Numerical Analysis of Coupled Circuit and Device Models

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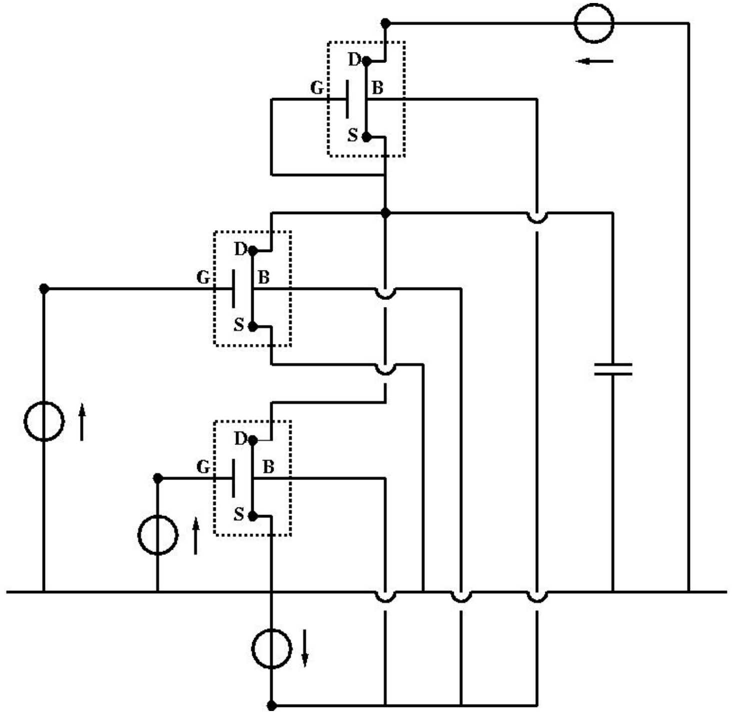
Overview

1. motivation
2. network modeling
3. device modeling
4. coupling of both systems
5. formulation as abstract differential-algebraic system
6. index für abstract DAEs
7. Galerkin approach for abstract DAEs

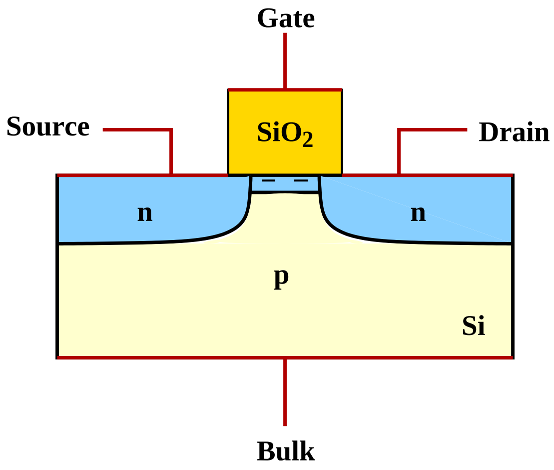
Circuit Modeling

- Kirchhoff's current law (KCL): $Ai = 0$
- Kirchhoff's voltage law (KVL): $A^{\mathsf{T}} e = u$
- circuit elements: $g(\frac{\mathrm{d}q(u,t)}{\mathrm{d}t}, \frac{\mathrm{d}\phi(i,t)}{\mathrm{d}t}, u, i, t) = 0$, e.g.:
  - capacitors: $i = C\frac{\mathrm{d}u}{\mathrm{d}t}, \quad i = \frac{\mathrm{d}q_C(u,t)}{\mathrm{d}t}$
  - inductors: $u = L\frac{\mathrm{d}i}{\mathrm{d}t}, \quad u = \frac{\mathrm{d}\phi_L(i,t)}{\mathrm{d}t}$
  - voltage sources: $u = v(t), \quad u = v(i, \hat{u}, t)$

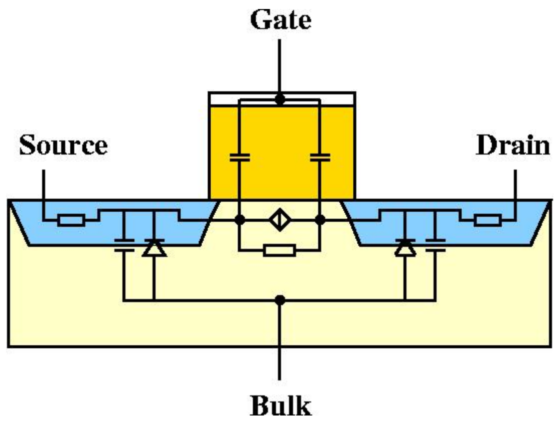


$\Rightarrow$ differential-algebraic equation (DAE) $f(\frac{\mathrm{d}q(x,t)}{\mathrm{d}t}, x, t) = 0$ with $x = \begin{pmatrix} i \\ e \\ u \end{pmatrix}$

Replacement Circuit Models for More Complex Elements



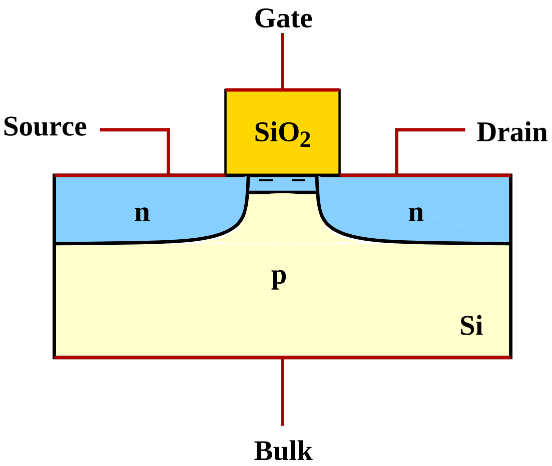
Device- $\longrightarrow$ **Simulation**



