Field-Circuit Coupling for Mechatronic Systems: Some Trends and Techniques

Stefan Kurz
Robert Bosch GmbH, Stuttgart

Now with the University of the German Federal Armed Forces, Hamburg
stefan.kurz@unibw-hamburg.de
Outline

- The Design Process
- Electric Circuit Elements
- Coils and de Rham Cohomology
- Setting of the Field-Circuit Problem
- Direct Coupling
  - mastered by Field Simulator
  - mastered by Circuit Simulator
- Summary
The Design Process

- System requirements specification
- System design specification
- Components requirements specification
- Components design specification
- Manufacturing of components
- Simulation not included in design process
- System test
- Components test

Iteration, Optimization
Main Categories of Field-Circuit Coupling

**Equivalent Circuit**
- System simulation on network level
- Field simulation to obtain parameters [14]

**Direct Coupling**
- Field and circuit equations are collected in one overall matrix

**Indirect Coupling**
- Keep both simulations separated
- Communication via coupling matrices [3]

**Mastered by Field Simulator**
- FE matrix augmented by circuit's contribution [18]

**Mastered by Circuit Simulator**
- FE equations represented as a multiport device [19]
The Circuit Concept of Voltage

- Voltmeters measure the line integral of the electric field along the path formed by the connecting leads.

- There are implicit limitations on the use of voltmeters: Indication should not depend appreciably on the exact position of the leads.
Definition of an Electric Circuit Element (ECE)

1. \( \exists \varphi \in \mathcal{F}^0(\Gamma) : \ t \vec{E} = d \varphi \)
2. \( \varphi = \varphi_k = \text{const} \) on \( S_k \),
   \( S_k \ldots \text{terminal connectors} \)
3. \( t d \vec{H} = t (\vec{j} + \partial_t D) \)
   \( = 0 \) on \( S_e \),
   \( S_e = \Gamma \setminus \sum S_k \)
   …insulating boundary

see: Munteanu & Ioan 2001 [14]
Terminal currents $i_k$ and voltages $u_k$

$$i_k = - \int_{\partial S_k} tH, \quad k = 1, \ldots, n$$

$$\rightarrow \sum_{k=1}^{n} i_k = 0 \quad \text{Kirchhoff’s current law}$$

$$u_k = \varphi_k - \varphi_n, \quad k = 1, \ldots, n \quad \text{Kirchhoff’s voltage law}$$

Received power

$$p_{\Gamma}(t) = - \int_{\Gamma} t(E \wedge H) = \sum_{k=1}^{n-1} u_k i_k$$

$$\rightarrow \int t(E \wedge H) + \sum u_k i_k \text{ gives a measure for the approximation of the ECE conditions (1) - (3)}$$
Stranded Conductor

- Winding density
  \[ \tau = \frac{w}{\sum} \mathrm{d}\sum \]
  \[ \int_{\sum} \tau = w \]

- Induced voltage
  \[ u_{\text{ind}} = u_{12} \bigg|_{i=0} \]
  \[ = \int_{\text{wire}} \partial_t A \]
  \[ = \int_{\Omega_{\text{coil}}} \partial_t A \wedge \tau \]
Coils and de Rham Cohomology

Solid Conductor

$\Omega_{\text{coil}}$

Highly conductive electrode, $w = 1$

$u_{12}$

i
Assume $\sigma = \text{const}$ throughout $\Omega_{\text{Coil}}$

Apply a DC voltage $U$ to the coil $\rightarrow I, \dot{j}_S$

Let $\tau = \frac{\dot{j}_S}{I}$ where $\int_{\Sigma} \tau = w$

Note that

$$d\tau = 0, \quad \delta\tau = 0, \quad t\tau = 0,$$

i.e. $\tau$ is a normal harmonic form.

$(\Sigma w)$ can be regarded as a basis for the relative 2-cycles of $\Omega_{\text{Coil}} \pmod{\partial\Omega_{\text{Coil}}}$.

$(\tau)$ is the dual basis, since $\int_{\Sigma} \tau = 1$
The current density $\mathbf{j}$ is divergence-free ($d \cdot \mathbf{j} = 0$) and has zero trace ($t \cdot \mathbf{j} = 0$)

\[ \mathbf{j} = d \mathbf{T} + i \mathbf{T}, \quad t \mathbf{T} = 0 \]

with the current vector potential $\mathbf{T}$ and the terminal current $i = \int_{w} \mathbf{j}$.

Power delivered by the source

\[ p(t) = i(t) \cdot u_{12}(t) \]

\[ = i(t) \cdot \left( i(t) \cdot \frac{1}{\kappa} \int_{\Omega_{\text{Coil}}} * \mathbf{T} \wedge \mathbf{T} + \frac{d}{dt} \int_{\Omega_{\text{Coil}}} \mathbf{A} \wedge \mathbf{T} \right) \]
Equivalent circuit diagram

\[ u_{12}(t) = R \cdot i(t) + \frac{d\psi}{dt} \]

\[ R = \frac{1}{\kappa} \int_{\Omega_{\text{Coil}}} *\tau \wedge \tau \]

\[ \psi = \int_{\Omega_{\text{Coil}}} A \wedge \tau \]

Circuit parameters \( R, \psi \) computable from a suitable basis of the space of normal harmonic forms.
A Simple Example


Circular coil, stranded conductor

Square aluminium plates

1/8 of the problem, used for computer model

coil

plate
Setting of the Problem

Response to a Step Voltage

![Graph showing computed and measured current over time.]

