Well-balanced schemes
for
nearly steady adiabatic flow

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Outline

- Introduction & Motivation
- Well-balanced schemes for hydrostatic states
- Well-balanced schemes for steady flow
- Conclusion

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- **Introduction & Motivation**
- Well-balanced schemes for hydrostatic states
- Well-balanced schemes for steady flow
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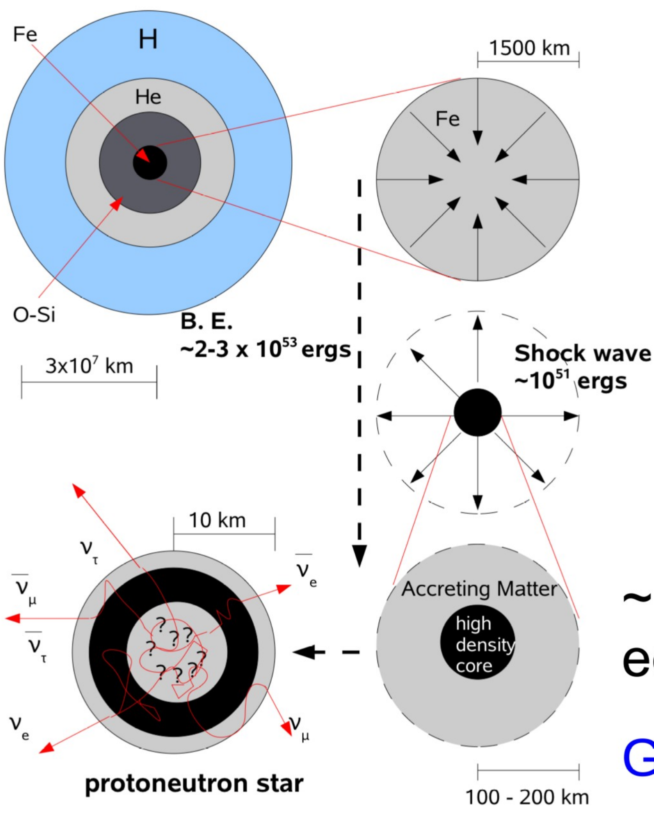
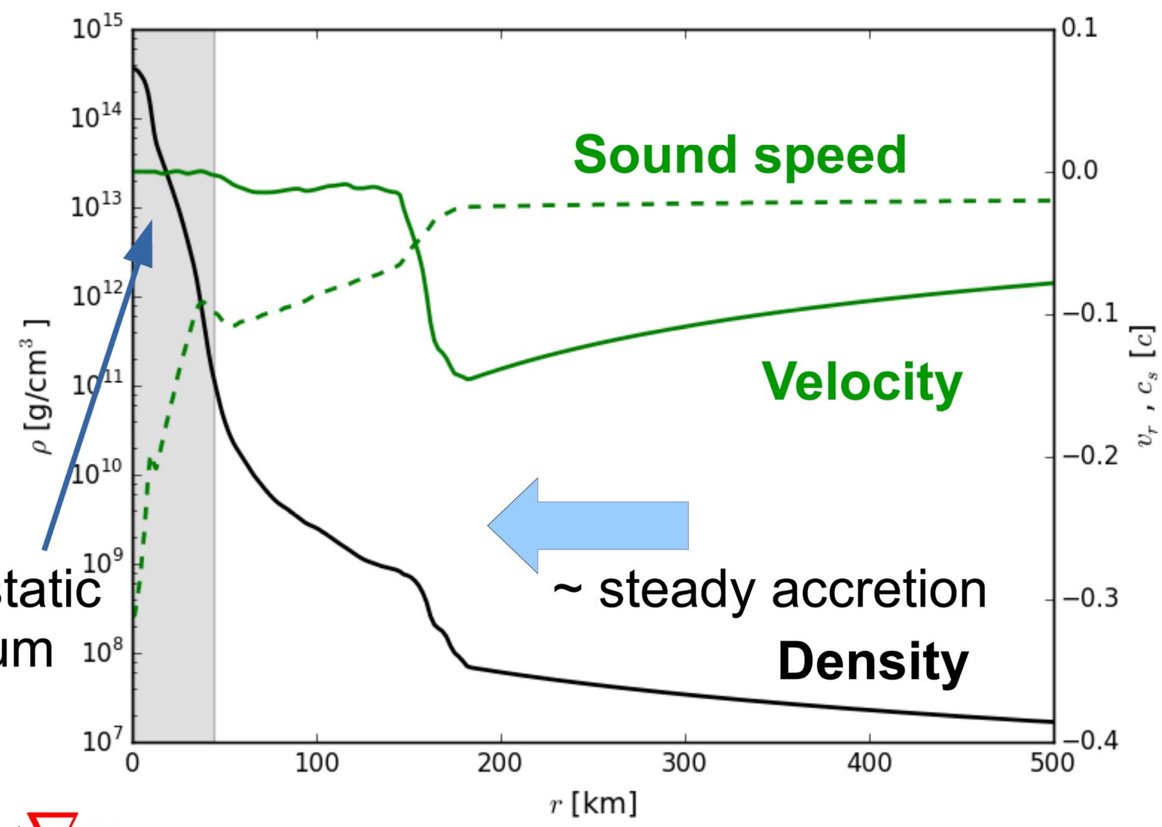
(Near) Steady states ubiquitous

- Core-collapse supernova





~ hydrostatic equilibrium

Gravity

$$-\rho\nabla\phi\approx\nabla p$$

Pressure

(Near) Steady states ubiquitous

- Waves in stellar atmospheres
  The waves amplitude may be much smaller when compared to the stratification stemming from gravity...
- Stellar “evolution”
  Stars evolve mostly quietly very close to a hydrostatic state...
  E.g., convection is only a small perturbation of the stationary state
- Climate modelling (on exoplanets)
  Atmospheric motions happen on a hydrostatic background
- Accretion phenomena
  Accumulation of mass onto a massive object (accretion discs, ...)
- ...

