Propagation of acoustic waves in fractal networks

Adrien SEMIN
Joint work with Patrick JOLY

MFO, 16 February 2009
Outline

- Introduction
- Equivalent boundary conditions
  - Notations
  - Helmholtz problem
  - Time-domain problem
- Numerical results
- Conclusion
Introduction

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These will permit us to construct equivalent boundary conditions.

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- Exploit self-similarity to construct equivalent boundary conditions (object of this talk)


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  \[
  e_{0,0} = \sum \\
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Definition of a self-similar $p$-adyc tree

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  \end{align*}$
- We define the generation $G_n$ as
  
  $G_n = \bigcup_{j=0}^{p^n-1} e_{n,j}$

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- We define the generation $G_n$ as
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  G_n = \bigcup_{j=0}^{p^n-1} e_{n,j}
  \]
- We define the partial tree $T_n$ as
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  T_n = \bigcup_{m=0}^{n} G_m
  \]
- Finally, we define the $p$-adyce tree $T$ as the limit of $T_n$ when $n$ tends to $\infty$. 

Example with $p=2$. 

\begin{itemize}
  \item [ ] \text{G}_0
  \item [ ] \text{e}_{0,0}
  \item [ ] \text{G}_1
  \item [ ] \text{e}_{1,0}
  \item [ ] \text{e}_{1,1}
  \item [ ] \text{G}_2
  \item [ ] \text{e}_{2,0}
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  \item [ ] \text{e}_{2,2}
  \item [ ] \text{e}_{2,3}
\end{itemize}
A model Helmholtz equation on $\mathcal{T}$

- On this domain, we define a weight $\mu : \mathcal{T} \rightarrow \mathbb{R}_+$ piecewise constant such that:

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\[ 
\begin{array}{c}
\text{Example with } p=2. \\
0
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For the $d$-geometric tree, we take $\mu_k = \alpha_k^{d-1}$.
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On tree $\mathcal{T}$, we want to solve the Helmholtz equation:

$$(\mu u')' + \omega^2 \mu u = 0 \quad \text{on} \quad \mathcal{T}$$

$u(0) = 1$

$Bu = 0 \quad \text{at infinity}$

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- The previous equation gives implicitly:
  \[
  u'' + \omega^2 u = 0 \quad \text{on each} \quad e_{n,j} \\
  u \quad \text{continuous at node} \quad M_{n,j} \\
  u'_{n,j}(M_{n,j}) = \sum_{k=0}^{p-1} \mu_{k+1}u'_{n+1,pj+k}(M_{n,j})
  \]

Example with $p=2$. 

On each $e_{n,j}$, $\mu$ is continuous at node $M_{n,j}$.
The non-classical condition $Bu=0$ represents a homogeneous Dirichlet or Neumann condition at «infinity» whose sense is given through a weak formulation of the problem.

This requires a functional framework.
Conditions at infinity

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- On tree $T$, we define the following weighted «broken» norms (depending on choose of weight $\mu$, i.e. depending on $(\mu_i)_{1 \leq i \leq p}$):

  \[
  \|u\|_{L^2_{\mu}(T)}^2 = \sum_{n \geq 0} \sum_{j=0}^{p^n-1} \mu_{n,j} \|u\|_{L^2(e_{n,j})}^2
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  \|u\|_{H^1_{\mu}(T)}^2 = \sum_{n \geq 0} \sum_{j=0}^{p^n-1} \mu_{n,j} \|u'\|_{L^2(e_{n,j})}^2
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  \|u\|_{H^1_{\mu}(T)}^2 = \|u\|_{L^2_{\mu}(T)}^2 + \|u\|_{H^1_{\mu}(T)}^2
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Moreover, we define the weighted Besov spaces $\mathcal{H}^1_{\mu}(\mathcal{T})$ and $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ as:

$$
\mathcal{H}^1_{\mu}(\mathcal{T}) = \left\{ v \text{ continuous such that } |v(0)|^2 + |v|^2_{H^1_{\mu}} < \infty \right\}
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\mathcal{H}^1_{\mu,0}(\mathcal{T}) = \text{closure of } \left\{ v \in \mathcal{H}^1_{\mu}(\mathcal{T}) \text{ such that } \exists n, v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}_n \right\}
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- We also define the Sobolev spaces $H^1_{\mu}(\mathcal{T})$ and $H^1_{\mu,0}(\mathcal{T})$. 
We denote the variational formulation of the Neumann problem as follow:

find \( u \in H^1_\mu(\mathcal{T}) \) such that \( u(0)=1 \) and

\[
\int_\mathcal{T} \mu u' v' - \omega^2 \int_\mathcal{T} \mu uv = 0, \quad \forall v \in H^1_\mu(\mathcal{T}) \text{ such that } v(0) = 0
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Weak formulations of the Helmholtz problems

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\]

Proposition (existence and uniqueness).

