordinate along the midline Γ_{m} and $s \in [-\varepsilon/2, \varepsilon/2]$ is the coordinate in thickness direction. Furthermore, let $S = s/\varepsilon$ the stretched coordinate, the parameter domain of (t, S) is called $\widehat{\Omega}$ and the internal solution is $U^\varepsilon_{\text{int}}(t, S) := u^\varepsilon_{\text{int}}(t, s)$. We are looking for an asymptotic of the solution with the form

$$U^\varepsilon_{\text{int}}(t, S) = \sum_{i=0}^{\infty} \varepsilon^i U^i_{\text{int}}(t, S) + \underset{\varepsilon\to\infty}{o}(\varepsilon^\infty) \quad . \tag{4}$$

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Joint problem We expand the values and normal derivative of $u^i_{\text{ext}}(t, \pm\varepsilon/2)$ by Taylor expansion around $s = \pm 0$ to obtain a Dirichlet and a Neumann transmission condition on the midline Γ_{m}. Together with the expansion of the Laplace operator and identifying terms of equal power of ε we find the coupled problems defining $u^i_{\text{ext}}(x)$ and $U^i_{\text{int}}(t, S)$

$$-\Delta u^i_{\text{ext}} = f\delta^i_0 \qquad \text{in } \Omega^0_{\text{ext}}, \tag{5a}$$

$$\partial^2_S U^i_{\text{int}}(t,S) = c_0 U^{i-1}_{\text{int}}(t,S) - \sum_{l=1}^{i} \mathbf{A}_l U^{i-l}_{\text{int}}(t,S) \qquad \text{in } \widehat{\Omega}, \tag{5b}$$

$$U^i_{\text{int}}(t, \pm\frac{1}{2}) - u^i_{\text{ext}}(t, \pm 0) = \sum_{l=1}^{i} \left(\pm\frac{1}{2}\right)^l \frac{1}{l!}\partial^l_s u^{i-l}_{\text{ext}}(t, \pm 0) \qquad \text{on } \Gamma_{\text{m}}, \tag{5c}$$

$$\partial_S U^i_{\text{int}}(t, \pm\frac{1}{2}) = \sum_{l=1}^{i} \left(\pm\frac{1}{2}\right)^{l-1} \frac{1}{(l-1)!}\,\partial^l_s u^{i-l}_{\text{ext}}(t, \pm 0), \qquad \text{on } \Gamma_{\text{m}}. \tag{5d}$$

Here δ^i_0 is the Kronecker symbol and $\mathbf{A}_l$ are differential operators in t and S resulting from the expansion of the Laplacian.

Theorem 2.1 (Existence, uniqueness and convergence) *This family of solution* (5) *is uniquely defined and the remainders* $r^{\varepsilon,N+1}_{\text{ext}}(x) := \sum_{i=0}^N \varepsilon^i u^i(x) - u^\varepsilon(x)$, $r^{\varepsilon,N+1}_{\text{int}}(t,s) := \sum_{i=0}^N \varepsilon^i U^i(t, \varepsilon S) - u^\varepsilon(t,s)$ *satisfies*

$$||r^{\varepsilon,N+1}||_{H^1(\Omega^\varepsilon_{\text{ext}})} + \sqrt{\varepsilon}||r^{\varepsilon,N+1}||_{H^1(\Omega^\varepsilon_{\text{int}})} \leq C_N \varepsilon^{N+1}. \tag{6}$$

Remark 2.2 The asymptotic expansion shows optimal order. The term $\sqrt{\varepsilon}$ results from the scaling of $\Omega^\varepsilon_{\text{int}}$.

3 Numerical results

A factorisation procedure can eliminate the internal field U^i_{int} and allows to compute the external solution u^i_{ext} alone. This leads to the transmission conditions of the first three orders

$$\begin{cases} \left[u^0_{\text{ext}}\right](t) = 0 \\ \left[\partial_n u^0_{\text{ext}}\right](t) - c_0\, u^0_{\text{ext}}(t,0) = 0 \end{cases} \qquad \begin{cases} \left[u^1_{\text{ext}}\right](t) = 0 \\ \left[\partial_n u^1_{\text{ext}}\right](t) - c_0 u^1_{\text{ext}}(t) = \dfrac{c_0^2}{6} u^0_{\text{ext}}(t,0) \end{cases}$$

$$\begin{cases} \left[u^2_{\text{ext}}\right](t) = -\dfrac{c_0}{24}k(t)u^0_{\text{ext}}(t,0) - \dfrac{c_0}{12}\left\{\partial_n u^0_{\text{ext}}\right\}(t) \\ \left[\partial_n u^2_{\text{ext}}\right](t) - c_0\left\{u^2_{\text{ext}}\right\}(t) = \dfrac{c_0^2}{6}u^1_{\text{ext}}(t) + \dfrac{c_0}{24}k(t)\left\{\partial_n u^0_{\text{ext}}\right\}(t) + c_0\left(\dfrac{7}{240}c_0^2 - \dfrac{\partial^2_t}{12}\right)u^0_{\text{ext}}(t). \end{cases}$$

Here we use the symbol $[\cdot](t)$ for the jump and $\{\cdot\}(t)$ for the mean value of the solution of both sides of the midline, and $k(t)$ is the curvature of the midline. For a circular domain with ellipsoidal sheet we were computing these asymptotic expansion solution $\sum_{i=0}^N \varepsilon^i u^i_{\text{ext}}(x)$ as well as the "exact" solution for each thickness (see Fig. 2 for $\varepsilon = 1/8$). We used discretisations by means of high order finite elements [5], in order that the discretisation error does not dominate the modeling error. The numerical results (see Fig. 3) confirm our theoretical results.

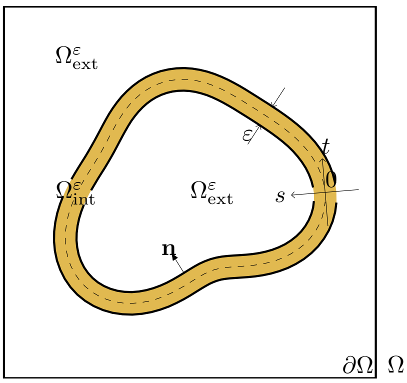

Fig. 1 Family of geometries.

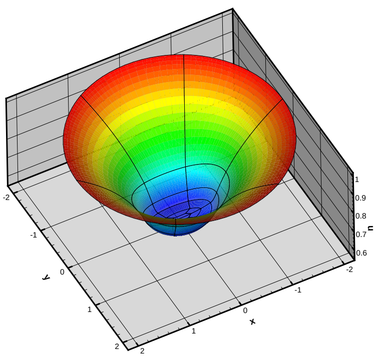

Fig. 2 Exact solution $u^\varepsilon_{\text{ext}}$ for $\varepsilon = 1/8$.

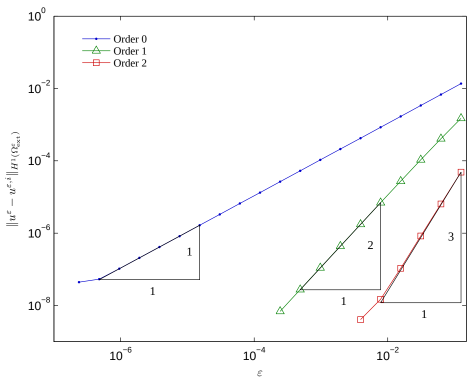

Fig. 3 Error in the H^1-norm.

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