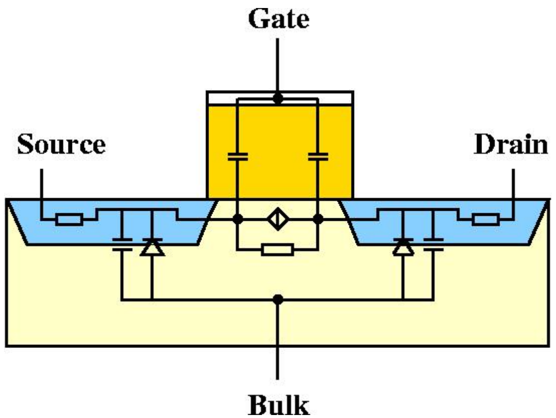
Advantages:

- resulting system is a differential-algebraic system
- fast simulation of the circuit is possible
- circuits with many transistors ($> 10^6$) can be simulated

Replacement Circuit Models for More Complex Elements



Device- $\longrightarrow$ Simulation



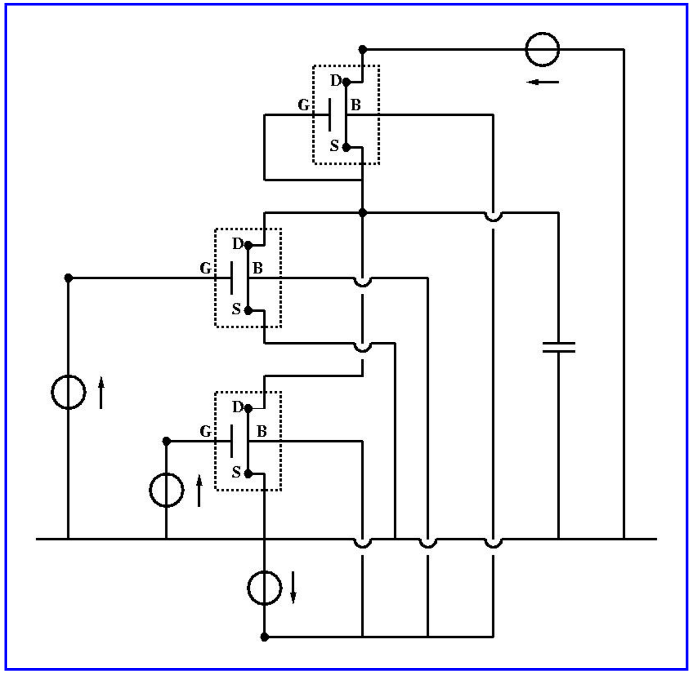
Disadvantages:

- interaction between circuit element and surrounding circuit might be insufficiently regarded (essential for high frequency circuits)

- more detailed models need a multitude of parameters (> 500 per transistor)

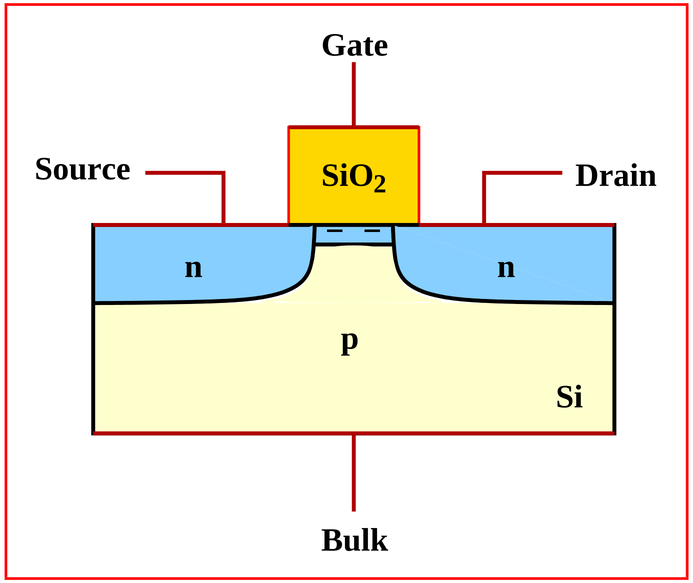
  $\Rightarrow$ parameter extraction is very time consuming
  $\Rightarrow$ parameter adjustment becomes problematic for optimal circuit design

Wish: Coupling of Circuit and Device Simulation



DAE

+



PDE

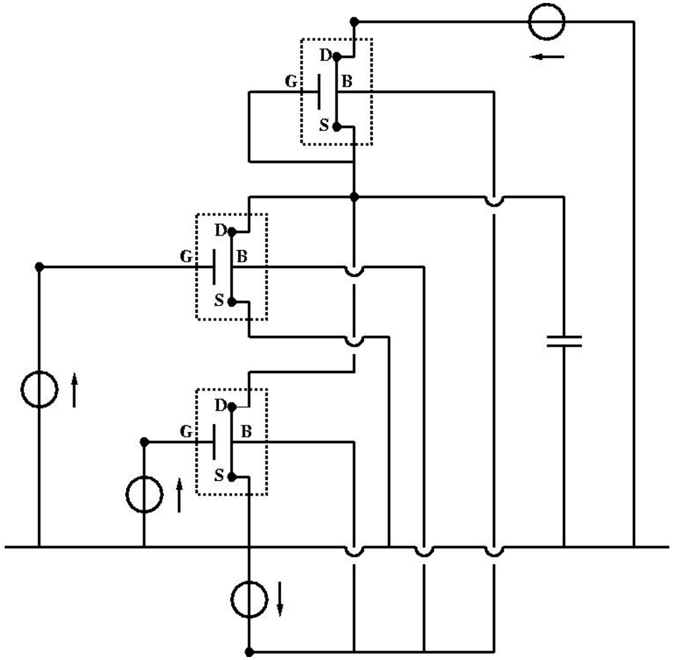
$\Rightarrow$ PDAE

Network Equations by Modified Nodal Analysis

$$A_C \frac{\mathrm{d}q(A_C^\top e, t)}{\mathrm{d}t} + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_S j_S = -A_I i_s$$

$$\frac{\mathrm{d}\phi(j_L, t)}{\mathrm{d}t} - A_L^\top e = 0$$

$$A_V^\top e = v_s$$



$$A = (A_C, A_R, A_L, A_V, A_I, A_S)$$

- e - nodal potentials
- j_L, j_V - currents of inductances and voltage sources
- j_S - currents of semiconductors

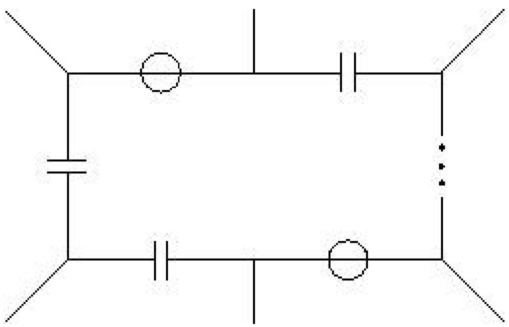
Index of Network DAEs

- DAE index is always ≤ 2.

