Bachelor Thesis Project/ Term Project

(Mathematics, Computational Science & Engineering)

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PUM Based Wave Ray Multigrid

Prerequisites. Knowledge about numerical methods for partial differential equations. Familiarity with multigrid methods is advantageous but not essential.

Problem description. We consider the following Helmholtz boundary value problem posed on a two-dimensional domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} \Delta u + k^2 &= 0 \quad \text{in } \Omega \,, \\ \frac{\partial u}{\partial \boldsymbol{n}} - iku &= g \quad \text{on } \Gamma_R \,, \\ u &= 0 \quad \text{on } \Gamma_D \,. \end{aligned} \tag{1}$$

Here, $k > 0$ is the wavenumber, and Γ_D and Γ_R are two well separated parts of the boundary, $\Gamma_R \cup \Gamma_D = \partial\Omega$, see Figure 1. The boundary value problem (1) has a variational formulation in $H^1_{\Gamma_D}(\Omega)$: $\mathsf{a}(u, v) = \ell(v)$ for all $v \in H^1_{\Gamma_D}(\Omega)$.

Discretization on a fine triangular mesh $\mathcal{T}_L$ is based on piecewise linear Lagrangian finite elements (space $\mathcal{S}^0_1(\mathcal{T}_L)$ with zero boundary conditions on Γ_D). Further we assume a hierarchy of nested meshes $\mathcal{T}_0 \prec \mathcal{T}_1 \prec \cdots \prec \mathcal{T}_L$ created by uniform, regular refinement.

Let $\mathcal{V}_l$ denote the set of vertices of $\mathcal{T}_l$ not lying on Γ_D. Write $b^l_{\boldsymbol{p}}$ for the piecewise linear nodal basis function ("tent function") associated with vertex $\boldsymbol{p} \in \mathcal{V}_l$. Write

$$\boldsymbol{d}^m_k = \begin{pmatrix} \cos \frac{2\pi}{m} k \\ \sin \frac{2\pi}{m} k \end{pmatrix} , \quad k = 0, \dots, m-1 \,, \tag{2}$$

and define the wave modulated partition of unity space according to

$$\begin{aligned} W_L &:= \mathcal{S}^0_1(\mathcal{T}_L) \,, \\ W_l &:= \text{span}\left\{ b^l_{\boldsymbol{p}}(\boldsymbol{x}) \exp(ik\boldsymbol{d}^{2^{L+1-l}}_k \cdot \boldsymbol{x}) : \ \boldsymbol{p} \in \mathcal{V}_l,\ k = 0, \dots, 2^{L+1-l} - 1 \right\} \,, \\ l &= 0, \dots, L-1 \,. \end{aligned} \tag{3}$$

The following two level algorithm (correction scheme) is proposed for solving the variational problem on W_l, $l \geq 1$:

1. Conduct a directional Gauss-Seidel relaxation.
2. Solve the residual equation on W_{l-1} to obtain a correction.

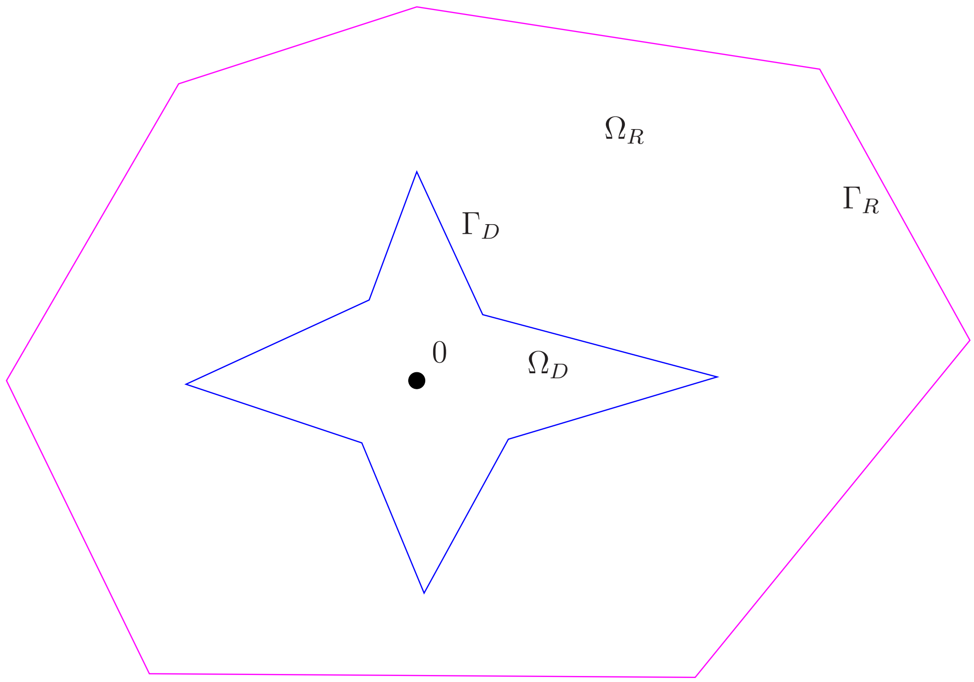

Figure 1: Geometric setting for scattering problem

3. Perform another directional Gauss-Seidel relaxation.

The directional Gauss-Seidel relaxation is based on the following ordering of the degrees of freedom:

- D.o.f. are first ordered according to the direction $\boldsymbol{d}$ they are associated with
- D.o.d. belonging to the same direction are partially ordered in *upstream fashion* with respect to that direction.

Issues. The spaces W_l are not nested, which entails finding a suitable approximate transfer of residuals and corrections. The ideas is to employ "adjacent directions".

Tasks. Implementation of the PUM based wave-ray multigrid algorithm in the framework of the MATLAB finite element library "LehrFEM", which is also used in courses on numerical methods for partial differential equations. The required data structures are all available in LehrFEM.

A detailed analysis of the complexity of the implementation should be carried out. Then the code should be used to test the convergence of the algorithm in various settings.

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