Dual Finite Element Formulations and Associated Global Quantities for Field-Circuit Coupling

Edge and nodal finite elements allowing natural coupling of fields and global quantities

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Constraints in partial differential problems

- Local constraints (on local fields)
  - Boundary conditions
    - i.e., conditions on local fields on the boundary of the studied domain
  - Interface conditions
    - e.g., coupling of fields between sub-domains
- Global constraints (functional on fields)
  - Flux or circulations of fields to be fixed
    - e.g., current, voltage, m.m.f., charge, etc.
  - Flux or circulations of fields to be connected
    - e.g., circuit coupling

Weak formulations for finite element models

Essential and natural constraints, i.e., strongly and weakly satisfied

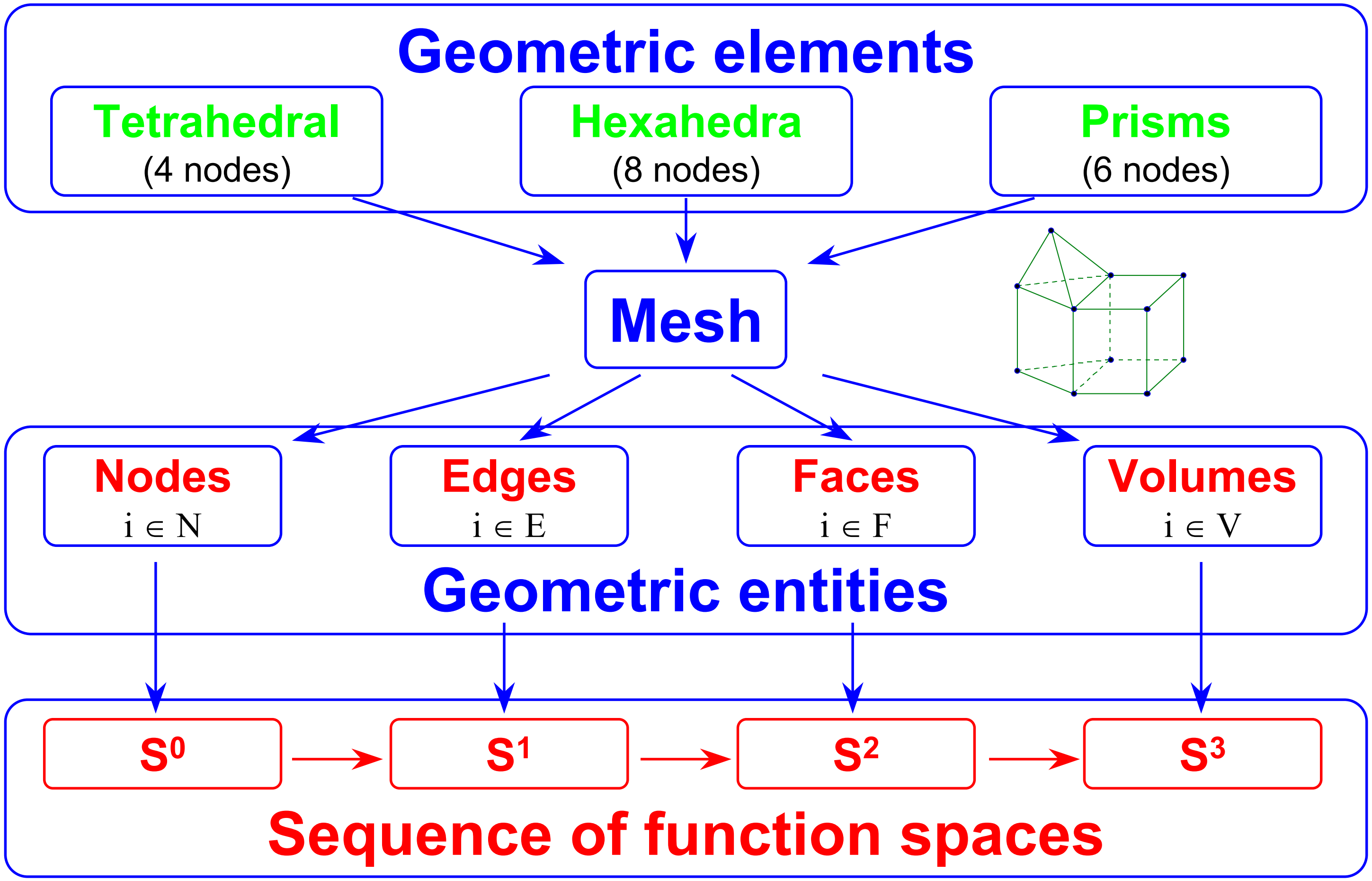
Constraints in electromagnetic systems

- Coupling of scalar potentials with vector fields
  - e.g., in h-ϕ and a-v formulations
- Gauge condition on vector potentials
  - e.g., magnetic vector potential a, source magnetic field h_s
- Coupling between source and reaction fields
  - e.g., source magnetic field h_s in the h-ϕ formulation, source electric scalar potential v_s in the a-v formulation
- Coupling of local and global quantities
  - e.g., currents and voltages in h-ϕ and a-v formulations (massive, stranded and foil inductors)
- Interface conditions on thin regions
  - i.e., discontinuities of either tangential or normal components

Interest for a "correct" discrete form of these constraints

Sequence of finite element spaces

Sequence of finite element spaces



Sequence of finite element spaces

	Functions	Properties	Functionals	Degrees of freedom
S^0	$\{s_i\ ,\ i \in N\}$	$s_i(x_j) = \delta_{ij} \quad \forall\, i,j \in N$	Point evaluation	Nodal value
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	$\int_j \mathbf{s}_i \cdot d\mathbf{l} = \delta_{ij} \quad \forall\, i,j \in E$	Curve integral	Circulation along edge
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	$\int_j \mathbf{s}_i \cdot \mathbf{n}\, ds = \delta_{ij} \quad \forall\, i,j \in F$	Surface integral	Flux across face
S^3	$\{s_i\ ,\ i \in V\}$	$\int_j s_i\, dv = \delta_{ij} \quad \forall\, i,j \in V$	Volume integral	Volume integral

→ **Nodal element**

→ **Edge element**

→ **Face element**

→ **Volume element**

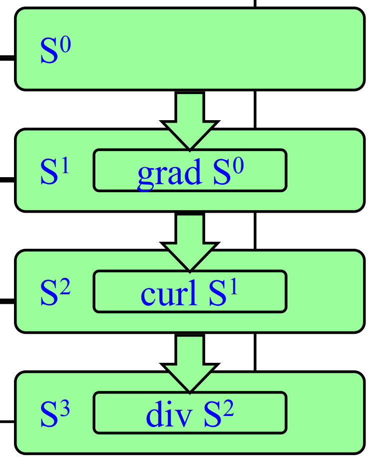
Bases

$$u_K = \sum_i \phi_i(u)\, s_i$$

Finite elements

Sequence of finite element spaces

	Base functions	Continuity across element interfaces	Codomains of the operators
S^0	$\{s_i\ ,\ i \in N\}$	value	
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	tangential component	$\mathrm{grad}\ S^0 \subset S^1$
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	normal component	$\mathrm{curl}\ S^1 \subset S^2$
S^3	$\{s_i\ ,\ i \in V\}$	discontinuity	$\mathrm{div}\ S^2 \subset S^3$



Conformity

Sequence

$$S^0 \xrightarrow{\mathrm{grad}} S^1 \xrightarrow{\mathrm{curl}} S^2 \xrightarrow{\mathrm{div}} S^3$$

Magnetodynamic problem with global constraints

Equations

$$\mathrm{curl}\ \mathbf{h} = \mathbf{j}$$

$$\mathrm{curl}\ \mathbf{e} = -\,\partial_t\, \mathbf{b}$$

$$\mathrm{div}\ \mathbf{b} = 0$$

Boundary conditions

$$\mathbf{n} \times \mathbf{h}\,\big|_{\Gamma_h} = 0\ , \quad \mathbf{n} \cdot \mathbf{b}\,\big|_{\Gamma_e} = 0$$

$$\int_{\gamma_i} \mathbf{e} \cdot d\mathbf{l} = V_i \quad , \qquad \int_{\Gamma_j^i} \mathbf{n} \cdot \mathbf{j}\ ds = I_i$$