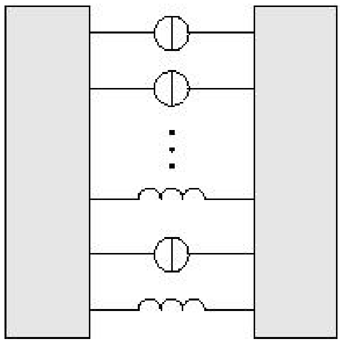
  [Günther/Feldmann 96, T. 97, Reissig 98, Estévez Schwarz/T. 00]

- DAE-Index $= 2$ $\Leftrightarrow$ (A_C, A_R, A_V) has not full row rank and $Q_C^\top A_V$ has not full column rank (Q_C projector onto $\ker A_C^\top$).

  $\Leftrightarrow$ The network has an LI-cutset or a CV-loop with at least one VS.



CV-loop

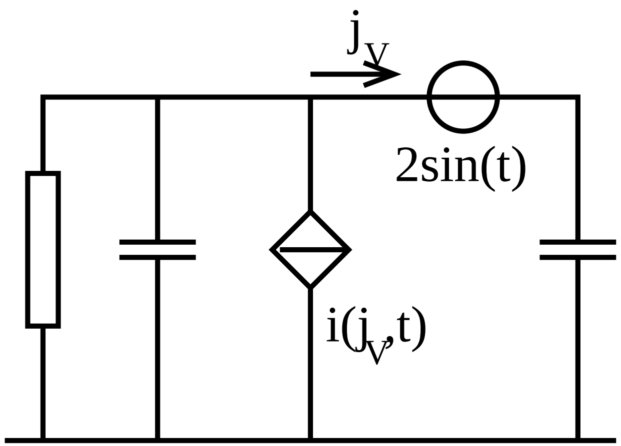


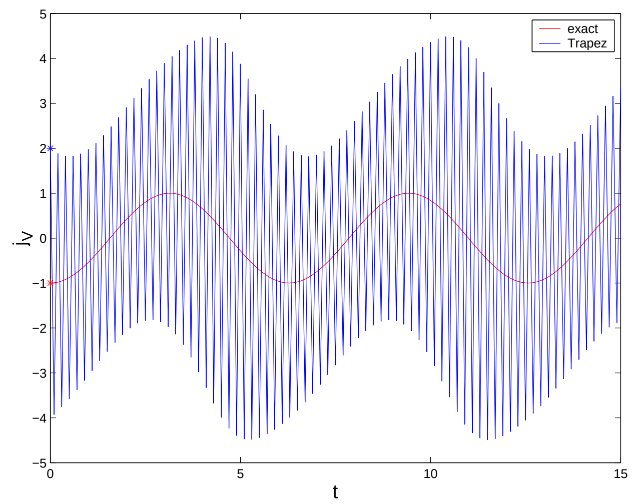
LI-cutset

Problems of the Simulation of DAEs with Higher Index

- Solution does not depend continuously on the initial data.
- Initial values have to fulfill (hidden) constraints.
- Simulation methods like BDF and trapezoidal rule can collapse.

Example: Integration with inconsistent initial value





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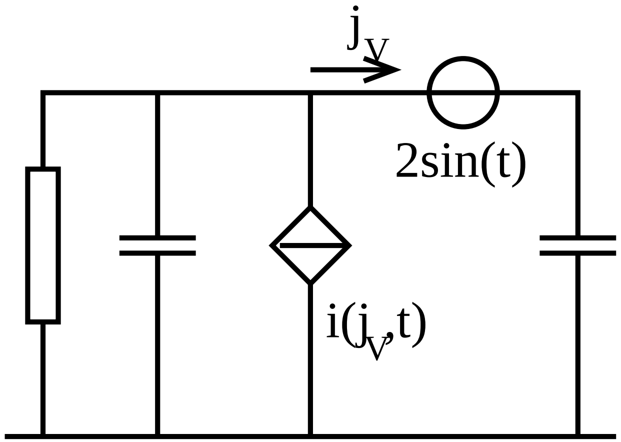

