Discrete compactness for the \( p \)-version of discrete differential forms

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Abstract

In this paper we prove the discrete compactness property for a wide class of \( p \) finite element approximations of non-elliptic variational eigenvalue problems in two and three space dimensions. In a very general framework, we find sufficient conditions for the \( p \)-version of a generalized discrete compactness property, which is formulated in the setting of discrete differential forms of order \( \ell \) on a polyhedral domain in \( \mathbb{R}^d \) (\( 0 < \ell < d \)). One of the main tools for the analysis is a recently introduced smoothed Poincaré lifting operator [M. Costabel and A. McIntosh, On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, Math. Z., (2009)]. In the case \( \ell = 1 \) our analysis shows that several widely used families of edge finite elements satisfy the discrete compactness property in \( p \) and hence provide convergent solutions to the Maxwell eigenvalue problem. In particular, Nédélec elements on triangles and tetrahedrons (first and second kind) and on parallelograms and parallelepipeds (first kind) are covered by our theory.

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Notations

Function spaces:

• \( L^2(\Omega) (\text{Lt}wo), L^2(\Omega_0) (\text{Lt}wo\{\Omega_{\text{ega}}\{0\}\}), \|u\|_{L^2(\Omega)} (\text{NLt}wo\{u\}) \)

• \( H^1(\Omega) (\text{H}one), H^1(\Omega_0) (\text{H}one\{\Omega_{\text{ega}}\{0\}\}), \bar{H}^1(\Omega) (\text{zbH}one), \|u\|_{H^1(\Gamma)} (\text{NHone}\{\Gamma\}\{u\}), |u|_{H^1(\Gamma)} (\text{SNHone}\{\Gamma\}\{u\}) \)

• \( H^s(D) (\text{H}m[D]\{s\}), \|u\|_{H^s(D)} (\text{NH}m[D]\{s\}\{u\}), \bar{H}^s(D) (\text{zbH}m[D]\{s\}) \)

• \( H(\text{div}, D) (\text{Hdiv}[D]), \bar{H}(\text{div}, D) (\text{zbH}div[D]), H(\text{div}0, D) (\text{kH}div[D]), \|u\|_{H(\text{div}, D)} (\text{Hdiv}[D]\{\bfu\}) \)

• \( H(\text{curl}, D) (\text{Hcurl}[D]), \bar{H}(\text{curl}, D) (\text{zbH}curl[D]), H(\text{curl}0, D) (\text{kH}curl[D]), \|u\|_{H(\text{curl}, D)} (\text{Hcurl}[D]\{\bfu\}) \)

Spaces of differential forms:

• \( C^\infty(D, \Lambda^\ell) (\text{FC}inf[D]\{\ell\}) \)

• \( (u, v)_{0,D} (\text{SPN}[D]\{\bfu\}\{\bfv\}): \text{inner product of differential forms on } D \text{ w.r.t Euclidean metric on } \mathbb{R}^d. \)

• \( L^2(\Omega, \Lambda^\ell) (\text{FL}t\omega\{\ell\}): \text{space of square integrable } l\text{-forms, norm } \|u\|_{L^2(\Omega, \Lambda^\ell)} (\text{NFL}t\omega\{\ell\}\{\bfu\}) \)

• \( H^s(D, \Lambda^\ell) (\text{FH}m[D]\{\ell\}\{s\}), \bar{H}^s(D, \Lambda^\ell) (\text{zbFH}m[D]\{\ell\}\{s\}): \text{Sobolev space of } l\text{-forms with coefficients in } H^s(D), \text{norm } \|v\|_{H^s(D, \Lambda^\ell)} (\text{NHF}m[D]\{\ell\}\{s\}\{\bfv\}) \)

• \( H(d_\ell, D) := \{v \in L^2(D, \Lambda^\ell) : d v \in L^2(D, \Lambda^{\ell+1})\} (\text{FHd[D]\{\ell\}}), \|v\|_{H(d_\ell, D)} (\text{NHFd[D]\{\ell\}\{\bfv\}}) \)

• \( \bar{H}(d_\ell, D) := \{v \in H(d_\ell, D) : \text{tr}_{\partial D} v = 0\} (\text{zbFHd[D]\{\ell\}}) \)

• \( H(d_\ell, 0, D) := \{v \in H(d_\ell, D) : d v = 0\} (\text{kFHd[D]\{\ell\}}) \)

• \( Y(d_\ell, D) := \{v \in H(d_\ell, D) : (v, u)_{0,D} = 0 \forall u \in H(d_\ell, 0, D)\} (\text{FXd[D]\{\ell\}}), \bar{Y}(d_\ell, D) (\text{zbFXd[D]\{\ell\}}) \)

• \( X(D, \Lambda^\ell) (\text{HX[D]\{\ell\}}) \text{ with norm } \|\|_{X(D, \Lambda^\ell)} (\text{NHX[D]\{\ell\}\{\bfcdot\}}): \text{space of smoother forms} \)
Various mesh related quantities:

- \( S(D, \Lambda^\ell)(\operatorname{HX}[D]\{\ell}\}) \) with norm \( \| \cdot \|_{S(D, \Lambda^{\ell-1})} \): space of even smoother forms

Spaces of discrete differential forms:

- \( P_p(\Lambda^\ell) \) \( (\operatorname{FPol}(p)\{\ell}\}) \): polynomials differential forms
- \( \mathcal{V}_p(\Lambda^\ell) \) \( (\operatorname{FTPol}(p)\{\ell}\}) \): polynomials differential forms
- \( \mathcal{V}_p^0(\{\ell}\}) \) \( (\operatorname{Wpl}(p)\{\ell}\}) \): space of discrete differential forms
- \( \mathcal{V}_p^0(d, \{\ell}\}) \) \( : \{ u_p \in \mathcal{V}_p^0(\{\ell}\}) : d u_p = 0 \) \( (\operatorname{kWpl}(p)\{\ell}\}) \)
- \( \mathcal{V}_p^0(\{\ell}\}) \) \( (\operatorname{zbWpl}(p)\{\ell}\}) \)
- \( \mathcal{V}_p^0(\{\ell}\}) \) \( (\operatorname{xpl}(p)\{\ell}\}) \)
- \( \mathcal{V}_p^0(\{\ell}\}) \) \( (\operatorname{PO}(p)\{\ell}\}) \): local commuting projector onto \( \mathcal{V}_p^0(\{\ell}\}) \), restriction to cell \( K \) is \( \pi_{p,K}^\ell(\operatorname{PO}(K)\{\ell}\}) \)

1 Introduction: Maxwell eigenvalue problem

The electric field of a standing electromagnetic wave in a closed cavity \( \Omega \subset \mathbb{R}^3 \) with perfectly electrically conducting (PEC) walls solves the variational eigenvalue problem: seek \( u \in H(\nabla, \Omega) \setminus \{0\}, \omega > 0 \) such that

\[
\left( \mu^{-1} \nabla u, \nabla v \right)_{L^2(\Omega)} = \omega^2 \left( \epsilon u, v \right)_{L^2(\Omega)} \quad \forall v \in H(\nabla, \Omega),
\]

with uniformly positive material tensors \( \mu = \mu(x), \epsilon = \epsilon(x) \). PMC (perfectly magnetically conducting) walls can be modelled through replacing \( \dot{H}(\nabla, \Omega) \) with \( H(\nabla, \Omega) \).

One aim of this paper is to prove the convergence for a large class of conforming spectral Galerkin finite element discretizations approximating the solutions of Maxwell eigenvalue problem (1.1). The finite element approximation of Maxwell

\[\text{(1.1)}\]

\[\text{eq:maxevp}\]
eigenvalues has been the object of intense investigations during the last decade. It was soon recognized that the problem of interest requires suitable finite element spaces which are generally termed edge finite elements (see [12, 36, 37]).

The first attempts to analyze the discretized eigenvalue problem have been made for the $h$-version of edge finite elements. We mention [32] as a pioneering work on lowest order edge finite elements, where the discrete compactness property (see [2]) has been indicated as a key ingredient for the analysis. Other relevant works on the subject are [7, 8, 11, 15, 16, 33, 35], and we refer the interested reader to [30, 34] and to the references therein for a review on this topic. The analysis presented in the references above covers the $h$-version for basically all known families of edge finite elements.

It soon turned out that the analysis of the $p$- and $hp$-versions of edge finite elements needed tools different from those developed for the $h$-version. In [10] the two-dimensional triangular case has been studied for the $hp$-version, but the analysis depends on a conjectured estimate which has only been demonstrated numerically. In [9] a rigorous proof for the $hp$-version of 2D rectangular edge elements has been proposed (allowing for one-irregular hanging nodes) which, in particular, contains the first proof of eigenvalue/eigenfunction convergence for the pure spectral method ($p$-version with one element) on a rectangle.

What paved the way for a successful attack on a general $p$-version analysis was the regularized Poincaré lifting recently introduced in [17]: it enjoys excellent continuity properties and at the same time respects discrete differential forms. We will describe this operator in more detail in Section 4.

In this paper we are going to show how the regularized Poincaré lifting can be combined with another recent invention, the projection based interpolation operators, see [18, 20], to clinch the analysis of the $p$-version of edge elements. This allows to prove the discrete compactness (and hence the convergence of the discrete eigensolutions) for a wide class of finite elements related to discrete differential forms: for (1.1) this includes, in particular, Nédélec elements on triangles and tetrahedrons (first and second kind) and on parallelograms and parallelepipeds (first kind).

As already mentioned, one of the key ingredients for the convergence analysis is the discrete compactness property. Much insight can be gained from investigating it in the more general framework of discrete differential forms (see [4] for a lucid introduction to this subject). In this setting, the proofs are more natural and simultaneously cover, in particular, two- and three-dimensional eigenvalue problems.