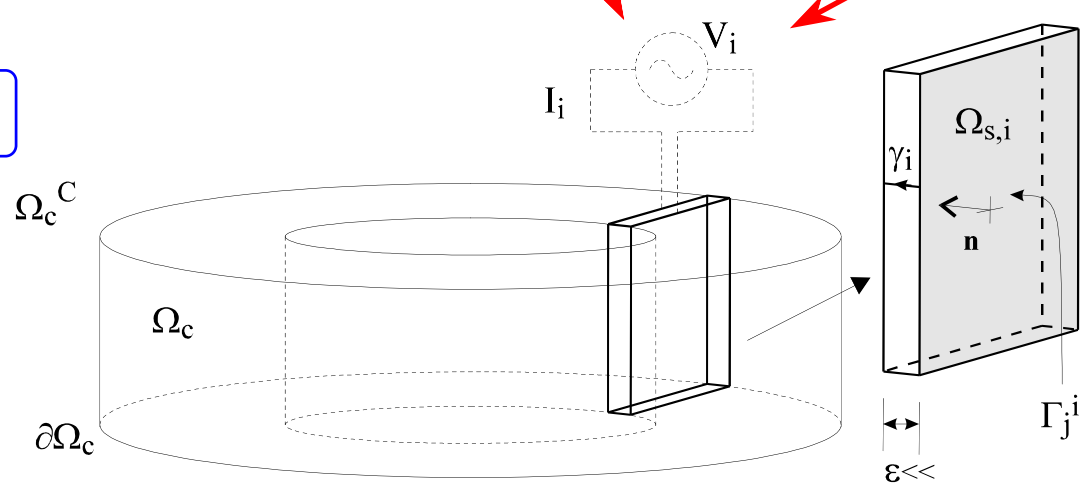
Voltage ***Current***

Global conditions for circuit coupling

Constitutive relations

$$\mathbf{b} = \mu\ \mathbf{h}$$

$$\mathbf{j} = \sigma\ \mathbf{e}$$



Inductor

Weak formulations

Green formulae

involved in weak formulations

Notations

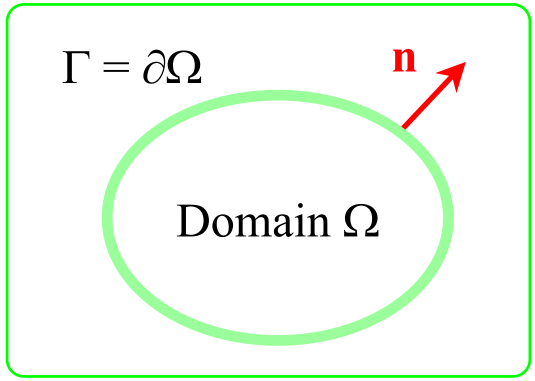
$$(\mathbf{a},\mathbf{b})_\Omega \overset{\text{def}}{=} \int_\Omega \mathbf{a}\cdot\mathbf{b}\,dv$$

$$\langle a,b\rangle_\Gamma \overset{\text{def}}{=} \int_\Gamma a\,b\,ds$$

grad - div formula

$$(\,\mathbf{u}\,,\,\text{grad v}\,)_\Omega + (\,\text{div}\,\mathbf{u}\,,\,\text{v}\,)_\Omega = <\mathbf{n}\cdot\mathbf{u}\,,\,\text{v}>_\Gamma$$

Weak global quantity of flux type



curl - curl formula

$$(\,\text{curl}\,\mathbf{u}\,,\,\mathbf{v}\,)_\Omega - (\,\mathbf{u}\,,\,\text{curl}\,\mathbf{v}\,)_\Omega = <\mathbf{n}\times\mathbf{u}\,,\,\mathbf{v}>_\Gamma$$

$$\mathbf{u},\,\mathbf{v} \in \mathbf{H}^1(\Omega),\ \text{v} \in H^1(\Omega)$$

Weak global quantity of circulation type

h-ϕ and t-ω weak formulations

Magnetic scalar potential in nonconducting regions Ω_c^C

ϕ

$$\mathbf{h_r} = -\operatorname{grad}\phi \quad \text{in } \Omega_c^C$$

with $\mathbf{h} = \mathbf{h}_s + \mathbf{h}_r$

Reaction magnetic field

Source magnetic field

h-ϕ magnetodynamic formulation

$$\partial_t(\mu\mathbf{h},\mathbf{h}')_\Omega + (\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{h}')_{\Omega_c} + <\mathbf{n}\times\mathbf{e}_s,\mathbf{h}'>_{\Gamma_e} = 0 \qquad \forall\mathbf{h}' \in F_{h\phi}(\Omega)$$

(1)

t-ω magnetodynamic formulation (similar)

How to couple local and global quantities ?

h, ϕ **V_i, I_i**

Current as a strong global quantity

Characterization of curl-conform vector fields : h or t

$$\mathbf{h} = \sum_{e \in E} h_e \, \mathbf{s}_e \,, \quad \mathbf{v} \in S^1(\Omega)$$

Coupling of edge end nodal finite elements

Explicit constraints for circulations and zero curl

$$\mathbf{h} = \sum_{k \in E_c} h_k \, \mathbf{s}_k + \sum_{n \in N_c{}^C} \phi_n \, \mathbf{v}_n + \sum_{i \in C} I_i \, \mathbf{c}_i$$

i.e. currents I_i

Basis functions

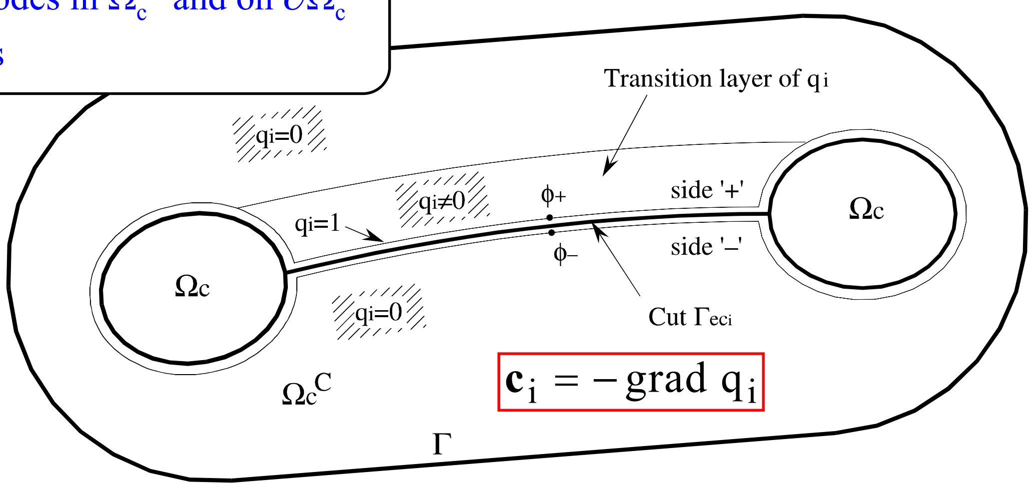
E_c : edges in Ω_c

$N_c{}^C$: nodes in $\Omega_c{}^C$ and on $\partial\Omega_c{}^C$

C : cuts

‘Circulation’ basis function, associated with a group of edges from a cut

$\rightarrow$ its circulation is equal to 1 along a closed path around Ω_c



$$\boxed{\mathbf{c}_i = -\operatorname{grad} q_i}$$

Elementary geometrical entities (nodes, edges) and global ones (groups of edges)