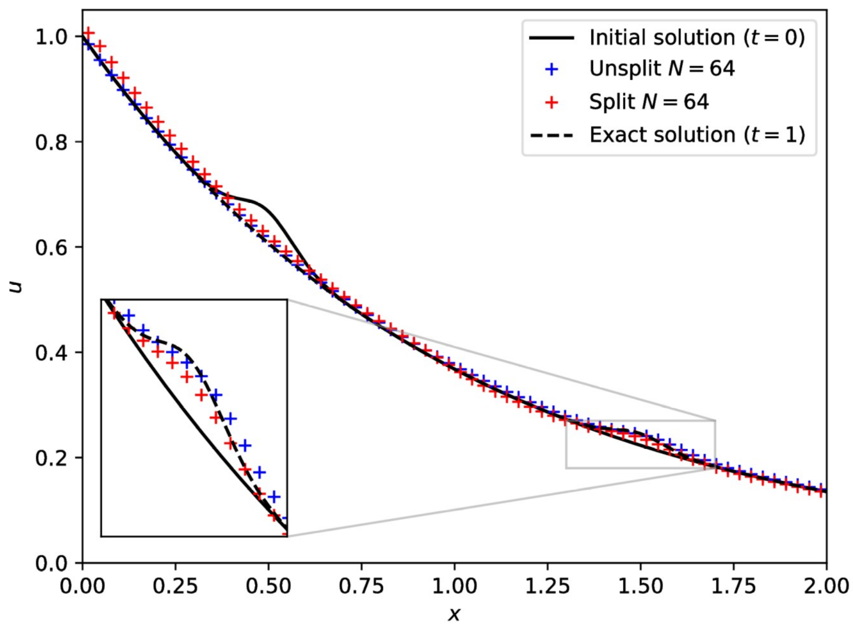
The issue

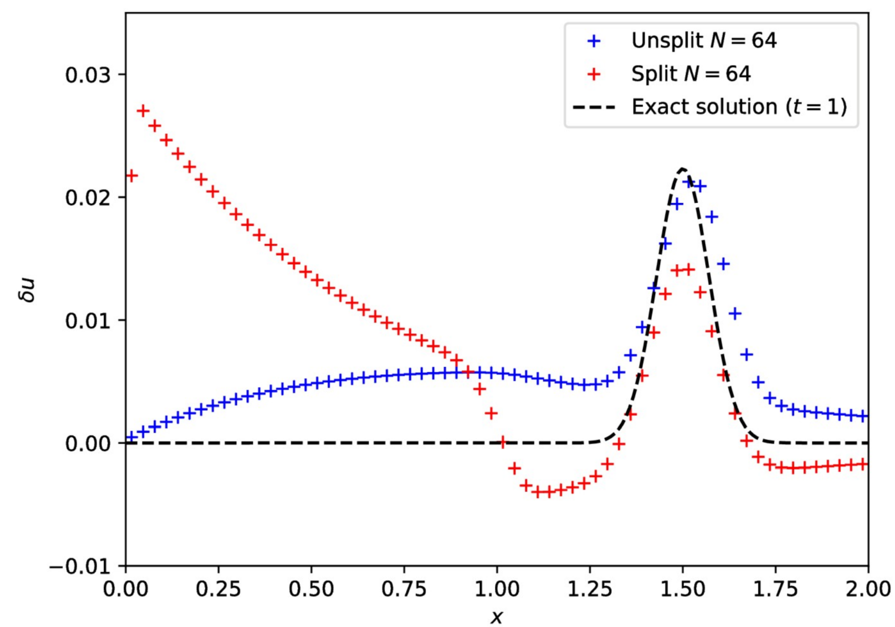
- Consider the simple example

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\lambda u \qquad u(x,t) = e^{-\lambda t}u_0(x-at)$$

Steady state: $$a\frac{\partial u}{\partial x} = -\lambda u \qquad u(x,t) = Ce^{-\lambda/a\ x}$$