The structure of the paper is as follows. We start with a generalization of (1.1) to eigenvalue problems set in Sobolev spaces of differential forms. Then we define the discrete compactness property and discuss its significance in the context of Galerkin discretization. It gives a crucial sufficient condition for asymptotic
convergence of eigenvalues and eigenvectors. Section 3 is the core of our paper and contains the description of our abstract assumptions. Having in mind the \( p \)-version of finite elements, we consider a fixed mesh \( M \) of a Lipschitz domain \( \Omega \subset \mathbb{R}^d \) and a sequence of spaces of discrete differential forms of order \( \ell \) (with \( 0 < \ell < d \)); we investigate the assumptions which imply the validity of the discrete compactness property for such a sequence of spaces. The abstract theory relies on the existence of a suitable Poincaré lifting operator which is presented in Section 4. Finally, Section 6 introduces concrete spaces of discrete differential forms, to which the abstract theory is applied for the cases \( d = 2 \) and \( d = 3 \), leading to the main result stated in Theorem 3.2. The analysis of a spectral edge element discretization of the Maxwell eigenvalue problem (1.1) is covered as case \( d = 3 \) and \( \ell = 1 \).

**Remark 1.1** In this paper, for ease of presentation, we will consider the material properties \( \epsilon \) and \( \mu \) constant and equal to the identity matrix. Indeed, the results obtained with \( \epsilon = \mu = 1 \) can be extended to more general situations with standard tools (see, in particular, Propositions 2.25, 2.26, and 2.27 of [15], and [31, Sect. 6], [30, Thm. 4.9]).

## 2 Differential forms and generalized Maxwell eigenvalue problem

The variational eigenvalue problem (1.1) turns out to be a member of a larger family of eigenvalue problems, when viewed from the perspective of differential forms. This more general perspective offers the benefit of a unified theoretical treatment of different kinds of eigenvalue problems, e.g., the scalar Laplace eigenproblem, Maxwell cavity eigenproblems in dimensions 2 and 3, the eigenproblem for the \textbf{grad div}-operator in dimension 3. This policy has had remarkable success in numerical analysis recently, cf. [3]. Thus, in this section we first recall some basic notion of differential forms. We refer the interested reader to [4, Sect. 2] for an introduction to this subject.

### 2.1 Function spaces of differential forms

Given a domain \( \Omega \subset \mathbb{R}^d \), we denote by \( C^\infty(\Omega, \Lambda^\ell) \), \( 0 \leq \ell \leq d \), the space of smooth differential forms on \( \Omega \) and by \( \Lambda(\Omega) \) the corresponding anti-commutative graded algebra

\[
\Lambda(\Omega) = \bigoplus_\ell C^\infty(\Omega, \Lambda^\ell).
\]
The **exterior derivative** is a graded linear operator of degree one

\[ d : \Lambda(\Omega) \to \Lambda(\Omega), \]

that is, for any \( \ell \) it is represented by \( d_\ell : C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell+1}) \).

We rely on the Hilbert spaces

\[ H(d_\ell, \Omega) := \{ v \in L^2(\Omega, \Lambda^\ell) : d_\ell v \in L^2(\Omega, \Lambda^{\ell+1}) \}, \quad (2.1) \]

where \( L^2(\Omega, \Lambda^\ell) \) is the space of differential \( \ell \)-forms on \( \Omega \) with square integrable coefficients in their canonical basis representation, see \([17, \text{Sect. 2}]\). Its inner product can be expressed as

\[ (u, v)_{0,\Omega} := \int_\Omega u \wedge \ast v, \quad u, v \in L^2(\Omega, \Lambda^\ell), \quad (2.2) \]

with \( \ast \) the Hodge star operator induced by the Euclidean metric on \( \mathbb{R}^d \), which maps \( \ell \)-forms to \((d - \ell)\)-forms. As above, \( \partial_\Omega \odot \) tags the subspaces of forms with vanishing trace \( \text{tr}_{\partial \Omega} \) on \( \partial \Omega \), which can also be obtained by the completion of compactly supported smooth \( \ell \)-forms with respect to the \( H(d_\ell, \Omega) \)-norm:

\[ \hat{H}(d_\ell, \Omega) := \{ v \in H(d_\ell, \Omega) : \text{tr}_{\partial \Omega} v = 0 \}. \quad (2.3) \]

The subspace of closed forms is the kernel of \( d_\ell \) and is denoted by \( \hat{H}(d_\ell 0, \Omega) \):

\[ \hat{H}(d_\ell 0, \Omega) := \{ v \in \hat{H}(d_\ell, \Omega) : d_\ell v = 0 \}. \quad (2.4) \]

Finally, we adopt the standard notation for the exterior co-derivative operator \( \delta_\ell := \ast d_{d-\ell} \ast \); in particular, we have \( \delta_\ell : C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell-1}) \).

### 2.2 Variational eigenvalue problems

After choosing bases for the spaces of alternating multilinear forms on \( \mathbb{R}^d \), vector fields ("vector proxies") \( \Omega \mapsto \mathbb{R}^{(\ell)} \) provide an isomorphic model for differential \( \ell \)-forms on \( \Omega \). Choosing the standard "Euclidean basis" the operators \( \ast, \delta, \text{tr}_{\partial \Omega} \) are incarnated by familiar operators of classical vector analysis, different for different dimension \( d \) and degree \( \ell \), see Table 1 and \([4, \text{Table 2.1}]\).

Hence, the eigenvalue problem (1.1) with \( \epsilon, \mu \equiv 1 \) is the special case \( d = 3, \ell = 1 \), of the following variational eigenvalue problem for differential \( \ell \)-forms,

\[ 0 \leq \ell < d; \text{ Seek } u \in \hat{H}(d_\ell, \Omega) \setminus \{0\}, \omega \in \mathbb{R}^+ \text{, such that } (d_\ell u, d_\ell v)_{0,\Omega} = \omega^2 (u, v)_{0,\Omega} \forall v \in \hat{H}(d_\ell, \Omega). \quad (2.5) \]

\[ \text{eq:evp} \]
Table 1: Identification between (operators on) differential forms and (operators on) Euclidean vector proxies in $\mathbb{R}^2$ and $\mathbb{R}^3$

<table>
<thead>
<tr>
<th>Differential form</th>
<th>Proxy representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell = 0$</td>
<td>$d_0$</td>
</tr>
<tr>
<td></td>
<td>$\text{tr}_{\partial \Omega} \phi$</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(d_0, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>$\text{grad}$</td>
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<tr>
<td></td>
<td>$\phi</td>
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<tr>
<td></td>
<td>$H^1(\Omega)$</td>
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<tr>
<td></td>
<td>$\text{grad}$</td>
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<tr>
<td></td>
<td>$\phi</td>
</tr>
<tr>
<td></td>
<td>$H^1(\Omega)$</td>
</tr>
<tr>
<td>$\ell = 1$</td>
<td>$d_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{tr}_{\partial \Omega} u$</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(d_1, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>$\delta_1$</td>
</tr>
<tr>
<td></td>
<td>$\text{curl}$</td>
</tr>
<tr>
<td></td>
<td>$(u \times n)</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(\text{curl}, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>$\text{div}$</td>
</tr>
<tr>
<td></td>
<td>$(q \cdot n)</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(\text{div}, \Omega)$</td>
</tr>
<tr>
<td>$\ell = 2$</td>
<td>$d_2$</td>
</tr>
<tr>
<td></td>
<td>$\text{tr}_{\partial \Omega} q$</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(d_2, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>$\delta_2$</td>
</tr>
<tr>
<td></td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$L^2(\Omega)$</td>
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<tr>
<td></td>
<td>$\rightarrow$ curl</td>
</tr>
<tr>
<td></td>
<td>$\text{div}$</td>
</tr>
<tr>
<td></td>
<td>$(q \cdot n)</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}(\text{div}, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>$\text{curl}$</td>
</tr>
</tbody>
</table>

A key observation is that the bilinear form $(u, v) \mapsto (d_\ell u, d_\ell v)|_{0, \Omega}$ has an infinite dimensional kernel $\hat{H}(d_\ell 0, \Omega)$ comprising all closed $\ell$-forms. It provides the invariant subspace associated with the essential spectrum $\{0\}$ of (2.5). This essential spectrum can be identified as the main source of difficulties confronted in the Galerkin discretization of (2.5).

On the other hand, any solution $u$ of (2.5) for $\omega \neq 0$ satisfies $(u, d_{\ell-1} \psi)|_{0, \Omega} = 0$ for all $\psi \in \hat{H}(d_{\ell-1} 0, \Omega)$. Thus the eigenfunctions corresponding to non-zero eigenvalues belong to the subspace

$$\hat{Y}(d_\ell, \Omega) := \{ v \in \hat{H}(d_\ell, \Omega) : (v, d_{\ell-1} \psi)|_{0, \Omega} = 0 \ \forall \psi \in \hat{H}(d_{\ell-1}, \Omega) \},$$

which means they belong to the kernel of $\delta_\ell$. This is the generalization of the divergence free constraint found for electric fields in the Maxwell case. From [38] we learn the following theorem.

**Theorem 2.1** For any $d \in \mathbb{N}$, $0 \leq l \leq d$, the embedding

$$\hat{Y}(d_\ell, \Omega) \hookrightarrow L^2(\Omega, \Lambda^l)$$

is compact.
By restricting the eigenvalue problem to $\mathcal{Y}(d_\ell, \Omega)$, we can therefore use Riesz-Schauder theory. This implies that (2.5) gives rise to an unbounded sequence of positive eigenvalues $\lambda^k = (\omega^k)^2$

$$\lambda^0 = 0 < \lambda^1 \leq \lambda^2 \leq \ldots, \quad \lambda^k \to \infty \ (k \to \infty), \quad (2.7)$$

with associated finite dimensional mutually $L^2(\Omega)$-orthogonal eigenspaces.