Voltage as a weak global quantity

Discrete weak formulation

$$\partial_t(\mu \mathbf{h},\mathbf{h}')_\Omega + (\sigma^{-1}\mathrm{curl}\,\mathbf{h},\mathrm{curl}\,\mathbf{h}')_{\Omega_c} + \langle \mathbf{n}\times\mathbf{e}_s,\mathbf{h}'\rangle_{\Gamma_e} = 0 \qquad \forall \mathbf{h}' \in F_{h\phi}(\Omega)$$

$$\mathbf{h} = \sum_{k\in E_c} h_k\,\mathbf{s}_k + \sum_{n\in N_c{}^C} \phi_n\,\mathbf{v}_n + \sum_{i\in C} I_i\,\mathbf{c}_i$$

system of equations (symmetrical matrix)

Test function $\mathbf{h}' = \mathbf{s}_k, \mathbf{v}_n$ $\rightarrow$ classical treatment, no contribution for $<\cdot>_{\Gamma_e}$

Test function $\mathbf{h}' = \mathbf{c}_i$ $\rightarrow$ contribution for $<\cdot>_{\Gamma_e}$

$$\langle \mathbf{n}\times\mathbf{e}_s,\mathbf{h}'\rangle_{\Gamma_h} = \langle \mathbf{n}\times\mathbf{e}_s,\mathbf{c}_i\rangle_{\Gamma_h} = \langle \mathbf{n}\times\mathbf{e}_s,-\mathrm{grad}\,q_i\rangle_{\Gamma_h} = \oint_{\gamma_i} \mathbf{e}_s\cdot d\mathbf{l} = V_i$$

Electromotive force

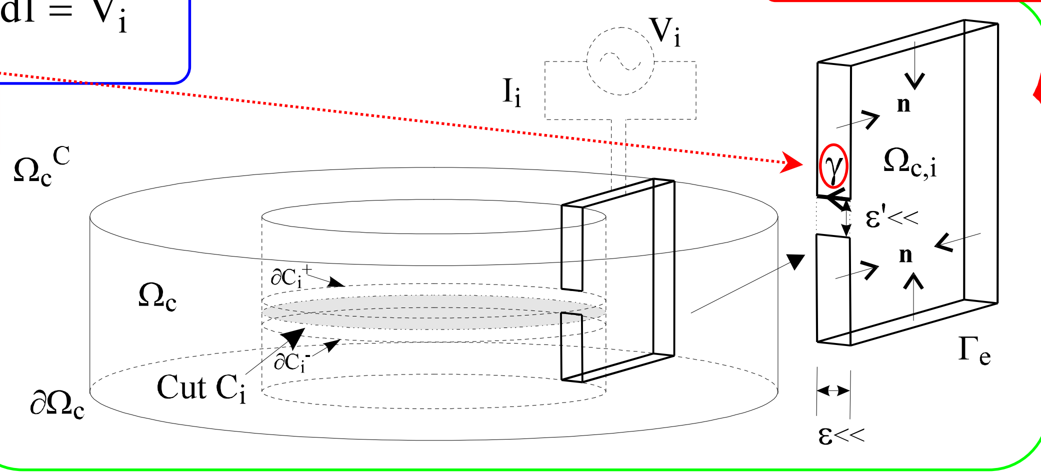
Weak global quantity

Voltage as a weak global quantity and circuit relations

$$\langle \mathbf{n} \times \mathbf{e}_s , \mathbf{c}_i \rangle_{\Gamma_h} = \oint_{\gamma} \mathbf{e}_s \cdot d\mathbf{l} = V_i$$

Electromotive force

Source of e.m.f.



in (1)

$$\partial_t(\mu\, \mathbf{h}, \mathbf{c}_i)_{\Omega} + (\sigma^{-1} \operatorname{curl} \mathbf{h}, \operatorname{curl} \mathbf{c}_i)_{\Omega_c} = -V_i$$

Weak circuit relation between V_i and I_i for inductor i

$$\text{“}\, \partial_t\, (\mathit{Magnetic\ Flux}) + \mathit{Resistance} \times \mathit{Current} = \mathit{Voltage} \,\text{”}$$

Natural way to compute a weak voltage !
Better than an explicit nonunique line integration

Massive and stranded inductors

Massive inductor — **Direct application**

Stranded inductor — **Additional treatment**

Tree technique ...

$$\mathbf{h} = \mathbf{h}_r + \sum_{j \in \Omega_s} I_{s,j}\, \mathbf{h}_{s,j} \qquad \mathbf{h} \in F_{h\phi h_s}(\Omega)$$

Number of turns

Reaction field

Source field due to a magnetomotive force N_j
(one basis function for each stranded inductor)

$$\partial_t(\mu \mathbf{h}, \mathbf{h}')_\Omega + (\sigma^{-1} \mathrm{curl}\,\mathbf{h}, \mathrm{curl}\,\mathbf{h}')_{\Omega_c} + (\sigma^{-1} \mathbf{j}_s, \mathrm{curl}\,\mathbf{h}')_{\Omega_w} + \langle \mathbf{n} \times \mathbf{e}_s, \mathbf{h}' \rangle_{\Gamma_e} = 0$$

$$\forall \mathbf{h}' \in F_{h\phi h_s}(\Omega)$$

$\mathbf{h}' = \mathbf{h}_{s,j}$

$$\partial_t(\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega + I_{s,j}\, (\sigma^{-1}\, \mathbf{j}_{s,j}, \mathrm{curl}\,\mathbf{h}_{s,j})_{\Omega_s} = -V_j$$

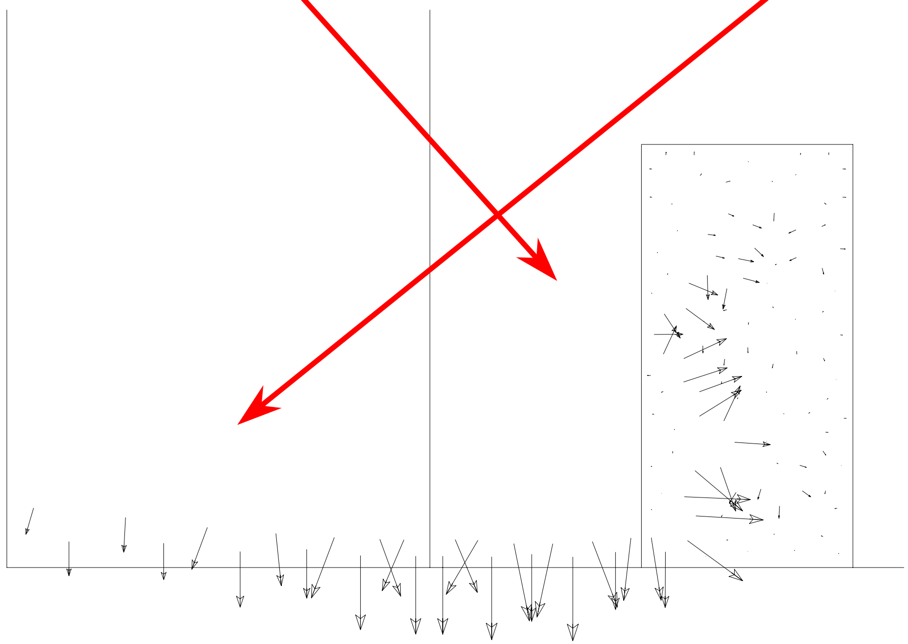
Weak circuit relation between V_j and I_j for stranded inductor j

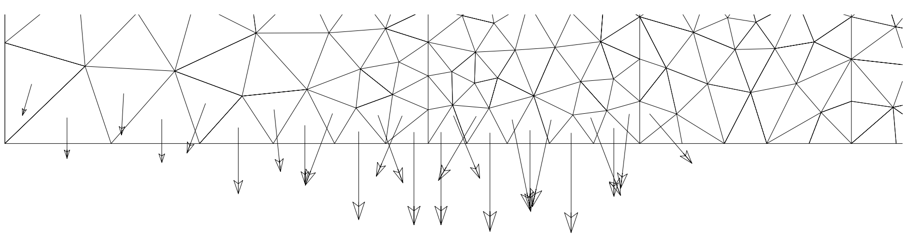
Natural way to compute the magnetic flux through all the wires !