First-order upwind with unsplit or split source term

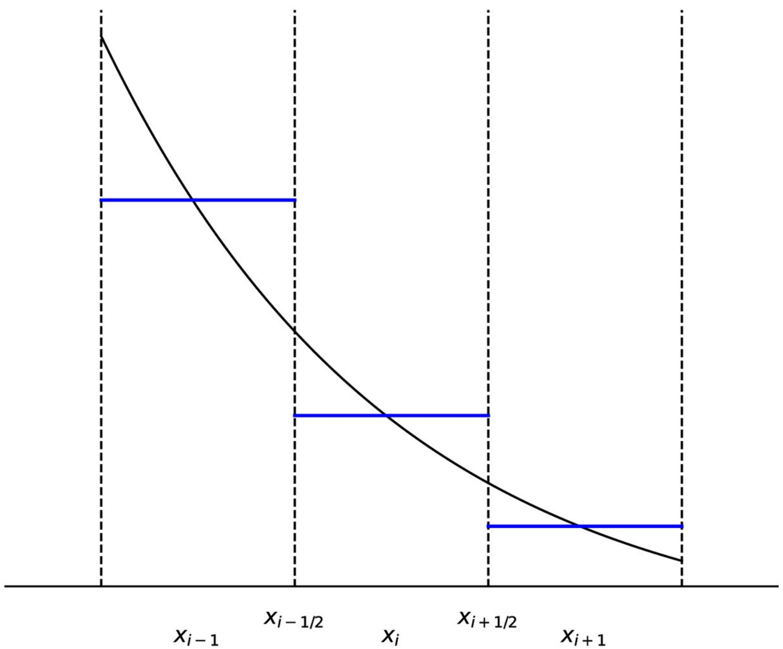


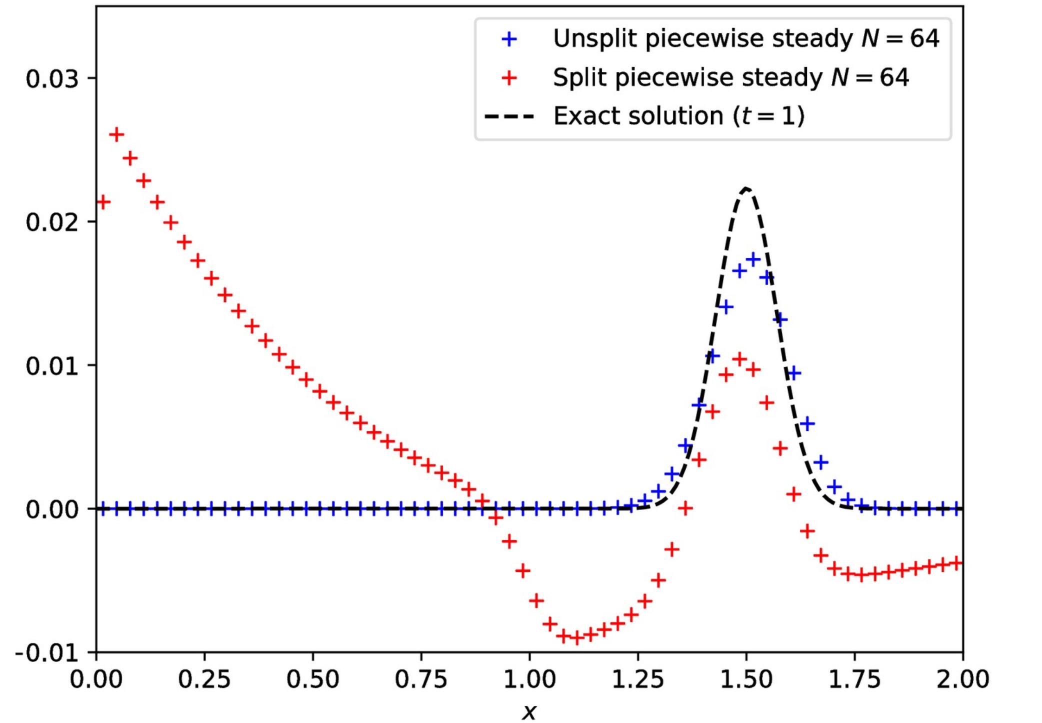


The issue

- The problem is that the usual (polynomial) reconstructions do not resolve the steady state very well...
- Use piecewise steady reconstruction instead

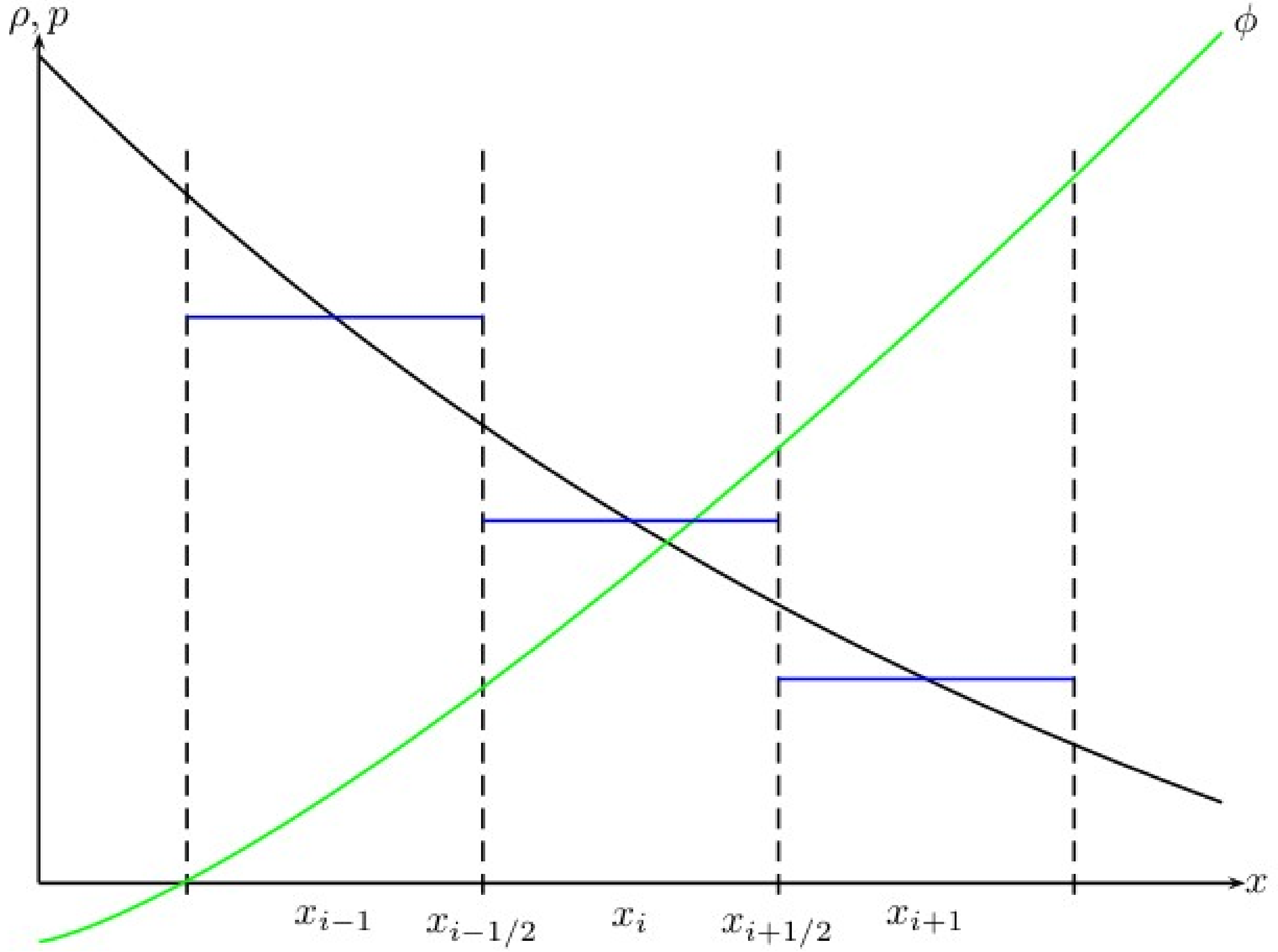
Liu (1979), Glaz & Liu (1984), van Leer (1984), ...





The issue

- The same problem for hydrostatic equilibrium... Piecewise steady reconstruction



Well-balanced schemes

- Preserve a discrete form of (certain) steady states exactly
- Many schemes developed for balance laws (shallow water, **Euler**, ...)