- Computed
- Measured
Setting of the Problem

Basic Equations

- Fundamental equation of the eddy current problem
  \[ d \nu \ast d \mathbf{A} + \sigma \ast \partial_t \mathbf{A} = \mathbf{j}_S \]

- Circuit equation for stranded conductor
  \[ \frac{d \psi}{dt} + R \cdot i = u_{12} \]

- Winding density
  \[ \tau = \frac{j_S}{i}, \quad d \tau = 0 \]

- Flux linkage
  \[ \psi = \int_{\Omega_{\text{Coil}}} \mathbf{A} \wedge \tau \]
Field-Circuit Coupling

\[
\begin{align*}
\text{d} \nu \ast \text{d} A + \sigma \ast \partial_t A - i \cdot \tau &= 0 \\
\int_{\Omega_{\text{Coil}}} (\partial_t A) \wedge \tau + R \cdot i &= u_{12}
\end{align*}
\]

PDE

\[\Rightarrow\] Space discretisation: Galerkin Edge-FEM

\[\Rightarrow\] Time discretisation: Implicit Euler

\[\Rightarrow\] Non-linear solver: Newton-Raphson

\[
\begin{pmatrix}
[J] \\
-\{U\}
\end{pmatrix}
\begin{pmatrix}
\{\delta A\} \\
\delta i
\end{pmatrix}
= 
\begin{pmatrix}
\{F_\delta\} \\
e_\delta
\end{pmatrix}
\]

Linear system
Slightly More General Problem: $p > 1$ Coils

- Linear system

\[
\begin{pmatrix}
[J] & -[U] \\
-[U]^T & -\Delta t [R]
\end{pmatrix}
\begin{pmatrix}
\{\delta A\} \\
\{\delta i\}
\end{pmatrix}
=
\begin{pmatrix}
\{F_\delta\} \\
\{e_\delta\}
\end{pmatrix}
\]

- Original system: Sparse, symmetric but indefinite \(\otimes\)

\[J \in \mathbb{R}^{n \times n}, \text{s.p.d.} \]
\[U \in \mathbb{R}^{n \times p} \]
\[R \in \mathbb{R}^{p \times p}, \text{s.p.d.} \]
\[p \ll n\]
The Schur Complement System

- Idea: Eliminate \( \{ \delta i \} \) by taking the Schur complement (Fetzer & Kurz 1998, De Gersem et.al. 2000 [8])

\[
\begin{bmatrix}
\bar{J} \\
-\left[ U \right]^T - \Delta t [R]
\end{bmatrix}
\begin{bmatrix}
\{ \delta A \} \\
\{ \delta i \}
\end{bmatrix}
= \begin{bmatrix}
\bar{J} \\
-\left[ U \right]^T - \Delta t [R]
\end{bmatrix}
\begin{bmatrix}
\{ \delta A \} \\
\{ \delta i \}
\end{bmatrix}
\begin{bmatrix}
\{ \delta e \}
\end{bmatrix}
\]

- Resulting system

\[
\{ \bar{F}_\delta \} = \{ F_\delta \} - \left[ U \right] \frac{1}{\Delta t} [R]^{-1} \{ \delta e \}
\]
Solution Strategy

\[
\begin{bmatrix}
\tilde{J}
\end{bmatrix} = \begin{bmatrix}
J
\end{bmatrix} + \begin{bmatrix}
U
\end{bmatrix} \frac{1}{\Delta t} \begin{bmatrix}
R
\end{bmatrix}^{-1} \begin{bmatrix}
U
\end{bmatrix}^T
\]

Field Jacobian...
- s.p.d.
- sparse

...augmented by circuit’s contribution
- symmetric, non-negative
- rank p

- As \( \begin{bmatrix} \tilde{J} \end{bmatrix} \) is s.p.d., the PCG method is the best choice for iterative solution.

- The Schur complement needs not to be explicitly formed to deal with matrix-by-vector products.

- The low rank perturbation does not much affect the original convergence rate.
  \((\text{Kurz & Rischmüller 2002})\)
What about More General Circuits?

- Situation more involved for general circuits: Stranded conductors, solid conductors, capacitors, inductors, resistors and sources joined together.