Stranded inductors - Source field

$$\mathbf{h} = \sum_{k \in E_c} h_k \, \mathbf{s}_k + \cancel{\sum_{n \in N_c} \phi_n \, \mathbf{v}_n} + \sum_{i \in C} I_i \, \mathbf{c}_i$$

Simplified source field





Projection method

Source

$$(\operatorname{curl} \mathbf{h}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}} = (\mathbf{j}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}}$$

$$\forall \mathbf{h}' \in F_h(\Omega_{s,j})$$

Electrokinetic problem

$$(\sigma^{-1} \operatorname{curl} \mathbf{h}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}} = 0$$

$$\forall \mathbf{h}' \in F_h(\Omega_{s,j})$$

Source = N_j

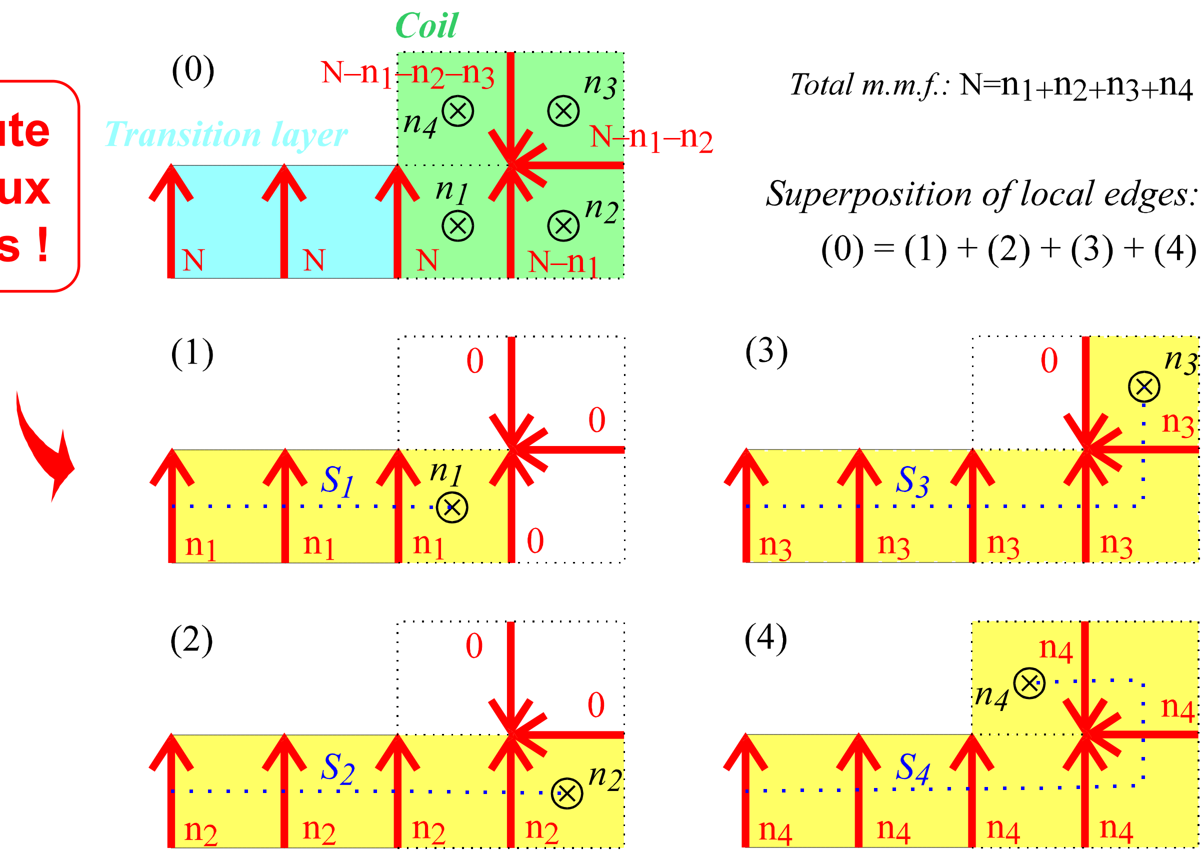
With gauge condition (tree) & boundary conditions

Stranded inductors - Magnetic flux

Physical and geometrical interpretation of the circuit relation

$(\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega$

Natural way to compute the magnetic flux through all the wires !



Total m.m.f.: $N = n_1 + n_2 + n_3 + n_4$

Superposition of local edges:

$(0) = (1) + (2) + (3) + (4)$

a-v weak formulation

Magnetic vector potential - Electric scalar potential

a **v**

$$\mathbf{e} = -\partial_t \mathbf{a} - \mathrm{grad}\, v \quad \text{in } \Omega_c \,, \quad \text{with} \quad \mathbf{b} = \mathrm{curl}\, \mathbf{a} \quad \text{in } \Omega$$

a-v magnetodynamic formulation

$$(\mu^{-1} \mathrm{curl}\, \mathbf{a}, \mathrm{curl}\, \mathbf{a}')_{\Omega} + (\sigma \partial_t \mathbf{a}, \mathbf{a}')_{\Omega_c} + (\sigma\, \mathrm{grad}\, v, \mathbf{a}')_{\Omega_c} - (\mathbf{j}_s, \mathbf{a}')_{\Omega_s} = 0\,, \quad \forall \mathbf{a}' \in F_a(\Omega) \qquad (1)$$

How to couple local and global quantities ?

a, v **V_i, I_i**

Voltage as a strong global quantity

With a' = grad v' in (1)

$$(\sigma \partial_t \mathbf{a}, \operatorname{grad} v')_{\Omega_c} + (\sigma \operatorname{grad} v, \operatorname{grad} v')_{\Omega_c} = \langle \mathbf{n} \cdot \mathbf{j}, v' \rangle_{\Gamma_j} \qquad \forall v' \in F_v(\Omega_c) \qquad (2)$$

Weak form of div j = 0

At the discrete level : implication only true when $\operatorname{grad} F_v(\Omega_c) \subset F_a(\Omega)$

OK with nodal and edge finite elements

Otherwise : consideration of the 2 formulations (1) and (2) with a penalty term for gauge condition

Voltage as a strong global quantity

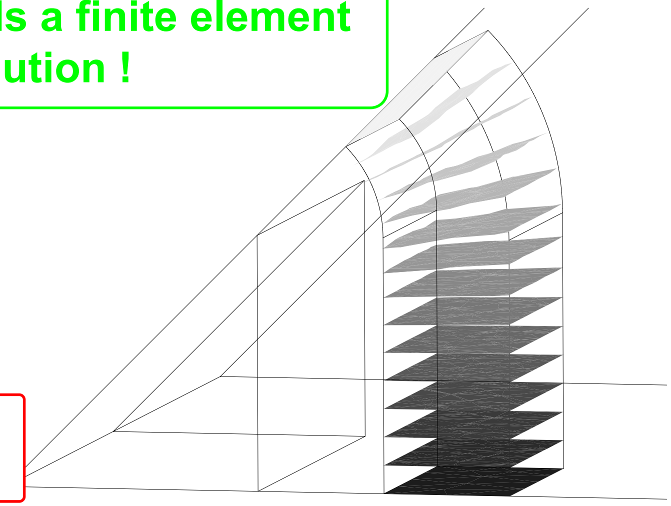
Unit source electric scalar potential v_0 (basis function for the voltage)

$$v = \sum_{i \in \Gamma_j} V_i \, v_0^{\,i}$$

Electrokinetic problem (physical field)