**Remark 2.2** Owing to the zero trace boundary conditions imposed on the functions in (2.5), it may be called a Dirichlet eigenvalue problem. Using $H(d_\ell, \Omega)$ as variational space would result in the corresponding Neumann eigenvalue problem. Its analysis runs utterly parallel to the Dirichlet case using the techniques presented below. □

### 2.3 Approximation of the eigenvalue problem and the role of discrete compactness

In the sequel we fix the degree $\ell$, $0 \leq \ell < d$, of the differential forms. Spaces of *discrete differential forms*

$$\mathcal{Y}_\ell^p \subset \mathcal{H}(d_\ell, \Omega), \quad \dim \mathcal{Y}_\ell^p < \infty,$$

lend themselves to a straightforward discretization of (2.5). In this section, $p \in \mathbb{N}$ stands for an abstract discretization parameter, and, sloppily speaking, large values of $p$ hint at trial/test spaces of high resolution.

We consider the approximation of the eigenvalue problem (2.5) by the Galerkin method: Find $u_p \in \hat{\mathcal{Y}}_\ell^p \setminus \{0\}$, $\omega \in \mathbb{R}_0^+$, such that

$$(d_\ell u_p, d_\ell v_p)_{0, \Omega} = \omega^2 (u_p, v_p)_{0, \Omega}, \quad \forall v \in \hat{\mathcal{Y}}_\ell^p. \quad (2.8)$$

Now, the key issue is convergence of eigenvalues and eigenvectors as $p \to \infty$, rigorously cast into the concept of *spectrally correct, spurious-free approximation* [15, Sect. 4]. Let us recall these notions in a few words for the case of self-adjoint nonnegative operators without continuous spectrum (which is the case here).

The spectral correctness of the approximation of an eigenvalue problem such as (2.5) by a sequence of finite rank eigenvalue problems (2.8) means that all eigenvalues and all eigenvectors of (2.5) are approached by the eigenvalues and eigenvectors of (2.8) as $p \to \infty$. If (2.5) has a compact resolvent (which is the case only when $\ell = 0$), the spectral correctness is an optimal notion: It implies that if $\{\lambda^k\}_{k \geq 1}$ and $\{\lambda^k_p\}_{k \geq 1}$ are the increasing eigenvalue sequences of (2.5) and (2.8) (with eigenvalues repeated according to their multiplicities), then

$$\lambda^k_p \to \lambda^k \quad \text{as} \quad p \to \infty \quad \forall k \geq 1, \quad (2.9)$$

**eq:conveig**
and the gaps between eigenspaces (correctly assembled according to multiplicities of the eigenvalues of (2.5)) tend to 0 as \( p \to \infty \).

If we face an eigenvalue problem for a self-adjoint non-negative operator with an infinite dimensional kernel, and otherwise discrete positive spectrum (which is the case for (2.5) for all \( \ell \geq 1 \)), the spectral correctness implies the same properties as above with the following modifications of the definitions: Now \( \{ \lambda^k \}_{k \geq 1} \) is the increasing sequence of positive eigenvalues of (2.5) (as specified in (2.7)) and, given a positive number \( \varepsilon < \lambda^1 \), \( \{ \lambda^k_p \}_{k \geq 1} \) is the increasing sequence of the eigenvalues of (2.8) larger than \( \varepsilon \) (still with repetitions according to multiplicities). With such conventions, the spectral correctness still implies convergence of eigenvalues (2.9) and eigenspaces as above. In this context, the spurious-free approximation means that there exists \( \varepsilon_0 > 0 \) such that all eigenvalues of (2.8) less than \( \varepsilon_0 \) are zero. Therefore, the spectrally correct, spurious-free approximation means that we have the convergence (2.9) and the corresponding convergence of eigenspaces if we define \( \{ \lambda^k_p \}_{k \geq 1} \) as the increasing sequence of the positive eigenvalues of (2.8).

There exist several different ways, all well studied and summarized in the literature of the last decade, for proving the convergence of the discrete eigenvalue problem (2.8) to the continuous eigenvalue problem (2.5): One can use a reformulation as an eigenvalue problem in mixed form as analyzed in [8], or one can use a regularization which gives an elliptic eigenvalue problem for the Hodge-Laplace operator as analyzed in [4], or one can follow the arguments of [15] and study the non-elliptic problem (2.5) directly.

Here we outline the latter approach, which employs the analysis of [26] of the approximation of eigenvalue problems of non-compact selfadjoint operators. Since [15] deals only with the Maxwell case, i.e. \( d = 3, \ell = 1 \), we examine the main arguments, in order to verify that they are also valid for the general case.

Let us define the solution operator \( A : L^2(\Omega, \Lambda^\ell) \to \mathring{H}(d\ell, \Omega) \) of the source problem corresponding to the eigenvalue problem (2.5) and its discrete counterpart \( A_p : \mathring{V}^{d\ell}_p \to \mathring{V}^{d\ell}_p \) by

\[
(d\ell \, Af, d\ell \, v)_{0,\Omega} + (Af, v)_{0,\Omega} = (f, v)_{0,\Omega} \quad \forall v \in \mathring{H}(d\ell, \Omega) \\
(d\ell \, A_pf, d\ell \, v)_{0,\Omega} + (A_pf, v)_{0,\Omega} = (f, v)_{0,\Omega} \quad \forall v \in \mathring{V}^{d\ell}_p
\]  

(2.10)  

Note that the operators \( A \) and \( A_p \) have the same eigenfunctions and the same eigenvalues (after a transformation) as the eigenvalue problems (2.5) and (2.8). Namely, (2.5) and (2.8) are equivalent to the relations

\[
u = (\omega^2 + 1)Au; \quad u_p = (\omega^2 + 1)A_pu_p.
\]  

(2.11)  

The infinite-dimensional eigenspace at \( \omega = 0 \) shows that \( A \) is not a compact operator.
Following [15], three conditions are identified that together are necessary and sufficient for a spectrally correct, spurious-free approximation of $A$ by $A_p$ or, equivalently, of the eigenvalue problem (2.5) by the discrete eigenvalue problem (2.8).

The first condition is rather natural. It states that the sequence of discrete spaces $(\hat{V}_p^\ell)_{p \in \mathbb{N}}$ is asymptotically dense in $\hat{H}(d, \Omega)$ (compare [15, Condition (CAS) – completeness of approximating subspaces]):

$$\lim_{p \to \infty} \inf_{v_p \in \hat{V}_p^\ell} \| v - v_p \|_{H(d, \Omega)} = 0 \quad \forall v \in \hat{H}(d, \Omega). \quad (2.12) \tag{eq:CAS}$$

The second condition, only relevant for $\ell > 0$, states that closed forms can be well approximated by discrete closed forms (compare [15, Condition (CDK) – completeness of discrete kernels]):

$$\lim_{p \to \infty} \inf_{z_p \in \hat{V}_p^\ell \cap \hat{H}(d, 0, \Omega)} \| z - z_p \|_{L^2(\Omega)} = 0 \quad \forall z \in \hat{H}(d, 0, \Omega). \quad (2.13) \tag{eq:CDK}$$

The third condition is the most intricate one and has been dubbed *discrete compactness*. For its formulation, we introduce the orthogonal complement of the space of discrete exact forms:

$$\hat{Y}_p^\ell := \{ u_p \in \hat{V}_p^\ell : (u_p, d_{\ell-1} \psi_p)_{0, \Omega} = 0 \quad \forall \psi_p \in \hat{V}_{p-1}^\ell \}. \quad (2.14) \tag{eq:Zpldef}$$

**Definition 2.3** Let us choose $\ell \in \{1, \ldots, d-1\}$. The *discrete compactness property* holds for a family $(\hat{V}_p^\ell)_{p \in \mathbb{N}}$ of finite dimensional subspaces of $\hat{H}(d, \Omega)$, if any sequence

$$ (u_p)_{p \in \mathbb{N}} \subset \hat{H}(d, \Omega) \quad \text{with} \quad u_p \in \hat{Y}_p^\ell \quad \text{and} \quad \| u_p \|_{H(d, \Omega)} \leq 1$$

contains a subsequence that converges in $L^2(\Omega, \Lambda^\ell)$.

The convergence proof is based on two lemmas, the first of which corresponds to [15, Theorem 4.12]. It implies, according to [26, Condition P1) and Theorems 2,4,5,6], the spectral correctness of the approximation.

**Lemma 2.4** If (2.12) and the discrete compactness property hold, then

$$\lim_{p \to \infty} \sup_{v_p \in \hat{V}_p^\ell : \| v_p \|_{H(d, \Omega)} = 1} \| A v_p - A_p v_p \|_{H(d, \Omega)} = 0. \quad (2.15) \tag{eq:CHN}$$

10
Proof. Note first that for \( \mathbf{v}_p \in \hat{\mathcal{V}}_p^\ell \cap \hat{H}(d, 0, \Omega) \) there holds \( A\mathbf{v}_p = \mathbf{v}_p = A_p \mathbf{v}_p \), so that by orthogonal decomposition of \( \hat{\mathcal{V}}_p^\ell \) one gets

\[
\sup_{\mathbf{v}_p \in \hat{\mathcal{V}}_p^\ell : \|\mathbf{v}_p\|_{H(d, \Omega)} = 1} \|A\mathbf{v}_p - A_p \mathbf{v}_p\|_{H(d, \Omega)} = \sup_{\mathbf{v}_p \in \hat{\mathcal{V}}_p^\ell : \|\mathbf{v}_p\|_{H(d, \Omega)} = 1} \|A\mathbf{v}_p - A_p \mathbf{v}_p\|_{H(d, \Omega)}.
\]

Furthermore, one has by definition of \( A \) and \( A_p \)

\[
\|A\mathbf{v}_p - A_p \mathbf{v}_p\|_{H(d, \Omega)} = \inf_{\mathbf{w}_p \in \hat{\mathcal{V}}_p^\ell} \|A\mathbf{v}_p - \mathbf{w}_p\|_{H(d, \Omega)}.
\]