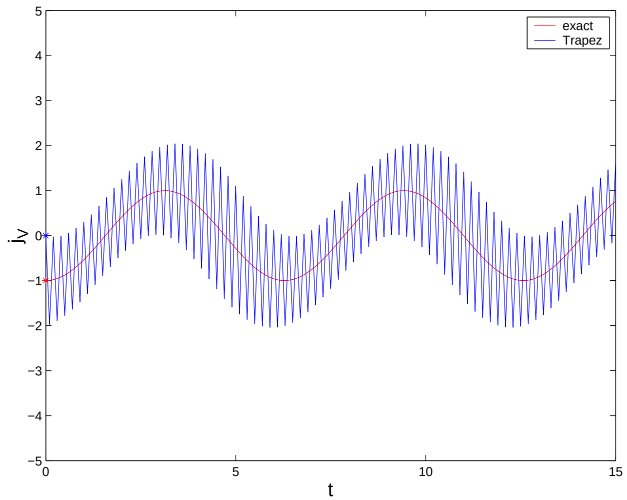

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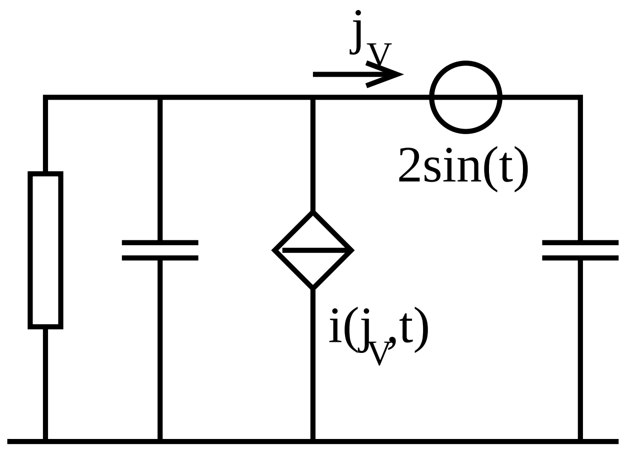
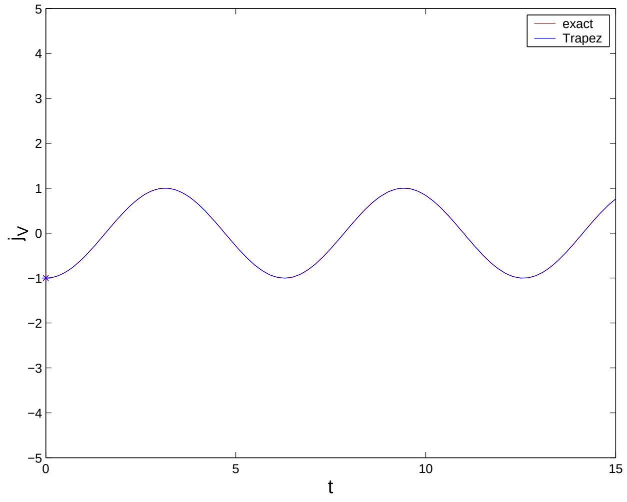
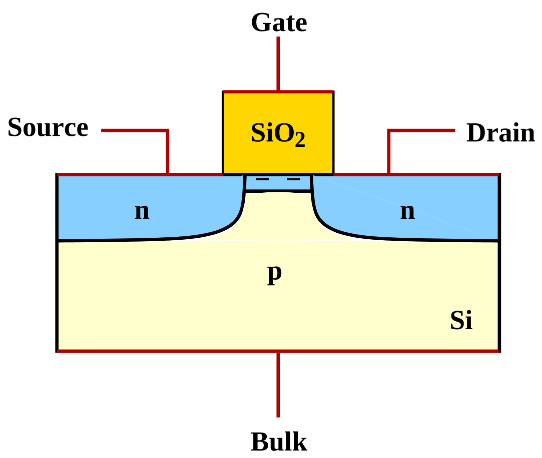




Semiconductor Equations (Drift Diffusion Model)



$$\begin{aligned}
\operatorname{div}(\varepsilon \operatorname{grad} V) &= q(n - p - N) \\
-\partial_t n + \frac{1}{q}\operatorname{div} J_n &= R(n, p, J_n, J_p) \\
\partial_t p + \frac{1}{q}\operatorname{div} J_p &= -R(n, p, J_n, J_p) \\
J_n &= q(D_n \operatorname{grad} n - \mu_n n \operatorname{grad} V) \\
J_p &= q(-D_p \operatorname{grad} p - \mu_p p \operatorname{grad} V)
\end{aligned}$$

- V - electrostatic potential
- n, p - electron and hole concentration
- J_n, J_p - current density of electrons and holes

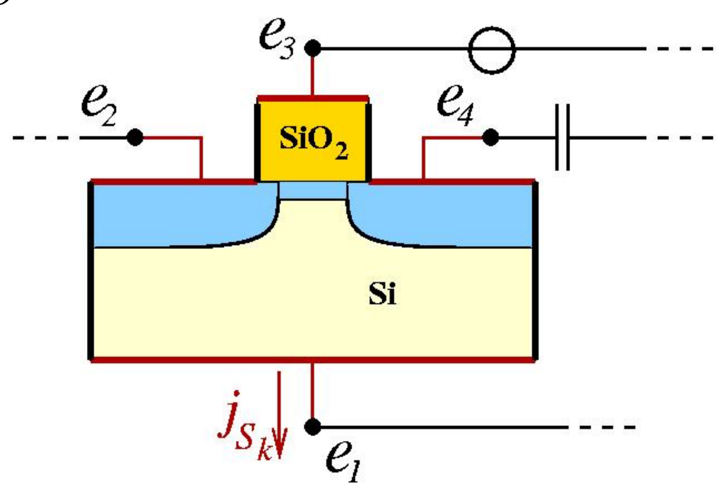
Boundary and Coupling Conditions

$$
\begin{aligned}
V &= e_l + c \cdot A_S^{\mathsf{T}} e + W && \text{on } \Gamma_{\mathsf{O}} \cup \Gamma_{\mathsf{S}} \\
\operatorname{grad} V \cdot \nu &= \alpha V - \alpha(e_l + c \cdot A_S^{\mathsf{T}} e) + \beta && \text{on } \Gamma_{\mathsf{MI}} \\
\operatorname{grad} V \cdot \nu &= 0 && \text{on } \Gamma_{\mathsf{I}}
\end{aligned}
$$

$$
\begin{aligned}
n &= n_0, & p &= p_0 && \text{on } \Gamma_{\mathsf{O}} \\
J_n \cdot \nu &= -q v_n (n - n_0), & J_p \cdot \nu &= q v_p (p - p_0) && \text{on } \Gamma_{\mathsf{S}} \\
J_n \cdot \nu &= -q R_{\mathsf{surf}}(n, p), & J_p \cdot \nu &= q R_{\mathsf{surf}}(n, p) && \text{on } \Gamma_{\mathsf{MI}} \\
J_n \cdot \nu &= 0, & J_p \cdot \nu &= 0 && \text{on } \Gamma_{\mathsf{I}}
\end{aligned}
$$

$$
j_{S_k} = \int_{\Gamma_k} (J_n + J_p - \varepsilon \operatorname{grad} \partial_t V) \cdot \nu \, \mathrm{d}\sigma
$$