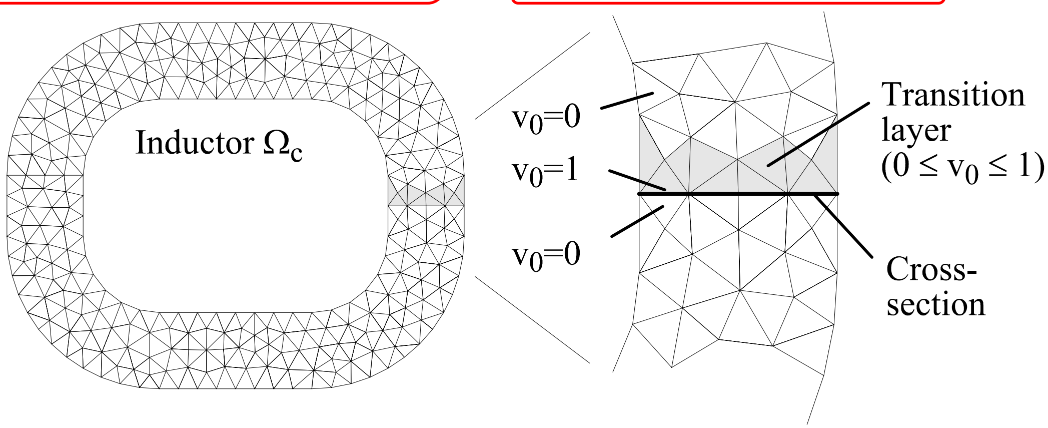
$$(\sigma \operatorname{grad} v_0 , \operatorname{grad} v')_{\Omega_c} = 0 \,, \quad \forall v' \in F_v(\Omega_c)$$

Needs a finite element resolution !



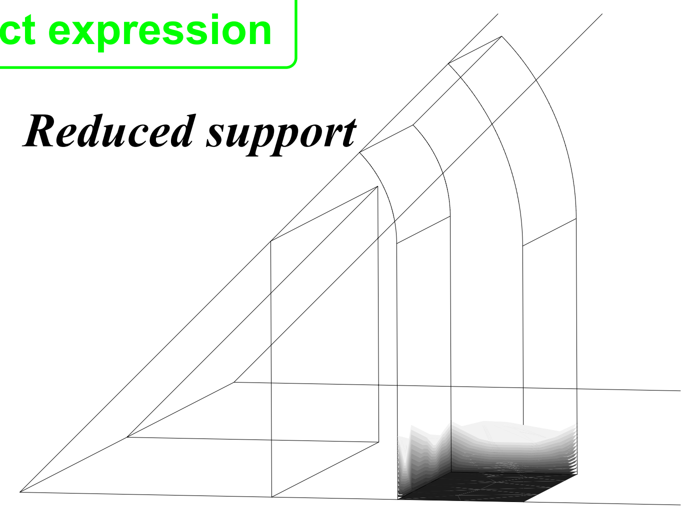
Generalized potential (nonphysical field)

$$v_0 = s^i = \sum_{n \in \Gamma_j^{\,i}} s_n$$



Direct expression

Reduced support



Current as a weak global quantity and circuit relations

for massive inductor i

$$<\mathbf{n}\cdot\mathbf{j}, s^i>_{\Gamma_j^i} = <\mathbf{n}\cdot\mathbf{j}, 1>_{\Gamma_j^i} = \int_{\Gamma_j^i} \mathbf{n}\cdot\mathbf{j}\, ds = I_i$$

in (2)

$$I_i = (\sigma\,\partial_t \mathbf{a}, \operatorname{grad} s^i)_{\Omega_c} + (\sigma \operatorname{grad} v, \operatorname{grad} s^i)_{\Omega_c}$$

$$I_i = (\sigma\,\partial_t \mathbf{a}, \operatorname{grad} s^i)_{\Omega_c} + V_i\, (\sigma \operatorname{grad} v_0{}^i, \operatorname{grad} s^i)_{\Omega_c}$$

Weak circuit relation between V_i and I_i for massive inductor i

Natural way to compute a weak current !
Better than an explicit nonunique surface integration

Circuit relation for stranded inductors

From the a-formulation

$$\cancel{I_j = (\sigma\, \partial_t \mathbf{a}, \mathrm{grad}\, v_{0,j})_{\Omega_{s,j}} + V_j (\sigma\, \mathrm{grad}\, v_{0,j}, \mathrm{grad}\, v_{0,j})_{\Omega_{s,j}}}$$

cannot be used

From the h-formulation

$$\partial_t (\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega + I_j (\sigma^{-1}\, \mathbf{j}_{s,j}, \mathrm{curl}\, \mathbf{h}_{s,j})_{\Omega_{s,j}} = -V_j$$

$$\partial_t (\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega = \partial_t (\mathbf{b}, \mathbf{h}_{s,j})_\Omega = \partial_t (\mathrm{curl}\, \mathbf{a}, \mathbf{h}_{s,j})_\Omega$$

$$\partial_t (\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega = \partial_t (\mathbf{a}, \mathrm{curl}\, \mathbf{h}_{s,j})_\Omega + \partial_t < \mathbf{n} \times \mathbf{a}, \mathbf{h}_{s,j} >_{\partial\Omega}$$

$$\partial_t (\mu\, \mathbf{h}, \mathbf{h}_{s,j})_\Omega = \partial_t (\mathbf{a}, \mathbf{j}_{s,j})_\Omega = \partial_t (\mathbf{a}, \mathbf{j}_{s,j})_{\Omega_{s,j}}$$

$$\boxed{\partial_t (\mathbf{a}, \mathbf{j}_{s,j})_{\Omega_{s,j}} + I_j (\sigma^{-1}\, \mathbf{j}_{s,j}, \mathbf{j}_{s,j})_{\Omega_{s,j}} = -V_j}$$

Weak circuit relation between V_j and I_j for massive inductor j

Equivalent current density (1)

Explicit distribution of the current density

$$\mathbf{j}_s = \frac{N_j}{S_j}\mathbf{t} \stackrel{\text{def}}{=} \mathbf{w} = \mathbf{w}\, I_{unit}$$

$$\partial_t \int_{\Omega_s} \mathbf{a}\cdot\mathbf{w}\; d\Omega_s + R\, I_j = -V_j$$

Equivalent current density (2)

Source electric scalar potential

Projection method

Source

$$(\sigma \operatorname{grad} v_0, \operatorname{grad} v')_{\Omega_{s,j}} = (-\mathbf{j}_{s0}, \operatorname{grad} v')_{\Omega_{s,j}}$$

$$\forall v' \in F_v(\Omega_{s,j})$$

Electrokinetic problem

Source

$$(\sigma \operatorname{grad} v_0, \operatorname{grad} v')_{\Omega_{s,j}} = \langle \mathbf{n} \cdot \mathbf{j}_s, v' \rangle_{\Gamma_{J,j}}$$

$$\forall v' \in F_v(\Omega_{s,j})$$

Tensorial conductivity

Source electric vector potential

Projection method

Source

$$(\operatorname{curl} \mathbf{h}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}} = (\mathbf{j}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}}$$

$$\forall \mathbf{h}' \in F_h(\Omega_{s,j})$$

Electrokinetic problem

$$(\sigma^{-1} \operatorname{curl} \mathbf{h}_{s,j}, \operatorname{curl} \mathbf{h}')_{\Omega_{s,j}} = 0$$