Assume now that (2.15) does not hold. Then there exists an \( \varepsilon > 0 \) and a sequence \((\mathbf{v}_p)\) with \( \mathbf{v}_p \in \hat{\mathcal{V}}_p^\ell \) satisfying \( \|\mathbf{v}_p\|_{H(d, \Omega)} = 1 \) and

\[
\|A\mathbf{v}_p - \mathbf{w}_p\|_{H(d, \Omega)} \geq \varepsilon \quad \forall p \in \mathbb{N}, \mathbf{w}_p \in \hat{\mathcal{V}}_p^\ell .
\] (2.16) eq:geeps

We can apply the discrete compactness property to the sequence \((\mathbf{v}_p)\) and obtain a subsequence converging in \( L^2(\Omega, \Lambda^\ell) \) to some \( \mathbf{v} \in L^2(\Omega, \Lambda^\ell) \). Since \( A : L^2(\Omega, \Lambda^\ell) \to \hat{H}(d, \Omega) \) is continuous, we find \( A\mathbf{v} \in \hat{H}(d, \Omega) \), and the approximation property (2.12) provides us with a sequence \((\mathbf{w}_p)\) with \( \mathbf{w}_p \in \hat{\mathcal{V}}_p^\ell \) that converges in \( \hat{H}(d, \Omega) \) to \( A\mathbf{v} \). Hence for the subsequence we obtain

\[
\|A\mathbf{v}_p - \mathbf{w}_p\|_{H(d, \Omega)} \leq \|A\mathbf{v}_p - A\mathbf{v}\|_{H(d, \Omega)} + \|A\mathbf{v} - \mathbf{w}_p\|_{H(d, \Omega)} \to 0 ,
\]

in contradiction with (2.16). \( \square \)

The second lemma corresponds to [15, Corollary 2.20]. It gives the discrete Friedrichs inequality (in [8] also called “ellipticity in the discrete kernel”), and it is easy to see that this implies that \( \omega = 0 \) is not a limit point of positive discrete eigenvalues, so that the spurious-free property of the approximation follows.

**Lemma 2.5** If (2.13) and the discrete compactness property hold, then there exists \( \alpha > 0 \) such that for all \( p \in \mathbb{N} \)

\[
\|d_{\ell} \mathbf{v}\|_{L^2(\Omega)} \leq \alpha \|\mathbf{v}\|_{L^2(\Omega)} \quad \forall \mathbf{v} \in \hat{\mathcal{V}}_p^\ell
\] (2.17) eq:DFI

Proof. Assume that (2.17) does not hold. Then there exists a sequence \((\mathbf{v}_p)\) with \( \mathbf{v}_p \in \hat{\mathcal{V}}_p^\ell \) satisfying

\[
\|\mathbf{v}_p\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \lim_{p \to \infty} \|d_{\ell} \mathbf{v}_p\|_{L^2(\Omega)} = 0 .
\] (2.18) eq:notDFI

The discrete compactness property can be applied to this sequence and gives a subsequence converging in \( L^2(\Omega, \Lambda^\ell) \) to some \( \mathbf{z} \in L^2(\Omega, \Lambda^\ell) \). From (2.18) follows
that the convergence actually takes place in \( \hat{H}(d_\ell, \Omega) \) and that \( z \in \hat{H}(d_\ell, \Omega) \). Therefore the approximation property (2.13) provides us with a sequence \( (z_p) \) with \( z_p \in \hat{Y}_p \cap \hat{H}(d_\ell, \Omega) \) that converges in \( L^2(\Omega, \Lambda^\ell) \) to \( z \). Hence for the subsequence we find

\[
\|v_p - z_p\|_{L^2(\Omega)} \leq \|v_p - z\|_{L^2(\Omega)} + \|z - z_p\|_{L^2(\Omega)} \to 0.
\]

This leads to a contradiction, because \( v_p \in \hat{Y}_p \) and \( z_p \in \hat{Y}_p \cap \hat{H}(d_\ell, \Omega) \) are \( L^2(\Omega) \)-orthogonal, hence for all \( p \)

\[
\|v_p - z_p\|_{L^2(\Omega)}^2 = \|v_p\|_{L^2(\Omega)}^2 + \|z_p\|_{L^2(\Omega)}^2 \geq 1.
\]

\[ \square \]

To summarize, Lemmas 2.4 and 2.5 together prove the following result.

\[ \text{thm:spurfree} \]

**Theorem 2.6** If the completeness of approximating subspaces (2.12), the completeness of discrete kernels (2.13) and the discrete compactness property hold, then (2.8) provides a spectrally correct, spurious-free approximation of the eigenvalue problem (2.5).

### 3 An abstract framework implying discrete compactness

In this section we fix a degree of differential forms

\[
\ell \in \{1, \ldots, d - 1\},
\]

and we formulate a set of hypotheses which allow us to prove the discrete compactness property. These hypotheses are organized in three groups:

1. standard assumptions relating to the finite element spaces \( \hat{Y}_p \) (Sect. 3.1),

2. assumptions on the existence and key properties of so-called lifting operators (Sect. 3.3),

3. hypotheses on projections onto \( \hat{Y}_p \) complying with the commuting diagram property and satisfying an approximation property (Sect. 3.4).

To state these assumptions we have to introduce intermediate spaces \( X \) and \( S \) of more regular forms

\[
\hat{Y}_p \subset X(\mathcal{M}, \Lambda^\ell) \subset \hat{H}(d_\ell, \Omega) \quad \text{and} \quad \hat{Y}_p^{-1} \subset S(\mathcal{M}, \Lambda^\ell) \subset \hat{H}(d_{\ell-1}, \Omega),
\]

allowing compact embedding arguments and precise notions of continuity of lifting and projection operators.
3.1 Discrete spaces

Our focus is on finite element spaces. For the sake of simplicity, we restrict ourselves to polyhedral Lipschitz domains $\Omega$. We assume that the finite dimensional trial and test spaces $\mathcal{V}_p^\ell$, $p \in \mathbb{N}$, are based on a fixed finite partition $\mathcal{M}$ of $\Omega$, composed of elements (cells) $K$:

$$\Omega = \bigcup_{K \in \mathcal{M}} K, \quad K \cap K' = \emptyset, \text{ if } K \neq K', \ K, K' \in \mathcal{M}.$$ 

For a cell $K \in \mathcal{M}$, let $\mathcal{F}_m(K)$ designate the set of $m$-dimensional facets of $K$: for $m = 0$ these are the vertices, for $m = 1$ the edges, for $m = d - 1$ the faces, and $\mathcal{F}_d(K) = \{K\}$.

We take for granted that the discrete spaces $\mathcal{V}_p^\ell$ can be assembled from local contributions in the sense that for each mesh cell $K \in \mathcal{M}$ there is a space $\mathcal{V}_p^\ell(K) \subset C^\infty(K, \Lambda^\ell)$ of smooth $\ell$-forms on $K$, such that

$$\mathcal{V}_p^\ell = \mathcal{V}_p^\ell(\mathcal{M}) := \{ v \in \mathcal{H}(d_\ell, \Omega) : v|_K \in \mathcal{V}_p^\ell(K) \forall K \in \mathcal{M} \}. \quad (3.1)$$

In other words, $\mathcal{V}_p^\ell$ can be defined by specifying the local spaces $\mathcal{V}_p^\ell(K)$ and requiring the continuity of traces across inter-element boundaries as well as the boundary conditions on $\partial \Omega$.

In the same fashion, we introduce a corresponding family $\mathcal{V}_p^{\ell-1} \subset \mathcal{H}(d_{\ell-1}, \Omega)$ of spaces of discrete $\ell - 1$-forms. We will see later on that as a consequence of further hypotheses, the local spaces $\mathcal{V}_p^{\ell-1}(K)$ and $\mathcal{V}_p^\ell(K)$ satisfy an exact sequence condition.

3.2 Spaces of more regular forms

We introduce a Hilbert space $X(\mathcal{M}, \Lambda^\ell) \subset \mathcal{H}(d_\ell, \Omega)$ that captures the extra regularity that distinguishes $\ell$-forms in the space $\mathcal{Y}(d_\ell, \Omega)$. We can think of this space as a space of “more regular” $\ell$-forms on $\Omega$.

**Assumption 1** The space $\mathcal{Y}(d_\ell, \Omega)$ (2.6) is continuously embedded in $X(\mathcal{M}, \Lambda^\ell)$:

$$\mathcal{Y}(d_\ell, \Omega) \hookrightarrow X(\mathcal{M}, \Lambda^\ell).$$

This means that with $C > 0$ depending only on $\Omega$

$$\|u\|_{X(\mathcal{M}, \Lambda^\ell)} \leq C \|u\|_{\mathcal{H}(d_\ell, \Omega)} \quad \forall u \in \mathcal{Y}(d_\ell, \Omega). \quad (3.2)$$

On the other hand, $X(\mathcal{M}, \Lambda^\ell)$ has to be small enough to maintain the compact embedding satisfied by $\mathcal{Y}(d_\ell, \Omega)$, cf. Thm. 2.1.
**Assumption 2** The space $X(\mathcal{M}, \Lambda^\ell)$ is compactly embedded in $L^2(\Omega, \Lambda^\ell)$:

$$X(\mathcal{M}, \Lambda^\ell)^{\text{comp}} \hookrightarrow L^2(\Omega, \Lambda^\ell).$$

As with the discrete spaces, the spaces $X(\mathcal{M}, \Lambda^\ell)$ are built from local contributions and will therefore depend on the mesh $\mathcal{M}$. We assume that for each mesh cell $K \in \mathcal{M}$ there are Hilbert spaces $X(K, \Lambda^\ell)$ so that:

$$X(\mathcal{M}, \Lambda^\ell) = \{ v \in \mathcal{H}(d_e, \Omega) : v|_K \in X(K, \Lambda^\ell) \ \forall K \in \mathcal{M} \},$$  \hspace{1cm} (3.3)  

and, in addition, the norm of $X(\mathcal{M}, \Lambda^\ell)$ is defined through local contributions:

$$\| u \|_{X(\mathcal{M}, \Lambda^\ell)}^2 = \| u \|_{\mathcal{H}(d_e, \Omega)}^2 + \sum_{K \in \mathcal{M}} \| u|_K \|_{X(K, \Lambda^\ell)}^2.$$  \hspace{1cm} (3.4)  