- Topological analysis (tree-cotree decomposition, Signal Flow Graph) yields a symmetric indefinite system with sparse FE matrix. (De Gersem et.al. 1998 [9])

$$
\begin{pmatrix}
[J] \\
[Q]^T \\
-[P]^T
\end{pmatrix}
\begin{pmatrix}
[Q] & -[P] \\
\Delta t[Y] & \Delta t[D] \\
\Delta t[D]^T & -\Delta t[Z]
\end{pmatrix}
\begin{pmatrix}
[U] \\
\{\delta A\} \\
\{\delta u_{tw}\} \\
\{\delta i_{in}\}
\end{pmatrix}
= 
\begin{pmatrix}
\{F_\delta\} \\
\{e_\delta\}
\end{pmatrix}
$$

- Elimination of the circuit part $[R]$ results in a s.p.d. Schur complement
The Schur Complement System

→ Eliminate \( \{\delta A\} \) by taking the Schur complement

\[
\begin{bmatrix}
\bar{R}
\end{bmatrix}
= 
\begin{bmatrix}
R
\end{bmatrix} + 
\frac{1}{\Delta t} 
\begin{bmatrix}
U
\end{bmatrix}^T
\begin{bmatrix}
J
\end{bmatrix}^{-1}
\begin{bmatrix}
U
\end{bmatrix}^T
\text{ s.p.d.}
\]

→ Resulting system

\[
\begin{bmatrix}
J & -U \\
0 & -\Delta t \bar{R}
\end{bmatrix}
\begin{bmatrix}
\{\delta A\}
\\
\{\delta i\}
\end{bmatrix}
= 
\begin{bmatrix}
\{F_\delta\}
\\
\{\tilde{e}_\delta\}
\end{bmatrix}
\]

\[
\{\tilde{e}_\delta\} = \{e_\delta\} + \begin{bmatrix}
U
\end{bmatrix}^T
\begin{bmatrix}
J
\end{bmatrix}^{-1} \{F_\delta\}
\]
\[
[R] \{i\}_{k+1}^{t+\Delta t} + \frac{1}{\Delta t} [L] \{\Delta i\}_{k+1} + \{v\}_k^{t+\Delta t} = \{u\}_k^{t+\Delta t}
\]

\[
[L] = [U]^T [J]^{-1} [U]
\]

... inductance matrix

\[
\{v\}_k^{t+\Delta t} = \frac{1}{\Delta t} [U]^T \left( \{\Delta A\}_{k+1} \right)_{\{i\} = \text{const}}
\]

... open circuit voltage that would be induced in the coils if the currents were kept constant

\(k\) ... index referring to the Newton-Raphson method
\(\Delta\) ... indicates Euler increments during \(\Delta t\)
Direct Coupling Mastered by Circuit Simulator

Equivalent Multiport Device \[^{[19]}\]

⇒ Circuit realisation of the second Schur complement equation

\[
\begin{align*}
\text{Circuit realisation of the second Schur complement equation} \\
\begin{array}{c}
\text{Equivalent Multiport Device} \\
\text{Circuit realisation of the second Schur complement equation}
\end{array}
\end{align*}
\]
How Do We Get the Parameters?

Assuming that the field solver is able to compute the coils’ flux linkages:

- Freeze the field Jacobian at its current value \( [J] = [J]^t_k \)

- Keep the currents constant and have the solver compute the flux increments \( \Delta \psi_{0,\nu} \) during the time step \( \Delta t \),

  \[
  \text{let } \quad v_\nu = \frac{\Delta \psi_{0,\nu}}{\Delta t}, \quad \nu = 1, \ldots, p.
  \]

- Increase successively each current \( i_\mu \) by one unit and have the solver compute the flux increments \( \Delta \psi_{\mu,\nu} \) in coil number \( \nu \),

  \[
  \text{let } \quad I_{\mu,\nu} = \Delta \psi_{\mu,\nu} - \Delta \psi_{0,\nu}, \quad \mu, \nu = 1, \ldots, p.
  \]

see: Demerdash & Nehl 1999 [6], Mc Dermott, Zhou & Gilmore 1997 [13].
Solution Strategy

\[
\begin{align*}
&\{i\}_t, \{i\}_0^{t+\Delta t} \\
&\{A\}_t, \{A\}_0^{t+\Delta t} \\
&\text{circuit solution} \\
&\text{p+1 field solutions} \\
&\text{Newton step}
\end{align*}
\]

\[
\begin{align*}
&\{i\}_{t+\Delta t} \\
&\{A\}_{t+\Delta t} \\
&\text{circuit solution} \\
&\text{p+1 field solutions} \\
&\text{Newton step}
\end{align*}
\]

\[
\begin{align*}
&\{i\}_{t+2\Delta t} \\
&\{A\}_{t+2\Delta t} \\
&\text{circuit solution} \\
&\text{p+1 field solutions} \\
&\text{Newton step}
\end{align*}
\]
Field-circuit coupling can create a link between simulation on the system and the components level of mechatronic systems.

The Electric Circuit Element provides a rigorous definition of terminal voltages and currents, according to Kirchhoff’s laws.

Direct coupling yields one overall matrix, which collects the field and the circuit equations.

The solution process can be mastered either by the field or by the circuit simulator.

Solution strategies for the resulting Schur complement systems have been discussed in both cases.