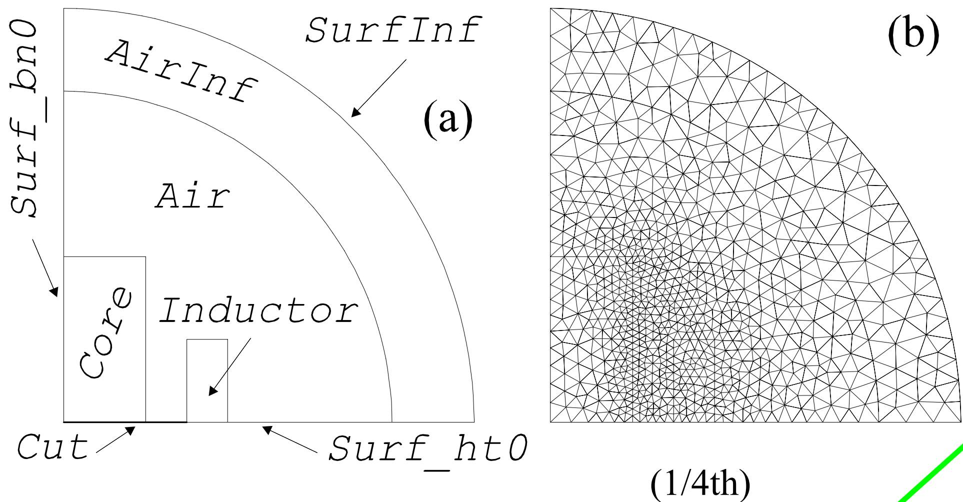
$$\forall \mathbf{h}' \in F_h(\Omega_{s,j})$$

Source = N_j

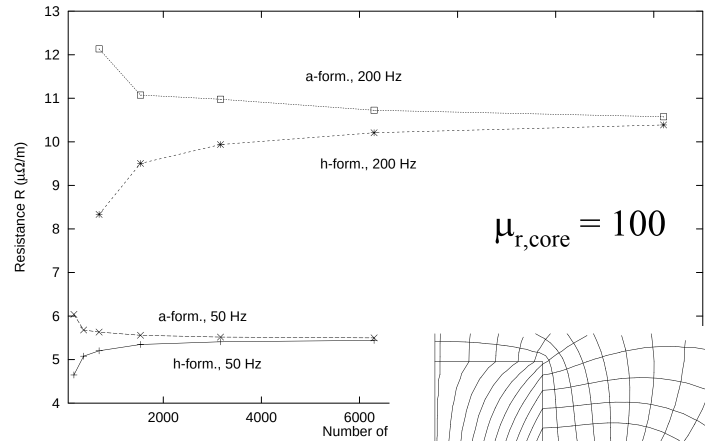
With gauge condition (tree) & boundary conditions

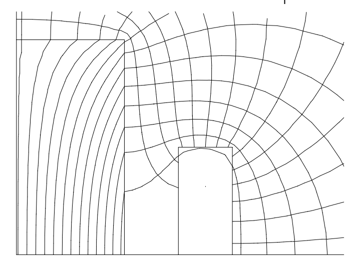
Application - Massive inductor

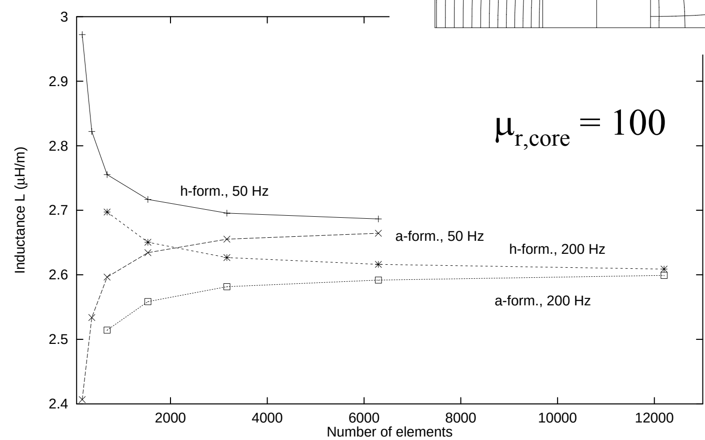
Inductor-Core system in air



$\mu_{r,core}$ = 1, 10, 100 , σ = 5.9 10^7m S/m
Frequency f = 50, 200 Hz







Computation of resistance and inducance

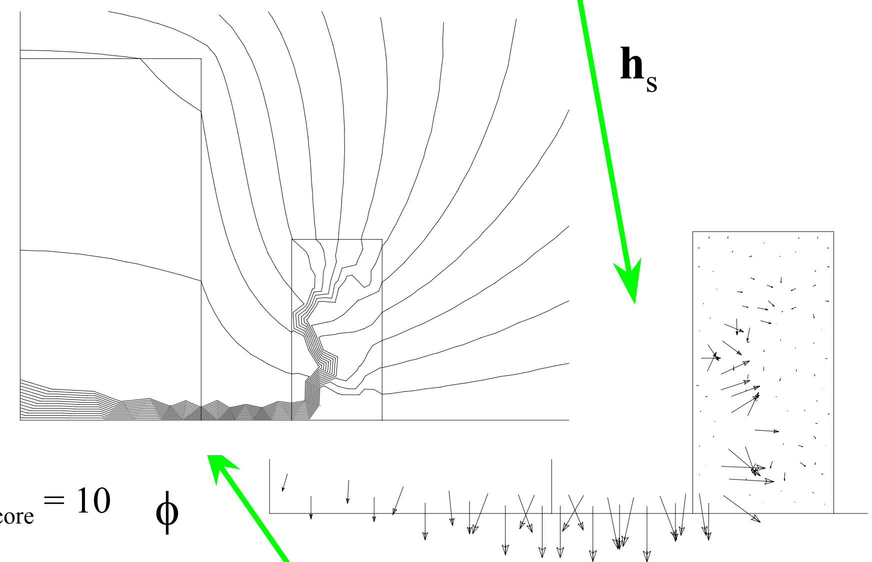
Complementarity between a-v and h-ϕ formulations $\rightarrow$ validation at global level

Application - Stranded inductor

Inductor-Core system in air

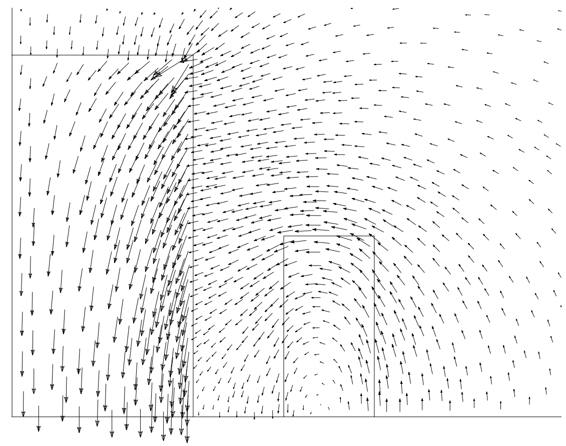
Computation of a source field

$\mathbf{h}_s$

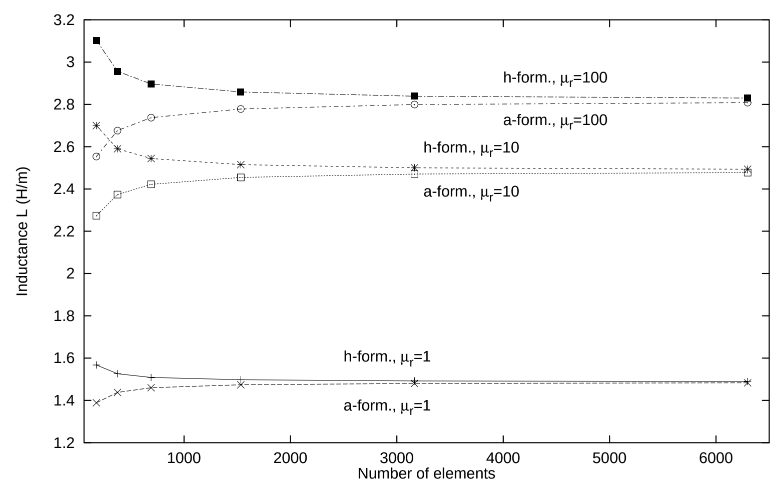


$\mu_{r,core} = 10$ ϕ

$\mu_{r,core} = 10$



$\mathbf{h}$

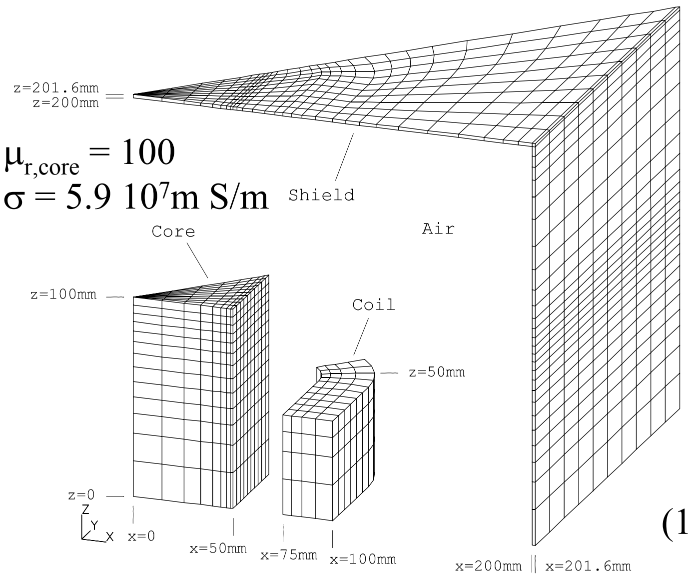


Computation of reaction field, total field and inducance

Complementarity between h-ϕ and a-v formulations $\rightarrow$ validation at global level