Finally, the local spaces have to be large enough to contain the discrete forms for any value of $p$:

$$\mathcal{V}_p^\ell(K) \subset X(K, \Lambda^\ell).$$  \hspace{1cm} (3.5)  

Correspondingly, we introduce a space $S(\mathcal{M}, \Lambda^{\ell-1}) \subset \mathcal{H}(d_{e-1}, \Omega)$ of “more regular potentials”. Similar to $X(\mathcal{M}, \Lambda^\ell)$, the spaces $S(\mathcal{M}, \Lambda^{\ell-1})$ are mesh-dependent and allow for a characterization through local Hilbert spaces $S(K, \Lambda^{\ell-1})$, $K \in \mathcal{M}$:

$$S(\mathcal{M}, \Lambda^{\ell-1}) = \{ \psi \in \mathcal{H}(d_{e-1}, \Omega) : \psi|_K \in S(K, \Lambda^{\ell-1}) \ \forall K \in \mathcal{M} \},$$  \hspace{1cm} (3.6)  

They are endowed with the norm

$$\| \phi \|_{S(\mathcal{M}, \Lambda^{\ell-1})}^2 = \| \phi \|_{\mathcal{H}(d_{e-1}, \Omega)}^2 + \sum_{K \in \mathcal{M}} \| \phi|_K \|_{S(K, \Lambda^{\ell-1})}^2.$$  \hspace{1cm} (3.7)  

The local spaces are large enough to contain the local discrete potential spaces:

$$\mathcal{V}_p^{\ell-1}(K) \subset S(K, \Lambda^{\ell-1}).$$  \hspace{1cm} (3.8)  

The following assumption establishes the connection between $X(\mathcal{M}, \Lambda^\ell)$ and $S(\mathcal{M}, \Lambda^{\ell-1})$.

**Assumption 3** The exterior derivative maps $S(\mathcal{M}, \Lambda^{\ell-1})$ continuously into $X(\mathcal{M}, \Lambda^\ell)$:

$$S(\mathcal{M}, \Lambda^{\ell-1}) \subset \{ \phi \in \mathcal{H}(d_{e-1}, \Omega) : d_{e-1} \phi \in X(\mathcal{M}, \Lambda^\ell) \},$$

and the image is maximal:

$$d_{e-1} S(\mathcal{M}, \Lambda^{\ell-1}) = d_{e-1} \mathcal{H}(d_{e-1}, \Omega) \cap X(\mathcal{M}, \Lambda^\ell).$$

To conclude this subsection, note that in the case of an element $K$ touching the boundary $\partial \Omega$, like for the discrete spaces $\mathcal{V}_p^\ell(K)$ and $\mathcal{V}_p^{\ell-1}(K)$, the local spaces $X(K, \Lambda^\ell)$ and $S(K, \Lambda^{\ell-1})$ are not obliged to comply with any boundary conditions.
3.3 Local liftings

A pair of linear mappings $R_{k,K} : C^\infty(K, \Lambda^k) \mapsto C^\infty(K, \Lambda^{k-1})$, $k = \ell, \ell + 1$, is called a lifting operator of degree $\ell$ if it fulfills

$$d_{\ell-1} \circ R_{\ell,K} + R_{\ell+1,K} \circ d_{\ell} = \text{id}_{\ell}. \quad (3.9)$$

This relation characterizes a “contracting homotopy” of the de Rham complex [5, Section 5.1.2].

Besides this algebraic relationship, our approach hinges on smoothing properties of the lifting operators, expressed by means of the local spaces $S(K, \Lambda^{\ell-1})$ of more regular potentials and $X(K, \Lambda^\ell)$ of more regular forms. The next assumption summarizes the continuity expected from the lifting operator.

**Assumption 4** For every $K \in \mathcal{M}$ there is a lifting operator $(R_{\ell,K}, R_{\ell+1,K})$ whose components can be extended to continuous mappings

$$R_{\ell+1,K} : L^2(K, \Lambda^{\ell+1}) \mapsto X(K, \Lambda^\ell) \quad \text{and} \quad R_{\ell,K} : X(K, \Lambda^\ell) \mapsto S(K, \Lambda^{\ell-1}),$$

and thus identity (3.9) holds on $X(K, \Lambda^\ell)$.

As a consequence, for each cell $K \in \mathcal{M}$, we have the exact sequence

$$S(K, \Lambda^{\ell-1}) \xrightarrow{d_{\ell-1}} X(K, \Lambda^\ell) \xrightarrow{d_{\ell}} L^2(K, \Lambda^{\ell+1}). \quad (3.10)$$

Finally, the local liftings have to be compatible with the local spaces of discrete differential forms:

**Assumption 5** The local operators $R_{\ell+1,K}$, when applied to exact local discrete $\ell + 1$-forms, yield local discrete $\ell$-forms, i.e.,

$$R_{\ell+1,K} \circ d_{\ell} : \mathcal{V}_p^\ell(K) \rightarrow \mathcal{V}_p^\ell(K).$$

3.4 Local projectors

As usual in methods based on discrete commuting diagrams we need projection operators $\pi_{p,K}^\ell$ onto discrete spaces for $\ell - 1$-forms and $\ell$-forms. For degree $\ell - 1$, our local spaces $S(K, \Lambda^{\ell-1})$ of more regular potentials can play the role of domains for the projectors $\pi_{p,K}^{\ell-1}$. For the degree $\ell$, by generalization of what we actually need in the case of dimension $d = 2$ and $d = 3$ for Maxwell, we define our projectors $\pi_{p,K}^\ell$ on smaller spaces than $X(K, \Lambda^\ell)$, namely spaces of forms with discrete exterior derivative

$$\tilde{X}_p(K, \Lambda^\ell) = \{ u \in X(K, \Lambda^\ell) : d_{\ell} u \in d_{\ell} \mathcal{V}_p^\ell(K) \}. \quad (3.11)$$

$$\tilde{X}_p(K, \Lambda^\ell) = \{ u \in X(K, \Lambda^\ell) : d_{\ell} u \in d_{\ell} \mathcal{V}_p^\ell(K) \}. \quad (3.11)$$
Following the approach of (3.6)-(3.7), we define the corresponding global space
\[ \tilde{X}_p(\mathcal{M}, \Lambda^\ell) = \{ u \in X(\mathcal{M}, \Lambda^\ell) : d_\ell u \in d_\ell \mathcal{V}_p^\ell \}, \]  
which is equipped with the \( X(\mathcal{M}, \Lambda^\ell) \)-norm.

**Assumption 6** There are local continuous linear projections
\[ \pi_{p,K}^{\ell-1} : S(K, \Lambda^{\ell-1}) \mapsto \mathcal{V}_p^{\ell-1}(K) \quad \text{and} \quad \pi_{p,K}^\ell : \tilde{X}_p(K, \Lambda^\ell) \mapsto \mathcal{V}_p^\ell(K) \]
for all mesh cells \( K \in \mathcal{M} \).

The standard commuting diagram property is as follows.

**Assumption 7** The projectors \( \pi_{p,K}^{\ell-1} \) and \( \pi_{p,K}^\ell \) are compatible with the exterior derivative in the sense that the diagram
\[
\begin{array}{ccc}
S(K, \Lambda^{\ell-1}) & \xrightarrow{d_{\ell-1}} & \tilde{X}_p(K, \Lambda^\ell) \\
\pi_{p,K}^{\ell-1} & \downarrow & \pi_{p,K}^\ell \\
\mathcal{V}_p^{\ell-1}(K) & \xrightarrow{d_{\ell-1}} & \mathcal{V}_p^\ell(K)
\end{array}
\]
commutes for every \( K \in \mathcal{M} \).

Let us note that, as a consequence of Assumptions 4 and 7 we find that the sequence
\[ \mathcal{V}_p^{\ell-1}(K) \xrightarrow{d_{\ell-1}} \mathcal{V}_p^\ell(K) \xrightarrow{d_\ell} d_\ell(\mathcal{V}_p^\ell(K)) \]
is exact.

Besides, the local projections at the level \( \ell - 1 \) are supposed to enjoy a crucial approximation property using the Hilbert space norms \( \| \cdot \|_{S(K, \Lambda^{\ell-1})} \).

**Assumption 8** There is a function \( \varepsilon_{\ell-1} : \mathbb{N} \mapsto \mathbb{R}^+ \) with \( \lim_{p \to \infty} \varepsilon_{\ell-1}(p) = 0 \) so that
\[ \| d_{\ell-1}(\phi - \pi_{p,K}^{\ell-1} \phi) \|_{L^2(K, \Lambda^\ell)} \leq \varepsilon_{\ell-1}(p) \| \phi \|_{S(K, \Lambda^{\ell-1})} \quad \forall \phi \in S(K, \Lambda^{\ell-1}) . \]

Finally, for the projections \( \pi_{p,K}^\ell \) we assume a natural condition of conformity: For all \( u \in \tilde{X}_p(K, \Lambda^\ell) \)
\[ \text{tr}_F u = 0 \quad \Rightarrow \quad \text{tr}_F \pi_{p,K}^\ell u = 0 \quad \forall F \in \mathcal{F}_m(K), \quad \ell \leq m \leq d , \]  
and the corresponding condition for the projections \( \pi_{p,K}^{\ell-1} \). This makes it possible to define global linear projections
\[ \tilde{\pi}_p : \tilde{X}_p(\mathcal{M}, \Lambda^\ell) \mapsto \mathcal{V}_p^\ell \quad \text{and} \quad \pi_{p,K}^{\ell-1} : S(\mathcal{M}, \Lambda^{\ell-1}) \mapsto \mathcal{V}_p^{\ell-1} \]
The continuity properties stated in Assumption 6 carry over to the global projectors. Additionally, as a consequence of Assumption 7 and (3.14), the global projectors \( \pi_{p}^{\ell-1} \) and \( \pi_{p}^{\ell} \) inherit the global commuting diagram property

\[
\begin{array}{c}
S(M, \Lambda^{\ell-1}) \xrightarrow{d_{\ell-1}} \tilde{X}_{p}(M, \Lambda^{\ell}) \\
\pi_{p}^{\ell-1} \downarrow \quad \quad \downarrow \pi_{p}^{\ell} \\
\tilde{\gamma}_{p}^{\ell-1} \xrightarrow{d_{\ell-1}} \tilde{\gamma}_{p}^{\ell}
\end{array}
\]  

(3.15) \text{eq:globalcdp}

3.5 Proof of the discrete compactness property

The estimate of Assumption 8 on “potentials” can be transported to \( \ell \)-forms with a discrete exterior derivative, that is, the elements of the space \( \tilde{X}_{p}(M, \Lambda^{\ell}) \), see (3.12).