For any frequency $\omega$ complex, Neumann and Dirichlet problems have a unique solution, called respectively $u_n(\omega, \cdot)$ and $u_d(\omega, \cdot)$.
We denote the variational formulation of the Neumann problem as follow: find $u \in H^1_\mu(T)$ such that $u(0)=1$ and

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**Theorem (Joly-Semin)**

We have the following equivalence:

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**Definition (\( \Lambda \))**

One defines the following two functions (depending of \( \omega \)):

\[
\Lambda_n(\omega) = u'_n(\omega,0) \\
\Lambda_\delta(\omega) = u'_\delta(\omega,0)
\]

- Propagation of acoustic waves in fractal networks
Construction of impedance condition

Assume that we know \( \Lambda \). If we cut on the first generation, one can replace resolution of Helmholtz equation on each subtree (red and blue on figure) by DtN conditions obtained on a \( p \)-adye tree of main length \( \alpha_i \):

\[
 u' = \alpha_i^{-1} \Lambda(\alpha_i \omega) u
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$$u' = \alpha_i^{-1} \Lambda(\alpha_i \omega) u$$

If we cut further (on the $n^{th}$ generation), one can use the DtN conditions obtained on each $p$-adic tree of main length $\ell_{n,j}$:

$$u' = \ell_{n,j}^{-1} \Lambda(\ell_{n,j} \omega) u$$
Assume that we know \( \Lambda \). If we cut on the first generation, one can replace resolution of Helmholtz equation on each subtree (red and blue on figure) by DtN conditions obtained on a \( p \)-adyc tree of main length \( \alpha_i \):

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u' = \ell_{n,j}^{-1} \Lambda(\ell_{n,j} \omega) u
\]

Since \( \ell_{n,j} \leq (\sup \alpha_i)^n \), it is sufficient to have a good approximation of \( \Lambda(\omega) \) for small \( \omega \).

---

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By a scaling argument, one has

$$u_i(\omega, s_i(x)) = u(\omega, 1) \cdot u(\alpha_i \omega, x)$$
We define the function $u_i$ as the restriction of $u$ to subtree $T_i$.

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Differentiating in $x$ gives

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Differentiating in $x$ gives

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Taking $x=0$ gives

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Taking \( x = 0 \) gives
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 \alpha_i u_i'(\omega, 1) = u(\omega, 1) \Lambda(\alpha_i \omega)
\]

Along the first branch:
\[
 u(\omega, x) = \cos(\omega x) + \frac{\Lambda(\omega)}{\omega} \sin(\omega x)
\]
We define the function $u_i$ as the restriction of $u$ to subtree $\mathcal{T}^i$.

By a scaling argument, one has

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Along the first branch:

$$u(\omega, x) = \cos(\omega x) + \frac{\Lambda(\omega)}{\omega} \sin(\omega x)$$

which gives

$$u(\omega, 1) = \cos(\omega) + \frac{\Lambda(\omega)}{\omega} \sin(\omega)$$

$$u'(\omega, 1) = \Lambda(\omega) \cos(\omega) - \omega \sin(\omega)$$
Functional relation on $\Lambda$

\[ \alpha_i u'_i(\omega, 1) = u(\omega, 1) \Lambda(\alpha_i \omega) \]

\[ u(\omega, 1) = \cos(\omega) + \frac{\Lambda(\omega)}{\omega} \sin(\omega) \]

\[ u'(\omega, 1) = \Lambda(\omega) \cos(\omega) - \omega \sin(\omega) \]
The node condition at $x=1$ writes

$$u'(\omega, 1) = \sum_{i=1}^{p} \mu_i u'_i(\omega, 1)$$
Functional relation on $\Lambda$

\[
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\]

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Conjecture

There exists two meromorphic functions satisfying this quadratic relation, and unicity is given knowing $\Lambda(0)$. 
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Theorem (existence and uniqueness).
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- Taking $\omega=0$ in $P$ leads to
  \[P_0, \quad \Lambda(0) = (1 + \Lambda(0)) \sum_{i=1}^{p} \frac{\mu_i \Lambda(0)}{\alpha_i}\]
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**Theorem (value of \( u'(0) \))**

For the Neumann problem, one gets:

\[
\Lambda_n = 0
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For the Dirichlet problem, one gets:

\[
\Lambda_0 = 1 - \sum_{i=1}^{p} \frac{\mu_i}{\alpha_i}
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For $\omega$ small, one has (using that the Helmholtz equation is even w.r.t $\omega$):

$$\Lambda(\omega) = \lambda_0 + \lambda_2 \omega^2 + O(\omega^4)$$
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- For ensuring stability, one would like $\lambda_0 \leq 0$ and $\lambda_2 \geq 0$. 