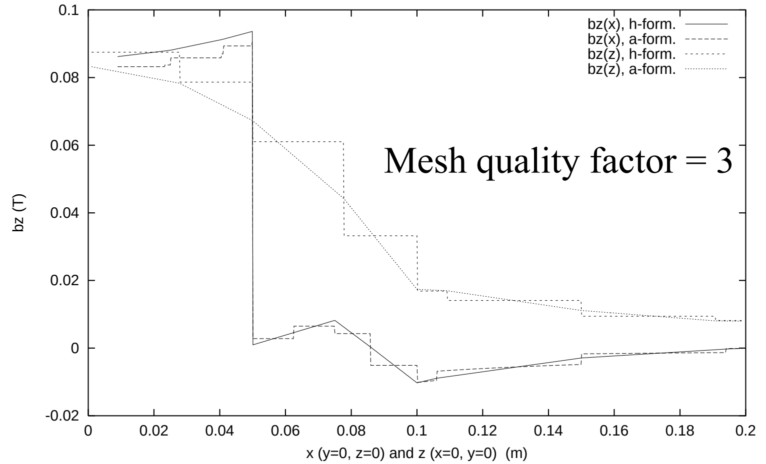
Application

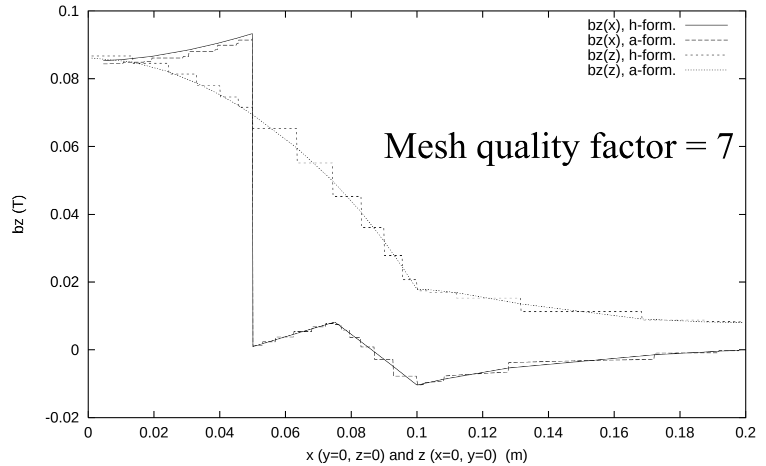
Inductor-Core system in air



$\mu_{r,core} = 100$
$\sigma = 5.9\ 10^7 m\ S/m$

(1/16th)





Enforcement of the current I_j

1 A

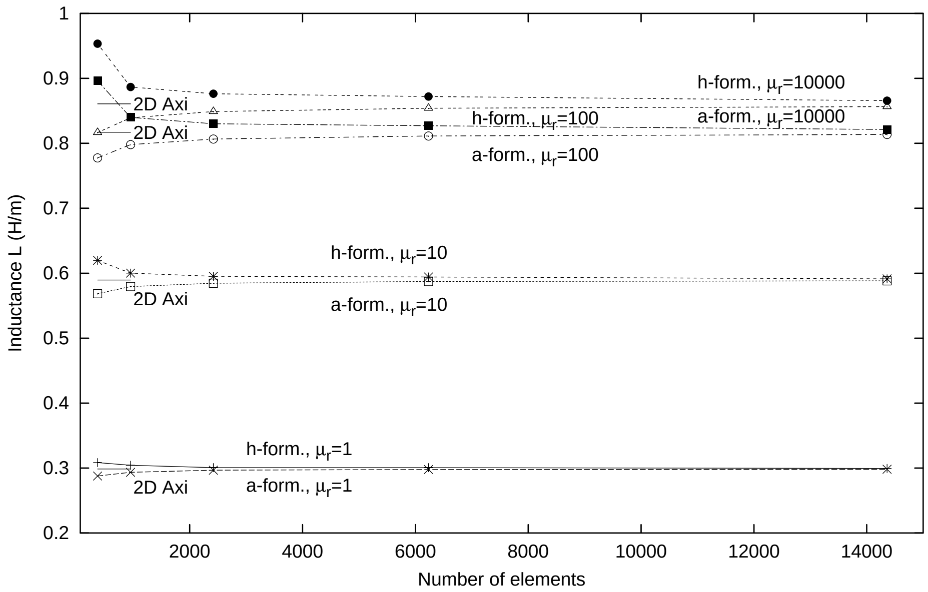
Complementarity between h-ϕ and a-v formulations → validation at local level

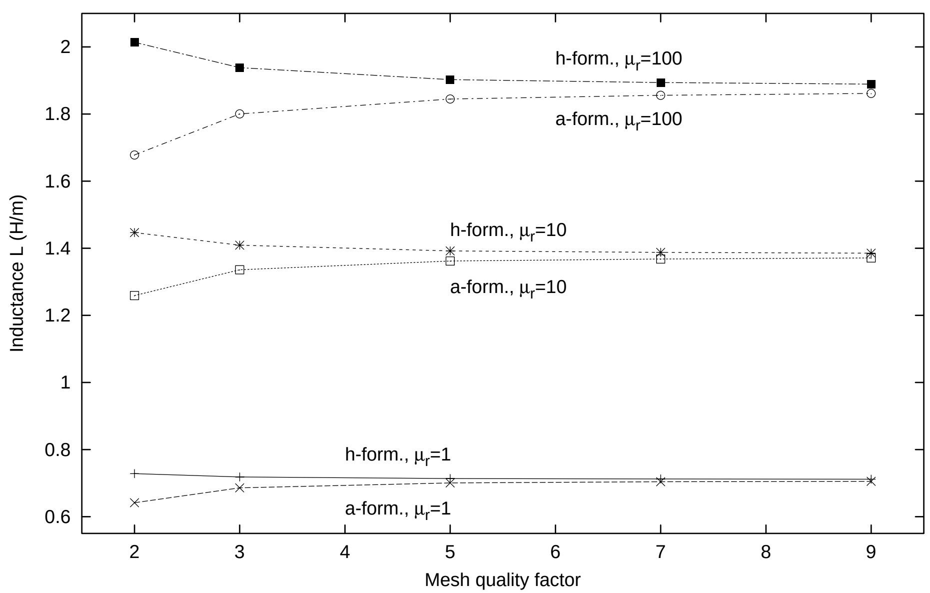
Application

Inductor-Core system in air

Computation of the inductance

3D coil





Axisymmetrical coil

Complementarity between a-v and h-ϕ formulations $\rightarrow$ validation at global level

Foil winding circuit relations

Circuit relation for one foil

$$\frac{N_f}{L_\alpha} I_i = (\sigma \partial_t \mathbf{a}, \mathrm{grad}\, v_{s,i})_{\Omega_{\beta\gamma}} + V_i (\sigma\, \mathrm{grad}\, v_{s,i}, \mathrm{grad}\, v_{s,i})_{\Omega_{\beta\gamma}}$$