E.g. Cargo & LeRoux (1994),LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010,2011), Xing & Shu (2013), Vides et al. (2013), Desveaux et al. (2014,2015), Chandrashekar & Klingenberg (2015), Ghosh & Constantinescu (2015,2016), Li & Xing (2015,2016,2018), Franck & Mendoza (2016), Chandrashekar & Zenk (2017), Berberich et al. (2018,2019,2021), Chertock et al. (2018), Gaburro et al. (2018), Qian et al. (2018), Popov et al. (2019), Thomann et al. (2019,2020), Klingenberg et al. (2019), Veiga et al. (2019), Varma &Chandrashekar (2019), Krause (2019), Padioleau et al. (2019), Kanbar et al. (2020), Castro & Pares (2020), Pares & Pares-Pulido (2021), Li & Gao (2021), Wu & Xing (2021), Edelmann et al. (2021), Gomez-Bueno et al. (2021), ...

See also Mellema et al. (1991,1995), Zingale et al. (2002), Kastaun (2006), Freytag et al. (2012), ...

Outline

- Introduction & Motivation
- **Well-balanced schemes for hydrostatic states**
- Well-balanced schemes for steady flow
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Hydrostatic equilibrium

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \rho \boldsymbol{v}) + \nabla p = -\rho \nabla \phi$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\, \boldsymbol{v}] = -\rho \boldsymbol{v} \cdot \nabla \phi$$

Gravitational potential

$\Rightarrow$

$$\nabla p = -\rho \nabla \phi$$

Describes only a mechanical equilibrium...
Density and pressure not uniquely determined

$$p = p(\rho, s) = p(\rho, T) = p(\rho, \vartheta)$$

s Entropy

T Temperature

ϑ Relevant thermodynamic quantity

However, note that not all thermal stratifications are physically stable

Isentropic well-balanced scheme

- Consider constant entropy profile

- Using the thermodynamic relation

$$\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}p}{\rho} \qquad h = e + \frac{p}{\rho}$$ Enthalpy

- Hydrostatic eq.

$$\frac{1}{\rho}\nabla p = \nabla h = -\nabla \phi$$

- Or simply

$$h + \phi = const$$

See e.g. Landau & Lifshitz, Vol. 6, Chap. 1

Isothermal well-balanced scheme

- Consider constant temperature profile

- Using the thermodynamic relation

$$\mathrm{d}g = \frac{\mathrm{d}p}{\rho} - s\mathrm{d}T \qquad g = h - Ts$$

Gibbs free energy

- Hydrostatic eq.

$$\frac{1}{\rho}\nabla p = \nabla g = -\nabla \phi$$

- Or simply

$$g + \phi = const$$

Barotropic well-balanced scheme

- Consider barotropic profile $p = p(\rho)$

- Using the barotropic relation

$$\mathrm{d}\Theta = \frac{\mathrm{d}p}{\rho} \qquad \Theta = \int \frac{\mathrm{d}p}{\rho}$$

- Hydrostatic eq.

$$\frac{1}{\rho}\nabla p = \nabla\Theta = -\nabla\phi$$

- Or simply

$$\Theta + \phi = const$$

Arbitrary strat. well-balanced scheme

- Integrate hydrostatic equilibrium from some reference point

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

$$p(x) = p_0 - \int_{x_0}^{x} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x$$

- Unfortunately, this is path dependent and does not generalize trivially to several dimensions

Well-balanced finite volume scheme (1)

- One-dimensional balance law

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{s}$$

- Finite volume method

$$\frac{\mathrm{d}\overline{\boldsymbol{U}}_i}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2} - \boldsymbol{F}_{i-1/2}\right) + \overline{\boldsymbol{S}}_i$$

- Steady states of interest

$$\frac{\partial}{\partial x}\boldsymbol{f}\left(\boldsymbol{u}_{eq}(x)\right) = \boldsymbol{s}\left(\boldsymbol{u}_{eq}(x)\right) \qquad \boldsymbol{U}_{eq}(x) = \boldsymbol{u}_{eq}(x) + \mathcal{O}(\Delta x^{\epsilon})$$

Well-balanced finite volume scheme (2)

- Piecewise steady reconstruction $\boldsymbol{U}_i(x) = \boldsymbol{U}_{eq,i}(x) + \delta\boldsymbol{U}_i(x)$

  Equilibrium part — Perturbation part

  - Equilibrium "projection"

$$\overline{\boldsymbol{U}}_i \longrightarrow \overline{\boldsymbol{U}}_{eq,i}$$

Projecting onto a cell average consistent with the steady states of interest

  - Match equilibrium

$$\frac{1}{\Delta x}\mathcal{Q}_i\left(\boldsymbol{U}_{eq,i}\right) = \overline{\boldsymbol{U}}_{eq,i}$$

