5.4. Eigenwerte und Eigenvektoren

Beispiel:

$$\underline{A} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

① Charakteristisches Polynom (Regel von Sarrus (5.2.H))

$$\begin{array}{ccccccccc} + & & + & & + & - & & - & & - \end{array}$$

$$\chi_A(\lambda) = \det \begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix} \begin{matrix} -2-\lambda & 1 \\ 1 & -2-\lambda \\ 0 & 1 \end{matrix}$$

$$= -(2+\lambda)^3 + (2+\lambda) + (2+\lambda) = -(2+\lambda)(\lambda^2 + 4\lambda + 2)$$

② $EW(\underline{A}) = \{\lambda \in \mathbb{R} : \chi_A(\lambda) = 0\} = \{-2, -2+\sqrt{2}, -2-\sqrt{2}\}$

$$\{\lambda_1, \lambda_2, \lambda_3\}$$

③ Berechnung von $\operatorname{Kern}(\underline{A} - \lambda_k \underline{I})$:

$\rightarrow$ Gausselimination (Satz 3.4.A (iv))

- $$\underline{A} + 2\underline{I} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \overset{\text{z.U.}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \overset{\text{z.U.}}{\rightsquigarrow} \underset{}{\overset{[\text{ZSF}]}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}}$$

$\rightarrow$ Rang = 2

$$\Rightarrow \quad \operatorname{Kern}(\underline{A} + 2\underline{I}) = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- $$\underline{A} + (2-\sqrt{2})\,\underline{I} = \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1/\sqrt{2} & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \overset{[\text{ZSF}]}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}}$$

$\rightarrow$ Rang = 2

$$\Rightarrow \quad \mathrm{Kern}(\underline{A} - \lambda_2 \underline{I}) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\}$$

• Analog: $\quad \mathrm{Kern}(\underline{A} - \lambda_3 \underline{I}) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \right\}$

$$\triangleright \quad \underline{S} = \underbrace{\begin{pmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}}_{\text{orthogonale Spalten}}, \quad \underline{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}$$

Beispiel:

$$\underline{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

① Charakteristisches Polynom (Regel von Sarrus (5.2.H))

$$\chi_A(\lambda) = \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} \begin{matrix} 2-\lambda & 1 \\ 1 & 2-\lambda \\ 1 & 1 \end{matrix}$$

$$= (2-\lambda)^3 + 1 + 1 - 3(2-\lambda)$$

② "Rate" Nullstelle $\lambda = 1$ & Polynomdivision

$$= ((1-\lambda)+1)^3 + 2 - 3((1-\lambda)+1)$$

$$= (1-\lambda)^3 + 3(1-\lambda)^2 + \cancel{3(1-\lambda)} + \cancel{1} + \cancel{2} - \cancel{3(1-\lambda)} - \cancel{3}$$

$$= (1-\lambda)^2(4-\lambda) \qquad \triangleright \quad EW(\underline{A}) = \underset{\{\lambda_1,\,\lambda_2\}}{\{1, 4\}}$$

③ Berechnung von $\text{Kern}(\underline{A} - \lambda_k \underline{I})$:

- $\underline{A} - \underline{I} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{z.U.}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \text{Rang} = 1$

$$\text{Kern}(\underline{A} - \underline{I}) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

- $\underline{A} - 4\underline{I} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{z.U.}} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{pmatrix}$

$$\rightsquigarrow \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \text{Rang} = 2$$

$$\text{Kern}(\underline{A} - 4\underline{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\triangleright$ $$\underline{S} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \underline{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$