- Γ_{O}, Γ_{S} - Ohmic and Schottky contacts
- Γ_{MI} - metal-insulator contacts
- Γ_{I} - insulator contacts
- Γ_k - contacts at the k-th terminal of the semiconductor



Homogenization

Let $f(x) = (f_1(x), ..., f_{b_S-1}(x))^\mathsf{T}$ and $g(x)$ be smooth functions on Ω with

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \Gamma_k \subseteq (\Gamma_\mathsf{O} \cup \Gamma_\mathsf{S} \cup \Gamma_\mathsf{MI}), \\ 0 & \text{if } x \in (\Gamma_\mathsf{O} \cup \Gamma_\mathsf{S} \cup \Gamma_\mathsf{MI}) \backslash \Gamma_k, \end{cases} \qquad \operatorname{grad} f_k \cdot \nu = 0 \text{ on } \Gamma$$

and

$$g = W \text{ on } \Gamma_\mathsf{O} \cup \Gamma_\mathsf{S}, \qquad \operatorname{grad} g \cdot \nu = 0 \text{ on } \Gamma_\mathsf{MI} \cup \Gamma_\mathsf{I}.$$

$$\tilde{V}(x,t) := V(x,t) - e_l(t) - f(x) \cdot A_S^\mathsf{T} e(t) - g(x)$$

$\Rightarrow$

$$\tilde{V} = 0 \text{ on } \Gamma_\mathsf{O} \cup \Gamma_\mathsf{S}, \quad \varepsilon \operatorname{grad} \tilde{V} \cdot \nu + \alpha \tilde{V} = \tilde{\beta} \text{ on } \Gamma_\mathsf{MI}, \quad \operatorname{grad} \tilde{V} \cdot \nu = 0 \text{ on } \Gamma_\mathsf{I}$$

with $\tilde{\beta} := \beta - \alpha g - \varepsilon \operatorname{grad} g \cdot \nu$.

Complete Coupled System

$$\begin{aligned} A_C \frac{\mathrm{d}q_C(A_C^\top e, t)}{\mathrm{d}t} + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_S j_S + A_I i_s &= 0 \\ \frac{\mathrm{d}\phi_L(j_L, t)}{\mathrm{d}t} - A_L^\top e &= 0 \\ A_V^\top e - v_s &= 0 \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\varepsilon \operatorname{grad} \tilde{V}) &= q(n - p - N) - \operatorname{div}(\varepsilon \operatorname{grad}(f \cdot A_S^\top e + g)) \\ -\partial_t n + \tfrac{1}{q} \operatorname{div} J_n &= R(n, p, J_n, J_p) \\ \partial_t p + \tfrac{1}{q} \operatorname{div} J_p &= -R(n, p, J_n, J_p) \end{aligned}$$

$$\begin{aligned} J_n &= q(D_n \operatorname{grad} n - \mu_n n \operatorname{grad}(\tilde{V} + f \cdot A_S^\top e + g)) \\ J_p &= q(-D_p \operatorname{grad} p - \mu_p p \operatorname{grad}(\tilde{V} + f \cdot A_S^\top e + g)) \end{aligned}$$

$$j_S = \int_\Gamma [(J_n + J_p) \cdot \nu\, \chi_1 - \varepsilon\, \partial_t \operatorname{grad} \tilde{V} \cdot \nu\, \chi_2]\, \mathrm{d}\sigma$$

$$\tilde{V} = 0 \text{ on } \Gamma_{\mathrm{O}} \cup \Gamma_{\mathrm{S}}, \quad \varepsilon \operatorname{grad} \tilde{V} \cdot \nu + \alpha \tilde{V} = \tilde{\beta} \text{ on } \Gamma_{\mathrm{MI}}, \quad \operatorname{grad} \tilde{V} \cdot \nu = 0 \text{ on } \Gamma_{\mathrm{I}}$$

+ boundary conditions for n and p as well as J_n and J_p

Coupled System as Abstract Differential-Algebraic System (I)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0$$

with $\mathcal{D}(u,t) = \begin{pmatrix} q_C(A_C^\top u_1, t) \\ \phi_L(u_2,t) \\ -\mathfrak{r}_1\, u_5 \\ u_6 \\ u_7 \end{pmatrix}$,

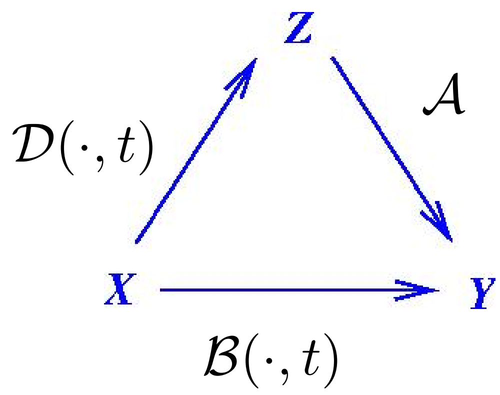
$$\mathcal{A} = \begin{pmatrix} A_C & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{pmatrix}, \quad \mathcal{B}(u,t) = \begin{pmatrix} A_R g(A_R^\top u_1, t) + A_L u_2 + A_V u_3 + A_S u_4 + A_I i_s(t) \\ -A_L^\top u_1 \\ A_V^\top u_1 - v_s(t) \\ \operatorname{div}(\varepsilon \operatorname{grad} u_5) - q(u_6 - u_7 - N) + \operatorname{div}(\varepsilon \operatorname{grad}(f \cdot A_S^\top u_1 + g)) \\ -\frac{1}{q}\operatorname{div} u_8 + R(u_6, u_7, u_8, u_9) \\ \frac{1}{q}\operatorname{div} u_9 + R(u_6, u_7, u_8, u_9) \\ u_8 - q(D_n \operatorname{grad} u_6 - \mu_n u_7 \operatorname{grad}(u_5 + f \cdot A_S^\top u_1 + g)) \\ u_9 - q(-D_p \operatorname{grad} u_7 - \mu_p u_7 \operatorname{grad}(u_5 + f \cdot A_S^\top u_1 + g)) \\ u_4 - \mathfrak{r}_2\,(u_8 + u_9) \end{pmatrix},$$

where $\mathfrak{r}_1 v := \int_\Gamma \varepsilon \operatorname{grad} v \cdot \nu\, \chi_2 \, \mathrm{d}\sigma$, $\mathfrak{r}_2 v := \int_\Gamma v \cdot \nu\, \chi_1 \, \mathrm{d}\sigma$ and

$$u(t) = (e(t), j_L(t), j_V(t), j_S(t), \tilde{V}(\cdot,t), n(\cdot,t), p(\cdot,t), J_n(\cdot,t), J_p(\cdot,t))$$