Quadrature rule

  - Reconstruct equilibrium perturbation

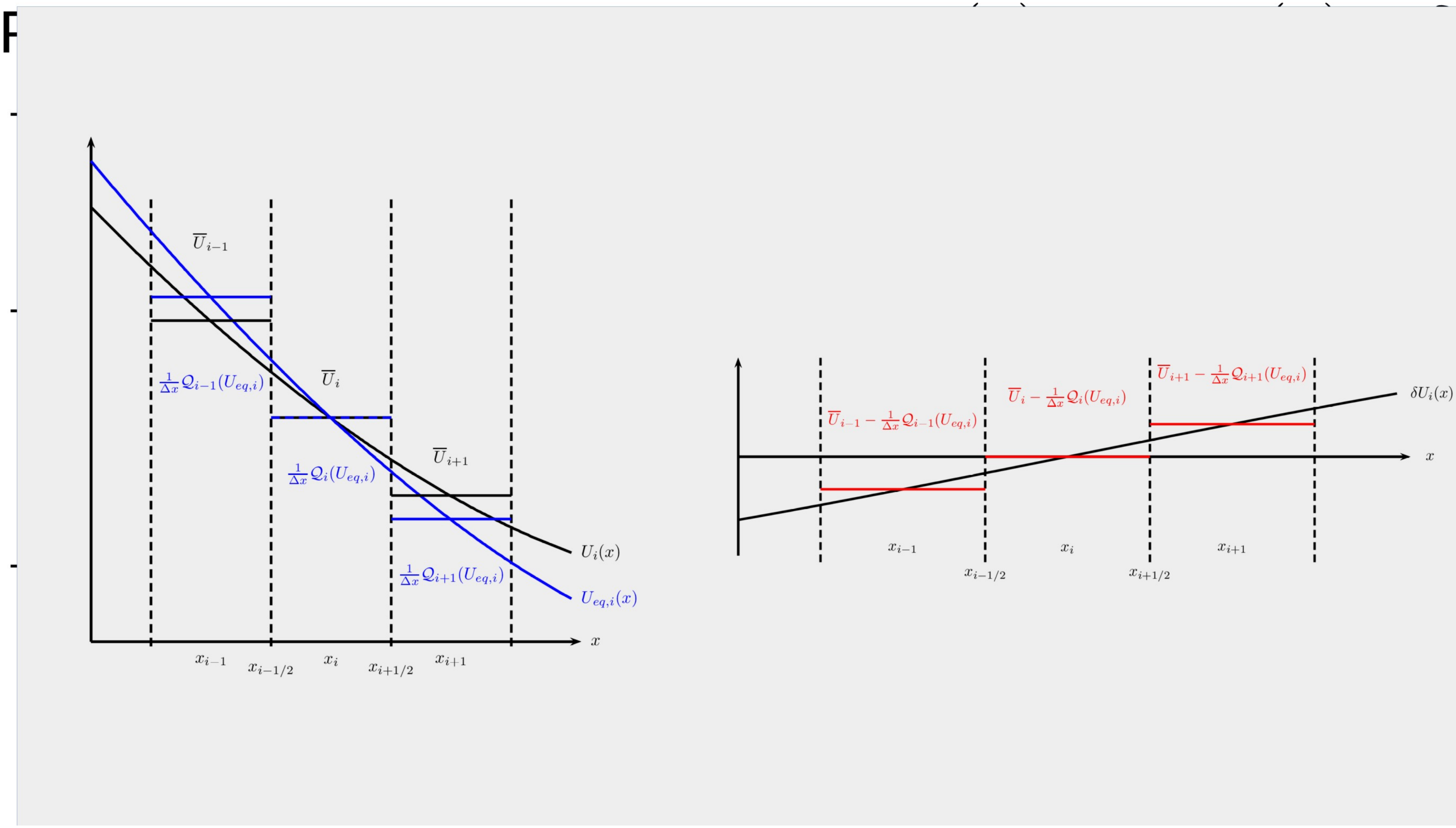
$$\delta\boldsymbol{U}_i(x) = \mathcal{R}\left(x; \left\{\overline{\boldsymbol{U}}_k - \frac{1}{\Delta x}\mathcal{Q}_k(\boldsymbol{U}_{eq,i})\right\}_{k\in S_i}\right)$$

Reconstruction procedure

Well-balanced finite volume scheme (2)

• F ... $U_i(x)$

- ... ion part
- ... nt with
-



Well-balanced finite volume scheme (3)

- Source term discretization

$$\overline{\boldsymbol{S}}_i = \frac{1}{\Delta x}\int_{\Omega_i} \boldsymbol{s}(\boldsymbol{U}_i)\mathrm{d}x$$

$$= \frac{1}{\Delta x}\int_{\Omega_i} \boldsymbol{s}(\boldsymbol{U}_{eq,i}) + \underbrace{\boldsymbol{s}(\boldsymbol{U}_i) - \boldsymbol{s}(\boldsymbol{U}_{eq,i})}\,\mathrm{d}x$$

Equilibrium part

Perturbation part

$$\approx \frac{1}{\Delta x}\boldsymbol{f}(\boldsymbol{U}_{eq,i}(x))\Big|_{x_{i-1/2}}^{x_{i+1/2}} + \frac{1}{\Delta x}\mathcal{Q}_i\left(\boldsymbol{s}(\boldsymbol{U}_i(x)) - \boldsymbol{s}(\boldsymbol{U}_{eq,i}(x))\right)$$