**Lemma 3.1 (Global projection error estimate)** Making Assumptions 4 through 8, the estimate

\[
\| u - \pi_{p}^{\ell} u \|_{L^{2}(\Omega, \Lambda^{\ell})} \leq C_{\ell-1}(p) \| u \|_{X(M, \Lambda^{\ell})} \quad \forall u \in \tilde{X}_{p}(M, \Lambda^{\ell}) ,
\]

holds true, with a constant \( C > 0 \) independent of \( p \).

**Proof.** Pick any \( u \in \tilde{X}_{p}(M, \Lambda^{\ell}) \). The locality of the projector \( \pi_{p}^{\ell} \), cf. (3.14), and (3.4) allows purely local considerations. Single out one cell \( K \in M \), still write \( u = u|_{K} \in \tilde{X}_{p}(K, \Lambda^{\ell}) \), and split \( u \) on \( K \) using (3.9) from Assumption 4:

\[
u = d_{\ell-1} R_{\ell,K} u + R_{\ell+1,K} d_{\ell} u = d_{\ell-1} \phi + R_{\ell+1,K} d_{\ell} u .
\]

(3.16) \text{eq:8}

with \( \phi := R_{\ell,K} u \). The continuity of \( R_{\ell,K} \) from Assumption 4 reveals that

\[
\| \phi \|_{S(K, \Lambda^{\ell-1})} \leq C \| u \|_{X(K, \Lambda^{\ell})} ,
\]

(3.17) \text{eq:9}

where here and below \( C \) will denote constants (possibly different at different occurrences) which depend neither on \( u \) nor on \( p \).

Thanks to identity (3.16) and the commuting diagram property from Assumption 7, we have

\[
\pi_{p,K}^{\ell} u = d_{\ell-1} \pi_{p,K}^{\ell-1} \phi + \pi_{p,K}^{\ell} R_{\ell+1,K} d_{\ell} u .
\]

(3.18) \text{eq:8b}
Recall that $u \in \tilde{X}_p(K, \Lambda^\ell)$ belongs to the domain of $\pi_{p,K}^\ell$ by Assumption 6. Further, as $u \in \tilde{X}_p(K, \Lambda^\ell)$, from Assumption 5 we infer that

\[ R_{\ell+1,K} d_{\ell} u \in V_{p,K}^\ell(K, \Lambda^\ell) . \tag{3.19} \]

Thus, owing to the identities (3.16), (3.18) and the projector property of $\pi_{p,K}^\ell$, the task is reduced to an interpolation estimate for $\pi_{p,K}^\ell-1$:

\[ (\text{Id} - \pi_{p,K}^\ell) u = d_{\ell-1} (\text{Id} - \pi_{p,K}^{\ell-1}) \phi + (\text{Id} - \pi_{p,K}^\ell) R_{\ell+1,K} d_{\ell} u . \quad \tag{3.20} \]

As a consequence, invoking Assumption 8,

\[ \|(\text{Id} - \pi_{p,K}^\ell) u\|_{L^2(K, \Lambda^\ell)} \leq \varepsilon_{\ell-1}(p) \|\phi\|_{S(K, \Lambda^{\ell-1})} \leq C \varepsilon_{\ell-1}(p) \|u\|_{X(K, \Lambda^\ell)} , \tag{3.21} \]

which furnishes a local version of the estimate. This estimate is uniform in $K \in \mathcal{M}$ because $\mathcal{M}$ is finite. Due to (3.4), squaring (3.21) and summing over all cells finishes the proof.

We are now in the position to prove the main result of this section.

**Theorem 3.2 (Discrete compactness)** Under Assumptions 1 through 8, the discrete compactness property of Definition 2.3 holds for the family $(\hat{\mathcal{V}}_p^\ell)_{p \in \mathbb{N}}$ of subspaces of $\tilde{H}(d_{\ell}, \Omega)$.

**Proof.** The proof resorts to the “standard policy” for tackling the problem of discrete compactness, introduced by Kikuchi [32, 33] for analyzing the $h$-version of Whitney-1-forms. It forms the core of most papers considering the issue of discrete compactness, see [10, Thm. 2], [9, Thm. 11], [30, Thm. 4.9], [25, Thm. 2], [9, Thm. 11], etc.

We consider a $H(d_{\ell}, \Omega)$-bounded sequence $(u_p)_{p \in \mathbb{N}}$ with members in $\hat{\mathcal{V}}_p^\ell$, that is,

\[ (i) \quad u_p \in \hat{\mathcal{V}}_p^\ell , \tag{3.22} \]

\[ (ii) \quad (u_p, d_{\ell-1} \psi_p)_{0,\Omega} = 0 \quad \forall \psi_p \in \hat{\mathcal{V}}_p^{\ell-1} , \tag{3.23} \]

\[ (iii) \quad \|u_p\|_{H(d_{\ell}, \Omega)} \leq 1 \quad \forall p \in \mathbb{N} . \tag{3.24} \]

We have to confirm that it possesses a subsequence that converges in $L^2(\Omega, \Lambda^\ell)$.
We start with the $L^2(\Omega, \Lambda^\ell)$-orthogonal projection of $u_p$ into $\hat{Y}(d_\ell, \Omega)$ parallel to $d_{\ell-1} \hat{H}(d_{\ell-1}, \Omega)$: let $\tilde{u}_p$ be the unique vector field in $\hat{H}(d_\ell, \Omega)$ with

$$
\tilde{u}_p = u_p + d_{\ell-1} \tilde{\phi}_p, \quad \tilde{\phi}_p \in \hat{H}(d_{\ell-1}, \Omega),
$$

(3.25)  

$$
(\tilde{u}_p, d_{\ell-1} \psi)_{0,\Omega} = 0 \quad \forall \psi \in \hat{H}(d_{\ell-1}, \Omega).
$$

(3.26)  

Obviously, the latter condition implies

$$
\tilde{u}_p \in \hat{Y}(d_\ell, \Omega).
$$

(3.27)  

Hence, by virtue of Assumption 1, the fact that $d_\ell \tilde{u}_p = d_\ell u_p$, and (3.12), $\tilde{u}_p$ satisfies

$$
\tilde{u}_p \in \tilde{Y}_p(\mathcal{M}, \Lambda^\ell), \quad \|\tilde{u}_p\|_{X(\mathcal{M}, \Lambda^\ell)} \leq C \|u_p\|_{H(d_\ell, \Omega)},
$$

(3.28)  

where $C > 0$ does not depend on $p$.

Since $d_{\ell-1} \tilde{\phi}_p = \tilde{u}_p - u_p \in X(\mathcal{M}, \Lambda^\ell)$, Assumption 3 implies that we may assume that $\tilde{\phi}_p \in S(\mathcal{M}, \Lambda^{\ell-1})$.

Thus we can use Nedélec’s trick [36] to obtain

$$
\|\tilde{u}_p - u_p\|^2_{L^2(\Omega, \Lambda^\ell)} = (\tilde{u}_p - u_p, \tilde{u}_p - \pi_p^\ell \tilde{u}_p + \pi_p^\ell \tilde{u}_p - u_p)_{0,\Omega}
$$

$$
= (\tilde{u}_p - u_p, \tilde{u}_p - \pi_p^\ell \tilde{u}_p)_{0,\Omega}.
$$

(3.29)  

This holds because from (3.25) and the projector property of $\pi_p^\ell$, we know

$$
\pi_p^\ell \tilde{u}_p - u_p = \pi_p^\ell u_p + \pi_p^\ell d_{\ell-1} \tilde{\phi}_p - u_p = \pi_p^\ell d_{\ell-1} \tilde{\phi}_p,
$$

and thanks to the commuting diagram property (3.15) (deduced from Assumption 7) combined with the orthogonality conditions (3.23) and (3.26), we find

$$
(\tilde{u}_p - u_p, \pi_p^\ell \tilde{u}_p - u_p)_{0,\Omega} = (\tilde{u}_p - u_p, d_{\ell-1} \pi_p^\ell \tilde{\phi}_p)_{0,\Omega} = 0.
$$

(3.30)  

Hence, appealing to Lemma 3.1, with $C > 0$ independent of $p$, we get

$$
\|\tilde{u}_p - u_p\|_{L^2(\Omega, \Lambda^\ell)} \leq \|\tilde{u}_p - \pi_p^\ell \tilde{u}_p\|_{L^2(\Omega, \Lambda^\ell)} \leq C \varepsilon_{\ell-1}(p) \|\tilde{u}_p\|_{X(\mathcal{M}, \Lambda^\ell)}
$$

$$
\leq C \varepsilon_{\ell-1}(p) \|u_p\|_{X(\mathcal{M}, \Lambda^\ell)} \rightarrow 0 \quad \text{for } p \rightarrow \infty.
$$

(3.31)  

From (3.28) we conclude that the sequence $(\tilde{u}_p)_{p \in \mathbb{N}}$ is uniformly bounded in $X(\mathcal{M}, \Lambda^\ell)$. By Assumption 2 it has a convergent subsequence in $L^2(\Omega, \Lambda^\ell)$. Owing to (3.31), the same subsequence of $(u_p)_{p \in \mathbb{N}}$ will converge in $L^2(\Omega, \Lambda^\ell).$  

\[ \square \]
3.6 Approximation of the eigenvalue problem

As discussed in Section 2.3, the discrete compactness property is the cornerstone of the proof of the convergence of the discrete generalized Maxwell eigenvalue problem (2.8).