Coupled System as Abstract Differential-Algebraic System (I)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0$$



$X := \bigtimes_{i=1}^{9} X_i$ with

$$X_1 = \mathbb{R}^{n-1}, \quad X_2 = \mathbb{R}^{n_L}, \quad X_3 = \mathbb{R}^{n_V}, \quad X_4 = \bigtimes_{l=1}^{n_s} \mathbb{R}^{k_l - 1}$$

$$X_5 = \{v \in \bigtimes_{l=1}^{n_s} H^2(\Omega_l) : \ v_l = 0 \text{ on } \Gamma_{l\mathsf{O}} \cup \Gamma_{l\mathsf{S}}\},$$

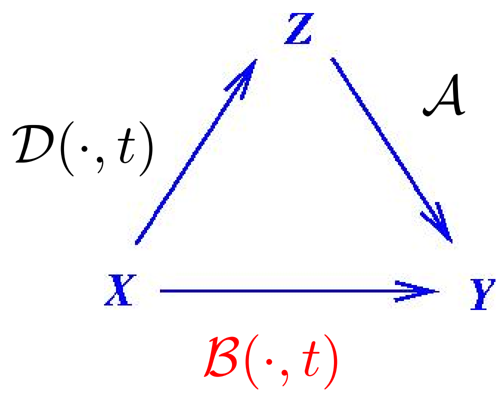
$$X_6 = X_7 = \bigtimes_{l=1}^{n_s} H^1(\Omega_l), \quad X_8 = X_9 = \bigtimes_{l=1}^{n_s} H(\mathrm{div}\,;\Omega_l).$$

$$Y := X_1 \times X_2 \times X_3 \times (\bigtimes_{l=1}^{n_s} L_2(\Omega_l))^5 \times X_4$$

$$Z := \mathbb{R}^{n_C} \times X_2 \times X_4 \times (\bigtimes_{l=1}^{n_s} H^1(\Omega_l))^2$$

Coupled System as Abstract Differential-Algebraic System (I)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0$$



- X, Y, Z - real Hilbert spaces
- $\mathcal{A}$, $\mathcal{D}(\cdot,t)$ continuous operators
- $\mathcal{B}(\cdot,t)$ is an unbounded operator!

Index for Abstract DAEs

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0 \qquad (1)$$

Assumptions: $\cdot$ $\exists$ Fréchet derivatives $\mathcal{B}_0(\cdot,t)$ and $\mathcal{D}_0(\cdot,t)$ of $\mathcal{B}(\cdot,t)$ and $\mathcal{D}(\cdot,t)$

$\cdot \ \ker \mathcal{A} \oplus \mathrm{im}\ \mathcal{D}_0(u,t) = Z$

$\cdot \ \ker \mathcal{G}_0(u,t)$ constant for $\mathcal{G}_0(u,t) := \mathcal{A}\mathcal{D}_0(u,t)$

(1) has **index 1** if $\exists$ projection operator $\mathcal{Q}_0 : \ X \to X$ onto $\ker \mathcal{G}_0(u,t)$ with

$$\mathcal{G}_1(u,t) := \mathcal{G}_0(u,t) + \mathcal{B}_0(u,t)\mathcal{Q}_0$$

injective and cl$(\mathrm{im}\ \mathcal{G}_1(u,t)) = Y$ for all $u \in X$, $t \in [t_0, T]$.

Index for Abstract DAEs

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0 \tag{1}$$

Assumptions: · $\exists$ Fréchet derivatives $\mathcal{B}_0(\cdot,t)$ and $\mathcal{D}_0(\cdot,t)$ of $\mathcal{B}(\cdot,t)$ and $\mathcal{D}(\cdot,t)$

· $\ker \mathcal{A} \oplus \mathrm{im}\ \mathcal{D}_0(u,t) = Z$

· $\ker \mathcal{G}_0(u,t)$ constant for $\mathcal{G}_0(u,t) := \mathcal{A}\mathcal{D}_0(u,t)$

(1) has **index 2** if $\exists$ projection operators $\mathcal{Q}_0 : \ X \to X$ onto $\ker \mathcal{G}_0(u,t)$ and $\mathcal{Q}_1 : \ X \to X$ onto $\ker \mathcal{G}_1(u,t)$ with codim(cl($\mathrm{im}\ \mathcal{G}_1(u,t)$)) > 0 and

$$\mathcal{G}_2(u,t) := \mathcal{G}_1(u,t) + \mathcal{B}_0(u,t)(\mathcal{I} - \mathcal{Q}_0)\mathcal{Q}_1$$

injective as well as cl($\mathrm{im}\ \mathcal{G}_2(u,t)$) $= Y$ for all $u \in X$, $t \in [t_0, T]$.

Index of the Coupled System

- DAE index is always ≤ 2.
- DAE index $= 2$ $\Leftrightarrow$ (A_C, A_R, A_V, A_S) has not full row rank and $Q_C^\mathsf{T}(A_V, A_S)$ has not full column rank, where Q_C projector is a onto $\ker A_C^\mathsf{T}$.

  $\Leftrightarrow$ The network has an LI-cutset or a CVS-loop.