Quadrature rule

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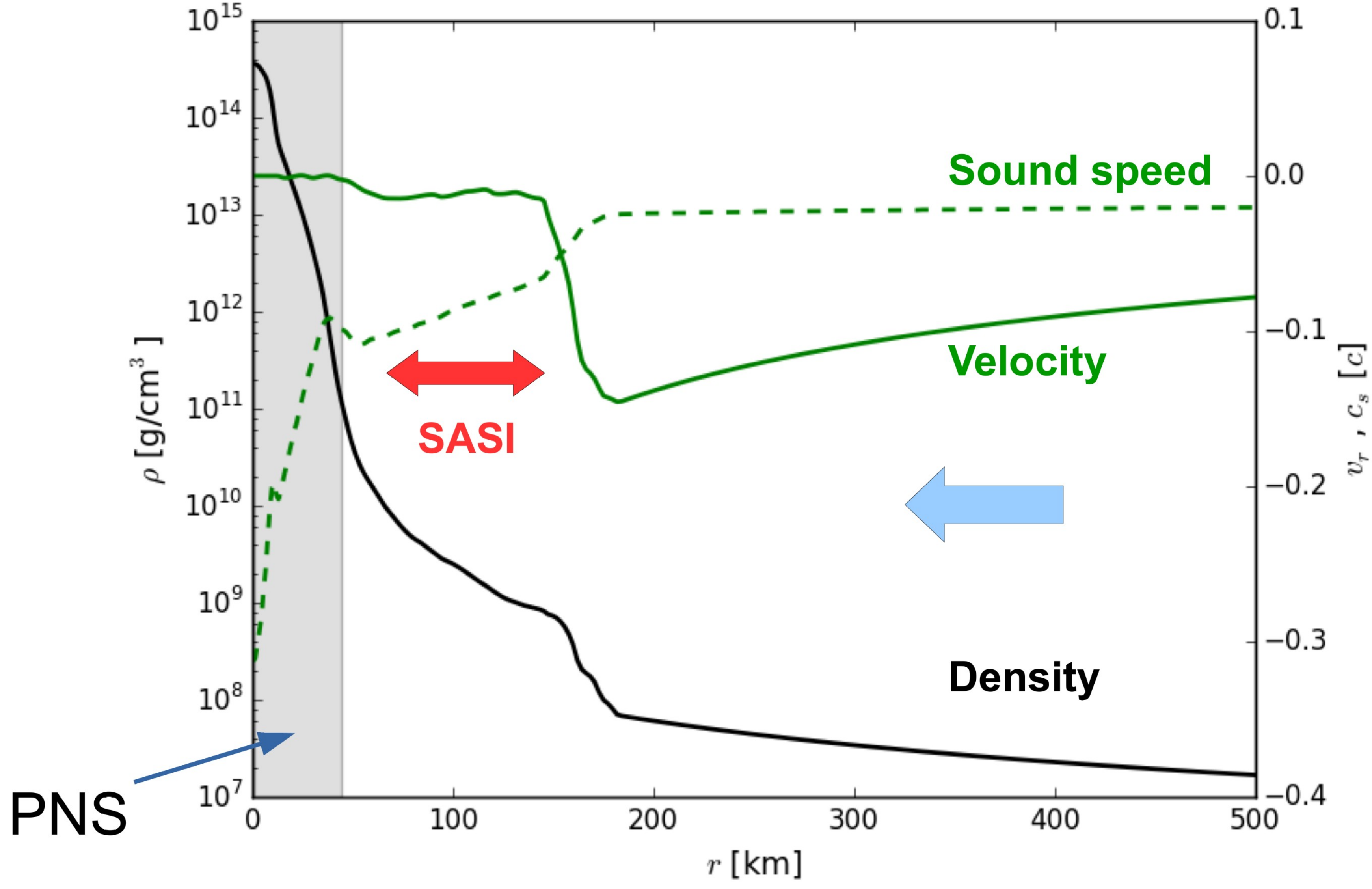
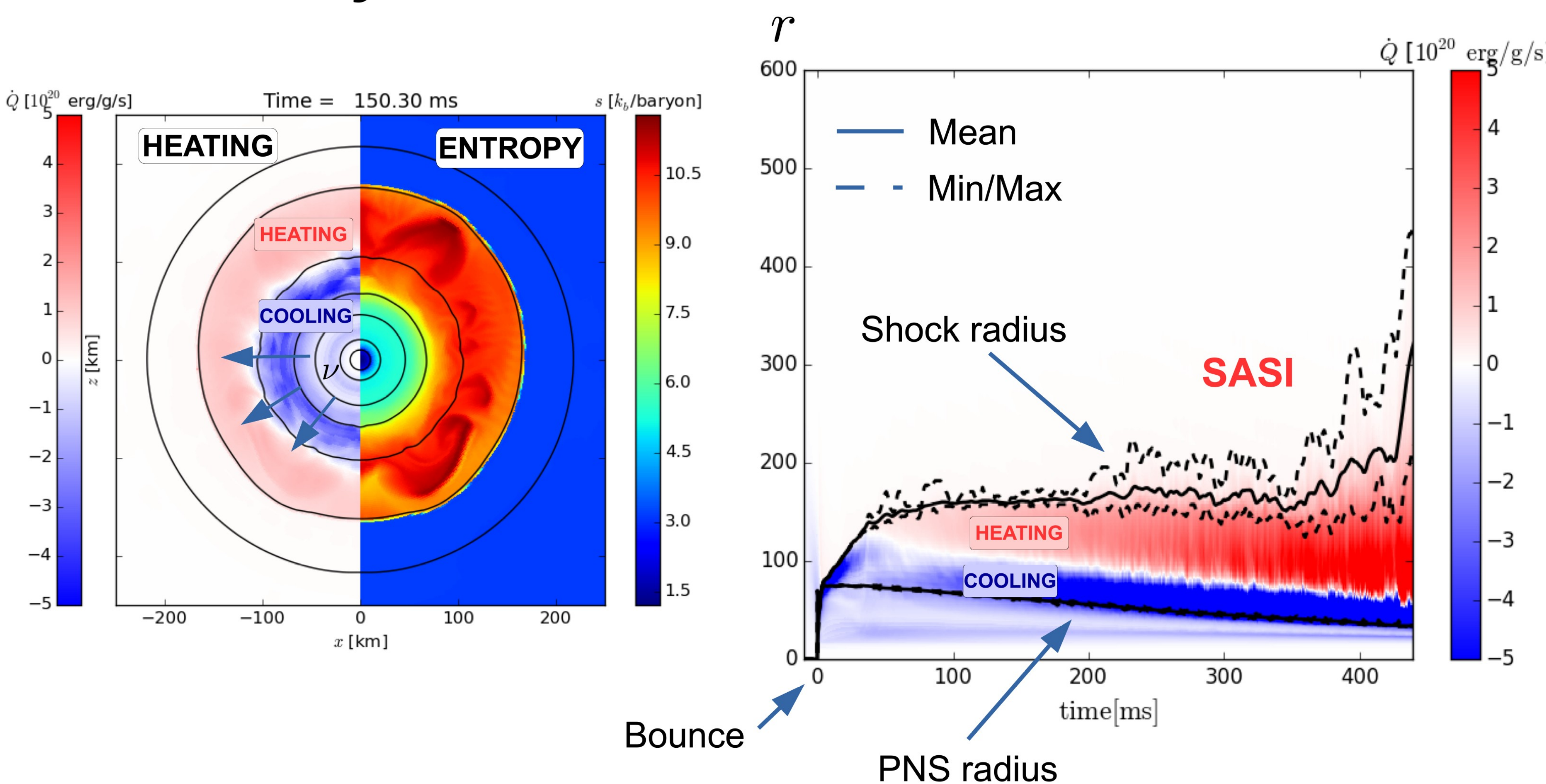
Core-collapse Supernova

- Steady accretion:



Core-collapse Supernova

- Steady accretion:



Standing Accretion Shock Instability (SASI)... See, e.g., Foglizzo et al. (2015) and refs therein