\# turns

Total thickness of all the foils

Voltage

Domain of the foil

$$V_i = V_i(\alpha)$$

Continuum for $V_i(\alpha)$ and weak form of circuit relation

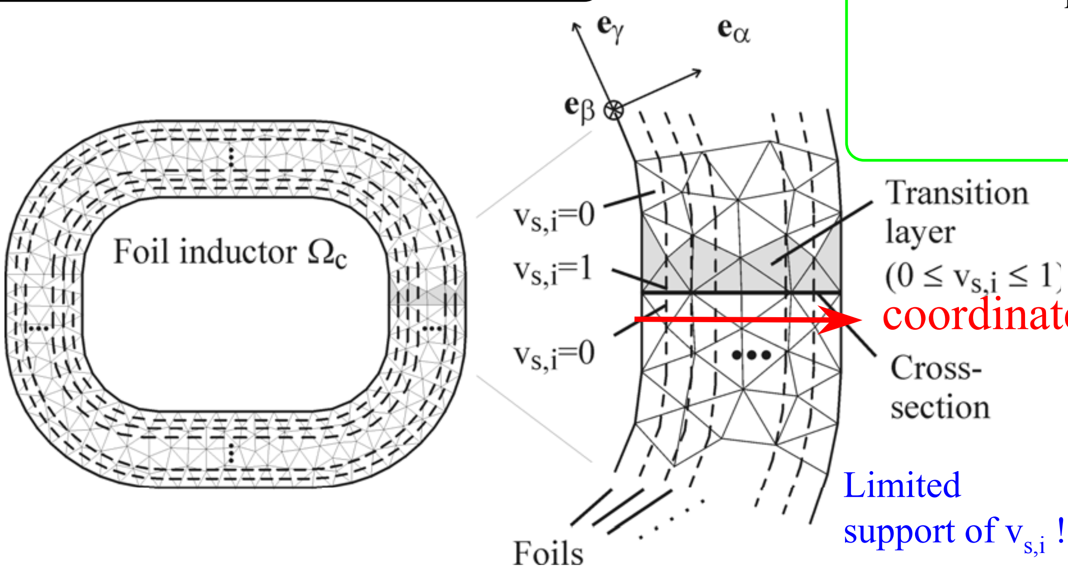
$$-\frac{N_f}{L_\alpha} I_i \int_{\Omega_\alpha} V'(\alpha)\, d\alpha + (\sigma \partial_t \mathbf{a}, V'(\alpha)\, \mathrm{grad}\, v_{s,i})_{\Omega_{\alpha\beta\gamma}}$$

$$+ V_i(\alpha) (\sigma\, \mathrm{grad}\, v_{s,i}, V'(\alpha)\, \mathrm{grad}\, v_{s,i})_{\Omega_{\alpha\beta\gamma}} = 0$$

$$\forall\, V'(\alpha) \in \mathrm{IR}([0, L_\alpha])$$

Whole foil region

Foil winding and its continuous representation

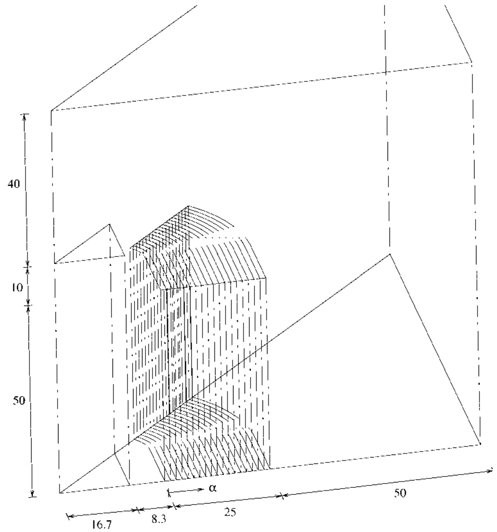


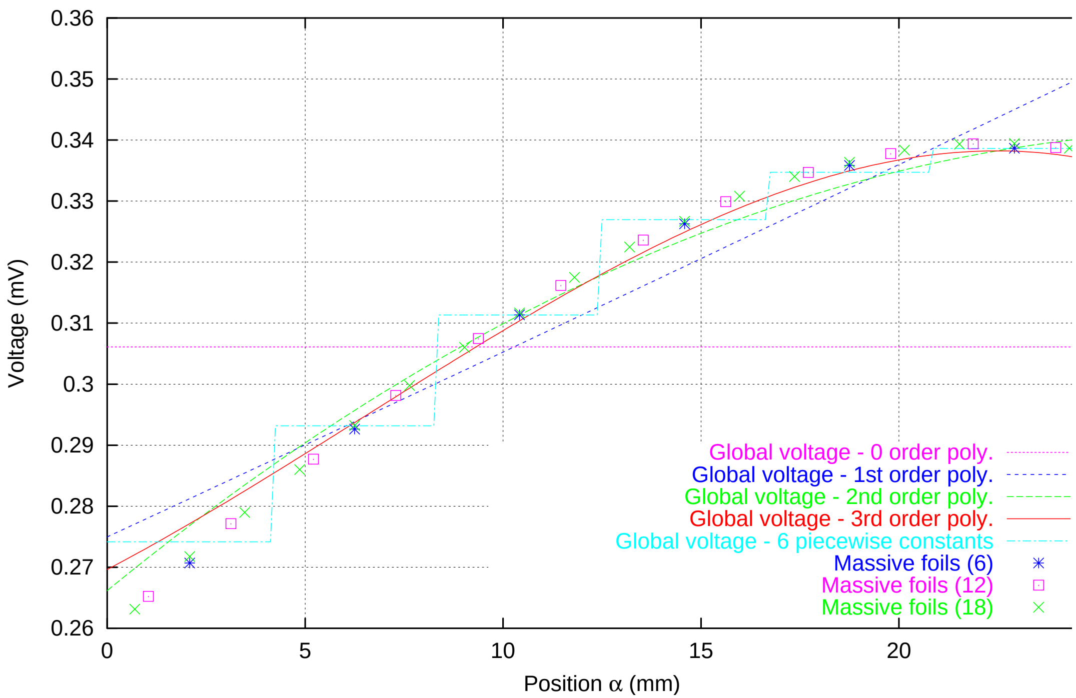
Need of basis functions for $V_i(\alpha)$ (polynomial or 1D finite element approximations)

! Anisotropic conductivity !

Spatially dependent global quantities

Foil Inductor-Core system in air





Voltage of the foils in an n-foil 3D winding (n = 6, 12, 18) and its continuum in the associated foil region approximated by complete and piecewise polynomials

Conclusions - Global quantities

- **General method for the definition of global quantities**
  - natural coupling between local quantities (scalar and vector fields) and global quantities (flux and circulation)
  - for various formulations of various physical problems
  - for all kinds of geometrical models (2D, 3D)
  - for linear or nonlinear material characteristics
  - for various finite elements (geometry and degree)
- **For efficient treatment of coupled problems**
  - within a finite element problem
  - through external lumped circuits

Conclusions - h-ϕ formulation

- h-ϕ magnetodynamic finite element formulations with massive and stranded inductors
- Use of edge and nodal finite elements for h and ϕ
  - Natural coupling between h and ϕ
  - Definition of current in a strong sense with basis functions either for massive or stranded inductors
  - Definition of voltage in a weak sense
  - Natural coupling between fields, currents and voltages

Conclusions - a-v formulation

- a-v_0 Magnetodynamic finite element formulation with massive and stranded inductors
- Use of edge and nodal finite elements for a and v_0
  - Definition of a source electric scalar potential v_0 in massive inductors in an efficient way (limited support)
  - Natural coupling between a and v_0 for massive inductors
  - Adaptation for stranded inductors: several methods
  - Natural coupling between local and global quantities, i.e. fields and currents and voltages