Coupled System as Abstract Differential-Algebraic System (II)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0 \qquad \text{with } \mathcal{A}^* v := \begin{pmatrix} A_C^\top v_1 \\ v_2 \\ A_S^\top v_4 \\ v_6 \\ v_7 \end{pmatrix},\ \mathcal{D}(u,t) = \begin{pmatrix} q_C(A_C^\top u_1, t) \\ \phi_L(u_2,t) \\ -\mathfrak{r}_1 u_5 \\ u_6 \\ u_7 \end{pmatrix},$$

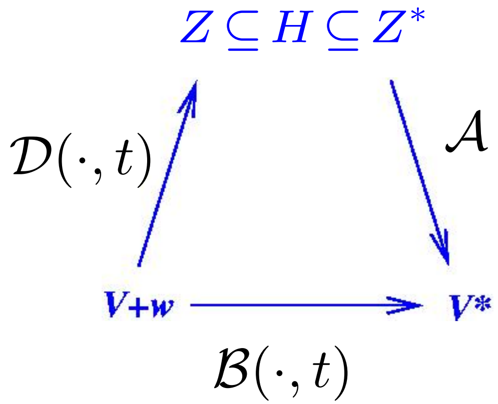
$$\begin{aligned}
\langle \mathcal{B}(u,t), v\rangle_V \;=\; & v_1^\top [A_R g(A_R^\top u_1, t) + A_L u_2 + A_V u_3 + A_S u_4 + A_I i_s(t)] \\
& - [v_2^\top A_L^\top + v_3^\top A_V^\top] u_1 + v_3^\top v_s(t) + v_4^\top u_4 - v_4^\top \mathfrak{r}_2 (J_n + J_p) \\
& + \int_\Omega \varepsilon \operatorname{grad}(u_5 + f \cdot A_S^\top u_1 + g) \cdot \operatorname{grad} v_5 \,\mathrm{d}x + \int_\Omega q(u_6 - u_7 - N) v_5 \,\mathrm{d}x \\
& + \tfrac{1}{q} \int_\Omega (J_n \cdot \operatorname{grad} v_6 - J_p \cdot \operatorname{grad} v_7)\,\mathrm{d}x + \int_\Omega R(u_6, u_7, J_n, J_p)(v_6 + v_7)\,\mathrm{d}x \\
& + \int_{\Gamma_{\mathsf{MI}}} (\alpha u_5 - \tilde{\beta}) v_5 \,\mathrm{d}\sigma + \int_{\Gamma_{\mathsf{MI}}} R_{\mathsf{surf}}(u_6, u_7)(v_6 + v_7)\,\mathrm{d}\sigma + \int_{\Gamma_{\mathsf{S}}} [v_n(u_6 - n_0) v_6 + v_p(u_7 - p_0) v_7]\,\mathrm{d}\sigma
\end{aligned}$$

where $\quad \mathfrak{r}_1 v := \int_\Gamma \varepsilon \operatorname{grad} v \cdot \nu\, \chi_2 \,\mathrm{d}\sigma, \qquad \mathfrak{r}_2 v := \int_\Gamma v \cdot \nu\, \chi_1 \,\mathrm{d}\sigma \qquad$ and

$$u(t) = (e(t), j_L(t), j_V(t), j_S(t), \tilde{V}(\cdot,t), n(\cdot,t), p(\cdot,t))$$

Coupled System as Abstract Differential-Algebraic System (II)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0$$



$V := \bigtimes_{i=1}^{7} V_i$ with

$$V_1 = \mathbb{R}^{n-1}, \quad V_2 = \mathbb{R}^{n_L}, \quad V_3 = \mathbb{R}^{n_V}, \quad V_4 = \bigtimes_{l=1}^{n_S} \mathbb{R}^{k_l - 1}$$

$$V_5 = \{v \in \bigtimes_{l=1}^{n_S} H^2(\Omega_l) : \ v_l = 0 \text{ on } \Gamma_{l\mathsf{O}} \cup \Gamma_{l\mathsf{S}}\},$$

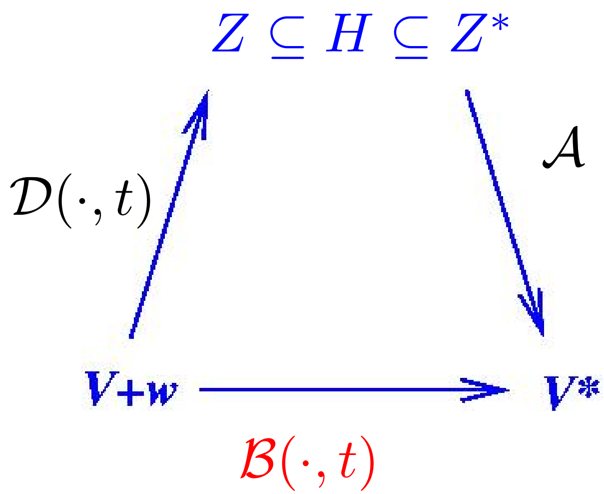
$$V_6 = V_7 = \{v \in \bigtimes_{l=1}^{n_S} H^1(\Omega_l) : \ v_l = 0 \text{ on } \Gamma_{l\mathsf{O}}\}.$$

$$Z := \mathbb{R}^{n_C} \times V_2 \times V_4 \times V_6 \times V_7$$

$$H := \mathbb{R}^{n_C} \times V_2 \times V_4 \times \bigtimes_{l=1}^{n_S} L_2(\Omega_l) \times \bigtimes_{l=1}^{n_S} L_2(\Omega_l)$$

Coupled System as Abstract Differential-Algebraic System (II)

$$\mathcal{A}\frac{d}{dt}\mathcal{D}(u(t),t) + \mathcal{B}(u(t),t) = 0$$



- $Z \subseteq H \subseteq Z^*$ evolution triple
- V real, separable, reflexive Banach space
- $\mathcal{A}$, $\mathcal{D}$ continuous operators
- $\mathcal{B}$ is bounded !

$$W^1_{2,\mathcal{D}}(t_0, T; V, Z, H) := \{u \in L_2(t_0, T; V) : \frac{d}{dt}\mathcal{D}(u(t),t) \in L_2(t_0, T; Z^*)\}$$

$$\|u\|_{W^1_{2,\mathcal{D}}} := \|u\|_{L_2(t_0,T;V)} + \|(\mathcal{D}(u,t))'\|_{L_2(t_0,T;Z^*)}$$