Steady flow

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STEADY →

Specific entropy

$$s = const$$

$$\rho v = m = const$$

$$\frac{v^2}{2} + h + \phi = const$$

Specific enthalpy

Along streamlines

https://commons.wikimedia.org/wiki/File:Leonhard_Euler.jpg

L. Euler

https://commons.wikimedia.org/wiki/File:Portr%C3%A4t_des_Daniel_Bernoulli.jpg

D. Bernoulli

Well-balanced scheme for steady flow

Perform piecewise steady reconstruction:

$$\left.\begin{aligned} \rho v = m = \bar{m}_i \\ \frac{v^2}{2} + h + \phi = \overline{Ber}_i \end{aligned}\right\} \frac{1}{2}\left(\frac{\bar{m}_i}{\rho_{0,i}(x)}\right)^2 + h(\rho_{0,i}(x), \bar{s}_i) + \phi(x) = \overline{Ber}_i$$

Actually: $p(\rho_{0,i}(x), \bar{s}_i)$

Solve (nonlinear!)

$$\rho_{0,i}(x) \;\;\&\;\; v_{0,i}(x) \;\;\&\;\; p_{0,i}(x)$$

$$\boldsymbol{w}_{i\pm1/2\mp} = \begin{bmatrix} \rho_{0,i}(x_{i\pm1/2}) \\ v_{0,i}(x_{i\pm1/2}) \\ p_{0,i}(x_{i\pm1/2}) \end{bmatrix}$$

Equilibrium reconstructed primitive variables

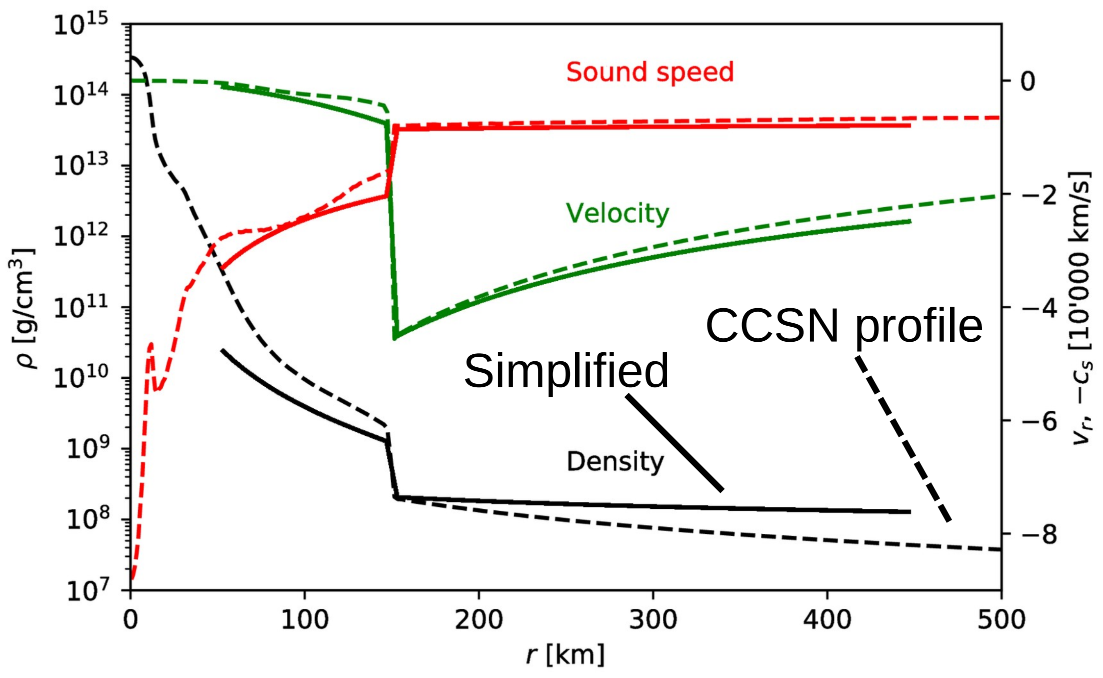
Analogous to shallow water & Euler-Poisson case:

See e.g. Gosse (2000), Jin (2001), Russo (2001),
Wen (2006), Noelle et al. (2007), Bouchut & Morales (2010),
Gosse (2013), Castro et al. (2013), ...

Simple SASI model

SASI in cylindrical coordinates, tab. multi-physics EoS (electrons, positrons, photons, nuclei, ...)

Similar to Yamasaki & Foglizzo (2008), Kazeroni et al. (2016)