Corollary 3.3 In addition to the hypotheses of Theorem 3.2, namely Assumptions 1 through 8, assume that the approximation property (2.12) holds and that the space \( X(\mathcal{M}, \Lambda^\ell) \cap \tilde{H}(d_\ell 0, \Omega) \) is dense in \( \tilde{H}(d_\ell 0, \Omega) \).

M.C.: These densities will have to be checked for the examples in sections 5&6.

Then (2.8) provides a spectrally correct, spurious-free approximation of the eigenvalue problem (2.5).

Proof. We use Theorem 2.6 from Section 2.3. Considering that the discrete compactness property is provided by Theorem 3.2, and that we assume the approximation property (2.12), we only need to show the approximation property (2.13), which concerns the approximation of closed forms by closed discrete forms.

Since we assumed the density of \( X(\mathcal{M}, \Lambda^\ell) \cap \tilde{H}(d_\ell 0, \Omega) \) in \( \tilde{H}(d_\ell 0, \Omega) \), it is sufficient to prove (2.13) for \( z \in X(\mathcal{M}, \Lambda^\ell) \cap \tilde{H}(d_\ell 0, \Omega) \). Such a \( z \) belongs to \( \tilde{X}_p(\mathcal{M}, \Lambda^\ell) \), and we can therefore apply Lemma 3.1, which shows that \( \pi_p^\ell z \to z \) in \( L^2(\Omega, \Lambda^\ell) \). We will have accomplished to show (2.13) with \( z_p = \pi_p^\ell z \), as soon as we show that \( d_\ell z_p = 0 \). Keeping in mind that \( z_p \in \dot{V}_p^\ell \subset H(d_\ell, \Omega) \), we see that it is sufficient to show the local relation \( d_\ell z_p = 0 \) for every \( K \in \mathcal{M} \). This follows finally as in (3.18) in the proof of Lemma 3.1, because \( d_\ell z = 0 \) implies

\[
\pi_{p,K}^\ell z = d_{\ell-1} \pi_{p,K}^{\ell-1} R_{\ell,K} z .
\]

Hence \( d_\ell z_p = d_\ell \pi_{p,K}^\ell z = d_\ell d_{\ell-1} \pi_{p,K}^{\ell-1} R_{\ell,K} z = 0 \), which ends the proof.

4 Regularized Poincaré lifting

In this section we describe the construction of a local lifting operator \( R_\ell \) that will satisfy Assumptions 4 and 5 in Section 3.3 for suitable spaces \( X(K, \Lambda^\ell) \), \( S(K, \Lambda^\ell) \) and \( \mathcal{V}_p^\ell(K) \). We follow the presentation in [17], where these operators are analyzed and where it is shown in particular that they are pseudodifferential operators of order \(-1\).
4.1 Definition

We consider a bounded domain \( D \subset \mathbb{R}^d \) that is star-shaped with respect to some subdomain \( B \subset D \), that is,
\[
\forall a \in B, \ x \in D : \ \{(1-t)a + tx, \ 0 < t < 1\} \subset D . \tag{4.1}
\]

For \( a \in B \) and \( 1 \leq \ell \leq d \), we define the Poincaré operator \( R_{\ell,a} \), acting on a differential form \( u \in C^\infty(D, \Lambda^\ell) \), by the path integral
\[
R_{\ell,a}u(x) = (x-a) \downarrow \int_0^1 t^{\ell-1} u(a + t(x-a)) \, dt , \quad x \in D . \tag{4.2}
\]

Here the symbol \( \downarrow \) denotes the contraction (also called “interior product”) of the vector field \( x \mapsto (x-a) \) with the \( \ell \)-form \( u \).

It is clear that \( R_{\ell,a} \) maps \( C^\infty(D, \Lambda^\ell) \) to \( C^\infty(D, \Lambda^{\ell-1}) \), and it has been shown (see [28] for proofs in the case \( d = 3 \)) that it can be extended to a bounded operator from \( L^2(D, \Lambda^\ell) \) to \( L^2(D, \Lambda^{\ell-1}) \). In order to define the regularized Poincaré operator \( R_{\ell} \), we choose a function
\[
\theta \in C^\infty_0(\mathbb{R}^d) , \quad \text{supp} \ \theta \subset B , \quad \int_B \theta(a) \, da = 1 ,
\]
and set
\[
R_{\ell}u(x) = \int_B \theta(a) R_{\ell,a}u(x) \, da . \tag{4.3}
\]

4.2 Regularity

The substitution \( y = a + t(x-a) , \ \tau = 1/(1-t) \) transforms the double integral in (4.2), (4.3) into
\[
R_{\ell}u(x) = \int_{\mathbb{R}^d} \int_1^\infty (\tau - 1)^{\ell-1} \tau^{d-\ell} \theta(x + \tau(y-x)) (x-y) \downarrow u(y) \, d\tau dy
\]
\[
= \int_{\mathbb{R}^d} k(y, y-x) \downarrow u(y) \, dy , \tag{4.4}
\]
where the kernel \( k(y, z) \) has an expansion into quasi-homogeneous terms:
\[
k(y, z) = -z \int_0^\infty s^{\ell-1}(s+1)^{d-\ell} \theta(y + sz) \, ds
\]
\[
= -\sum_{j=0}^{d-\ell} \binom{d-\ell}{j} \frac{z^j}{|z|^{d-j}} \int_0^\infty r^{d-j-1} \theta(y + r \frac{z}{|z|}) \, dr . \tag{4.5}
\]
The operator $R_\ell$ is therefore a weakly singular integral operator. In [17, Section 3.3], the following result is shown.

**Proposition 4.1** For $1 \leq \ell \leq d$, the operator $R_\ell$ is a pseudodifferential operator of order $-1$ on $\mathbb{R}^d$. It is well defined on $C^\infty(D, \Lambda^\ell)$, and it maps $C^\infty(D, \Lambda^\ell)$ to $C^\infty(D, \Lambda^{\ell-1})$ and $C^\infty(D, \Lambda^\ell)$ to $C^\infty(D, \Lambda^{\ell-1})$, and for any $s \in \mathbb{R}$ it has an extension as a bounded operator

$$R_\ell : H^s(D, \Lambda^\ell) \to H^{s+1}(D, \Lambda^{\ell-1}).$$

Here, $H^s(D, \Lambda^\ell)$ is the Sobolev space of $\ell$-forms on $D$ of order $s$.

### 4.3 Lifting property

The lifting property (3.9) is a consequence of the following identity, which is a special case of “Cartan’s magic formula” for Lie derivatives and for a flow field generated by the dilations with center $a$.

$$\frac{d}{dt}(t^\ell u(a + t(x - a))) = d_{\ell-1}\left((t^{\ell-1}(x - a)) \lrcorner u(a + t(x - a))\right) + t^\ell(x - a) \lrcorner d_\ell u(a + t(x - a)) \quad (4.6) \tag{eq:Cartan}$$

Here $u$ is an $\ell$-form. The result is

$$d_{\ell-1} R_\ell u + R_{\ell+1} d_\ell u = u \quad (1 \leq \ell \leq d - 1);$$

$$R_1 d_0 u = u - (\theta, u)_{0,D} \quad (\ell = 0);$$

$$d_{d-1} R_d u = u \quad (\ell = d). \tag{4.7} \tag{eq:dR+Rd=1}\$$

These relations are valid for all $u \in C^\infty(\mathbb{R}^d, \Lambda^\ell)$ and by extension for all $u \in H^s(D, \Lambda^\ell)$, $s \in \mathbb{R}$.

The perfect match of (4.7) with (3.9) from Assumption 4 suggests that the regularized Poincaré lifting $R_\ell$ provides suitable local liftings as stipulated in Assumption 4. To this end, we can choose as spaces of “more regular forms”

$$X(K, \Lambda^\ell) := H(d_\ell, K) \cap H^r(K, \Lambda^\ell),$$

$$S(K, \Lambda^{\ell-1}) := H^r(d_{\ell-1}, K) \quad \text{and} \quad S(K, \Lambda^\ell) := H^r(d_\ell, K), \tag{4.8} \tag{eq:XS}$$

for some $0 < r \leq 1$, where we denote by $H^r(d_k, K)$ the space

$$H^r(d_k, K) := \{v \in H^r(K, \Lambda^k) : d_k v \in H^r(K, \Lambda^{k+1})\}.$$

Let us recall that the global spaces $X(\mathcal{M}, \Lambda^\ell)$, $S(\mathcal{M}, \Lambda^{\ell-1})$ and $S(\mathcal{M}, \Lambda^\ell)$ are determined by their local definition on the mesh cells $K$, cf. (3.3) and (3.6). The
construction of $R_\ell$ entails a constraint on the cell shapes. This is satisfied for standard finite element meshes, where the cells usually are convex polyhedra.

**Assumption 9** Every cell $K \in \mathcal{M}$ is a star-shaped polyhedron.

We point out that the choice of $r$ in (4.8) is governed by Assumption 1. Also note that whenever we opt for (4.8), Rellich’s theorem ensures Assumption 2, because the mesh is kept fixed.

**Lemma 4.2** Assumption 9 and the choice (4.8) for the spaces $X(K, \Lambda^\ell)$ and $S(K, \Lambda^{\ell-1})$ imply Assumptions 2, 3 and 4.