Assumptions

$$\begin{aligned} \mathcal{A}\frac{d}{dt}(\mathcal{D}u(t)) + \mathcal{B}(t)u(t) &= q(t) \\ \mathcal{D}u(t_0) &= z_0 \in Z \end{aligned}$$

- $\mathcal{A} = \mathcal{D}^*$, $\mathcal{D}$ is linear, continuous and surjective
- $\mathcal{B}(t)$ is linear, uniformly bounded and strongly monotone $\forall\, t \in [t_0, T]$
- $z_0 \in Z$, $q \in L_2(t_0, T; Z^*)$
- $\{w_1, w_2, ...\}$ basis in V, $\{z_1, z_2, ...\}$ basis in Z with

$$\forall\, n \in \mathbb{N}\; \exists\, m_n \in \mathbb{N} : \quad \{\mathcal{D}w_1, ..., \mathcal{D}w_n\} \subseteq \{z_1, ..., z_{m_n}\}$$

- $(z_{n_0}) \in Z$: $\quad z_{n_0} \to z_0$ in Z with $z_{n_0} \in \text{span}\{\mathcal{D}w_1, ..., \mathcal{D}w_n\}$

Galerkin Approach

$$\langle \mathcal{A}[\mathcal{D}u(t)]', v\rangle_V + \langle \mathcal{B}(t)u(t), v\rangle_V = \langle q(t), v\rangle_V \qquad \forall\, v \in V$$

$$u_n(t) = \sum_{i=1}^{n} c_{in}(t) w_i$$

Galerkin equations: $\forall\, i = 1, ..., n$

$$\begin{aligned} \langle \mathcal{A}[\mathcal{D}u_n(t)]', w_i\rangle_V + \langle \mathcal{B}(t)u_n(t), w_i\rangle_V &= \langle q(t), w_i\rangle_V \\ \mathcal{D}u_n(t_0) &= z_{n0} \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \Big(\sum_{j=1}^{n} [c_{jn}(t)\mathcal{D}w_j]' | \mathcal{D}w_i \Big)_H + \sum_{j=1}^{n} \langle \mathcal{B}(t)w_j, w_i\rangle_V c_{jn}(t) &= \langle q(t), w_i\rangle_V \\ \mathcal{D}u_n(t_0) &= z_{n0} \end{aligned}$$

Galerkin Equations

$$
\begin{aligned}
A(Dc_n(t))' + B(t)c_n(t) &= r(t) \\
Dc_n(t_0) &= D\alpha_n
\end{aligned}
$$

with

$$
c_n(t) = \begin{pmatrix} c_{1n}(t) \\ \vdots \\ c_{nn}(t) \end{pmatrix}, \quad r(t) = \begin{pmatrix} \langle q(t), w_1 \rangle_V \\ \vdots \\ \langle q(t), w_n \rangle_V \end{pmatrix},
$$

and

$$
A = (a_{ik})_{\substack{i=1,\ldots,n \\ k=1,\ldots,m}}, \quad D = (d_{kj})_{\substack{k=1,\ldots,m, \\ j=1,\ldots,n}} \quad B(t) = (b_{ij}(t))_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}
$$

with

$$
\mathcal{D}w_i = \sum_{k=1}^{m_n} a_{ik} z_k
$$

and

$$
d_{kj} = (\mathcal{D}w_j | z_k)_H \qquad \text{and} \qquad b_{ij}(t) = \langle \mathcal{B}(t) w_j, w_i \rangle_V
$$

for $i, j = 1, \ldots, n$ and $k = 1, \ldots, m_n$

Properties of the Resulting DAE

$$
\begin{aligned}
A(Dc_n(t))' + B(t)c_n(t) &= r(t) \\
Dc_n(t_0) &= D\alpha_n
\end{aligned}
$$

1. The DAE has a proper formulated leading term, i.e.
$$\ker A \oplus \mathrm{im}\ D = \mathbb{R}^{m_n}.$$
2. $(\mathrm{im}\ A)^{\perp} = \ker D$
3. AD is positive semidefinite, $B(t)$ is positive definite.
4. The DAE has maximal the index 1.

$$
\begin{aligned}
&\|c_n\|_{L^2([t_0,T],\mathbb{R}^n)} + \|Dc_n\|_{C([t_0,T],\mathbb{R}^{m_n})} \\
&\quad + \|(Dc_n)'\|_{L^2([t_0,T],\mathbb{R}^{m_n})} \leq C\left(\|D\alpha_n\| + \|r\|_{L^2([t_0,T],\mathbb{R}^n)}\right).
\end{aligned}
$$

Existence and Uniqueness of Solutions of the ADAS

Assumptions:

- $\ker \mathcal{D}$ splits V, i.e. $\exists$ projection operator $\mathcal{Q}: \ V \to V$ with $\mathrm{im}\ \mathcal{Q} = \ker \mathcal{D}$
- basis $\{w_1, w_2, ...\}$ of V such that

$$w_i \in \mathrm{im}\ I - \mathcal{Q} \quad \text{for odd } i,$$
$$w_i \in \mathrm{im}\ \mathcal{Q} \qquad\quad \text{for even } i.$$

The ADAS

$$\mathcal{A}\frac{d}{dt}(\mathcal{D}u(t)) + \mathcal{B}u(t) = q(t), \qquad \mathcal{D}u(t_0) = z_0 \ \in Z$$

has exactly one solution $u \in W^1_{2,\mathcal{D}}(t_0, T; V, Z, H)$.

Summary

- network model $\rightarrow$ DAE
- semiconductor model $\rightarrow$ system of parabolic and elliptic PDEs
- coupling over boundary conditions and integral relations
- index is always ≤ 2 ($= 2$, if there is a CVS-loop or an LI-cutset)
- Galerkin method converges for linear abstract DAEs of index 1 with monotone operators if the basis is chosen appropriately.
- Do we have convergence also for the nonlinear coupled system?
- Which index have the Galerkin equations if the network has CVS-loops or LI-cutsets?
- How should we choose the basis functions for abstract DAEs with higher index?