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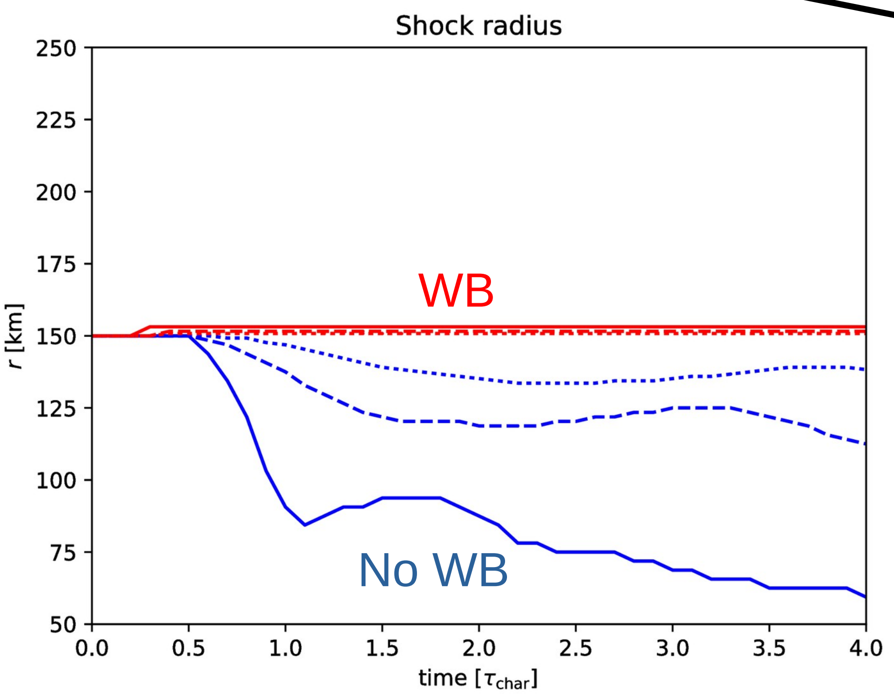
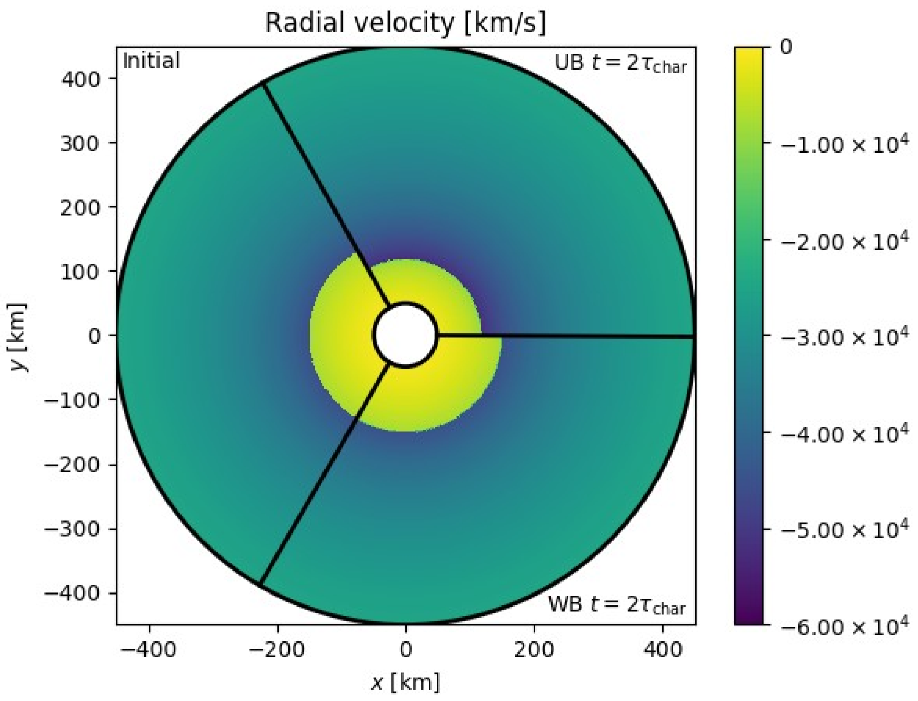
$$(N_r, N_\varphi) = (128, 256), (256, 512), (512, 1024)$$





Simple SASI model

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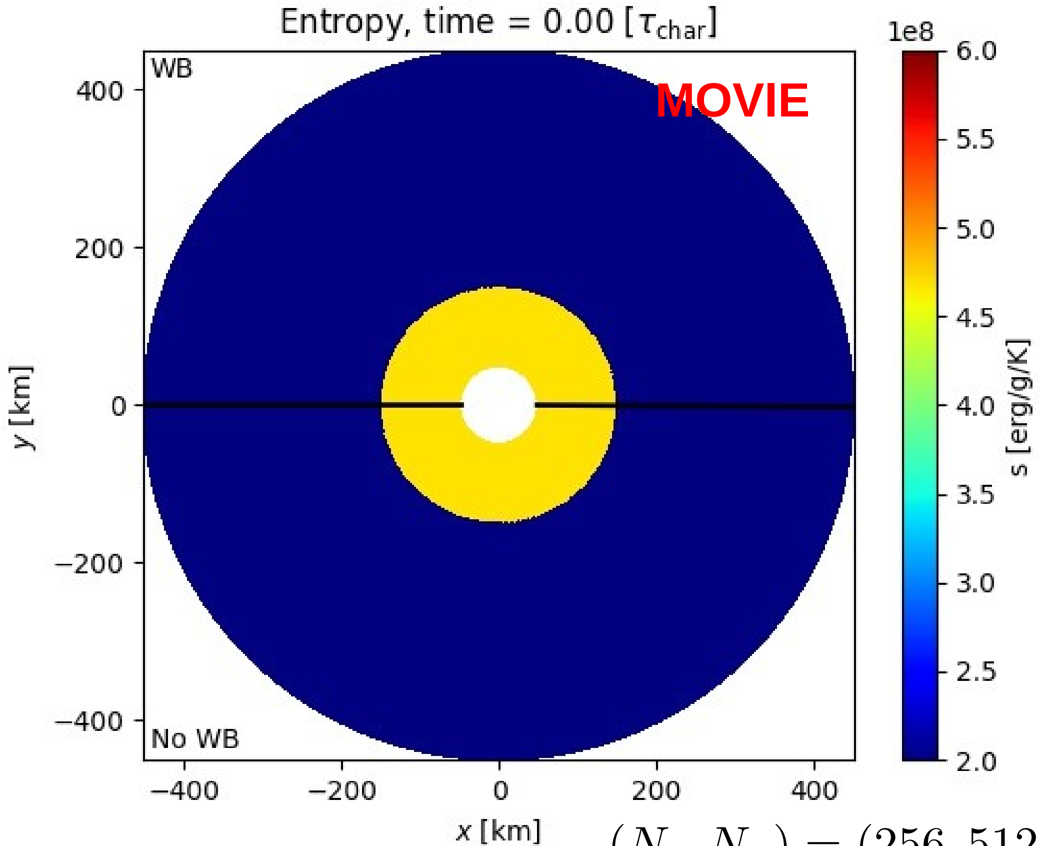
Add density perturbations!



$$(N_r, N_\varphi) = (256, 512)$$

See Foglizzo et al. (2012) for the shallow water analogue and experimental device!

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Conclusion

- High-order well-balanced schemes for barotropic hydrostatic equilibrium   Grosheintz-Laval & Käppeli (2019)
- High-order well-balanced schemes for arbitrary thermal stratifications   Berberich et al. (2021)
- Second-order well-balanced schemes for steady adiabatic flow   Grosheintz-Laval & Käppeli (2020)
- Ongoing
  - Geometric source terms in orthogonal curvilinear coordinates
  - Equilibria with magnetic fields (magnetars, tokamaks, Grad–Shafranov)
  - Rotating equilibria (rotating stars, accretion disk)
  - Multi-dimension...

Extension to mD challenging