**Proof.** The only fact which remains to be proved is the maximality relation

$$d_{\ell-1} S(\mathcal{M}, \Lambda^{\ell-1}) = d_{\ell-1} H(d_{\ell-1}, \Omega) \cap X(\mathcal{M}, \Lambda^\ell).$$

The inclusion $\subset$ holds by definition. Let us prove the converse inclusion. Let $u \in d_{\ell-1} H(d_{\ell-1}, \Omega) \cap X(\mathcal{M}, \Lambda^\ell)$. Thus $u = d_{\ell-1} \phi$ with $\phi \in H(d_{\ell-1}, \Omega)$. Since $u \in L^2(\Omega, \Lambda^\ell)$, using [17, Cor. 4.7] we obtain that there exists $\psi \in H^1(\Omega, \Lambda^{\ell-1})$ such that $u = d_{\ell-1} \psi$. In particular, $\psi|_K$ belongs to $H^r(K, \Lambda^\ell)$ for all $K$, and, since $u|_K$ belongs to $H^r(d_{\ell-1}, K)$, we finally find that $\psi|_K \in H^r(d_{\ell-1}, K)$. \hfill $\Box$

### 4.4 Preservation of polynomial forms

From the definition (4.2) it is clear that the Poincaré operator $R_{\ell,a}$ maps differential forms with polynomial coefficients to differential forms with polynomial coefficients. The same holds for the regularized Poincaré operator $R_\ell$ by (4.3). If we want $R_\ell$ to map a space $P(\Lambda^\ell)$ of differential forms of order $\ell$ with polynomial coefficients into a space $P(\Lambda^{\ell-1})$ of differential forms of order $\ell - 1$, it is sufficient to require the following two properties, see [17, Proposition 4.2].

**Proposition 4.3** Assume that $P(\Lambda^\ell)$ and $P(\Lambda^{\ell-1})$ are finite-dimensional spaces of differential forms satisfying
(i) The space $P(\Lambda^\ell)$ is invariant with respect to dilations and translations, that is

For any $t \in \mathbb{R}, a \in \mathbb{R}^n$: If $u \in P(\Lambda^\ell)$, then $(x \mapsto u(tx + a)) \in P(\Lambda^\ell)$.

(ii) The interior product $x \cdot : u \mapsto x \cdot u$ maps $P(\Lambda^\ell)$ to $P(\Lambda^{\ell-1})$. Then $R_\ell$ maps $P(\Lambda^\ell)$ into $P(\Lambda^{\ell-1})$.

For the compatibility Assumption 5 to hold, it is therefore sufficient to make the following assumption about the local polynomial space $V_\ell(K)$. 

23
Assumption 10

(i) The space $\mathcal{V}^\ell_p(K)$ is invariant with respect to dilations and translations.

(ii) The differential operator $x \uplus d : u \mapsto x \uplus d u$ maps $\mathcal{V}^\ell_p(K)$ to $\mathcal{V}^\ell_p(K)$.

To summarize:

Assumptions 9, 10, (4.8)  $\Rightarrow$  Assumptions 2, 3, 4, and 5

5  Discrete differential forms

Now we introduce concrete spaces of discrete differential forms. We merely summarize the constructions that have emerged from research in differential geometry (the “Whitney-forms” introduced in [40]) and finite element theory ("Raviart-Thomas elements" of [39] and “Nedeléc finite elements” of [36, 37]”). These schemes were later combined into the concept of discrete differential forms [12, 29]. A Survey and details are given in [4, 5, 13, 30].

Fundamental is the notion of polynomial differential forms. For an ordered $\ell$-tuple $I = (i_1, \ldots, i_\ell)$, $i_1 < i_2 < \ldots < i_\ell$, $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, d\}$, let

$$dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_\ell},$$

where $dx_j$, $j = 1, \ldots, d$, are the co-ordinate 1-forms in Euclidean space $\mathbb{R}^d$. The space $P_p(\Lambda^\ell)$ of polynomial $\ell$-forms on $\mathbb{R}^d$ is defined as

$$P_p(\Lambda^\ell) := \{ u = \sum_I u_I dx_I : u_I \in P_p(\mathbb{R}^d) \},$$

where $\sum_I$ indicates summation over all ordered $\ell$-tuples, and $P_p(\mathbb{R}^d)$ is the space of $d$-variate polynomials of total degree $\leq p$. We remark that for $d \in \{2, 3\}$ polynomial forms possess polynomial vector proxies.

5.1  Simplicial meshes

Let $\mathcal{M}$ be a conforming simplicial finite element mesh covering $\Omega \subset \mathbb{R}^d$. As elaborated in [4, Sect. 3 & 4] the following choices

$$\mathcal{V}^\ell_p(K) := P_{p-1}(\Lambda^\ell)|_K + P_{p-1}(\Lambda^{\ell+1})|_K \uplus x,$$  \hspace{1cm} (5.1)  eq:ned1

and

$$\mathcal{V}^\ell_p(K) := P_p(\Lambda^\ell)|_K,$$  \hspace{1cm} (5.2)  eq:ned2
of local spaces, through (3.1), gives rise to meaningful global finite element spaces of discrete differential forms.

By construction both Assumption 9 and Assumption 10 are satisfied in this setting, as well as Assumption 11.

### 5.2 Tensor product meshes

Let $\mathcal{M}$ be a conforming finite element mesh of $\Omega$ whose cells are affine images of the unit hypercube $\hat{K}$ in $\mathbb{R}^d$: for $K \in \mathcal{M}$ the we write $\Phi_K : \hat{K} \mapsto K$ for the associated unique affine mapping. We generalize the construction of [36]: on the cube we define

$$
\mathcal{V}_p^\ell(\hat{K}) := \left\{ \tilde{v} = \sum_I u_I d x_I, \quad u_I(x) = \prod_{j=1}^d u_{I,j}(x_j), \quad u_{I,j} \in \left\{ \begin{array}{ll} \mathcal{P}_{p-1}, & \text{if } j \in I, \\
\mathcal{P}_p, & \text{if } j \notin I \end{array} \right\} \right\}.
$$

The local spaces are obtained by affine pullback

$$
\mathcal{V}_p^\ell(K) := (\Phi_K^{-1})^* \mathcal{V}_p^\ell(\hat{K}). \quad (5.3)
$$

This affine tensor product construction also complies with Assumption 9, Assumption 10, and Assumption 11.

### 6 Special cases

We adopt the discrete spaces from Sect. 5 along with the regularized Poincaré lifting from Sect. 4. Making the choice (4.8), in order to establish the discrete compactness property from Definition 2.3,

it remains to verify assumptions 1 and assumptions 6, 7, and 8 for judiciously chosen local projectors $\pi_{p,K}^\ell$.

The right projection operators for spectral approximation are so-called projection based interpolation operators, see [14, 18–20, 24] and . Variants for any $\ell$ and $d$ are available and they are designed to commute in the sense of (??) [30, Sect. 3.5].

Unfortunately, here we have to abandon the framework of general $\ell$ and $d$, because both regularity results (Assumption 1) and the analysis of projection operators (Assumption 8) are not (yet?) available for general $\ell$ and $d$. We have to discuss them for special choices of $\ell$ and $d$ separately, relying on a wide array of sophisticated results from the literature.
Theorem 6.1 (Convergence of spectral Galerkin approximations) For \(d = 2, 3\), \(0 < l < d\), the spectral Galerkin discretization of (2.5) based on any of the families of discrete differential forms introduced in Sect. 5 offers a spectrally correct approximation.

Proof. To begin with, we focus on the discrete compactness property and verify the assumptions 1, 6, 7, and 8 for \(d = 2\) and \(d = 3\) separately.

\(d = 2\), \(l = 1\): In terms of vector proxies we find the correspondence

\[
Y(d_1, \Omega) \sim \hat{H}(\text{div}, \Omega) \cap H(\text{curl}, \Omega). \tag{6.1}
\]

Regularity theorems for boundary value problems for \(-\Delta\) on the polygon confirm the existence of \(\varepsilon = \varepsilon(\Omega) > 0\) such that

\[
\hat{H}(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \subset H^{\varepsilon + 1/2}(\Omega), \tag{6.2}
\]

in the sense of continuous embedding, see [27, Sect. 3.2]. This suggests to choose \(r = \varepsilon + 1/2\) in (4.8) and Assumption 1 will hold true.

Commuting local projection based interpolation operators \(\pi_{p,K}^1\) and \(\pi_{p,K}^0\) have been proposed for triangles in [19] and for quadrilaterals in [20]. For \(r = \varepsilon + 1/2\) and (4.8) they meet Assumptions 6 and 7. Assumption 8 holds with \(\varepsilon_0(p) = C\sqrt{p}^{-1/2}\) and \(C > 0\) depending only on the shape-regularity of the cells [6, Thm. 4.1].

\(d = 3\): We have the vector proxy incarnation

\[
Y(d_\ell, \Omega) \sim \begin{cases} 
\hat{H}(\text{curl}, \Omega) \cap H(\text{div}, \Omega), & \text{for } \ell = 1, \\
\hat{H}(\text{div}, \Omega) \cap H(\text{curl}, \Omega), & \text{for } \ell = 2.
\end{cases} \tag{6.3}
\]

Citing results from [1], we find \(\varepsilon = \varepsilon(\Omega) > 0\) and a continuous embedding

\[
\hat{H}(\text{curl}, \Omega) \cap H(\text{div}, \Omega), \hat{H}(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \subset H^{\varepsilon + 1/2}(\Omega). \tag{6.4}
\]

Therefore, for \(r = \varepsilon + 1/2\) in (4.8) Assumption 1 is satisfied.

The essential commuting local projection based interpolation operators \(\pi_{p,K}^m\), \(m = 0, 1, 2\), have been introduced in [20]. By construction they comply with Assumption 7. Assumption 6 for the spaces from (4.8) and \(r = \varepsilon + 1/2\) is a consequence of Sobolev embedding theorems. Invoking new polynomials extension theorems from [21–23] the analysis of [20, Sect. 6] furnishes the estimate of Assumption 8 with \(\varepsilon_m(p) = C\frac{\log(p+1)}{\sqrt{p}}\) and \(C > 0\) depending only on shape-regularity of the mesh cells.

Finally, we remark that (2.12) and (2.13) can be immediately concluded from the density of smooth compactly supported functions in \(\hat{H}(d_\ell, \Omega)\) and the existence of local projectors with properties as specified in Sect. 3.4. \(\square\)
References