Adrien SEMIN
Propagation of acoustic waves in fractal networks

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE
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This condition means that $L^2$-norm of constant function is finite.
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Neumann case ($d$-geometric tree)

$\lambda_2 = \frac{1}{1 - \sum_{i=1}^{p} \alpha_i^d}$

Stability of Neumann case ($d$-geometric tree)

$\lambda_2 > 0$ if the Hausdorff dimension of the tree is strictly lesser than $d$.

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\lambda_0 &= \frac{1 - \sum_{i=1}^{p} \alpha_i^{d-2}}{\sum_{i=1}^{p} \alpha_i^{d-2}} \\
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\end{align*}
\]
Numerical simulations (Dirichlet problem)

- We solve the time-domain wave equation on $\mathcal{T}_n$ for different values of $n$, with outgoing condition at entrance of the tree, and various conditions at the end of the tree:

\[
\begin{align*}
  u &= 0 \quad \text{(Dirichlet condition)} \\
  \frac{\partial u}{\partial n} &= \lambda_0 u \quad \text{(First order impedance condition)} \\
  \frac{\partial u}{\partial n} &= \lambda_0 u - \lambda_2 \frac{\partial^2 u}{\partial t^2} \quad \text{(Second order impedance condition)}
\end{align*}
\]
Dyadic symmetric tree, $\alpha_1 = \alpha_2 = 0.6$, angle between slots is taken $\pi/2$ for convenience, $d=2$.

Cauchy data is a Gaussian spreading right.

A constant spacestep $h$ along the tree.

“Exact” computation is done with taking 20 generations and Dirichlet condition.

We look at the value of the solution (as a function of time) in the middle of the longest slot.
Computation done with 7 generations (left) and 9 generations (right).

Condition used at boundary is Dirichlet condition.

Exact solution is red lined, approximated solution is blue dotted.
Computation done with 7 generations (left) and 9 generations (right).

Condition used at boundary is first order impedance condition.

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Computation done with 7 generations (left) and 9 generations (right).

Condition used at boundary is second order impedance condition.

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### Numerical simulations - error estimates

<table>
<thead>
<tr>
<th>Number of generations</th>
<th>Dirichlet condition</th>
<th>First order imp. condition</th>
<th>Second order imp. condition</th>
<th>Gain with first order</th>
<th>Gain with second order</th>
<th>Number of d.o.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.429</td>
<td>0.320</td>
<td>0.123</td>
<td>1.34</td>
<td>3.05</td>
<td>198598</td>
</tr>
<tr>
<td>6</td>
<td>0.370</td>
<td>0.205</td>
<td>0.050</td>
<td>1.80</td>
<td>7.35</td>
<td>258318</td>
</tr>
<tr>
<td>7</td>
<td>0.217</td>
<td>0.075</td>
<td>0.028</td>
<td>2.89</td>
<td>7.63</td>
<td>329982</td>
</tr>
<tr>
<td>8</td>
<td>0.083</td>
<td>0.018</td>
<td>0.013</td>
<td>4.53</td>
<td>6.45</td>
<td>415978</td>
</tr>
<tr>
<td>9</td>
<td>0.023</td>
<td>0.0031</td>
<td>0.0028</td>
<td>7.47</td>
<td>8.09</td>
<td>519174</td>
</tr>
</tbody>
</table>

$L^2$-error between sismograph of exact solution and sismigraph of approximated solution, with respect of number of generations and condition at boundary of the tree. Number of d.o.f. of exact simulation is $\approx 4.5 \times 10^6$.

Green values: round problems with double precision programs. Actually testing with quadruple precision programs.
Conclusion and perspectives

- Results are validated in case of symmetric tree, are in progress when $\mu_1 \neq \mu_2$.
- Need to compute poles and zeros of function $\Lambda$ and establish their relationship with eigenvalues of Laplacian operator on the tree,
- Need to prove that we are able to take some Taylor development of $\Lambda(\omega)$ with respect to $\omega=0$. A priori, it is not possible when $\mathcal{H}_\mu^1(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$ (this intuition is guided by the behaviour of the Sobolev spaces when we take a monoadyc tree, i.e. $p=1$).
Thank you for your attention